Performance bounds for feedback control of non-minimum phase MIMO systems with arbitrary delay structure

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Abstract

This paper derives 2-norm performance bounds for the feedback control of discrete-time MIMO non-minimum phase (NMP) systems with arbitrary delay structure. Also, the associated optimal controller, in Youla parameterization form, is explicitly obtained. The derivation of those results uses a special interactor matrix to extract the delays. It is shown that this interactor is unique and a building algorithm is also proposed.

1 Introduction

The study of fundamental limitations and performance bounds in control systems has been an important research topic since Bode's pioneer work (see, for example, [1], [2], [3] and [4]). The study of those topics is justified since they can provide answers to the following fundamental questions:

- What limitations are inescapable to all possible linear control system designs for a given linear model?
- What is the best achievable performance considering some specific class of controllers and performance indexes?

The significance of these questions is manyfold. One not so evident issue is that the answer to them provide a benchmark against which, for instance, nonlinear control performance can be compared and measured.

Answers to the first question have more than sixty years and were first introduced by Bode's sensitivity integrals. In that work it was shown that, for a given continuous time stable SISO plant, there is a trade-off between achieving small sensitivity in different frequency bands. Those results have

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been extended to the unstable and NMP SISO case [5] and to the discrete time case [6]. They rely entirely upon the analytic properties of the sensitivity and complementary sensitivity functions and they are expressed using Poisson or Cauchy type integrals. Those results state that delays, unstable poles and NMP zeros are sources of limitations. Extensions of those limitations to the MIMO case can be found in [7] and [3]. In those works it is found that, in the MIMO case, not only the unstable poles and NMP zeros are relevant to the limitations, but also their directions. In a later work [4] those results were complemented with the computation of bounds for the sensitivity and the complementary sensitivity peaks, which are specially relevant for robust control design.

In the above mentioned work the negative effect that unstable poles and NMP zeros have in the performance of the control system is recurrent. However, those results cannot be used to compute the best achievable performance for the control of a given process model. In other words, they define what cannot be achieved, but they can not be used to determine the best (in some appropriate sense) controller for a given plant.

Results related to the second question (performance bounds) are more recent and the research has followed two different although connected paths: the *deterministic* and the *stochastic* approaches.

We will firstly review the advances in the *deterministic* framework.

Many of the results available in the deterministic case use a 2-norm based performance index and, therefore, they admit time domain interpretation using Parseval's relations. Abundant references can be found in [1] and [8] for the SISO case. An advantage of this approach is that it can be linked to optimal synthesis techniques and hence, they can be used to obtain a controller capable of achieving that *best* performance [2].

Results for the MIMO case have been presented in [9], where the best achievable performance for a continuous time MIMO process is evaluated using a quadratic performance index. The case of one and two degrees of freedom controllers are considered there. Again, the best performance is limited by the unstable poles, NMP zeros and the delays of the system. It is worth noting that in [9] only very special structures of time delays are considered. Namely, only measurement delays. This particular structure is not flexible enough to encompass multivariable models with delayed interactions which are frequent in industrial processes.

In [10] performance limitations are computed introducing an explicit control effort penalization in one-degree-of-freedom control architectures. This leads to the conclusion that, in this case, not only unstable poles, NMP zeros and delays limit the achievable performance, but also the resonant poles and other plant features. In [11] those results are extended to two-degree-of-freedom control architectures.

Performance limitations for discrete time MIMO plant have been considered in [12], where a natural extension of [9] is presented. Again, only very restrictive delay structures are explicitly considered.

Comprehensive surveys of the research using an stochastic setting can be found in [13] and [14]. Most of the work reviewed in those references is based upon the research reported in [15]. In the latter work, focused on SISO systems, minimum variance control (MVC) [16] is used as a benchmark and closed loop data is used to evaluate loop performance. Several indexes have been proposed to compare the performance of a given control loop to the idealized minimum variance strategy [17] [18] [19] [20]. The relevance of those indexes is that they can be estimated online using, in many cases, only normal operating data plus knowledge of the delay of the system [13], [14]. The results in the references cited above (with the exception of the compilation work [13] [14]) are restricted to SISO systems.

A key feature of the MVC-based approach is that classical MVC [16] can not stabilize NMP plants [21], [13], [14] and therefore, the idealized minimum variance performance measure is an unachievable lower bound for the control of NMP plants. This is highly significant since the difference between the idealized minimum variance performance and the achievable variance may be very large for processes with NMP zeros near $(1,0)$ [22]. This originates in the classical design tradeoff due to NMP zeros [2], which gets worse as the zeros get closer to $(1,0)$.

In the MIMO case, minimum variance performance bounds require knowledge of an interactor matrix of the process [13] [14] which captures the system delay structure. Once this matrix has been computed, one can extend SISO performance measures to the MIMO case as proposed, for instance, in [13], in [14] and in the references therein. It is worth pointing out that the interactor matrix is highly non-unique as discussed, for example, in [23]. Therefore, the selected structure for it may have significant impact on the computation of the performance bounds depending on the metrics being used. In particular, the unitary interactor introduced in [24] is useful when considering quadratic performance measures, since it allows to compute the best performance bound [25] [26].

In the MVC-based approach, significant work is devoted to the identification of the interactor matrix and the estimation of the performance bound using closed loop data. However no achievable bound, in closed form, is computed for systems with NMP zeros. Also, to the best of the authors' knowledge, the MVC based strategy to asses and to monitor performance does not deliver a controller in closed form. This feature limits the designer's ability to analytically asses the effects of plant features such as NMP zeros location, delay structure and signal directionality.

The results presented in this paper are based upon a special case of the interactor matrix [24],[27],[28],[2] although the proofs proceed along different paths to those of the cited works.

A this point of the discussion it should be made clear that the two mentioned approaches mostly encountered in the literature regarding performance limitations are not completely separated ones. As a matter of fact, the SISO minimum variance control problem considering an ARIMA(n,1,p) model for the disturbance can be stated as the problem of finding a stabilizing controller that achieves the minimum 2 norm of the (weighted) error due to a step change in the reference, and therefore *the solution of both problems are closely related.*

In this paper we follow a deterministic approach and the main contributions are:

- A closed form expression for the best **achievable** performance in the linear control of a stable NMP discrete time MIMO plant with an **arbitrary delay structure**, i.e. where delays appear not only in the measurements but also in the channel interactions.
- A closed form expression for the controller which achieves that performance bound.
- The description of the above mentioned performance bound and controller in a way that highlights the interplay between performance, NMP zeros, delays and directionality.
- Technical issues regarding uniqueness and building of a class of interactors

The paper is organized as follows: in section §2 we recall some definitions and we review basic ideas. Section §3 describes useful zero factorizations and defines unitary interactors. Those interactors are the key to derive the main results. The most important result of this section is that there is a unique unitary interactor with unity DC-gain, for which a simple way to compute is proposed and illustrated through an example. In section §4 we compute performance bounds in the case of stable, discrete-time MIMO plants with arbitrary delay structures. Finally, section §5 presents some concluding remarks.

2 Definitions

This section introduces basic definitions and the notation used throughout the paper. For any complex number z, $\bar{z}$ represents its conjugate. Given $v \in \mathbb{C}^{n\times 1}$, v^H denotes its conjugate transpose; for

$W \in \mathbb{C}^{n \times n}$, its conjugate transposed (hermitian) matrix is defined as W^H and for a rational transfer matrix $M(z) \in \mathbb{C}^{n \times n}$ we define the operation $(\cdot)^{\sim}$ as

$$M^{\sim}(z) = M^H \left(\frac{1}{\bar{z}} \right) \tag{1}$$

which reduces to

$$M^{\sim}(z) = M^T \left(\frac{1}{\bar{z}} \right) \tag{2}$$

in the real rational case, i.e. when $M(z) \in \mathbb{R}^{n \times n}$, $\forall z \in \mathbb{R}$. Note that in either case, $(\cdot)^{\sim}$ reduces to $(\cdot)^H$ when $z = e^{j\omega}$.

