

New Approximation to the One-sided Radial Crossing Minimization

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Abstract

In this paper, we study a crossing minimization problem in a *radial drawing* of a graph. Radial drawings have strong application in social network visualization, for example, displaying *centrality* indices of actors [20]. The main problem of this paper is called the *one-sided radial crossing minimization* between two concentric circles, named *orbits*, where the positions of vertices in the outer orbit are fixed. The main task of the problem is to find the vertex ordering of the free orbit and edge routing between two orbits in order to minimize the number of edge crossings. The problem is known to be NP-hard [1], and the first polynomial time 15-approximation algorithm was presented in [9].

In this paper, we present a new 3α -approximation algorithm for the case when the free orbit has no leaf vertex, where α represents the approximation ratio of the one-sided crossing minimization problem in a horizontal drawing. Using the best known result of $\alpha = 1.4664$ [13], our new algorithm can achieve 4.3992-approximation.

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1 Introduction

1.1 Horizontal Drawings and Radial Drawings

Given a bipartite graph $G = (V, W, E)$ with two parallel straight-lines L_1 and L_2 , a 2-layered *horizontal drawing* consists of placing vertices in the vertex set V on L_1 and placing vertices in the vertex set W on L_2 . In a horizontal drawing, each edge is represented by a straight-line segment joining the two end-vertices, as shown in Fig. 1(a).

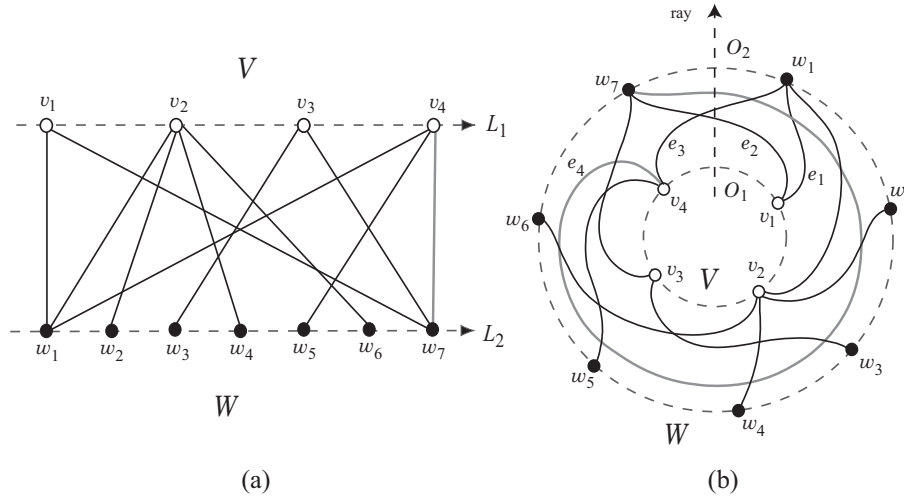


Figure 1: (a) A horizontal drawing $D = (\pi, \sigma)$ of a bipartite graph $G = (V, W, E)$; (b) A radial drawing $D = (\pi, \sigma, \psi)$ of the graph G with two orbits O_1 and O_2 , where $\psi(e_1) = 0$, $\psi(e_2) = 1$, $\psi(e_3) = -1$, and $\psi(e_4) = 1$ for $e_1 = (v_1, w_1)$, $e_2 = (v_1, w_7)$, $e_3 = (v_4, w_1)$, and $e_4 = (v_4, w_7)$.

Given a bipartite graph $G = (V, W, E)$ with two concentric circles called orbits O_1 and O_2 , a 2-layered radial drawing consists of placing vertices in V on O_1 and placing vertices in W on O_2 . Note that in a radial drawing, each edge is drawn as a simple curve in the area between O_1 and O_2 , see Fig. 1(b).

In a horizontal drawing, the embedding is fully determined by the vertex orderings of V and W . However, for radial drawings, it is necessary to know where the orderings start (without loss of generality, clockwise) and end on each orbit. For this purpose, we introduce a *ray* that indicates this borderline between the first and last vertices in the orderings. The ray is a simple curve that intersects each orbit exactly once, as shown in Fig. 1(b), where a ray is typically drawn as a straight-line segment. An edge $e \in E$ is called a *cut edge* in a radial drawing if it intersects the ray of the drawing.

Both 2-layered horizontal and radial drawings are fundamental building blocks for drawing graphs with multi-layers (or levels) in order to display *hier-*

archical relationships in graphs, according to the *Sugiyama* method [18, 19] and the radial drawing framework [1], an adaptation of the Sugiyama method to radial drawings. Moreover, radial drawing is one of the most popular drawing conventions in social network visualization for displaying *centralities*. Centrality indices are one of the fundamental concepts in social network analysis for identifying important actors in the social network [20]. In a radial drawing of a social network, the most prominent (or influential) actor is placed in the center, and then other actors are embedded on the concentric circles, based on their relative centrality values; i.e., more important actors are placed on inner circles, and less important actors are placed on the outer circles [1].

1.2 Crossing Minimization in Horizontal Drawings

There is a rich literature on minimizing edge crossings of drawings in Graph Drawing (see [3, 11, 15]). In specific applications such as VLSI layout design, drawings with fewer edge crossings can lead to a product with less cost and higher reliability. Further, human experiments suggested that minimizing edge crossing is one of the most important aesthetics criteria for understanding a drawing of a graph [16].

Given a bipartite graph $G = (V, W, E)$ with two parallel lines L_1 and L_2 , and a fixed ordering of vertices in W on L_2 , the *one-sided horizontal crossing minimization* (OHCM) problem asks to find an ordering of vertices in V on L_1 such that the number of edge crossings between two layers is minimized. Note that the OHCM problem is one of the major step in the Sugiyama method: Sugiyama *et al.* [19] first assigned vertices into multiple layers (levels), and then repetitively performed the one-sided horizontal crossing minimization from the top layer to the bottom layer and vice versa in order to minimize edge crossings between adjacent layers in horizontal drawings.

The crossing minimization problem in horizontal drawings is well studied. The OHCM problem was proved to be NP-complete [7, 8], and a number of heuristics, approximation algorithms and exact algorithms have been proposed [3, 7, 10, 11, 13, 18, 19]. Jünger and Mutzel [10] presented integer linear programming algorithms. For heuristics such as *Adjacency-Exchange*, *Split*, and *Sifting*, see [3, 11].

