ON THE DIRECTLY AND SUBDIRECTLY IRREDUCIBLE MANY-SORTED ALGEBRAS

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Abstract. A theorem of general algebra asserts that every finite algebra can be represented as a product of a finite family of finite directly irreducible algebras. In this paper we show that the many-sorted counterpart of the above theorem is also true, but under the condition of requiring, in the definition of directly reducible many-sorted algebra, that the supports of the factors be included in the support of the many-sorted algebra. Moreover, we show that the theorem of Birkhoff according to which every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras, is also true for the many-sorted algebras.

1. Introduction

Some theorems of ordinary universal algebra can not be automatically generalized to world of many-sorted universal algebra, see e.g.,[3] and [4] for the case of a representation theorem of Birkhoff-Frink, or [5] for that one of the injectivity of the insertion of the generators in the relatively free many-sorted algebras.

Our main aim in this paper is to prove, in the third section, that, under a mild condition on the supports of the factors in the definition of the concept of directly reducible many-sorted algebra, every finite many-sorted algebra can also be represented as a product of a finite family of finite directly irreducible many-sorted algebras. In addition, in the fourth section, for completeness, we show that the many-sorted counterpart of the well-known theorem of Birkhoff about the representation of every single-sorted algebra as a subdirect product of subdirectly irreducible single-sorted algebras, is also true for the many-sorted algebras.

In the second section we define those notions and constructions from the theory of many-sorted sets and algebras which are indispensable in order to attain the above indicated goals.

2. Many-sorted signatures, algebras, homomorphisms, subalgebras, products, congruences, and quotients.

In this section we begin by defining for an arbitrary, but fixed, set of sorts S, those concepts of the theory of S-sorted sets which we need in order to state the notions of many-sorted signature, algebra, subalgebra, homomorphism from a many-sorted algebra to another, product of a family of many-sorted algebras, and congruence on a many-sorted algebra.

Definition 1. Let S be a set of sorts.

(1) A *word on S* is a mapping $w\colon n \longrightarrow S$, for some $n \in \mathbb{N}$. We denote by $S^\star$ the underlying set of the free monoid on S, i.e., the set $\bigcup_{n\in\mathbb{N}} S^n$ of all mappings from the finite ordinals to S. Moreover, we call the unique mapping $\lambda\colon \varnothing \longrightarrow S$, the *empty word on S*.

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(2) An S-*sorted set* A is a function $(A_s)_{s\in S}$ from S to $\boldsymbol{\mathcal{U}}$, where $\boldsymbol{\mathcal{U}}$ is a Grothendieck universe, fixed once and for all, and the *support of* A, denoted by $\mathrm{supp}(A)$, is the set $\{\, s \in S \mid A_s \neq \varnothing \,\}$. An S-sorted set A is *finite* if $\mathrm{supp}(A)$ is finite and, for every $s \in \mathrm{supp}(A)$, A_s is finite, or, what is equivalent, if $\coprod A$ is finite. If A and B are S-sorted sets, then $A \subseteq B$ if, for every $s \in S$, $A_s \subseteq B_s$ and $A \subseteq_{\mathrm{fin}} B$ if A is finite and $A \subseteq B$. Moreover, we denote by $\mathrm{Sub}(A)$ the set of all S-sorted sets X such that, for every $s \in S$, $X_s \subseteq A_s$. Finally, given a set I and an I-indexed family $(A^i)_{i\in I}$ of S-sorted sets, we denote by $\prod_{i\in I} A^i$ the S-sorted set such that, for every $s \in S$,

$$\left(\textstyle\prod_{i\in I} A^i\right)_s = \textstyle\prod_{i\in I} A^i_s,$$

by $\bigcup_{i\in I} A^i$ the S-sorted set such that, for every $s \in S$,

$$\left(\textstyle\bigcup_{i\in I} A^i\right)_s = \textstyle\bigcap_{i\in I} A^i_s,$$

and if I is nonempty, by $\bigcap_{i\in I} A^i$ the S-sorted set such that, for every $s \in S$,

$$\left(\textstyle\bigcap_{i\in I} A^i\right)_s = \textstyle\bigcap_{i\in I} A^i_s.$$

(3) Given a sort $t \in S$ we call *delta of Kronecker in* t, the S-sorted set $\delta^t = (\delta^t_s)_{s\in S}$ defined, for every $s \in S$, as:

$$\delta^t_s = \begin{cases} 1, & \text{if } s = t; \\ \varnothing, & \text{otherwise.} \end{cases}$$