We say that a rational transfer matrix $M(z) \in \mathbb{C}^{n \times n}$ is unitary if and only if

$$M^{\sim}(z) M(z) = I \tag{3}$$

Note that $M(z)$ unitary implies that $M(e^{j\omega})$ is unitary in the traditional sense for all $\omega \in \mathbb{R}$, i.e. $M^H(e^{j\omega}) M(e^{j\omega}) = I$.

$\mathcal{L}_2$ is defined as the Hilbert space of all matrix functions $M(z)$ measurable over the unit circle, i.e. for $|z| = 1$, with inner product defined by

$$\langle M, G \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} trace \left\{ M^H(e^{j\omega}) G(e^{j\omega}) \right\} d\omega \tag{4}$$

for all M and G in $\mathcal{L}_2$. The norm induced by (4) is known as the 2-norm. It can be shown that the subspace $\mathcal{H}_2 \subset \mathcal{L}_2$, of all matrix functions that are analytic outside the unit circle ($|z| > 1$), and the subspace $\mathcal{H}_2^{\perp} \subset \mathcal{L}_2$, of all matrix functions that are analytic inside the unit circle ($|z| < 1$), form an orthogonal subspace pair.

3 Zero factorizations in discrete-time MIMO systems

Zero factorization in MIMO systems can be carried out in several ways using, for example, the so called interactor matrices and z-interactors [29], [2].

Recall that a MIMO system with transfer function $G_o(z)$ has a zero at $z = z_o$ if an only if $G_o(z_o)$ is singular.

In the sequel we will consider the factorization of NMP zeros in discrete time MIMO systems. It is important to note that zeros of a discrete time systems include finite zeros and zeros at infinity. The zeros at infinity describe the relative degree and, hence, they capture the delay structure of the system.

The interactor matrices defined in [28] and [29], for example, have the important property of being unique and triangular, but it can be seen that they are not necessarily unitary (see eg. Example 25.2 in [2]). This makes such interactors unsuitable for the derivations in section §4.

To compute the performance bounds we need two key results. The first one is given in the following lemma.

Lemma 1. *Consider a real rational transfer matrix $G(z) \in \mathbb{C}^{n \times n}$. Suppose that $G(z)$ has a zero at $z = c$ with multiplicity m_c associated to the unitary direction η_c (there may be more zeros at other locations). Define*

$$\hat{G}(z) = L_c(z) G(z) \tag{5}$$

where

$$L_c(z) = \frac{1-c}{1-\bar{c}}\frac{1-\bar{c}z}{z-c}\eta_c\eta_c^H + U_cU_c^H = \begin{bmatrix}\eta_c & U_c\end{bmatrix}\begin{bmatrix}\frac{1-c}{1-\bar{c}}\frac{1-\bar{c}z}{z-c} & 0 \\ 0 & I_{n-1}\end{bmatrix}\begin{bmatrix}\eta_c^H \\ U_c^H\end{bmatrix} \tag{6}$$

and U_c *is chosen so that* $\begin{bmatrix}\eta_c & U_c\end{bmatrix}$ *is unitary. Then,*

1. $\hat{G}(z)$ *has* $m_c - 1$ *zeros at* $z = c$ *associated to the direction* η_c *and, possibly, one additional zero at* $z = \frac{1}{\bar{c}}$ *associated to* η_c*.*
2. *Except for the previous fact,* $\hat{G}(z)$ *shares its poles and zeros with* $G(z)$*, but not necessarily the associated directions.*
3. $L_c(z)$ *is unitary and has unity DC-gain, i.e.* $L_c(1) = I$*.*
4. $L_c(z)$ *is biproper.*

Proof

Please see [12].

□□□

Note that from Lemma 1

$$G(z) = L_c^{-1}(z)\hat{G}(z); \qquad \text{with} \quad L_c^{-1}(z) = \frac{1-\bar{c}}{1-c}\frac{z-c}{1-\bar{c}z}\eta_c\eta_c^H + U_cU_c^H \tag{7}$$

which means that $L_c(z)$ extracts one of the zeros of $G(z)$ in $z = c$ associated to η_c. Furthermore, provided that $\eta_c^H\, G(z)$ is analytical at $z = 1/\bar{c}$, the effect of $L_c(z)$ is to replace one NMP zero at $z = c$ in $G(z)$ by its *stable reflection*.

A second key result is given in Lemma 2. This lemma captures the idea that the delay structure can be visualized as the structure of system zeros at infinity.

Lemma 2. *Suppose that* $G(z)$ *has a zero at infinity with multiplicity* m_∞ *associated to the unitary direction* η_∞ *(note that there may be more zeros at other locations). Define*

$$\tilde{G}(z) = L_\infty(z)G(z) \tag{8}$$

where

$$L_\infty(z) = z\eta_\infty\eta_\infty^T + U_\infty U_\infty^T = \begin{bmatrix}\eta_\infty & U_\infty\end{bmatrix}\begin{bmatrix}z & 0 \\ 0 & I_{n-1}\end{bmatrix}\begin{bmatrix}\eta_\infty^T \\ U_\infty^T\end{bmatrix} \tag{9}$$

and U_∞ *is chosen so that* $\begin{bmatrix}\eta_\infty & U_\infty\end{bmatrix}$ *is unitary. Then,*

1. $\tilde{G}(z)$ *has* $m_\infty - 1$ *zeros at infinity associated to the direction* η_∞ *and, possibly, one zero in* $z = 0$ *associated to* η_∞*.*

2. *Except by the previous fact, $\tilde{G}(z)$ shares its poles and zeros with $G(z)$, but not necessarily their directions.*

3. *$L_\infty(z)$ is unitary and has unity DC-gain, i.e. $L_\infty(1) = I$.*

4. *$L_\infty(z)$ is a polynomial matrix.*

Proof

We will proceed by parts:

1. $G(z)$ has a zero with multiplicity m_∞ at infinity associated to the direction η_∞ iff

$$\eta_\infty^T G(z) = \sum_{i=1}^{n} \eta_{\infty_i} G_{i*}(z) = \frac{1}{\prod_{i=1}^{m_\infty}(z-p_i)} \Delta(z) \tag{10}$$

where $G_{i*}(z)$ denotes the $i-$th row of $G(z)$, η_{∞_i} the $i-$th component of η_∞ and $\{p_i\}_{i=1,\cdots m}$, are poles of $\eta_\infty^T G(z)$. $\Delta(z)$ is a row vector such that $\Delta(\infty) \neq 0$ and $\Delta(\infty)$ has finite entries. Therefore,

$$\begin{aligned} \eta_\infty^T \tilde{G}(z) &= \eta_\infty^T L_\infty(z) G(z) \\ &= \eta_\infty^T \left(z\eta_\infty \eta_\infty^T + U_\infty U_\infty^T \right) G(z) \\ &= z\eta_\infty^T G(z) \end{aligned} \tag{11}$$

which shows that $\tilde{G}(z)$ has a zero in $z = 0$ with left direction η_∞, if and only if the row vector $\eta_\infty^T G(z)$ is analytical at $z = 0$.

Also, from (10) and (11)

$$\begin{aligned} \eta_\infty^T \tilde{G}(z) &= z \sum_{i=1}^{n} \eta_{\infty_i} G_{i*}(z) \\ &= z \frac{1}{\prod_{i=1}^{m_\infty}(z-p_i)} \Delta(z) \\ &= \frac{1}{\prod_{i=1}^{m_\infty - 1}(z-p_i)} \frac{z}{z-p_{m_\infty}} \Delta(z) = \frac{1}{\prod_{i=1}^{m_\infty - 1}(z-p_i)} \bar{\Delta}(z) \end{aligned} \tag{12}$$

which shows that $\tilde{G}(z)$ has $m_\infty - 1$ zeros at *infinity* with left direction η_∞.

2. Note that (9) is the Smith decomposition of $L_\infty(z)$ (see [30]), hence $L_\infty(z)$ has only one zero at the origin and no poles. Therefore, at most one zero will be added to $G(z)$ to form $\tilde{G}(z)$. It was shown above that the effect of $L_\infty(z)$ is to cancel one zero at infinity of $G(z)$ and, possibly, to substitute it by a zero at $z = 0$. Therefore, the rest of poles and zeros of $G(z)$ are also poles and zeros of $\tilde{G}(z)$, although not necessarily with the same directions.