For a bipartite graph $G = (V, W, E)$ and a given fixed ordering σ of vertices in W , a well known lower bound $LB(G, \sigma)$ on the minimum number of crossings over all orderings π of vertices in V is obtained by summing up $\min\{\chi(u, v; G), \chi(v, u; G)\}$ over all vertex pairs $u, v \in V$, where $\chi(u, v; G)$ denotes the number of crossings generated by the edges incident to u and v when u precedes v in an ordering of V . Eades and Wormald [7] proposed a *Median method*, which produces a 3-approximate solution based on the lower bound $LB(G, \sigma)$. Note that the *Barycenter method* by Sugiyama *et al.* [19] is an $O(\sqrt{n})$ -approximation algorithm [7]. Currently, the best known approximation algorithm for the OHCM problem, given by Nagamochi [13] delivers a drawing with at most $1.4664LB(G, \sigma)$ crossings.

1.3 Crossing Minimization in Radial Drawing

Given a bipartite graph $G = (V, W, E)$ with two orbits O_1 and O_2 , and a fixed ordering of vertices in W on O_2 , the *one-sided radial crossing minimization* (ORCM) problem asks to find an ordering of vertices in V on O_1 and a routing of edges between two orbits in order to minimize the number of edge crossings. Note that the ORCM problem is the major step in the radial drawing framework, an adaptation of the Sugiyama method to radial drawings, presented by Bachmaier [1].

Note that the problem of crossing minimization in radial drawing is more challenging, as it involves both *vertex ordering* and *edge routing* problems. That is, even if the orderings of vertices in both orbits are fixed, we still need to decide how to route (i.e., clockwise or counter-clockwise) each edge around the inner orbit in order to minimize the number of edge crossings.

The crossing minimization problem in radial drawings has not been well studied. The ORCM problem was proved to be NP-hard, and heuristics such as *Cartesian barycenter heuristic*, *Cartesian median heuristic*, and *Radial sifting heuristic* were presented with experimental results [1].

Note that the OHCM problem admits the lower bound $LB(G, \sigma)$. However, no such effective lower bound is known to the ORCM problem. Therefore, it is natural to investigate whether the freedom of routing edges around the inner orbit makes the ORCM problem hard to approximate or not.

In our companion paper, we presented the first polynomial time constant factor approximation algorithm [9]. More specifically, we have proved the following result.

Theorem 1 [9] *There is a 15-approximation algorithm for one-sided crossing minimization in a radial drawing. It runs in*

$$O(|E| \min\{|V|, |W|\} \min\{|V||W|, |E| \log(\min\{|V|, |E|})\})$$

time. □

The main idea in deriving Theorem 1 was to reduce a given instance of the ORCM problem to that of the OHCM problem. For this, it was shown that any radial drawing D of a given graph G can be modified so that the resulting radial drawing D' of G has a *crossing-free edge* (i.e., an edge which has no crossing on it), whereas the increased number of edge crossings is at most twice the number of edge crossings in D . Hence the task was to solve the second problem, which requires to find a radial drawing with the minimum number of edge crossings among radial drawings with a prescribed crossing-free edge. Since a direct reduction of the second problem to the OHCM problem may require to solve exponentially many instances of the OHCM problem with auxiliary bipartite graphs, the second problem was *approximately* formulated as the third problem, which requires to solve a polynomial number of instances of the OHCM problem. It was shown that a *new modified median method* to the third problem gave a 5-approximate solution to the second problem, implying that the original problem is 15-approximable.

1.4 Our Contribution

In this paper, we present a new 3α -approximation algorithm for the ORCM problem when the free orbit has no leaf vertex, where α represents the approximation ratio of the algorithm for the OHCM problem. More specifically, the following theorem summarizes our main result.

Theorem 2 *Given a bipartite graph $G = (V, W, E)$ with $\deg(v) \geq 2$ for each $v \in V$, there is a 3α -approximation algorithm for the ORCM problem, where α represents the approximation ratio of the algorithm for the OHCM problem. The algorithm runs in $O(|E| \cdot T(|V|, |E|))$ time, where $T(|V|, |E|)$ represents the running time of the approximation algorithm for the OHCM problem. \square*

To prove Theorem 2, we follow a similar idea to that used in [9], of converting a radial drawing problem into that of a horizontal drawing problem. However, our new algorithm has the following new contribution:

- Using the current best known result of $\alpha = 1.4664$ [13] (see Lemma 3), our new algorithm can achieve 4.3992-approximation ratio in $O(|V|^2|E|^2)$ time. This is a significant improvement over the 15-approximation ratio given in [9].
- We prove that, if the free orbit has no leaves, then any radial drawing D can be modified so that the resulting radial drawing D' has a crossing-free edge with a *special property* while keeping the increased number of edge-crossings at most twice the original number of edge crossings in D . The special property will be explained in more detail in Section 4. Roughly speaking, we use *two edges* (see Fig. 3), instead of the single edge used in [9].
- We formulate a problem of finding a radial drawing with the minimum number of edge crossings among radial drawings that have a prescribed crossing-free edge with the special property. With the special property, the problem can be directly reduced to a polynomial number of instances of the OHCM problem. By computing an α -approximate solution to each of these instances using *any α -approximation algorithm* for the OHCM problem, we obtain a 3α -approximate solution to the original instance of the ORCM problem.

This paper is organized as follows. In Section 2, we formally define a horizontal drawing and a radial drawing of a bipartite graph, and review basic properties. Section 3 investigates a property of radial drawings which have at least one crossing-free edge. In Section 4, we prove that among such drawings, there is a 3-approximate solution to the ORCM problem. Section 5 proves that the problem of finding an optimal radial drawing with a crossing-free edge is α -approximable if the free orbit has no leaf vertex, by reducing the problem to the OHCM problem. The results in Sections 4 and 5 imply Theorem 2. In Section 6, we conclude with some open problems in radial drawings.