(4) An S-sorted set A is *subfinal* if, for every $s \in S$, $\mathrm{card}(A_s) \leq 1$.
(5) If A and B are S-sorted sets, an S-*sorted mapping from* A *to* B is an S-indexed family $f = (f_s)_{s\in S}$, where, for every s in S, f_s is a mapping from A_s to B_s.
(6) An S-*sorted equivalence on* A is a subset Φ of $A \times A$ such that, for every $s \in S$, Φ_s is an equivalence on A_s. We denote by $\mathrm{Eqv}(A)$ the set of S-sorted equivalences on the S-sorted set A and by $\mathbf{Eqv}(\mathbf{A})$ the ordered set $(\mathrm{Eqv}(A), \subseteq)$. Moreover, A/Φ, the S-*sorted quotient set of* A *modulus* Φ, is $(A_s/\Phi_s)_{s\in S}$

For every set of sorts S, the support of an S-sorted set A is a subset of S, hence it really a mapping $\mathrm{supp}\colon \boldsymbol{\mathcal{U}}^S \longrightarrow \mathrm{Sub}(S)$. In the following proposition we gather together some useful properties of the mapping supp.

Proposition 1. *Let S be a set of sorts, A, B be two S-sorted sets, $(A^i)_{i\in I}$ a family of S-sorted sets, and Φ an S-sorted equivalence on an A. Then the following properties hold:*

(1) *If* $A \subseteq B$, *then* $\mathrm{supp}(A) \subseteq \mathrm{supp}(B)$.
(2) $\mathrm{supp}((\varnothing)_{s\in S}) = \varnothing$.
(3) $\mathrm{supp}(\bigcup_{i\in I} A^i) = \bigcup_{i\in I} \mathrm{supp}(A^i)$.
(4) *If I is nonempty,* $\mathrm{supp}(\bigcap_{i\in I} A^i) = \bigcap_{i\in I} \mathrm{supp}(A^i)$.
(5) $\mathrm{supp}(\prod_{i\in I} A^i) = \bigcap_{i\in I} \mathrm{supp}(A_i)$.
(6) $\mathrm{supp}(A) - \mathrm{supp}(B) \subseteq \mathrm{supp}(A - B)$.
(7) $\mathrm{Hom}(A, B) \neq \varnothing$ *iff* $\mathrm{supp}(A) \subseteq \mathrm{supp}(B)$.
(8) $\mathrm{supp}(A) = \mathrm{supp}(A/\Phi)$.

Following this we define the concepts of many-sorted signature, algebra, and homomorphism.

Definition 2. A *many-sorted signature* is a pair (S, Σ), where S is a set of sorts and Σ an S-*sorted signature*, i.e., a function from $S^\star \times S$ to $\boldsymbol{\mathcal{U}}$ which sends a pair $(w, s) \in S^\star \times S$ to the set $\Sigma_{w,s}$ of the formal operations of arity w, sort (or coarity)

s, and rank (or biarity) (w, s). Sometimes we will write $\sigma\colon w \longrightarrow s$ to indicate that the *formal operation* σ belongs to $\Sigma_{w,s}$. From now on, to shorten notation, we will write $\mathbf{\Sigma}$ instead of (S, Σ).

Definition 3. Let $\mathbf{\Sigma}$ be a many-sorted signature. Then

(1) The $S^{\star} \times S$-sorted set of the *finitary operations on* an S-sorted set A, denoted by $\mathrm{HOp}_S(A)$, is

$$(\mathrm{Hom}(A_w, A_s))_{(w,s)\in S^{\star}\times S},$$

where $A_w = \prod_{i\in|w|} A_{w_i}$, with $|w|$ denoting the length of the word w.

(2) A *structure of* $\mathbf{\Sigma}$*-algebra on* an S-sorted set A is a family $F = (F_{w,s})_{(w,s)\in S^{\star}\times S}$, where, for $(w, s) \in S^{\star} \times S$, $F_{w,s}$ is a mapping from $\Sigma_{w,s}$ to $\mathrm{Hom}(A_w, A_s)$. For a pair $(w, s) \in S^{\star} \times S$ and a formal operation $\sigma \in \Sigma_{w,s}$, in order to simplify the notation, the operation from A_w to A_s corresponding to σ under $F_{w,s}$ will be written as F_σ instead of $F_{w,s}(\sigma)$.

(3) A $\mathbf{\Sigma}$*-algebra* is a pair (A, F), abbreviated to $\mathbf{A}$, where A is an S-sorted set and F a structure of $\mathbf{\Sigma}$-algebra on A.