3. To prove this, it suffices to note that

$$\begin{aligned} L_\infty^\sim(z) L_\infty(z) &= \left(z^{-1}\eta_\infty \eta_\infty^T + U_\infty U_\infty^T \right) \left(z\eta_\infty \eta_\infty^T + U_\infty U_\infty^T \right) \\ &= \eta_\infty \eta_\infty^T \eta_\infty \eta_\infty^T + z^{-1}\eta_\infty \eta_\infty^T U_\infty U_\infty^T + zU_\infty U_\infty^T \eta_\infty \eta_\infty^T + U_\infty U_\infty^T U_\infty U_\infty^T \\ &= I \end{aligned} \tag{13}$$

where we have used the fact that η_∞ and $\begin{bmatrix}\eta_\infty & U_\infty\end{bmatrix}$ are unitary. For the same reason,

$$L_\infty(1) = \left\{z\eta_\infty\eta_\infty^T + U_\infty U_\infty^T\right\}\Big|_{z=1} = \eta_\infty\eta_\infty^T + U_\infty U_\infty^T = I \tag{14}$$

4. Straightforward from (9).

□□□

From Lemma 2 we have that

$$G(z) = L_\infty^{-1}(z)\tilde{G}(z); \qquad \text{with} \quad L_\infty^{-1}(z) = \frac{1}{z}\eta_\infty\eta_\infty^T + U_c U_c^T \tag{15}$$

which says that $L_\infty(z)$ extracts one of the zeros of $G(z)$ in $z = \infty$, i.e. the relative degree is reduced by one.

3.1 Unitary interactors

In this section Lemma 2 is used to find an unitary interactor matrix with unity DC-gain. It will be also shown that an interactor with those properties is unique.

Definition 1. *Given any real, proper and nonsingular transfer matrix $G_o(z) \in \mathbb{C}^{n\times n}$, then any polynomial transfer matrix, $\xi_g(z)$, that satisfies*

$$\lim_{z\to\infty} \xi_g(z)G_o(z) = K \tag{16}$$

where K is a real non-singular matrix (and therefore only with finite entries), will be called an interactor matrix of $G_o(z)$.

The above definition is equivalent to saying that a polynomial transfer matrix, $\xi_g(z)$, is a left interactor matrix for another transfer matrix, $G_o(z)$, if their product, $\xi_g(z)G_o(z)$ is biproper. Also note that, in general, the interactor matrix is non unique [23], but it can be made unique if some special constraints are imposed [29], [31], [2]. Note that the previous definition allows the interactor to be NMP, but as we will see in the following sections, it is preferable to choose an interactor matrix with all its zeros in the stability region.

The following result is extracted from [24]:

Theorem 1. *Given any proper and non-singular transfer matrix $G_o(z) \in \mathbb{C}^{n\times n}$, then there exists an unitary interactor, $\xi_{gU}(z)$, which satisfies*

$$det\{\xi_{gU}(z)\} = kz^m \tag{17}$$

where m is the relative degree of $G_o(z)$ and k is an appropriated real number. In the sequel, $\xi_{gU}(z)$ will be called unitary interactor.

$\xi_{gU}(z)$ is non-unique, but if $\xi_{gU1}(z)$ and $\xi_{gU2}(z)$ are unitary interactors, then there exists a constant real unitary matrix T such that

$$\xi_{gU1}(z) = T\xi_{gU2}(z) \tag{18}$$

Proof

See [24] and [27] where an explicit algorithm to build unitary interactors is given.

□□□

It is worth noting that T **must be constant** in order to satisfy (18) with $\xi_{gU1}(z)$ and $\xi_{gU2}(z)$ interactors of $G_o(z)$.

It must be also noted that the unitary interactor defined above has no especial structure. In particular it is in general non triangular, as Wolovich and Falb's interactor is [28]. It has, however, the key property of being unitary. The relevance of a unitary interactor $L(z)$ in the framework of this paper is that

$$||L(z)F(z)||_2 = ||F(z)||_2 \tag{19}$$

for any $F(z) \in \mathcal{L}_2$. This fact will be used in the following section.

From Theorem 1 we have the following corollary:

Corolary 1. *Given any real, proper and non-singular transfer matrix $G_o(z) \in \mathbb{C}^{n\times n}$, then there exists a unique unitary interactor with unity DC-gain.*

Proof

From Theorem 1 we have that there exist unitary interactors $\xi_{gU}(z)$ for $G_o(z)$. It is clear that since $\xi_{gU}(z)$ is unitary for all z, $\xi_{gU}(1)$ is unitary and given that the product of unitary matrices is unitary, it follows that

$$\bar{\xi}_{gU}(z) = \xi_{gU}^{-1}(1)\xi_{gU}(z) \tag{20}$$

is a unitary interactor with unity DC-gain. Let us assume that there exists another unitary interactor with unity DC-gain $\bar{\xi}_{gU2}(z) \neq \bar{\xi}_{gU}(z)$, then there exists an unitary T such that

$$\bar{\xi}_{gU2}(z) = T\bar{\xi}_{gU}(z) \Rightarrow \bar{\xi}_{gU2}(1) = T\bar{\xi}_{gU}(1) \Rightarrow I = T \Rightarrow \bar{\xi}_{gU2}(z) = \bar{\xi}_{gU}(z) \tag{21}$$

which contradicts our assumption. Therefore, there exists only one unitary interactor with unity DC-gain.

□□□

The unitary interactor with unity DC-gain described by corollary 1 can be built as follows:

Lemma 3. *Consider any real proper transfer matrix $G_o(z) \in \mathbb{C}^{n\times n}$ with relative degree equal to m (that is, with m zeros at infinity)[1]. The unitary interactor with unity DC-gain is given by*

$$\hat{\xi}_{gU}(z) = L_m(z)\cdots L_1(z) = \prod_{i=1}^{m} L_{m-i+1}(z) \tag{22}$$

where we consider the following auxiliary definition

$$G_i(z) = L_i(z)G_{i-1}(z) \quad , i = 1, \cdots, m \tag{23}$$

with $G_0(z) = G_o(z)$ and where $L_i(z)$ is defined in (9) *(see Lemma 2) making $\eta_\infty = \eta_i$, where η_i is the direction associated to the first zero of $G_{i-1}(z)$ at infinity (the ordering is arbitrary).*

[1]It should be noted that, if the transfer matrix is proper, zeros at infinity are not cancelled by poles at infinity (otherwise, the properness assumption would not hold) and therefore, the number of zeros at infinity that can be seen in $det\{\cdot\}$ (relative degree) is equal to the total number of zeros at infinity.

Proof

To prove our assertion it suffices to prove that the matrix given by (22) corresponds to the interactor matrix defined in definition 1 with the properties in theorem 1 and in corollary 1.

- From Lemma 2, $L_i(z)$ is an unitary matrix for all i, therefore, $\prod_{i=1}^{m} L_{m-i+1}(z)$ is also unitary.
- From Lemma 2, $L_i(1) = I$ for all i and therefore $\prod_{i=1}^{m} L_{m-i+1}(1) = I$.
- $L_i(z)$ is clearly a polynomial matrix and hence $\prod_{i=1}^{m} L_{m-i+1}(z)$ is also polynomial.
- From Lemma 2 we have that $det\{L_i(z)\} = z$, therefore, $det\left\{\prod_{i=1}^{m} L_{m-i+1}(z)\right\} = z^m$ and property (17) is verified with $k = 1$.
- It remains to prove that $\prod_{i=1}^{m} L_{m-i+1}(z)$ satisfies (16). We first write

$$G_1(z) = L_1(z)G_o(z) \tag{24}$$

and, in agreement with Lemma 2, we have that $G_1(z)$ shares poles and zeros with $G_o(z)$ except one zero at infinity that has been replaced (or cancelled) by a zero at the origin. This implies that the relative degree (number of zeros at infinity) of $G_1(z)$ is smaller, in one unit, than the relative degree of $G_o(s)$.