2 Preliminaries

Let $G = (V, W, E)$ be a simple bipartite graph with vertex sets V and W and an edge set E . For a vertex $v \in V$ in G , let $E(v; G)$ denote the set of edges incident to v , $N(v; G)$ denote the set of *neighbors* of v (i.e., vertices adjacent to v in G), and $d(v; G)$ denote the degree of a vertex v (i.e., $d(v; G) = |E(v; G)| = |N(v; G)|$). A subgraph $G' = (V', E')$ of $G = (V, E)$ is *induced* by V' if E' is given by $E' = \{e \in E \mid \text{both end-vertices of } e \text{ belong to } V'\}$; G' will be denoted by $G[V']$. Throughout the paper, we assume that $d(u; G) \geq 1$ for all $u \in V \cup W$. In this paper, a *bijection* τ on a finite set T means a bijective function $\tau : T \rightarrow \{0, 1, \dots, |T| - 1\}$.

2.1 Horizontal Drawings

Let π and σ be bijections on V and W , respectively. A pair of π and σ defines a horizontal drawing of G in the plane in such a way that, for two parallel horizontal lines L_1 and L_2 , the vertices in V (respectively, W) are arranged on L_1 (respectively, L_2) according to π (respectively, σ), and each edge is displayed by a straight-line segment joining the end-vertices. For any choice of the coordinates of the points representing the vertices in $V \cup W$ in a horizontal drawing of G defined by (π, σ) , two edges $(v, w), (v', w') \in E$ intersect properly (or create a *crossing*) if and only if $(\pi(v) - \pi(v'))(\sigma(w) - \sigma(w'))$ is negative.

Lemma 1 [14] *The number $\chi(D; G)$ of crossings in a 2-layered horizontal drawing $D = (\pi, \sigma)$ of a bipartite graph $G = (V, W, E)$ can be computed in $O(\min\{|V||W|, |E| \log |V|\})$ time.*

Given a bipartite graph $G = (V, W, E)$ and a bijection σ on W , the OHCM problem asks to find a bijection π on V that minimizes the number of crossings in a horizontal drawing (π, σ) of G . For an ordered pair of two vertices $u, v \in V$, let $\chi(u, v; G)$ denote the number of crossings generated by the edges incident to u and the edges incident to v if $\pi(u) < \pi(v)$ holds in a horizontal drawing $D = (\pi, \sigma)$. The total number $\chi(D; G)$ of edge crossings of $D = (\pi, \sigma)$ of G is given by

$$\chi(D; G) = \sum_{u, v \in V: \pi(u) < \pi(v)} \chi(u, v; G).$$

From the formula of $\chi(D; G)$, we observe the following lower bound on the minimum $\chi(D; G)$.

Lemma 2 [7] *Given a bipartite graph $G = (V, W, E)$ and a bijection σ on W , let $LB(G, \sigma) = \sum_{u, v \in V} \min\{\chi(u, v; G), \chi(v, u; G)\}$, and $\chi_h^*(G, \sigma) = \min\{\chi(D; G) \mid D = (\pi, \sigma), \text{ and } \pi \text{ is a bijection on } V\}$. Then it holds $LB(G, \sigma) \leq \chi_h^*(G, \sigma)$.*

It is known that $\chi_h^*(G, \sigma) \leq 1.4664LB(G, \sigma)$ always holds, and there is an instance (G, σ) with $\chi_h^*(G, \sigma) = 1.1818LB(G, \sigma)$ [13].

Lemma 3 [13] *Given a bipartite graph $G = (V, W, E)$ and a bijection σ on W , a bijection π on V such that $\chi(D; G) \leq 1.4664LB(G, \sigma)$ holds for its horizontal drawing $D = (\pi, \sigma)$ can be found in $O(|V|^2|E|)$ time.*

2.2 Radial Drawings

Let O_1 and O_2 be two orbits with the common center in the plane, where O_1 is the inner orbit and O_2 is the outer orbit. The positions of vertices in V (respectively, W) are defined as a bijection π on V (respectively, σ on W), where positions $0, 1, \dots, |V| - 1$ (respectively, $0, 1, \dots, |W| - 1$) appear in this order when we traverse O_1 (respectively, O_2). Each edge is drawn as a simple curve in the area between O_1 and O_2 .

In horizontal drawings with two layers, a crossing between two edges only depends on the orderings of the end-vertices. In radial drawings, however, it is also necessary to consider the direction in which the edges are wound around the inner orbit. Moreover, edges can also be wound around the inner orbit multiple times.

We consider a radial drawing with the minimum number of edge crossings, and hence we assume that a given radial drawing satisfies the following two conditions:

(C1) Every two edges e and e' cross each other at most once.

(C2) No two edges e and e' incident to the same vertex cross each other.

This is because otherwise we can reduce the number of crossings by edges e and e' in (C1) by two (respectively, (C2) by one) without increasing the number of crossings between any other two edges. We also assume that an edge e may cross the ray only in the same direction, clockwise or counter-clockwise; If e crosses the ray clockwise and counter-clockwise simultaneously, then we can reduce two crossings between the ray and e , without increasing the number of crossings between any other two edges.

For each edge $e \in E$ in a radial drawing, let c_e be the crossings of e with the ray. The *offset* of e in the drawing is then defined to be $\psi(e) = c_e$ if e crosses the ray counter-clockwise and $\psi(e) = -c_e$ otherwise, where $\psi(e) = 0$ if e has no crossing with the ray. The sign of $\psi(e)$ reflects the mathematical direction of rotation, see Fig 1(b). For simplicity, $\psi(e)$ for an edge $e = (u, v)$ may be denoted by $\psi(u, v)$. Thus, we can formally define a *radial drawing* D as a pair of vertex orderings (π, σ) and the edge offsets ψ (i.e., $D = (\pi, \sigma, \psi)$).

We are now ready to describe edges crossings in a radial drawing D . Let $\chi(e_1, e_2; D)$ denote the number of crossings between two edges $e_1, e_2 \in E$. Let $\text{sgn} : \mathbb{R} \rightarrow \{-1, 0, 1\}$ denote the signum function.

Lemma 4 [1] *Let $D = (\pi, \sigma, \psi)$ be a radial drawing of a bipartite graph $G = (V, W, E)$. Then the number of crossings between two edges $e_1 = (v_1, w_1), e_2 = (v_2, w_2) \in E$ is*

$$\chi(e_1, e_2; D) = \max \left\{ 0, |\psi(e_2) - \psi(e_1) + \frac{b-a}{2}| + \frac{|a| + |b|}{2} - 1 \right\},$$

where $a = \text{sgn}(\pi(v_1) - \pi(v_2))$ and $b = \text{sgn}(\sigma(w_1) - \sigma(w_2))$.