(4) A $\mathbf{\Sigma}$*-homomorphism* from $\mathbf{A}$ to $\mathbf{B}$, where $\mathbf{B} = (B, G)$, is a triple $(\mathbf{A}, f, \mathbf{B})$, abbreviated to $f\colon \mathbf{A} \longrightarrow \mathbf{B}$, where f is an S-sorted mapping from A to B such that, for every $(w, s) \in S^{\star} \times S$, $\sigma \in \Sigma_{w,s}$, and $(a_i)_{i\in|w|} \in A_w$ we have that

$$f_s(F_\sigma((a_i)_{i\in|w|})) = G_\sigma(f_w((a_i)_{i\in|w|})),$$

where f_w is the mapping $\prod_{i\in|w|} f_{w_i}$ from A_w to B_w which sends $(a_i)_{i\in|w|}$ in A_w to $(f_{w_i}(a_i))_{i\in|w|}$ in B_w.

We denote by $\mathbf{Alg}(\mathbf{\Sigma})$ the category of $\mathbf{\Sigma}$-algebras.

Sometimes, to avoid any confusion, we will denote the structures of $\mathbf{\Sigma}$-algebra of the $\mathbf{\Sigma}$-algebras $\mathbf{A}$, $\mathbf{B}$, ..., by $F^{\mathbf{A}}$, $F^{\mathbf{B}}$, ..., respectively, and the components of $F^{\mathbf{A}}$, $F^{\mathbf{B}}$, ..., as $F^{\mathbf{A}}_\sigma$, $F^{\mathbf{B}}_\sigma$, ..., respectively.

Next we define the concept of subalgebra of a many-sorted algebra.

Definition 4. Let $\mathbf{A}$ be a $\mathbf{\Sigma}$-algebra and $X \subseteq A$.

(1) Let σ be such that $\sigma\colon w \longrightarrow s$, i.e., a formal operation in $\Sigma_{w,s}$. We say that X is *closed under the operation* $F_\sigma\colon A_w \longrightarrow A_s$ if, for every $a \in X_w$, $F_\sigma(a) \in X_s$.

(2) We say that X is a *subalgebra* of $\mathbf{A}$ if X is closed under the operations of $\mathbf{A}$. We denote by $\mathrm{Sub}(\mathbf{A})$ the set of all subalgebras of $\mathbf{A}$.

Following this we recall the concept of product of a family of many-sorted algebras.

Definition 5. Let $(\mathbf{A}^i)_{i\in I}$ be a family of $\mathbf{\Sigma}$-algebras, where, for $i \in I$, $\mathbf{A}^i = (A^i, F^i)$.

(1) The *product* of $(\mathbf{A}^i)_{i\in I}$, denoted by $\prod_{i\in I} \mathbf{A}^i$, is the $\mathbf{\Sigma}$-algebra $(\prod_{i\in I} A^i, F)$ where, for every $\sigma\colon w \longrightarrow s$ in Σ, F_σ is defined as

$$F_\sigma \begin{cases} (\prod_{i\in I} A^i)_w & \longrightarrow \prod_{i\in I} A^i_s \\ (a_\alpha \mid \alpha \in |w|) & \longmapsto (F^i_\sigma(a_\alpha(i) \mid \alpha \in |w|))_{i\in I} \end{cases}$$

(2) The i-th canonical projection, pr^i, is the homomorphism from $\prod_{i\in I} \mathbf{A}^i$ to $\mathbf{A}^i$ defined, for every $s \in S$, as follows

$$\mathrm{pr}^i_s \begin{cases} \prod_{i\in I} A^i_s & \longrightarrow A^i_s \\ (a_i \mid i \in I) & \longmapsto a_i \end{cases}$$

We define next the concept of subfinal many-sorted algebra, since it will be used in the following section in an essential way.

Definition 6. A $\mathbf{\Sigma}$-algebra $\mathbf{A}$ is subfinal if, for every $\mathbf{\Sigma}$-algebra $\mathbf{B}$, there is at most a homomorphism from $\mathbf{B}$ to $\mathbf{A}$.

We point out that the subfinal many-sorted algebras are subobjects of the final many-sorted algebra, therefore their underlying many-sorted sets are subfinal.

We define now the concepts of many-sorted congruence on a many-sorted algebra and of many-sorted quotient algebra of a many-sorted algebra modulo a many-sorted congruence.