If we write now

$$G_2(z) = L_2(z)G_1(z) = L_2(z)L_1(z)G_o(z) \tag{25}$$

and following the previous argument, it turns out that the relative degree of $G_2(z)$ is smaller than the relative degree of $G_o(z)$ in two units. Therefore, generalizing the analysis we have that

$$G_m(z) = L_m(z)G_{m-1}(z) = L_m(z)\cdots L_1(z)G_o(z) \tag{26}$$

is a biproper matrix (i.e. without zeros at infinity) and, therefore, non-singular for $z \to \infty$.

The previous discussion allows us to conclude that (22) defines the unique unitary interactor with unity DC-gain of $G_o(z)$.

□□□

The previous ideas are next illustrated with an example:

Example 1. *Consider the continuous time plant model given by*

$$G_{o_c}(s) = \begin{bmatrix} \dfrac{1.116e^{-7s}}{s+0.2231} & 0 & \dfrac{0.8926e^{-5s}}{s+0.2231} \\ \dfrac{0.5579e^{-6s}}{s+0.2231} & \dfrac{1.116}{s+0.2231} & \dfrac{1.339e^{-4s}}{s+0.2231} \\ 0 & \dfrac{0.5579}{s+0.2231} & \dfrac{0.6694e^{-4s}}{s+0.2231} \end{bmatrix} \tag{27}$$

Considering a sampling interval of $1[s]$ *and a zero-order hold at the input of the plant, the corresponding discrete time model is found to be*

$$G_o(z) = \begin{bmatrix} \dfrac{5}{(5z-4)z^7} & 0 & \dfrac{4}{(5z-4)z^5} \\ \dfrac{2.5}{(5z-4)z^6} & \dfrac{5}{5z-4} & \dfrac{6}{(5z-4)z^4} \\ 0 & \dfrac{2.5}{5z-4} & \dfrac{3}{(5z-4)z^4} \end{bmatrix} \tag{28}$$

Thus

$$det\{G_o(z)\} = \frac{25}{z^{11}(5z-4)^3} \tag{29}$$

which implies that $G_o(z)$ *has* $m = 14$ *zeros at infinity, i.e. it has relative degree equal to* 14, *and no NMP zeros. Note that there are* 11 *poles at the origin which take the delays of the plant model into account.*

The unitary interactor with unity DC-gain will be constructed following the procedure outlined in the proof of Lemma 3:

$$\lim_{z\to\infty} G_o(z) = E_o = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{30}$$

Therefore the left null space of E_o *has as basis* $\{\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T, \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T, \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T\}$ *and choosing arbitrarily* $\eta_1 = \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T$, $L_1(z)$ *can be formed as follows*

$$L_1(z) = z\eta_1\eta_1^T + \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{U_1} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{31}$$

therefore

$$G_1(z) = L_1(z)G_o(z) = \begin{bmatrix} \dfrac{5}{(5z-4)z^7} & 0 & \dfrac{4}{(5z-4)z^5} \\ \dfrac{2.5}{(5z-4)z^5} & \dfrac{5z}{5z-4} & \dfrac{6}{(5z-4)z^3} \\ 0 & \dfrac{2.5}{5z-4} & \dfrac{3}{(5z-4)z^4} \end{bmatrix} \tag{32}$$

Note that

$$det\{G_1(z)\} = \frac{25}{(5z-4)^3 z^{10}} \tag{33}$$

what implies that $G_1(z)$ *has* 13 *zeros at infinity, i.e., it has relative degree equal to* 13. *Note that this verifies that* $L_1(z)$ *cancels one of the zeros at infinity of* $G_o(z)$ *with its zero at the origin.*

Proceeding in a similar fashion, one can construct the 14 *factors that form the unitary interactor of* $G_o(z)$*. The procedure is easily implemented automatically using any computation software. Note that*

$$G_{14}(z) = \begin{bmatrix} \frac{5}{z(5z-4)} & 0 & \frac{4z}{5z-4} \\ \frac{0.5z^6+2}{(5z-4)z^5} & \frac{5z}{5z-4} & \frac{6}{z^3(5z-4)} \\ \frac{-(z-1)(z+1)(z^2+1+z)(z^2-z+1)}{(5z-4)z^5} & \frac{2.5z}{5z-4} & \frac{3}{z^3(5z-4)} \end{bmatrix} \tag{34}$$

and therefore

$$det\{G_{14}(z)\} = \frac{25z^3}{(5z-4)^3} \tag{35}$$

which implies that the relative degree of $G_{14}(z)$ *is zero and, therefore,* $G_{14}(z)$ *is biproper and the procedure stops.*

From (22) *the unitary interactor with unity DC-gain is then given by*

$$\begin{aligned}\hat{\xi}_{gU}(z) &= \prod_{i=1}^{14} L_{14-i+1}(z) \\ &= \begin{bmatrix} z^6 & 0 & 0 \\ 0 & 0.2z(z^6+4) & -2/5z(z-1)(z+1)(z^2+1+z)(z^2-z+1) \\ 0 & -2/5z(z-1)(z+1)(z^2+1+z)(z^2-z+1) & 0.2z(4z^6+1) \end{bmatrix}\end{aligned} \tag{36}$$

□□□

3.2 Unitary zero-interactors

This section provides a precise expression for a unitary zero-interactor for any square transfer matrix.

Definition 2. *Consider a proper and non-singular transfer matrix* $G_o(z) \in \mathbb{C}^{n\times n}$*, with no zeros on the unit circle, but having q NMP finite zeros, denoted jointly by* $\{c_i\}_{i=1,\cdots,q}$*,* $|c_i| > 1$*, then any minimum phase biproper matrix,* $\xi_c(z)$*, will be called a zero-interactor of* $G_o(z)$ *if it satisfies*

$$\lim_{z\to c_i} \xi_c(z)G_o(z) = K_i \quad \forall i \in \{1, \cdots, q\} \tag{37}$$

where K_i *is a non-singular real matrix.*

The previous definition is equivalent to saying that any matrix, $\xi_c(z)$, is a (left) zero-interactor of a transfer matrix, $G_o(z)$, if $\xi_c(z)G_o(z)$ has all its zeros inside the unit circle. Note that the definition can be applied to minimum phase zeros as well, but only the NMP zero-interactor will prove useful in what follows.

Expressions for certain classes of zero-interactors can be found in [29] and in [2].

The following lemma provides an explicit expression for the unitary zero-interactor of a given transfer matrix:

Lemma 4. *Consider a transfer matrix* $G_o(z) \in \mathbb{C}^{n\times n}$ *with* q *finite NMP zeros. One unitary zero-interactor with unity DC-gain is given by*

$$\hat{\xi}_{cU}(z) = L_{cq}(z)\cdots L_{c1}(z) = \prod_{i=1}^{q} L_{cq-i+1}(z) \tag{38}$$

where the following auxiliary definition is considered

$$G_i(z) = L_{ci}(z)G_{i-1}(z) \quad , i = 1, \cdots, q \tag{39}$$

and $L_{ci}(z)$ *is defined in* (6) *(see Lemma 1) with* $\eta_c = \eta_i$*, where* η_i *is the direction of the first NMP zero of* $G_{i-1}(z)$ *(the ordering is arbitrary).*

Proof

To prove the result we will proceed as in the proof of Lemma 3. It suffices to prove that the matrix given by (38) satisfies all the properties of the interactor defined in definition 2, that it is unitary and that it has unity DC-gain.

From Lemma 1 we have that

- $L_{ci}(z)$ is a unitary matrix for all i, therefore, $\prod_{i=1}^{q} L_{cq-i+1}(z)$ is also unitary.
- $L_{ci}(1) = I$ for all i and therefore, $\prod_{i=1}^{q} L_{cq-i+1}(1) = I$.
- $det\{L_{ci}(z)\} = \frac{1-c_i}{1-\bar{c}_i}\frac{1-\bar{c}_i z}{z-c_i}$ which implies that

$$det\left\{\prod_{i=1}^{q} L_{cq-i+1}(z)\right\} = \prod_{i=1}^{q} \frac{1-c_i}{1-\bar{c}_i}\frac{1-\bar{c}_i z}{z-c_i} \tag{40}$$

  The last equation and the fact that each of the factors $L_{ci}(z)$ is realizable, imply that $\prod_{i=q}^{1} L_i(z)$ is biproper.