We define

$$\begin{aligned} \chi(e; D) &= \sum_{e' \in E - \{e\}} \chi(e, e'; D) \text{ for } e \in E, \\ \chi(D; G) &= \sum_{e, e' \in E: e \neq e'} \chi(e, e'; D), \end{aligned}$$

where $\chi(D; G) = \frac{1}{2} \sum_{e \in E} \chi(e; D)$ holds. We may write $\chi(D; G)$ as $\chi(D)$ if the underlying graph G is clear from the context.

Given a radial drawing and an edge e^* , let $\psi(e; e^*)$, $e \in E - \{e^*\}$, denote the offset of e when e^* is regarded as the ray. For example, $\psi(e_2; e_3) = \psi(e_4; e_3) = 1$ and $\psi(e_1; e_3) = 0$ hold for edges $e_1 = (v_1, w_1)$, $e_2 = (v_1, w_7)$, $e_3 = (v_4, w_1)$, and $e_4 = (v_4, w_7)$ in Fig 1(b).

Let $D = (\pi, \sigma, \psi)$ be a radial drawing satisfying (C1) and (C2). For each edge $e = (v, w)$, let $\chi^-(e; D)$ be the number of edge crossings between e and edges $e' = (v', w')$ with $\psi(e'; e) = -1$. Similarly, let $\chi^+(e; D)$ denote the number of edge crossings between $e = (v, w)$ and edges $e' = (v', w')$ with $\psi(e'; e) = 1$. Obviously $\chi(e; D) = \chi^-(e; D) + \chi^+(e; D)$ holds.

In this paper, we consider the ORCM problem, i.e., the problem of finding a radial drawing $D = (\pi, \sigma, \psi)$ of a bipartite graph $G = (V, W, E)$ that minimizes $\chi(D)$, when a bijection σ on W is fixed. Let $\chi_r^*(G, \sigma)$ denote the optimal value, i.e., the minimum number of edge crossings over all radial drawings of G with specified positions σ of W .

3 Radial Drawings with Crossing-free Edges

An edge e is called *crossing-free* in a radial drawing D if $\chi(e; D) = 0$. Let $D = (\pi, \sigma, \psi)$ be a radial drawing with crossing-free edge $\hat{e} = (\hat{v}, \hat{w}) \in E$ ($\hat{v} \in V$, $\hat{w} \in W$).

Consider offset $\psi(v, w)$ of an edge $e = (v, w) \in E - \{\hat{e}\}$ with $v \in V$ and $w \in W$. If e is not adjacent to \hat{e} , then the offset of e is uniquely determined by the positions of v and w in π and σ , because it does not cross edge \hat{e} in D . More precisely, it is given by

$$\psi(v, w) = \begin{cases} 0 & \text{if } (\pi(v) - \pi(\hat{v}))(\sigma(w) - \sigma(\hat{w})) > 0, \\ 1 & \text{if } \pi(v) < \pi(\hat{v}) \text{ and } \sigma(w) > \sigma(\hat{w}), \\ -1 & \text{if } \pi(v) > \pi(\hat{v}) \text{ and } \sigma(w) < \sigma(\hat{w}). \end{cases} \quad (1)$$

On the other hand, if e is adjacent to \hat{e} , then it has two possible ways of drawing without crossing \hat{e} , and hence the offset of e cannot be determined only by (π, σ) (see Fig. 2(a) and (b)). We can assume that no two edges in $E(\hat{v}; G)$ (respectively, $E(\hat{w}; G)$) cross each other by (C2). There are $d(\hat{v}; G)$ such drawings for $E(\hat{v}; G)$ (respectively, $d(\hat{w}; G)$ such drawings for $E(\hat{w}; G)$).

Thus, $E(\hat{v}; G) - \{\hat{e}\}$ is partitioned into two subsets $E_{\hat{v}}^{right}$ and $E_{\hat{v}}^{left}$, called the *right-set* and *left-set* (each of which may be empty), where edges in $E_{\hat{v}}^{right}$

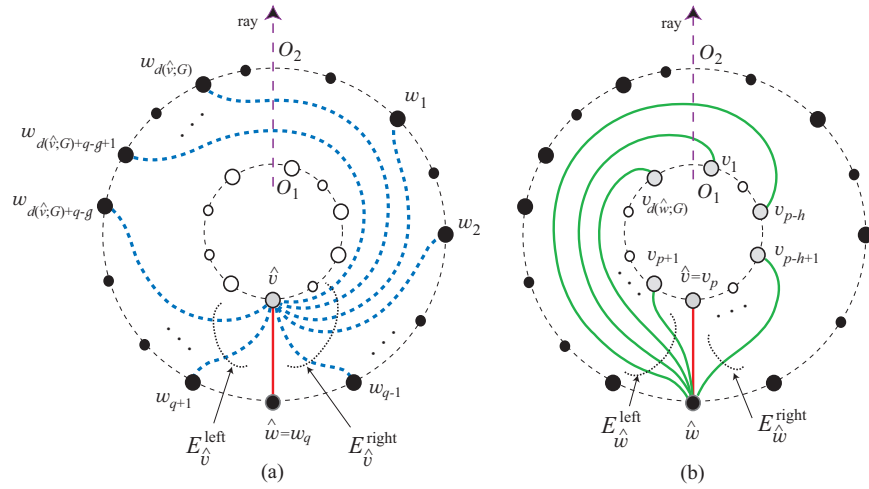


Figure 2: (a) A partition of $E(\hat{v}; G)$, where $g > q$; (b) A partition of $E(\hat{w}; G)$, where $p > h$.

(respectively, $E_{\hat{v}}^{left}$) are drawn so that they enter \hat{v} on the right of \hat{e} (respectively, the left of \hat{e}). We define the right-set $E_{\hat{w}}^{right}$ and the left-set $E_{\hat{w}}^{left}$ analogously. Note that there are $d(\hat{v}; G) - 1$ choices of $E_{\hat{v}}^{left}$ over a fixed ordering σ on W , whereas there are $2^{d(\hat{w}; G)-1}$ choices of $E_{\hat{w}}^{left}$ over all possible orderings π on V .