Definition 7. Let $\mathbf{A}$ be a $\mathbf{\Sigma}$-algebra and Φ an S-sorted equivalence on A. We say that Φ is an *S-sorted congruence on* $\mathbf{A}$ if, for every $(w,s) \in (S^{\star} - \{\lambda\}) \times S$, $\sigma\colon w \longrightarrow s$, and $a, b \in A_w$ we have that

$$\frac{\forall i \in |w|, a_i \equiv_{\Phi_{w(i)}} b_i}{F_\sigma(a) \equiv_{\Phi_s} F_\sigma(b)}$$

We denote by $\mathrm{Cgr}(\mathbf{A})$ the set of S-sorted congruences on $\mathbf{A}$ and by $\mathbf{Cgr}(\mathbf{A})$ the ordered set $(\mathrm{Cgr}(\mathbf{A}), \subseteq)$.

Definition 8. Let $\mathbf{A}$ be a $\mathbf{\Sigma}$-algebra and $\Phi \in \mathrm{Cgr}(\mathbf{A})$. The *many-sorted quotient algebra of* $\mathbf{A}$ *modulus* Φ, $\mathbf{A}/\Phi$, is the $\mathbf{\Sigma}$-algebra $((A_s/\Phi_s)_{s\in S}, F)$ where, for every $\sigma\colon w \longrightarrow s$ in Σ, the operation $F_\sigma\colon (A/\Phi)_w \longrightarrow A_s/\Phi_s$ is defined, for every $([a_i]_{\Phi_{w(i)}})_{i\in|w|} \in (A/\Phi)_w$, as follows

$$F_\sigma \begin{cases} (A/\Phi)_w & \longrightarrow A_s/\Phi_s \\ ([a_i]_{\Phi_{w(i)}})_{i\in|w|} & \longmapsto [F_\sigma(a_i \mid i \in |w|)]_{\Phi_s} \end{cases}$$

3. Directly irreducible many-sorted algebras.

In this section we show that every finite many-sorted algebra is isomorphic to a finite product of finite directly irreducible many-sorted algebras.

Unlike that which happens for single-sorted algebras, there exists subfinal, but not final, many-sorted algebras that are isomorphic to products of nonempty families of nonsubfinal many-sorted algebras, and this is so because the supports of the factors can strictly contain the support of the product. This suggest that in the definition of directly reducible many-sorted algebra we should require that the supports of the factors of the product be included in the support of the many-sorted algebra under consideration. This additional condition will allow us to obtain the theorem about the representation of a finite many-sorted algebra as a product of a finite family of finite directly irreducible many-sorted algebras.

Definition 9. Let $\mathbf{A}$ be a $\mathbf{\Sigma}$-algebra. We say that $\mathbf{A}$ is *directly reducible* if $\mathbf{A}$ is isomorphic to a product of two nonsubfinal $\mathbf{\Sigma}$-algebras such that their supports are included in that of $\mathbf{A}$. If $\mathbf{A}$ is not directly reducible, then we will say that $\mathbf{A}$ is *directly irreducible.*

Obviously, every subfinal $\mathbf{\Sigma}$-algebra is directly irreducible. Moreover, every finite $\mathbf{\Sigma}$-algebra $\mathbf{A}$ such that, for some $S \in S$, $\mathrm{card}(A_s)$ is a prime number is also directly irreducible.

As for single-sorted algebras, we define the factorial congruences on a many-sorted algebra, from which we will obtain a characterization of the directly irreducible many-sorted algebras.

Definition 10. Let Φ and Ψ be two congruences on a $\mathbf{\Sigma}$-algebra $\mathbf{A}$. We say that Φ and Ψ are a *pair of factorial congruences on* $\mathbf{A}$ if they satisfy the following conditions:

$$\begin{aligned}\Phi \wedge \Psi &= \Delta_{\mathbf{A}},\\ \Phi \circ \Psi &= \Psi \circ \Phi,\\ \Phi \vee \Psi &= \nabla_{\mathbf{A}}.\end{aligned}$$

Proposition 2. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two* $\mathbf{\Sigma}$*-algebras. Then the kernels of the canonical projections from* $\mathbf{A}\times\mathbf{B}$ *to* $\mathbf{A}$ *and* $\mathbf{B}$*, denoted by* $\mathrm{Ker}(\mathrm{pr}_0)$ *and* $\mathrm{Ker}(\mathrm{pr}_1)$*, respectively, are a pair of factorial congruences on* $\mathbf{A}\times\mathbf{B}$*.*