At this point, it only remains to prove that $\prod_{i=q}^{1} L_{cq-i+1}(z)$ satisfies (37). To that end we write

$$G_1(z) = L_{c1}(z)G_o(z) \tag{41}$$

From Lemma 1, we have that $G_1(z)$ shares the poles and zeros of $G_o(z)$ except for one zero at $z = c_1$ of $G_o(z)$ that has been replaced by a zero at $z = \frac{1}{\bar{c}_1}$ in $G_1(z)$, provided that $\eta_{c1}G_o(z)$ is analytical at $z = \frac{1}{\bar{c}_1}$.

Writing now

$$G_2(z) = L_{c2}(z)G_1(z) = L_{c2}(z)L_{c1}(z)G_o(z) \tag{42}$$

and repeating the previous argument, it is clear that $G_2(z)$ has the same poles and zeros of $G_o(z)$ except for two zeros, originally at $z = c_1$ y $z = c_2$ in $G_o(z)$, that were replaced by zeros at $z = \frac{1}{\bar{c}_1}$ and $z = \frac{1}{\bar{c}_2}$ in $G_2(z)$, provided that $\eta_{c1}G_o(z)$ is analytical at $z = \frac{1}{\bar{c}_1}$ and $\eta_{c2}G_1(z)$ is analytical at $z = \frac{1}{\bar{c}_2}$. Therefore,

$$G_q(z) = L_{cq}(z)G_{q-1}(z) = L_{cq}(z)\cdots L_{c1}(z)G_o(z) \tag{43}$$

is a matrix without NMP zeros. Hence $\hat{\xi}_{cU}(z)G_o(z)$ is non-singular at $z = c_i, \forall i$.

The previous discussion allows us to conclude that (38) defines a unitary zero-interactor with unity DC-gain.

□□□

The relevance of a unitary zero-interactor $L_c(z)$ in the framework of this paper is that

$$||L_c(z)F(z)||_2 = ||F(z)||_2 \tag{44}$$

for any $F(z) \in \mathcal{L}_2$. This fact will be used in the following section to derive performance bounds in MIMO control systems.

4 Performance bounds in MIMO systems

In this section we compute the minimum value of the cost functional

$$J = \sum_{k=0}^{\infty} e^T(k)e(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} E^H(e^{j\omega})E(e^{j\omega})d\omega = ||E(z)||_2^2 \tag{45}$$

where $e(k)$ denotes the tracking error of a one degree of freedom control loop, for a step reference signal, $r(k)$, applied at $k = 0$. This means that $r(k) = v\mu(k)$ with $v \in \mathbb{R}^{n\times 1}$, where $\mu(k)$ denotes the unit step function. Using the Youla parameterization of all stabilizing controllers (see, for example, [2]) it is possible to re-write J as

$$J = \left|\left|(I - G_o(z)Q(z))\frac{vz}{z-1}\right|\right|_2^2 \tag{46}$$

where Q is the Youla parameter.

4.1 Case I: stable, minimum phase plant

Theorem 2. *Consider a proper stable minimum-phase transfer matrix $G_o(z) \in \mathbb{C}^{n\times n}$ with relative degree (number of zeros at infinity) equal to m and such that $G_o(1)$ is non-singular.*

1. *The optimal causal Youla parameter that stabilizes the plant and that minimizes J, as defined in (46), is given by*

$$Q_{opt}(z) = \arg\min_{Q\in S} J = \tilde{G}_o^{-1}(z) \tag{47}$$

where S denotes the set of all stable and proper rational transfer matrices $K(z) \in \mathbb{C}^{n\times n}$ and $\tilde{G}_o(z)$ satisfies

$$\tilde{G}_o(z) = \hat{\xi}_{gU}(z)G_o(z) \tag{48}$$

where $\hat{\xi}_{gU}(z)$ is the unitary interactor matrix with unity DC-gain for $G_o(z)$.

2. *The minimum cost J is given by*

$$J_{opt} = \min_{Q\in S} J = v^H \sum_{i=1}^{m} \eta_i\eta_i^H v = \sum_{i=1}^{m} |\eta_i^H v|^2 \tag{49}$$

where η_i corresponds to the direction of the first zero at infinity of $G_{i-1}(z)$, with $G_{i-1}(z)$ defined in Lemma 3.

Proof

We will proceed by parts:

1. Using (48) in (46) we have that

$$J = \left\| (I - \hat{\xi}_{gU}^{-1}(z)\tilde{G}_o(z)Q(z)) \frac{vz}{z-1} \right\|_2^2$$

$$J = \left\| z\hat{\xi}_{gU}^{-1}(z)(\hat{\xi}_{gU}(z) - \tilde{G}_o(z)Q(z)) \frac{v}{z-1} \right\|_2^2 \tag{50}$$

Then, using the fact that $\hat{\xi}_{gU}(z)$ and zI_n are unitary transfer matrices and using elementary 2-norm properties, (50) can be written as

$$J = \left\| (\hat{\xi}_{gU}(z) - \tilde{G}_o(z)Q(z)) \frac{v}{z-1} \right\|_2^2 \tag{51}$$

Note that since $\hat{\xi}_U$ has unity DC-gain it is necessary that $Q(1) = \tilde{G}_o^{-1}(1) = G_o^{-1}(z)$ (the last equality follows from (48)) in order to J to be finite. Equation (51) leads to

$$J = \left\| \underbrace{\left(\hat{\xi}_{gU}(z) - I_n\right) \frac{v}{z-1}}_{A(z)} + \underbrace{\left(I_n - \tilde{G}_o(z)Q(z)\right) \frac{v}{z-1}}_{B(z)} \right\|_2^2 \tag{52}$$

From Theorem 1, $\hat{\xi}_U(z)$ is a polynomial matrix and, therefore, has not finite poles. Moreover, $\hat{\xi}_U(1) = I \Rightarrow \hat{\xi}_{gU}(z) - I_n = (z-1)\tilde{N}(z)$. Therefore, $A(z)$ is analytical over and inside the unit circle, which means that $A(z) \in \mathcal{H}_2^{\perp}$ (see [12] y [32]). On the other hand, $\tilde{G}_o(z)$ is stable since $G_o(z)$ is itself stable and $\hat{\xi}_U(z)$ does not add poles to G_o when constructing $\tilde{G}_o(z)$ (see (48)). Moreover, Q must be stable in order to have a stable control loop (since G_o is stable, this suffices to guarantee stability) and since $Q(1) = \tilde{G}_o^{-1}(1) = G_o^{-1}(1) \Rightarrow (I_n - \tilde{G}_o(z)Q(z)) = (z-1)\tilde{\tilde{N}}(z)$, it is clear that $B(z)$ has all its poles inside the unit circle and, therefore, $B(z) \in \mathcal{H}_2$. Therefore,

$$J = \left\| \left(\hat{\xi}_{gU}(z) - I_n\right) \frac{v}{z-1} \right\|_2^2 + \left\| \left(I_n - \tilde{G}_o(z)Q(z)\right) \frac{v}{z-1} \right\|_2^2 \tag{53}$$

From (53) it is clear that the minimum cost is achieved if

$$Q_{opt}(z) = \tilde{G}_o^{-1}(z) \tag{54}$$

Note that given (48) and (16), $\tilde{G}_o(z)$ is biproper. Furthermore, since $G_o(z)$ is minimum phase and the zeros of $\tilde{G}_o(z)$ are the zeros of $G_o(z)$ plus up to m zeros at the origin (see (17)), it is clear that $\tilde{G}_o(z)$ is minimum phase. Hence, $\tilde{G}_o^{-1}(z)$ is stable and biproper. Note that it is verified that $Q_{opt}(1) = \tilde{G}_o^{-1}(1)$ what validates our previous assumption regarding the DC-gain of Q_{opt}.

This discussion allows us to conclude that (54) is the optimal Youla parameter.