For $g \in \{1, 2, \dots, d(\hat{v}; G)\}$ and $h \in \{1, 2, \dots, d(\hat{w}; G)\}$, we denote by $\psi_{\pi, \sigma, \hat{e}, g, h}$ the offsets of E such that \hat{e} is crossing-free, $|E_{\hat{v}}^{right}| = g - 1$, and $|E_{\hat{w}}^{right}| = h - 1$. More precisely, $\psi_{\pi, \sigma, \hat{e}, g, h}$ is given as follows.

Let $w_1, w_2, \dots, w_{d(\hat{v}; G)}$ be the neighbors of \hat{v} in G , where $\sigma(w_1) < \sigma(w_2) < \dots < \sigma(w_{d(\hat{v}; G)})$ and $w_q = \hat{w}$. Note that there are exactly $q - 1$ neighbors of \hat{v} on the outer orbit between the ray and $w_q = \hat{w}$. Hence if $g - 1 \geq q - 1$ then the last $(g - 1) - (q - 1)$ neighbors of \hat{v} (i.e., $w_{d(\hat{v}; G)+q-g+1}, w_{d(\hat{v}; G)+q-g+2}, \dots, w_{d(\hat{v}; G)}$) are joined to \hat{v} via edges with offsets 1 (see Fig. 2(a)). Thus, if $g \geq q$ then

$$\psi_{\pi, \sigma, \hat{e}, g, h}(\hat{v}, w_i) = \begin{cases} 0 & \text{for } 1 \leq i \leq d(\hat{v}; G) + q - g, \\ 1 & \text{for } d(\hat{v}; G) + q - g + 1 \leq i \leq d(\hat{v}; G) \end{cases}$$

On the other hand, if $g - 1 < q - 1$ then the first $(g - 1) - (q - 1)$ neighbors of \hat{v} (i.e., w_1, w_2, \dots, w_{g-q}) are joined to \hat{v} via edges with offsets -1 ; i.e., if $g < q$, then

$$\psi_{\pi, \sigma, \hat{e}, g, h}(\hat{v}, w_i) = \begin{cases} -1 & \text{for } 1 \leq i \leq q - g, \\ 0 & \text{for } q - g + 1 \leq i \leq d(\hat{v}; G). \end{cases}$$

The offsets $\psi_{\pi, \sigma, \hat{e}, g, h}(e)$ of edges e incident to \hat{w} also can be given analogously (see Fig. 2(b)). For each edge $e \in E$ not adjacent to \hat{e} , its offset $\psi_{\pi, \sigma, \hat{e}, g, h}(e)$ is defined by (1).

4 Modifying Offsets

In our companion paper [9], we have shown the following result.

Theorem 3 [9] *For any radial drawing $D = (\pi, \sigma, \psi)$ of $G = (V, W, E)$, there are offsets ψ' of E such that the radial drawing $D' = (\pi, \sigma, \psi')$ of G has at least one crossing-free edge and satisfies*

$$\chi(D') \leq 3\chi(D).$$

□

As we do not know which edge is the crossing-free edge in such a drawing D' in the theorem, we need to treat each of all the edges as the crossing-free edge $\hat{e} = (\hat{v}, \hat{w})$ in D' , and define a horizontal layout based on \hat{e} . However, as observed in the previous section, the number of possible partitions of the edge set $E(\hat{w}; G)$ we need in order to define the horizontal layout may be *exponential*.

In this section, we show that, if the inner orbit has no leaf vertices, then there always exists a radial drawing D' with a crossing-free edge $\hat{e} = (\hat{v}, \hat{w})$ with $E_{\hat{w}}^{right} = \emptyset$. This new special property of $E_{\hat{w}}^{right} = \emptyset$ enables us to *uniquely* determine the partition of the edge set $E(\hat{w}; G)$.

To derive this new result, we need a new method for bounding the number of edges adjacent to a crossing-free edge \hat{e} that appear on the right (or left) side of \hat{e} . In order to apply the technique for bounding the number of crossings on some edge $\hat{e} = (\hat{v}, \hat{w})$ in the proof of Theorem 3, we count the number of crossings on an edge $\tilde{e} = (\hat{v}, \tilde{w})$ adjacent to \hat{e} together with the number of crossings on \hat{e} . Note that the number of edges in $E_{\tilde{w}}^{right}$ is bounded from above by the number of crossings on an edge $\tilde{e} = (\hat{v}, \tilde{w})$ if $\sigma(\tilde{w}) < \sigma(\hat{w})$ (see Fig. 3(a)). Using this new technique, we can obtain a stronger result than Theorem 3.

Theorem 4 *Let $G = (V, W, E)$ be a bipartite graph with $\deg(v; G) \geq 2$ for all $v \in V$. For any radial drawing $D = (\pi, \sigma, \psi)$ of G , there are offsets ψ' of E such that the radial drawing $D' = (\pi, \sigma, \psi')$ of G has a crossing-free edge $\hat{e} = (\hat{v}, \hat{w})$ with*

$$E_{\hat{w}}^{right} = \emptyset$$

and satisfies

$$\chi(D') \leq 3\chi(D).$$

Proof: Without loss of generality, we assume that D satisfies (C1) and (C2). For each vertex $v \in V$, we consider the cyclic order of the neighbors of v according to the bijection σ on W . We say that two edges incident to v are σ -adjacent if they appear consecutively in the cyclic order.

For two σ -adjacent edges $e = (v, w), e' = (v, w')$ at a vertex $v \in V$, where e appears after e' around v when we visit the edges incident to v , we define

$$c(e; D) = \chi^+(e; D) + \chi^-(e'; D).$$

Then we have

$$\sum_{e \in E} c(e; D) = 2\chi(D)$$

because an edge crossing between edges a and b is counted in either $\chi^+(a; D) + \chi^-(b; D)$ or $\chi^+(b; D) + \chi^-(a; D)$.

Let $k = \min\{c(e; D) \mid e \in E\}$, where it holds

$$k \leq \sum_{e \in E} c(e; D) / |E| = 2\chi(D) / |E|.$$

Let $\hat{e} = (\hat{v}, \hat{w})$ and $\tilde{e} = (\tilde{v}, \tilde{w})$ be the σ -adjacent edges that attain k . Thus $\hat{e} \in \operatorname{argmin}\{c(e; D) \mid e \in E\}$ and $k = c(\hat{e}; D) = \chi^+(\hat{e}; D) + \chi^-(\tilde{e}; D)$.