Proposition 3. *If* Φ *and* Ψ *is a pair of factorial congruences on* $\mathbf{A}$*, then we have that* $\mathbf{A} \cong \mathbf{A}/\Phi \times \mathbf{A}/\Psi$*.*

Proof. Let $f\colon A \longrightarrow A/\Phi \times A/\Psi$ be the S-sorted mapping defined, for every $s \in S$ and $a \in A_{s\in S}$, as $f_s(a) = ([a]_{\Phi_s}, [a]_{\Psi})$. It is obvious that f is a homomorphism. Moreover, if $f_s(a) = f_s(b)$, then $(a, b) \in \Phi_s$ and $(a, b) \in \Psi_s$, hence f is injective. Finally, if $a,b \in A_s$, then, because the congruences are such that $\Phi\circ\Psi = \Psi\circ\Phi$, there exists an $c \in A_s$ such that $(a, c) \in \Phi_s$ and $(c, b) \in \Psi_s$, hence $f_s(c) = ([a]_{\Phi_s}, [a]_{\Psi})$ and f is surjective. □

Proposition 4. *Let* $\mathbf{A}$ *be a* $\mathbf{\Sigma}$*-algebra. Then* $\mathbf{A}$ *is directly irreducible if and only if* $\Delta_{\mathbf{A}}$ *and* $\nabla_{\mathbf{A}}$ *is the only pair of factorial congruences on* $\mathbf{A}$*.*

Theorem 1. *Every finite* $\mathbf{\Sigma}$*-algebra is isomorphic to a product of a finite family of finite directly irreducible* $\mathbf{\Sigma}$*-algebras.*

Proof. Let $\mathbf{A}$ be a finite $\mathbf{\Sigma}$-algebra. If $\mathrm{card}(\coprod_{s\in S} A_s) = 0$, then $\mathbf{A}$ is irreducible. Let $\mathbf{A}$ be such that $\mathrm{card}(\coprod_{s\in S} A_s) = n+1$, with $n \geq 0$, and let us assume the theorem for every finite $\mathbf{\Sigma}$-algebra $\mathbf{B}$ such that $\mathrm{card}(\coprod_{s\in S} B_{s\in S}) \leq n$. If $\mathbf{A}$ is directly irreducible, then we are finished. Otherwise, we have that $\mathbf{A} \cong \mathbf{A}^0 \times \mathbf{A}^1$, with $\mathbf{A}^0$ and $\mathbf{A}^1$ nonsubfinal $\mathbf{\Sigma}$-algebras and such that, for $i = 0, 1$, $\mathrm{supp}(A^i) \subseteq \mathrm{supp}(A)$.

Let $\mathbf{A}^i{\restriction}T$ be, for $i = 0, 1$ and $T = \mathrm{supp}(A) = \mathrm{supp}(A^0)\cap\mathrm{supp}(A^1)$, the $\mathbf{\Sigma}$-algebra $(A^i{\restriction}T, F^{\mathbf{A}^i{\restriction}T})$, where $A^i{\restriction}T$, for every $s \in S$, is defined as

$$(A^i{\restriction}T)_s = \begin{cases} A^i_s, & \text{if} s \in T;\\ \varnothing, & \text{otherwise,}\end{cases}$$

and $F^{\mathbf{A}^i{\restriction}T}$ is defined, for every $(w, s) \in S^\star \times S$, as

$$F^{\mathbf{A}^i{\restriction}T}_{w,s}\begin{cases} \Sigma_{w,s} \longrightarrow \mathrm{Hom}((A^i{\restriction}T)_w, (A^i{\restriction}T)_s)\\ \quad\sigma \quad\longmapsto \begin{cases} F^{\mathbf{A}^i}(\sigma), & \text{if } \mathrm{Im}(w) \subseteq T \text{ and } s \in T;\\ \alpha_{A_s} : \varnothing \longrightarrow A_s, & \text{if } \mathrm{Im}(w) \nsubseteq T,\end{cases}\end{cases}$$

where α_{A_s} is the unique mapping from $\varnothing$ to A_s. The definition of the many-sorted structure is sound since, for $\sigma\colon w \longrightarrow s$, both $\mathrm{Im}(w) \subseteq T$ and $s \notin T$ can not occur.