2. Replacing (47) in (53) the optimal cost can be expressed as

$$J_{opt} = \min_{Q \in S} J = \left\| \frac{\hat{\xi}_{gU}(z) - I_n}{z-1} v \right\|_2^2 = v^H \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{\hat{\xi}_{gU}(e^{j\omega}) - I_n}{e^{j\omega} - 1} \right)^H \frac{\hat{\xi}_{gU}(e^{j\omega}) - I_n}{e^{j\omega} - 1} d\omega \right] v \quad (55)$$

using the definition $z = e^{j\omega}$ and Cauchy's residue theorem (see, for example, [1]), (55) can be written as

$$J_{opt} = v^H \left[\sum_{i=1}^{k} Res_{z=z_i} \underbrace{\frac{1}{z} \left(\frac{\hat{\xi}^{\sim}_{gU}(z) - I_n}{z^{-1} - 1} \right) \left(\frac{\hat{\xi}_{gU}(z) - I_n}{z-1} \right)}_{O(z)} \right] v \quad (56)$$

where $\{z_i\}_{i=1\cdots k}$ denotes the set of all poles of $O(z)$ inside the unit circle. It is straightforward to prove, using the fact that the unitary interactor $\hat{\xi}_{gU}(z)$ is a polynomial matrix and has unity DC-gain, that the only poles of $O(z)$ inside the unit circle are at the origin. This implies that it suffices to calculated the residues of $O(z)$ at the origin to evaluate (56).

Note that

$$\begin{aligned} O(z) &= \frac{\hat{\xi}^{\sim}_{gU}(z) - I_n}{1-z} \frac{\hat{\xi}_{gU}(z) - I_n}{z-1} \\ &= \frac{\overbrace{\hat{\xi}^{\sim}_{gU}(z)\hat{\xi}_{gU}(z)}^{I_n} - \hat{\xi}^{\sim}_{gU}(z) - \hat{\xi}_{gU}(z) + I_n}{(1-z)(z-1)} \\ &= \frac{-2I_n}{(z-1)^2} + \frac{\hat{\xi}_{gU}(z)}{(z-1)^2} + \frac{\hat{\xi}^{\sim}_{gU}(z)}{(z-1)^2} \end{aligned} \quad (57)$$

where the unitarity property of the interactor has been used. But, given the explicit expression of the interactor given by (22) and the properties of each of it factors (see lema 2), it is clear that the first two summands in (57) have no poles at the origin and, consequently, their residues at $z = 0$ are zero. It only remains to evaluate the residues of the third term in (57).

Using (22), (6) and the results mentioned in the last paragraphs, one has that

$$\begin{aligned} \sum_{i=1}^{k} Res_{z=z_i} \frac{1}{z} \left(\frac{\hat{\xi}^{\sim}_{gU}(z) - I_n}{z^{-1} - 1} \right) \left(\frac{\hat{\xi}_{gU}(z) - I_n}{z-1} \right) &= Res_{z=0} \frac{\hat{\xi}^{\sim}_{gU}(z)}{(z-1)^2} \\ &= Res_{z=0} \frac{\prod_{i=1}^{m} \begin{bmatrix} \eta_i & U_i \end{bmatrix} \begin{bmatrix} \frac{1}{z} & 0 \\ 0 & I_{n-1} \end{bmatrix} \begin{bmatrix} \eta_i^T \\ U_i^T \end{bmatrix}}{(z-1)^2} \\ &= Res_{z=0} \frac{1}{z^m (z-1)^2} \prod_{i=1}^{m} \{ \eta_i \eta_i^T + z(I_n - \eta_i \eta_i^T) \} \end{aligned} \quad (58)$$

Note that the right side of (58) has at least relative degree equal to 2. This implies, in accordance to Lemma 1 in [8], that the sum of residues at each singularity of that expression equals zero. Therefore,

$$Res_{z=0}\frac{1}{z^m(z-1)^2}\prod_{i=1}^{m}\left\{\eta_i\eta_i^H + z(I_n-\eta_i\eta_i^H)\right\} = -Res_{z=1}\frac{1}{z^m(z-1)^2}\prod_{i=1}^{m}\left\{\eta_i\eta_i^H + z(I_n-\eta_i\eta_i^H)\right\} \tag{59}$$

But if we define $\Lambda_i(z) = \left\{\eta_i\eta_i^H + z(I_n-\eta_i\eta_i^H)\right\}$, then

$$Res_{z=1}\frac{1}{z^m(z-1)^2}\prod_{i=1}^{m}\Lambda_i(z) = \lim_{z\to 1}\frac{d\left[z^{-m}\prod_{i=1}^{m}\Lambda_i(z)\right]}{dz} \tag{60}$$

It can be verified that $\Lambda_i(z)|_{z=1} = I_n$ and $\left.\frac{d\Lambda_i(z)}{dz}\right|_{z=1} = I_n - \eta_i\eta_i^H$. Moreover, applying elementary derivative properties it follows that

$$\frac{d\left[z^{-m}\prod_{i=1}^{m}\Lambda_i(z)(z)\right]}{dz} = -mz^{-m-1}\prod_{i=1}^{m}\Lambda_i(z) + z^{-m}\sum_{j=1}^{m}\left\{\left(\prod_{i=1}^{j-1}\Lambda_i(z)\right)\frac{d\Lambda_j(z)}{dz}\left(\prod_{i=j+1}^{m}\Lambda_i(z)\right)\right\} \tag{61}$$

what, in accordance to the properties of $\Lambda_i(z)$ mentioned above, allows us to conclude that the limiting operation in (60) can be written as

$$\begin{aligned}\lim_{z\to 1}\frac{d\left[z^{-m}\prod_{i=1}^{m}\Lambda_i(z)(z)\right]}{dz} &= -mI_n + \sum_{j=1}^{m}\left\{I_n(I_n-\eta_j\eta_j^H)I_n\right\}\\ &= -\sum_{j=1}^{m}\eta_j\eta_j^H\end{aligned} \tag{62}$$

what jointly with (60), (59), (58) and (56) implies that

$$J_{opt} = v^H\left[\sum_{j=1}^{m}\eta_j\eta_j^H\right]v \tag{63}$$

which proves the result.

□□□

It is important to note that the previous result cannot be derived using Wolovich and Falb interactor matrices [28], [2] since that interactor is not unitary. This confirms that the Wolovich and Falb interactor is not as suitable as other versions of the interactor matrix as confirmed in [26] in the context of minimum variance control.

The result of Theorem 2 can be particularized to the case of considering a step change in the $i-$th channel. In this case, the minimum 2-norm of the error (see (46)) is given by

$$J_i = \left\|(I-G_o(z)Q(z))\frac{\epsilon_i z}{z-1}\right\|_2^2 \tag{64}$$

where ϵ_i is the $i-$th canonic basis vector of $\mathbb{C}^{n\times 1}$. If one is interested in considering the sum of the 2-norm of the errors resulting of the application of successive step changes in each channel, the corresponding cost can be written as

$$J_F = \sum_{i=1}^{n} J_i \tag{65}$$

The minimum value of J_F can be found using Theorem 2 as shown by the following corollary:

Corolary 2. *Consider a stable minimum phase and proper transfer matrix $G_o(z) \in \mathbb{C}^{n\times n}$ with relative degree (number of zeros at infinity) equal to m and such that $G_o(1)$ is non singular. Then,*

$$J_{F_{opt}} = \min_{Q\in S} J_F = m \tag{66}$$

Proof

According to Theorem 2 the minimum value of the cost J_i is given by

$$J_{i_{opt}} = \min_{Q\in S} J_i = \sum_{j=1}^{m} |\eta_j^H \epsilon_i|^2 \tag{67}$$

Denoting the $i-$th component of η_j by η_{j_i}, (67) allows to write the following

$$\begin{aligned} J_{F_{opt}} = \min_{Q\in S} J_F &= \sum_{i=1}^{n}\sum_{j=1}^{m} |\eta_j^H \epsilon_i|^2 \\ &= \sum_{j=1}^{m}\sum_{i=1}^{n} |\eta_j^H \epsilon_i|^2 \\ &= \sum_{j=1}^{m} \underbrace{\left(\sum_{i=1}^{n} |\eta_{j_i}|^2\right)}_{||\eta_j||^2} \end{aligned} \tag{68}$$

But η_j is unitary, what implies that $J_{F_{opt}} = m$ and the proof is completed.