Let \hat{E} be the set of edges that cross \hat{e} , and \tilde{E} be the set of edges incident to \tilde{w} that cross \tilde{e} (i.e., $\tilde{E} = E_{\tilde{w}}^{\text{right}}$ in D). See Fig. 3(a). Note that $\hat{E} \cap \tilde{E} = \emptyset$ (by (C2)) and $|\hat{E} \cup \tilde{E}| \leq k$.

We now modify the offset $\psi(e)$ of each edge $e \in \hat{E} \cup \tilde{E}$ so that the resulting radial drawing $D' = (\pi, \sigma, \psi')$ satisfies the theorem. Let $\psi'(e) = \psi(e)$ for all $e \in E - (\hat{E} \cup \tilde{E})$.

We redraw each edge $e \in \hat{E}$ by changing its offset $\psi(e)$ into a new offset $\psi'(e)$ so that it no longer crosses \hat{e} . See Fig. 3(b). More precisely, $\psi'(e) = \operatorname{sgn}(\psi(e))(|\psi(e)| - 1)$ for $e \in \hat{E}$ with $|\psi(e)| > 0$, and $\psi'(e) = -\psi(e; \hat{e})$ for $e \in \hat{E}$ with $\psi(e) = 0$.

We redraw each edge $e \in \tilde{E}$ by changing its offset $\psi(e)$ into a new offset $\psi'(e)$ so that it belongs to $E_{\tilde{w}}^{\text{left}}$ in the resulting drawing. More precisely, let $\psi'(e) = \psi(e) + 1$ for each $e \in \tilde{E}$.

Let ψ' be the resulting offsets of E , and let $D' = (\pi, \sigma, \psi')$, wherein $E_{\tilde{w}}^{\text{right}} = \emptyset$ holds. We easily see that $\chi(e, \hat{e}; D') = 0$ for all edges $e \in E - \{\hat{e}\}$ and that $\chi(e, e'; D') = \chi(e, e'; D)$ for all edges $e, e' \in \hat{E} \cup \tilde{E}$.

Now we show that $\chi(e, e'; D') \leq \chi(e, e'; D) + 1$ holds for all edge pairs $e \in \hat{E} \cup \tilde{E}$ and $e' \in E - (\hat{E} \cup \tilde{E})$. By Lemma 4, the increased number of crossings between two edges e and e' is at most $\chi(e, e'; D') - \chi(e, e'; D) \leq |(\psi'(e) - \psi'(e')) - (\psi(e) - \psi(e'))|$, since the positions π of V and σ of W remain unchanged. For each edge $e \in \hat{E} \cup \tilde{E}$, we have $|(\psi'(e) - \psi'(e')) - (\psi(e) - \psi(e'))| = 1$ since $|\psi'(e) - \psi(e)| = 1$ holds. This implies that $\chi(e, e'; D') - \chi(e, e'; D) \leq |(\psi'(e) - \psi'(e')) - (\psi(e) - \psi(e'))| = 1$ for all edge pairs $e \in \hat{E} \cup \tilde{E}$ and $e' \in E - (\hat{E} \cup \tilde{E})$. Hence

$$\begin{aligned} \chi(D') &= \sum_{e, e' \in E} \chi(e, e'; D') \\ &\leq \sum_{e \in \hat{E} \cup \tilde{E}} \sum_{e' \in E - (\hat{E} \cup \tilde{E})} (\chi(e, e'; D) + 1) \leq \chi(D) + |\hat{E} \cup \tilde{E}| |E - (\hat{E} \cup \tilde{E})|. \end{aligned}$$

Since $|\hat{E} \cup \tilde{E}| \leq k \leq 2\chi(D) / |E|$, we have

$$\chi(D') \leq \chi(D) + k|E| \leq \chi(D) + 2\chi(D) = 3\chi(D).$$

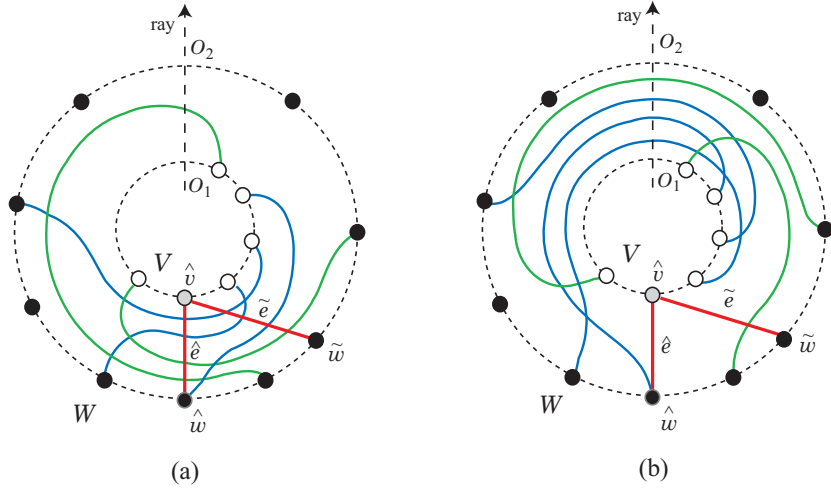


Figure 3: (a) Edges \hat{e} and \tilde{e} in a radial drawing D ; (b) Edges in $\hat{E} \cup \tilde{E}$ redrawn.

This proves the theorem. □

By Theorem 4, a given graph G contains an edge $\hat{e} = (\hat{v}, \hat{w}) \in E$ ($\hat{v} \in V$, $\hat{w} \in W$) such that $\chi(D) \leq 3\chi_r^*(G, \sigma)$ holds for some radial drawing $D = (\pi, \sigma, \psi)$ of G in which \hat{e} is crossing-free and $E_{\hat{w}}^{right} = \emptyset$.

Our aim is now to consider the problem of finding a radial drawing D with crossing-free edge \hat{e} that minimizes the number of edge crossings for fixed positions σ of W and a specified edge \hat{e} . Let $\mathcal{D}_{\hat{e}}^{right}$ be the set of all radial drawings with the positions σ in which edge $\hat{e} = (\hat{v}, \hat{w}) \in E$ ($\hat{v} \in V$, $\hat{w} \in W$) is crossing-free and $E_{\hat{w}}^{right} = \emptyset$. Then we prove the following results.