From this it follows that $\mathbf{A} \cong \mathbf{A}^0{\restriction}T \times \mathbf{A}^1{\restriction}T$ and, for $i = 0, 1$, that $\mathrm{card}(A^i{\restriction}T) < \mathrm{card}(A)$, hence, by the induction hypothesis, we can assert that

$$\begin{aligned}\mathbf{A}^0{\restriction}T &\cong \mathbf{B}^0 \times \cdots \times \mathbf{B}^{p-1}\\ \mathbf{A}^1{\restriction}T &\cong \mathbf{C}^0 \times \cdots \times \mathbf{C}^{q-1}\end{aligned}$$

where, for $j \in p$ and $h \in q$, $\mathbf{B}^j$ and $\mathbf{C}^k$ are directly irreducible. Therefore,

$$\mathbf{A} \cong \mathbf{B}^0 \times \cdots \times \mathbf{B}^{p-1} \times \mathbf{C}^0 \times \cdots \times \mathbf{C}^{q-1}.$$

□

4. Subdirectly irreducible algebras.

In this last section we extend to the many-sorted algebras that theorem of Birkhoff according to which every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras. To this end we begin by defining the concept of subdirect product of a family of many-sorted algebras.

Definition 11. A $\mathbf{\Sigma}$-algebra $\mathbf{A}$ is a *subdirect product of a family of* $\mathbf{\Sigma}$*-algebras* $(\mathbf{A}^i)_{i\in I}$ if it satisfies the following conditions

(1) $\mathbf{A}$ is a subalgebra of $\prod_{i\in I}\mathbf{A}^i$.
(2) For every $i\in I$, $\mathrm{pr}^i\restriction\mathbf{A}$ is surjective.

On the other hand, we will say that an embedding $f\colon \mathbf{A}\longrightarrow\prod_{i\in I}\mathbf{A}^i$ is a *subdirect embedding* if $f[\mathbf{A}]$ is a subdirect product of $(\mathbf{A}^i)_{i\in I}$.

Proposition 5. *Let* $\mathbf{A}$ *be a* $\mathbf{\Sigma}$*-algebra and* $(\Phi^i)_{i\in I}$ *a family of congruences on* $\mathbf{A}$*. Then* $\mathbf{A}/\bigcap_{i\in I}\Phi^i$ *can be subdirectly embedded into* $\prod_{i\in I}\mathbf{A}/\Phi^i$*.*

Proof. Let f^i be, for every $i\in I$, the unique homomorphism from $\mathbf{A}/\bigcap_{i\in I}\Phi^i$ into $\mathbf{A}/\Phi^i$ such that $f^i\circ\mathrm{pr}^{\bigcap_{i\in I}\Phi^i}=\mathrm{pr}^{\Phi^i}$. Then the unique homomorphism $\langle f^i\rangle_{i\in I}\colon \mathbf{A}/\bigcap_{i\in I}\Phi^i\longrightarrow\prod_{i\in I}\mathbf{A}/\Phi^i$ determined by the universal property of the product, is a subdirect embedding. □

Corollary 1. *Let* $\mathbf{A}$ *be a* $\mathbf{\Sigma}$*-algebra and* $(\Phi^i)_{i\in I}$ *a family of congruences on* $\mathbf{A}$ *such that* $\bigcap_{i\in I}\Phi^i=\Delta_{\underline{A}}$*. Then* $\langle\mathrm{pr}^i\rangle_{i\in I}\colon\mathbf{A}\longrightarrow\prod_{i\in I}\mathbf{A}/\Phi^i$ *is a subdirect embedding.*

Definition 12. Let $\mathbf{A}$ be a $\mathbf{\Sigma}$-algebra. We say that $\mathbf{A}$ is *subdirectly irreducible* if, for every subdirect embedding f of $\mathbf{A}$ into the cartesian product $\prod_{i\in I}\mathbf{A}^i$ of a nonempty family of $\mathbf{\Sigma}$-algebras $(\mathbf{A}^i)_{i\in I}$, there exists an index $i\in I$ such that the homomorphism $\mathrm{pr}^i\circ f\colon\mathbf{A}\longrightarrow\mathbf{A}^i$ is injective.