□□□

Example 2. *Consider the plant of example 1 given by* (28) *and its unitary interactor with unity DC-gain given by* (36)*. According to Theorem 2 the optimum Youla parameter is given by*

$$\begin{aligned} Q_{opt}(z) &= \tilde{G}_o^{-1}(z) \\ &= G_{14}^{-1}(z) = \begin{bmatrix} 0 & \dfrac{2z-1.6}{z} & \dfrac{-4z+3.2}{z} \\ \dfrac{-1.5z+1.2}{z^5} & \dfrac{(4z^2+11)(0.2z-0.16)}{z^7} & \dfrac{(z^6-11)(0.4z-0.32)}{z^7} \\ \dfrac{1.25z-1}{z} & \dfrac{-2.5z+2}{z^3} & \dfrac{5z-4}{z^3} \end{bmatrix} \end{aligned} \tag{69}$$

and the minimum cost by

$$J_{opt} = v^H \sum_{i=1}^{14} \eta_i \eta_i^H v = v^H \begin{bmatrix} 6 & 0 & 0 \\ 0 & 11/5 & -12/5 \\ 0 & -12/5 & 29/5 \end{bmatrix} v \tag{70}$$

Figure 1 shows the temporal evolution of the error considering $v = \begin{bmatrix} 1 & -1 & 0.5 \end{bmatrix}^H$*. The cost obtained upon direct evaluation of the left expression in* (45) *using the simulation results, is given by* $J_{sim} = 12.05$ *which is equal to the cost obtained evaluating* (70) *in this case.*

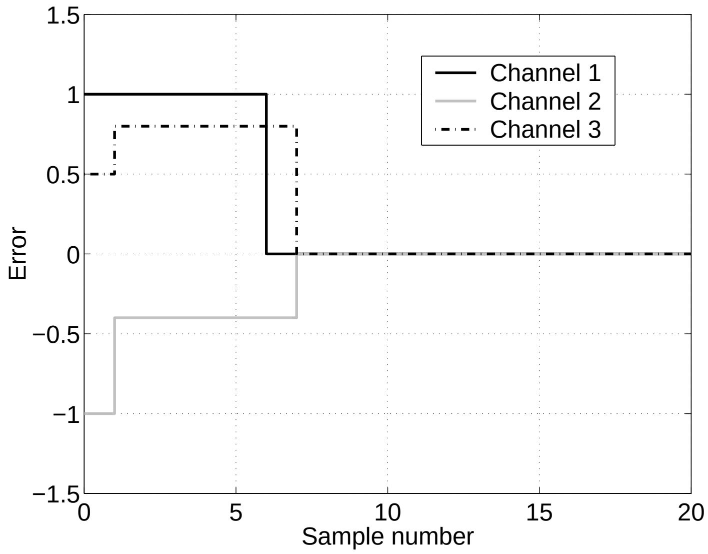

Figure 1: Error evolution using the optimal controller and $r(k) = [1 \quad -1 \quad 0.5]^H \mu(k)$.

□□□

4.2 Case II: stable, NMP plant

In this subsection we will extend the result of Theorem 2 to the NMP case.

Theorem 3. *Consider a proper stable NMP transfer matrix* $G_o(z) \in \mathbb{C}^{n\times n}$ *with relative degree (number of zeros at infinity) equal to* m*,* q *NMP zeros denoted jointly as* $\{c_i\}_{i=1\cdots q}$ *and such that* $G_o(1)$ *is non singular. Then,*

1. *The optimal causal Youla parameter that stabilizes the plant and achieves a finite cost* J*, as defined in* (46)*, is given by*

$$Q_{opt}(z) = \arg\min_{Q\in S} J = \tilde{G}_o^{-1}(z) \tag{71}$$

where S *denotes the set of all stable and proper rational transfer matrices* $K(z) \in \mathbb{C}^{n\times n}$ *and* $\tilde{G}_o(z)$ *satisfies*

$$\tilde{G}_o(z) = \hat{\xi}_{cU}(z)\hat{\xi}_{gU}(z)G_o(z) \tag{72}$$

where $\hat{\xi}_{gU}(z)$ is the unitary interactor matrix with unity DC-gain of $G_o(z)$ and $\hat{\xi}_{cU}(z)$ is the unitary zeros-interactor defined in Lemma 4 for $\hat{\xi}_{gU}(z)G_o(z)$.

2. *The minimum cost J is given by*

$$J_{opt} = \min_{Q\in S} J = v^H\left[\sum_{i=1}^{m}\eta_i^g\eta_i^{gT} + \sum_{i=1}^{q}\frac{|c_i|^2-1}{|1-c_i|^2}\eta_i^c\eta_i^{cH}\right]v = \sum_{i=1}^{m}|\eta_i^{gT}v|^2 + \sum_{i=1}^{q}\frac{|c_i|^2-1}{|1-c_i|^2}|\eta_i^{cH}v|^2 \quad (73)$$

where η_i^g corresponds to the direction of the first zero at infinity of $G_{i-1}(z)$, with $G_{i-1}(z)$ defined in Lemma 3 considering $G_o(z)$ as the plant whose zeros at infinity should be removed. η_i^c corresponds to the direction of the first finite NMP zeros of $G_{i-1}(z)$, with G_{i-1} defined in Lemma 4, considering $\hat{\xi}_{gU}(z)G_o(z)$ as the plant whose finite NMP zeros should be removed.

Proof

The proof goes along the same lines as in the proof of Theorem 2:

1. Substituting (72) in (46) one gets

$$\begin{aligned} J &= \left\|(1-[\hat{\xi}_{cU}(z)\hat{\xi}_{gU}(z)]^{-1}\tilde{G}_o(z)Q(z))\frac{vz}{z-1}\right\|_2^2 \\ &= \left\|z[\hat{\xi}_{cU}(z)\hat{\xi}_{gU}(z)]^{-1}(\hat{\xi}_{cU}(z)\hat{\xi}_{gU}(z)-\tilde{G}_o(z)Q(z))\frac{v}{z-1}\right\|_2^2 \\ &= \left\|\underbrace{(\hat{\xi}_{cU}(z)\hat{\xi}_{gU}(z)-I_n)\frac{v}{z-1}}_{A(z)}+\underbrace{(I-\tilde{G}_o(z)Q(z))\frac{v}{z-1}}_{B(z)}\right\|_2^2 \end{aligned} \quad (74)$$

Given the properties of the interactor matrices involved in (74), $A(z)$ has finite DC-gain since the factor $z-1$ is simplified. Therefore, all the poles of $A(z)$ are outside the unit circle. Note that some of them are at infinity. This implies that $A(z) \in \mathcal{H}_2^{\perp}$. On the other hand, $B(z)$ is stable with finite DC-gain because $Q(z)$ must be chosen with the inverse DC-gain of the plant and this is the DC-gain of $\tilde{G}_o(z)$. Moreover, $B(z)$ is biproper and stable so that $B(z) \in \mathcal{H}_2$. Therefore,

$$J = \left\|(\hat{\xi}_{cU}(z)\hat{\xi}_{gU}(z)-I_n)\frac{v}{z-1}\right\|_2^2 + \left\|(I-\tilde{G}_o(z)Q(z))\frac{v}{z-1}\right\|_2^2 \quad (75)$$

which clearly implies that

$$Q_{opt}(z) = \arg\min_{Q\in S} J = \tilde{G}_o^{-1}(z) \quad (76)$$

2. From (76) y (75),

$$\begin{aligned} J_{opt} = \min_{Q\in S} J &= \left\|(\hat{\xi}_{cU}(z)\hat{\xi}_{gU}(z)-I_n)\frac{v}{z-1}\right\|_2^2 \\ &= v^H\left[\sum_{i=1}^{k} Res_{z=z_i}\left\{\frac{-2I_n}{(z-1)^2}+\frac{\hat{\xi}_{cU}(z)\hat{\xi}_{gU}(z)}{(z-1)^2}+\frac{\left(\hat{\xi}_{cU}(z)\hat{\xi}_{gU}(z)\right)^{\sim}}{(z-1)^2}\right\}\right]v \end{aligned} \quad (77)$$

where we have followed the same procedure we used to prove Theorem 2 and $\{z_i\}_{i=1\cdots,k}$ corresponds to the set of stable poles of each of the quantities whose residues must be evaluated in (77). Given the properties of the unitary interactor matrices involved, it is clear that the first and second summand in (77) have no poles inside the unit circle and therefore, their residues are zero. This implies that

$$J_{opt} = v^H \left[\sum_{i=1}^{k} Res_{z=z_i} \frac{\left(\hat{\xi}_{cU}(z)\hat{\xi}_{gU}(z) \right)^{\sim}}{(z-1)^2} \right] v \tag{78}$$