Theorem 5 For positions σ of W in a bipartite graph $G = (V, W, E)$ and an edge $\hat{e} = (\hat{v}, \hat{w}) \in E$ ($\hat{v} \in V$, $\hat{w} \in W$), a radial drawing $D_0 \in \mathcal{D}_{\hat{e}}^{right}$ such that

$$\chi(D_0) \leq \alpha \min\{\chi(D) \mid D \in \mathcal{D}_{\hat{e}}^{right}\}$$

can be obtained in $O(T(|V|, |E|) \cdot d(\hat{v}; G))$ time (α represents the approximation ratio and $T(|V|, |E|)$ represents the running time of the algorithm for the OHCM problem). □

From this and Lemma 3, the following holds.

Theorem 6 For positions σ of W in a bipartite graph $G = (V, W, E)$ and an edge $\hat{e} = (\hat{v}, \hat{w}) \in E$ ($\hat{v} \in V$, $\hat{w} \in W$), a radial drawing $D_0 \in \mathcal{D}_{\hat{e}}^{right}$ such that

$$\chi(D_0) \leq 1.4664 \min\{\chi(D) \mid D \in \mathcal{D}_{\hat{e}}^{right}\}$$

can be obtained in $O(|V|^2|E|d(\hat{v}; G))$ time. □

By Theorem 4, if we apply Theorem 6 to all edges \hat{e} in E , and choose the best drawing D_0 among the resulting drawings, then $\chi(D_0) \leq \alpha \min\{\chi(D) \mid \hat{e} \in E, D \in \mathcal{D}_{\hat{e}}^{right}\} \leq 3\alpha\chi_r^*(G, \sigma)$ holds. The entire running time is

$$O\left(\sum_{e=(v,w) \in E} d(v; G) \cdot T(|V|, |E|)\right) = O(|E| \cdot T(|V|, |E|)).$$

Thus, this implies Theorem 2.

5 Reduction from Radial Drawings to Horizontal Drawings

This section proves Theorem 6. In the rest of this section, we derive some conditions on offsets $\psi : E \rightarrow \{-1, 0, 1\}$ such that $D = (\pi, \sigma, \psi) \in \mathcal{D}_{\hat{e}}^{right}$ minimizes $\chi(D)$ for given positions π and σ .

From Theorem 4, to prove Theorem 6, it suffices to consider only radial drawings in the form of $D_{g,h} = (\pi, \sigma, \psi_{\pi, \sigma, \hat{e}, g, h})$ ($g \in \{1, 2, \dots, d(\hat{v}; G)\}$, $h = 1$), which we denote by $D_g = (\pi, \sigma, \psi_{\pi, \sigma, \hat{e}, g})$ ($g \in \{1, 2, \dots, d(\hat{v}; G)\}$). By definition, the following holds.

Lemma 5 *For positions π of V and σ of W in a bipartite graph $G = (V, W, E)$ and an edge $\hat{e} = (\hat{v}, \hat{w}) \in E$ with $\hat{v} \in V$ and $\hat{w} \in W$, let $\psi_{\pi, \sigma, \hat{e}, g}$ be the offsets with respect to π, σ, \hat{e} , and $g \in \{1, 2, \dots, d(\hat{v}; G)\}$. Then a radial drawing $D^* \in \mathcal{D}_{\hat{e}}^{right}$ with the minimum crossing number is given as*

$$D_g^* = (\pi^*, \sigma, \psi_{\pi^*, \sigma, \hat{e}, g}) \tag{2}$$

for some $g \in \{1, \dots, d(\hat{v}; G)\}$.

Consider the drawing D_g^* in (2). Based on G , choice of \hat{e} and a partition of $E(\hat{v}; G) - \{\hat{e}\}$ in D_g^* , we first define a graph G_A and a horizontal drawing D_A as follows.

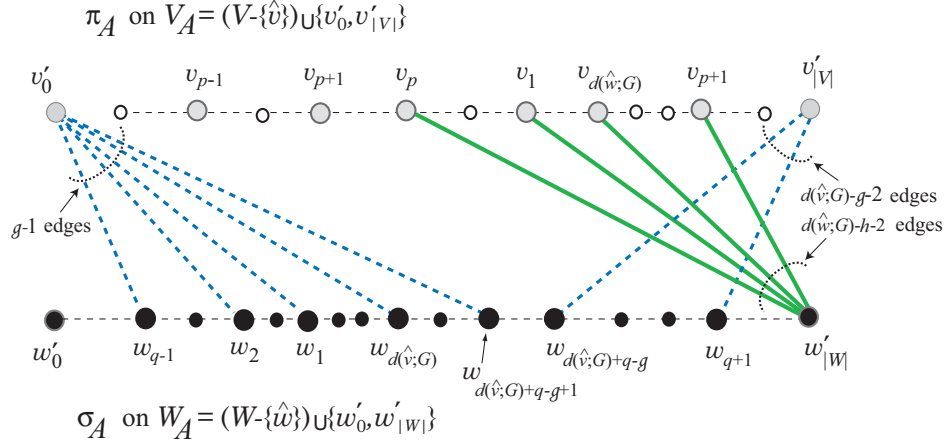
Remove edge $\hat{e} = (\hat{v}, \hat{w})$ from G . Then split vertex \hat{v} into two new vertices v'_0 and $v'_{|V|}$, changing end-vertex \hat{v} of each edge (\hat{v}, w_i) in the right-set (respectively, the left-set) of $E(\hat{v}; G) - \{\hat{e}\}$ to v'_0 (respectively, $v'_{|V|}$). Analogously, split vertex \hat{w} into w'_0 and $w'_{|W|}$, changing end-vertex \hat{w} of each edge (v_i, \hat{w}) in the left-set $E_{\hat{w}}^{left} = E(\hat{w}; G) - \{\hat{e}\}$ to $w'_{|W|}$. We denote the resulting bipartite graph by

$$G_A = (V_A = (V - \{\hat{v}\}) \cup \{v'_0, v'_{|V|}\}, W_A = (W - \{\hat{w}\}) \cup \{w'_0, w'_{|W|}\}, E_A). \tag{3}$$

See Fig. 4.