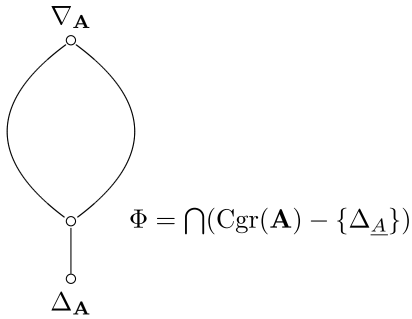
Proposition 6. *A* $\mathbf{\Sigma}$*-algebra* $\mathbf{A}$ *is subdirectly irreducible if and only if* $\mathbf{A}$ *is subfinal or there exists a minimum congruence in* $\mathrm{Cgr}(\mathbf{A})-\{\Delta_{\underline{A}}\}$*.*

Proof. If $\mathbf{A}$ is not subfinal and $\mathrm{Cgr}(\mathbf{A})-\{\Delta_{\underline{A}}\}$ has not a minimum congruence, then $\bigcap(\mathrm{Cgr}(\mathbf{A})-\{\Delta_{\underline{A}}\})=\Delta_{\underline{A}}$. Let $I=$ $mathrmCgr(\mathbf{A})-\{\Delta_{\underline{A}}\}$ be, then the canonical mapping $\langle\mathrm{pr}^{\Phi}\rangle_{\Phi\in I}\colon\mathbf{A}\longrightarrow\prod_{\Phi\in I}\mathbf{A}/\Phi$ is, by the Corollary 1, a subdirect embedding and since, for every $\Phi\in I$, the canonical projections $\mathrm{pr}^{\Phi}\colon\mathbf{A}\longrightarrow\mathbf{A}/\Phi$ are not injectives, it follows that $\mathbf{A}$ is not subdirectly irreducible. Therefore, if $\mathbf{A}$ is subdirectly irreducible, then $\mathbf{A}$ is subfinal or there exists a minimum congruence in $\mathrm{Cgr}(\mathbf{A})-\{\Delta_{\underline{A}}\}$.

If $\mathbf{A}$ is subfinal, then it is subdirectly irreducible, since if f is a subdirect embedding of $\mathbf{A}$ into the cartesian product $\prod_{i\in I}\mathbf{A}^i$ of a nonempty family of $\mathbf{\Sigma}$-algebras $(\mathbf{A}^i)_{i\in I}$, then, for every $i\in I$, pr^i is surjective and $\mathrm{supp}(A)=\mathrm{supp}(\prod_{i\in I}A^i)=\mathrm{supp}(A^i)$, hence $\mathbf{A}\cong\prod_{i\in I}\mathbf{A}^i\cong\mathbf{A}^i$, for every $i\in I$.

Finally, let us suppose that there exists a minimum congruence Φ in $\mathrm{Cgr}(\mathbf{A})-\{\Delta_{\underline{A}}\}$, hence, necessarily, $\Phi=\bigcap(\mathrm{Cgr}(\mathbf{A})-\{\Delta_{\mathbf{A}}\})(\neq\Delta_{\mathbf{A}})$ and $\mathbf{A}$ is not subfinal. Therefore we can choose a sort $s\in S$ and a pair $(a,b)\in\Phi_s$ such that $a\neq b$. Let $f\colon\mathbf{A}\longrightarrow\prod_{i\in I}\mathbf{A}^i$ be a subdirect embedding of $\mathbf{A}$ into the cartesian product $\prod_{i\in I}\mathbf{A}^i$ of the $\mathbf{\Sigma}$-algebras $\mathbf{A}^i$. Then there exists an index $i\in I$ such that $(\mathrm{pr}^i_s\circ f_s)(a)\neq(\mathrm{pr}^i_s\circ f_s)(b)$, since, otherwise, $f_s(a)=f_s(b)$ and therefore $a=b$, which is a contradiction. From this follows that $(a,b)\notin\mathrm{Ker}(\mathrm{pr}^i_s\circ f_s)$ and, since $(a,b)\in\Phi_s$, that $\Phi\nsubseteq\mathrm{Ker}(\mathrm{pr}^i\circ f)$, thus $\mathrm{Ker}(\mathrm{pr}^i\circ f)=\Delta_{\underline{A}}$ and, consequently, $\mathrm{pr}^i\circ f\colon\mathbf{A}\longrightarrow\mathbf{A}^i$ is injective. From this we can assert that $\mathbf{A}$ is subdirectly irreducible. □

Remark. If for a $\mathbf{\Sigma}$-algebra $\mathbf{A}$ the lattice $(\mathrm{Cgr}(\mathbf{A}) - \{\Delta_{\underline{A}}\}, \subseteq)$ has a minimum Φ, then the lattice $\mathbf{Cgr}(\mathbf{A})$ has the form:



where $\nabla_{\mathbf{A}}$ is $A \times A$, the maximum congruence on $\mathbf{A}$. The congruence Φ, called the *monolith of* $\mathbf{A}$ and denoted by $\mathrm{M}^{\mathbf{A}}$, has the property that $\mathrm{M}^{\mathbf{A}} = \mathrm{Cg}_{\mathbf{A}}(\delta^{s,(a,b)})$, for every $s \in S$ and every $(a,b) \in \mathrm{M}^{\mathbf{A}}_s$, with $a \neq b$, where $\delta^{s,(a,b)}$ is the S-sorted set which has as s-th coordinate the set $\{(a,b)\}$ and as t-th coordinate, for $t \neq s$, the empty set, and $\mathrm{Cg}_{\mathbf{A}}$ the generated congruence operator for $\mathbf{A}$.