Using (22) and (38) one can write

$$\frac{\left(\hat{\xi}_{cU}(z)\hat{\xi}_{gU}(z) \right)^{\sim}}{(z-1)^2} =$$
$$\frac{\prod_{j=1}^{m} \left\{ \begin{bmatrix} \eta_j^g & U_j^g \end{bmatrix} \begin{bmatrix} \frac{1-z}{z} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_j^{gT} \\ U_j^{gT} \end{bmatrix} + I_n \right\} \prod_{j=1}^{q} \left\{ \begin{bmatrix} \eta_j^c & U_j^c \end{bmatrix} \begin{bmatrix} \frac{1-\bar{c}_i}{1-c_i}\frac{z-c_i}{1-\bar{c}_i z} - 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_j^{cH} \\ U_j^c \end{bmatrix} + I_n \right\}}{(z-1)^2} \tag{79}$$

where it becomes apparent that the first factor of the right side numerator of (79) has all its poles at the origin and, since $|c_i| > 1$, the second factor has all its poles inside the unit circle. Proceeding as is the proof of Theorem 2, it can be verified that the relative degree of (79) is equal to two and therefore, one can use the same ideas that showed useful in that proof. Proceeding in this way, it is straightforward to obtain

$$J_{opt} = v^H \left[\sum_{i=1}^{m} \eta_i^g \eta_i^{gH} + \sum_{i=1}^{q} \frac{|c_i|^2 - 1}{|1-c_i|^2} |\eta_i^{cH}|^2 \right] v \tag{80}$$

which completes the proof.

□□□

5 Discussion of the Results

Expressions (49) and (73) can be interpreted as a measure of how far $\mathbf{Q}(z)$ is from the perfect inverse of the plant model $G_o(z)$, in a direction v. Thus, cero cost is equivalent to perfect inversion. It is known that the process characteristics that prevent perfect (stable and feasible) inversion are the process delays (zeros at infinity) and its finite NMP zeros. In the first case, it is impossible to build a physically realizable inverse and in the second, the resulting inverse would be unstable. The results of theorem 3 are then sensible: the optimal cost, as measured by (45) is bounded from below by the presence of finite and non-finite NMP zeros.

The analysis of the closed form expressions of the minimal costs (49) and (73) also shows that there is a strong dependence upon the direction of the reference. This means that not only the presence of some non invertible characteristic will increase the functional value. Indeed, there is an interplay between directionality, NMP zeros and delays. This is a key issue, and it implies that experiments poorly designed might obscure some of the process main features. Consider for instance a process with only one finite NMP zero. If one selects a reference with a direction that is orthogonal to the NMP zero direction, the particular cost will be equal to the cost achieved in the case of a process that shares the delay structure of the first plant, but that is minimum phase. This is not surprising since directionality is a key issue [2] in MIMO systems.

Finally it is interesting to examine more carefully the effect of the location of the finite NMP zeros on the value of the functional (45). According to theorem 3 (see (73)) the general term that weights the product between the zero direction and the reference (or output disturbance) direction is

$$P(c_i) = \frac{|c_i|^2 - 1}{|1 - c_i|^2} \tag{81}$$

This function tends to infinity as $c_i \to 1$ and to zero if $c_i \to -1$. This means that only zeros near $z = 1$ will cause a high rising of the minimum functional value, but other zeros, although having magnitude near one, won't have an important effect. This can be best viewed in figure 2 where P_{c_i} is plotted considering $c_i = re^{j\theta}$ with $r \in (1, 3]$ and $\theta \in [-\pi, \pi]$.
The result shown in Figure 2 is in complete agreement with [22] and other classical results (see e.g. [2]): NMP zeros near one are very hard to deal with, in particular they makes a process very hard to control.

6 Conclusions

This paper presents the computation of an achievable performance bound for the feedback control of stable MIMO systems with arbitrary delay structure and NMP ceros. The result also includes a closed form expression for the Youla controller which allows to achieve that performance bound. The expression for the bound explicitly includes the key plant features which determine and limit the control performance.

The main results rely entirely on the unitary property of interactors and zero-interactors. As a by-product, we have proven the uniqueness of the unitary interactor, which in some way completes the available results on that topic.

Future work in the areas covered by the paper should include the discussion of performance limits in the full decentralized case in order to evaluate, for example, the consequential performance deterioration.

Acknowledgments

The authors gratefully acknowledge the support received through grants Fondecyt 1040313 and UTFSM 230425. They are also grateful for helpful discussions with Juan I. Yuz.

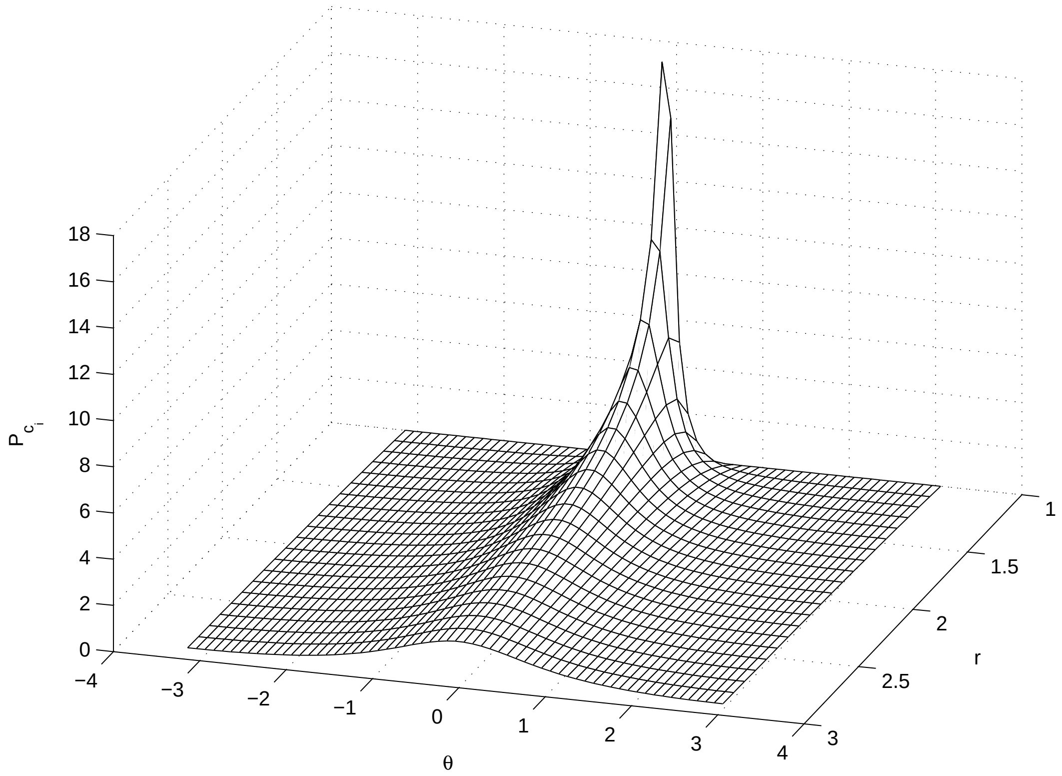

Figure 2: $P(c_i)$ as a function of r and θ, where $c_i = re^{j\theta}$.

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