Let π_A and σ_A be the bijections on V_A and W_A respectively, such that

$$\begin{aligned} \pi_A(v'_0) &= 0, \pi_A(v'_{|V|}) = |V|, \pi_A(v) = \pi^*(v) - \pi^*(\hat{v}) \pmod{|V|} \text{ for } v \in V - \{\hat{v}\}, \\ \sigma_A(w'_0) &= 0, \sigma_A(w'_{|W|}) = |W|, \sigma_A(w) = \sigma(w) - \sigma(\hat{w}) \pmod{|W|} \text{ for } w \in W - \{\hat{w}\}. \end{aligned} \tag{4}$$


 Figure 4: Horizontal drawing $D_A = (\pi_A, \sigma_A)$ of G_A .

Then the number of edge crossings in horizontal drawing $D_A = (\pi_A, \sigma_A)$ of G_A is equal to that of D_g^* , i.e.,

$$\chi(D_A; G_A) = \chi(D_g^*; G).$$

Note that the construction of G_A depends on the right-set of $E(\hat{v}; G) - \{\hat{e}\}$ in drawing D_g^* . There are at most $d(\hat{v}; G)$ ways of constructing G_A for all possible D_g^* , $1 \leq g \leq d(\hat{v}; G)$, since there are at most $d(\hat{v}; G)$ choices of $E_{\hat{v}}^{left}$ over a fixed bijection σ on W . For each choice of G_A , we find a horizontal drawing D'_A which is an α -approximate solution.

Thus, by choosing the best horizontal drawing D'_0 among these drawings D'_A , we see that the radial drawing $D_0 \in \mathcal{D}_{\hat{e}}^{right}$ corresponding to D'_0 satisfies $\chi(D_0; G) \leq \alpha \min\{\chi(D_g^*) \mid 1 \leq g \leq d(\hat{v}; G)\} \leq \alpha \chi(D^*)$. We easily see that such D_0 can be computed in $O(T(|V|, |E|) \cdot d(\hat{v}; G))$ time. This proves Theorem 6.

The algorithm is formally described as follows.

Algorithm RADIAL_CM_APX(G, σ, \hat{e})

Input: A bipartite graph $G = (V, W, E)$ with a bijection σ on W and an edge $\hat{e} = (\hat{v}, \hat{w}) \in E$ ($\hat{v} \in V, \hat{w} \in W$).

Output: A radial drawing $D_0 = (\pi, \sigma, \psi) \in \mathcal{D}_{\hat{e}}^{right}$ of G in Theorem 6.

1. **for** each subset $S \subseteq E(\hat{v}; G) - \{\hat{e}\}$ of edges which appear consecutively after \hat{e} in the clockwise order around v **do**
 - (a) Construct the bipartite graph G_A in (3) and the bijection σ_A on W in (4) by using S as the right-set of $E(\hat{v}; G) - \{\hat{e}\}$.

/* (G_A, σ_A) is an instance of the one-sided horizontal crossing minimization */

- (b) Find an α -approximate solution $D'_A = (\pi'_A, \sigma_A)$ to (G_A, σ_A) (for example, by Lemma 3).

endfor

2. Choose the best drawing D'_0 among the above horizontal drawings D'_A .
3. Output the radial drawing $D_0 = (\pi, \sigma, \psi) \in \mathcal{D}_e^{right}$ of G that corresponds to D'_0 and its right-set $S \subseteq E(\hat{v}; G) - \{\hat{e}\}$.

6 Conclusion

In this paper, we present a new 3α -approximation algorithm to the one-sided crossing minimization problem in radial drawings when the free orbit has no leaf vertex, where α represents the approximation ratio of the one-sided crossing minimization problem in horizontal drawing. Using the best known result of $\alpha = 1.4664$ [13], our algorithm achieves 4.3992-approximation. This is a significant improvement over the first 15-approximation [9].

We conclude with some open problems in radial drawings.

1. *Exact algorithm:* Design exact algorithms for one-sided radial crossing minimization problem. Jünger and Mutzel presented an integer linear programming algorithm for the one-sided horizontal crossing minimization problem [10].
2. *Fixed parameter tractability:* Is one-sided radial crossing minimization problem fixed parameter tractable? See [4, 5] for fixed parameter algorithms for the one-sided horizontal crossing minimization problem.
3. *Two-sided Radial Crossing Minimization problem (TRCM):* Design exact algorithms and approximation algorithms for the two-sided radial crossing minimization problem.

The *Two-sided Horizontal Crossing Minimization* (THCM) problem which asks to find orderings of W on L_2 and V on L_1 to minimize the number of edge crossings are well studied. Shahrokhi *et al.* [17] proved that the problem is $O(\log n)$ -approximable, if the maximum degree over the minimum degree is bounded by a constant. Zheng and Buchheim presented a quadratic programming approach for the THCM problem [21].

4. *Computing the best offset for the Fixed-Fixed case:* To the best of our knowledge, the complexity of the problem of finding an optimal offset for given positions of vertices in the inner and outer orbits remains open.

More specifically, it is not known whether the problem of testing a given bipartite graph $G = (V, W, E)$ with specified positions π of V and σ of W admits an offset ψ of E such that $\chi(D = (\pi, \sigma, \psi); G) \leq K$ for a

given integer K is NP-hard or not. However, it is not difficult to see that a 3-approximation algorithm for the problem can be obtained by using Theorem 3 and Lemma 5.

5. *Planarization*: Planarization, i.e., determining whether a bipartite graph G has a biplanar subgraph with at least K edges, in radial drawings would be another interesting research area to be investigated, since the counterpart in horizontal drawings have been well studied [6, 12].

The planarization problem in horizontal drawings is NP-complete for both two-sided and one-sided cases, and can be solved in polynomial time for the case of given two fixed orderings [6]. For an integer programming solution for the two-layer planarization problem in horizontal drawings, see [12].

Recently, it has been shown that the planarization problem in radial drawings is NP-hard for both two-sided and one-sided cases, and can be solved in polynomial time for the case of given two fixed orderings [2]. Heuristics for the planarization problem in radial drawings are also suggested in [2]. However, we are not aware of any approximation algorithms for the problem.

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