We define next the simple many-sorted algebras, that are a special kind of subdirectly irreducible algebra.

Definition 13. Let $\mathbf{A}$ be a $\mathbf{\Sigma}$-algebra. We say that $\mathbf{A}$ is *simple* if A is subfinal or $\mathrm{Cgr}(\mathbf{A})$ has exactly two congruences. Moreover, we say that a congruence Φ on $\mathbf{A}$ is *maximal* if the interval $[\Phi, \nabla_{\underline{A}}]$ in the lattice $\mathbf{Cgr}(\mathbf{A})$ has exactly two congruences.

As for single-sorted algebras, also many-sorted algebras it is true that the quotient many-sorted algebra of a many-sorted algebra by a congruence on it is simple if and only if the congruence is maximal or the congruence is the maximum congruence on the many-sorted algebra.

Proposition 7. *Let* $\mathbf{A}$ *be a* $\mathbf{\Sigma}$*-algebra and* Φ *a congruence on* $\mathbf{A}$*. Then* $\mathbf{A}/\Phi$ *is simple if and only if* Φ *is a maximal congruence on* $\mathbf{A}$ *or* $\Phi = \nabla_{\underline{A}}$*.*

In the following proposition we gather together some relations between the simple, the subdirectly irreducible, and the directly irreducible many-sorted algebras.

Proposition 8. *Every simple many-sorted algebra is subdirectly irreducible and every subdirectly irreducible many-sorted algebra is directly irreducible.*

We prove next, as was announced in the introduction of this paper, the many-sorted counterpart of the well-known theorem of Birkhoff about the representation of every single-sorted algebra as a subdirect product of subdirectly irreducible single-sorted algebras, is also true for the many-sorted algebras.

Theorem 2 (Birkhoff)**.** *Every many-sorted algebra is isomorphic to a subdirect product of a family of subdirectly irreducible many-sorted algebras.*

Proof. Since the subfinal $\mathbf{\Sigma}$-algebras are subdirectly irreducibles, it is enough to consider nonsubfinal $\mathbf{\Sigma}$-algebras. Let $\mathbf{A}$ be a nonsubfinal $\mathbf{\Sigma}$-algebra and

$$I = \textstyle\bigcup_{s \in S}(\{s\} \times (A_s^2 - \Delta_{A_s}))$$

that is nonempty, because $\mathbf{A}$ is nonsubfinal. Then, for every $(s,(a,b)) \in I$, making use of the lemma of Zorn, there exists a congruence $\Phi^{(s,(a,b))}$ on $\mathbf{A}$ such that $\Phi^{(s,(a,b))} \cap \delta^{s,(a,b)} = (\varnothing)_{s \in S}$ and maximal with that property. Moreover, the congruence $\Phi^{(s,(a,b))} \vee \mathrm{Cg}_{\mathbf{A}}(\delta^{s,(a,b)})$ is the minimum in $[\Phi^{(s,(a,b))}, \nabla_{A}] - \{\Phi^{(s,(a,b))}\}$. Therefore, in the lattice $\mathbf{Cgr}(\mathbf{A}/\Phi^{(s,(a,b))})$, the congruence $\Phi^{(s,(a,b))} \vee \mathrm{Cg}_{\mathbf{A}}(\delta^{s,(a,b)})$ is the monolith of $\mathbf{A}/\Phi^{(s,(a,b))}$, that is subdirectly irreducible.

Since $\bigcap\{\Phi^{(s,(a,b))} \mid (s,(a,b)) \in I\} = \Delta_{\underline{A}}$, we have, finally, that $\mathbf{A}$ can be subdirectly embedded in $\prod(\mathbf{A}/\Phi^{(s,(a,b))})_{(s,(a,b))\in I}$, which is a product of subdirectly irreducible $\mathbf{\Sigma}$-algebras. □

Corollary 2. *Every finite many-sorted algebra is isomorphic to a subdirect product of a finite family of finite subdirectly irreducible many-sorted algebras.*

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