49th IEEE Conference on Decision and Control
December 15-17, 2010
Hilton Atlanta Hotel, Atlanta, GA, USA

An interpolation-based approach to $\mathcal{H}_\infty$ model reduction of dynamical systems

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***Abstract*— We introduce an interpolatory approach to $\mathcal{H}_\infty$ model reduction for large-scale dynamical systems. Guided by the optimality conditions of [26] for best uniform rational approximants on the unit disk, our proposed method uses the freedom in choosing the d-term in the reduced order model to enforce $2r+1$ interpolation conditions in the right-half plane for any given reduction order, r. $2r$ of these points are initialized by the Iterative Rational Krylov Algorithm of [16]; and then the d-term is chosen to minimize the $\mathcal{H}_\infty$ error for this initial set of interpolation points. Several numerical examples illustrate the effectiveness of the proposed method. It consistently yields better results than balanced truncation. In all cases examined its performance is very close to or better than that of Hankel norm approximation. For the special case of state-space symmetric systems, important properties are established. Finally, we examine $\mathcal{H}_\infty$ model reduction from a potential theoretic perspective and present a second methodology for choosing interpolation points.**

I. Introduction

The model reduction problem seeks to replace a given set of differential equations with a much smaller set while keeping the behavior of both systems similar in an appropriately defined sense. The need for model reduction results from the desire for high accuracy in mathematical models of complex physical phenomena that lead to a large number of differential equations, i..e. large-scale dynamical systems. In such large-scale settings, the resulting computational burden can be overwhelming Hence, model reduction, i.e. reducing the number of differential equations while preserving the original system behavior, becomes a necessary component in such large-scale settings. For an overview, see [1].

In this paper, we consider single input/single output (SISO) linear dynamical systems in state-space form:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}\,u(t), \quad y(t) = \boldsymbol{c}^T\boldsymbol{x}(t) \tag{1}$$

where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^n$. In (1) $\boldsymbol{x}(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, are its *state*, *input*, and *output*, respectively. The transfer function of the system is $H(s) = \boldsymbol{c}^T(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}$. Using the standard abuse of notation, we denote both the system and its transfer function by H. The dimension of the state vector $\boldsymbol{x}$ is called the order of $H(s)$. Hence, $H(s)$ in (1) has order n. By $\mathcal{H}_\infty^k(\jmath\mathbb{R})$, we denote the set of rational functions of order at most k, which are bounded and analytic in the closed right half of the complex plane and we assume that $H \in \mathcal{H}_\infty^n(\jmath\mathbb{R})$. In this setting, the goal of model reduction is to construct another system

$$\dot{\boldsymbol{x}}_r(t) = \boldsymbol{A}_r\boldsymbol{x}_r(t) + \boldsymbol{b}_r u(t), \;\; y_r(t) = \boldsymbol{c}_r^T\boldsymbol{x}_r(t) + d_r u(t) \tag{2}$$

of smaller dimension $r \ll n$, with $\boldsymbol{A}_r \in \mathbb{R}^{r\times r}$ and $\boldsymbol{b}_r$, $\boldsymbol{c}_r \in \mathbb{R}^r$ such that $y_r(t)$ approximates $y(t)$ well in a certain norm for all bounded inputs $u(t)$. Note that without loss of generality, we have assumed that, unlike for the reduced-order model (ROM) $H_r(s)$, the d-term for the original system $H(s)$ is 0. The general case of the non-zero d term for $H(s)$ can be recovered by replacing the reduced-order quantity d_r by $d_r - d$ throughout the text.

In this paper, we will focus on model reduction in the $\mathcal{H}_\infty$ norm: The $\mathcal{H}_\infty$ norm of the SISO dynamical system (1) is defined as $\|H\|_{\mathcal{H}_\infty} = \max_{\omega\in\mathbb{R}} \mid H(\jmath\omega) \mid$. The $\mathcal{H}_\infty$ norm is the L^2 induced operator norm of the underlying convolution operator for H. So to ensure that the output error $y(t)-y_r(t)$ is small in a root mean square sense for $t > 0$ uniformly over all inputs $u(t)$ with bounded energy, one needs to ensure that the $\mathcal{H}_\infty$ norm of the error system $H-H_r$ is small. This leads to the optimal $\mathcal{H}_\infty$ model reduction problem: For the full-order model $H \in \mathcal{H}_\infty^n(\jmath\mathbb{R})$, and a given reduced order $r < n$, find $H_r \in \mathcal{H}_\infty^r(\jmath\mathbb{R})$, that solves

$$\min_{\hat{H}_r\in\mathcal{H}_\infty^r(\jmath\mathbb{R})} \left\| H - \hat{H}_r \right\|_{\mathcal{H}_\infty} . \tag{3}$$

This problem is an active area of research [2]. For a given r, a lower bound on the achievable error is given by the $(r+1)^{\text{th}}$ Hankel singular value of H. A conceptual solution is given in [17]. However, the method of [17] is not computationally feasible since it assumes the knowledge of the minimum of (3). On the other hand, gramian-based model reduction methods such as balanced truncation [21], [22] and optimal Hankel norm approximation [11] are known to yield small approximation errors in the $\mathcal{H}_\infty$ norm [15], [1] and are commonly used in obtaining satisfactory $\mathcal{H}_\infty$ approximants. In this paper, we will present a new framework in the interpolation setting for tackling the optimal $\mathcal{H}_\infty$ approximation problem. By connecting ideas from interpolatory $\mathcal{H}_2$ model reduction [16], realization theory [19], and complex Chebyshev approximation [26], we develop a numerically efficient interpolation-based method for the optimal $\mathcal{H}_\infty$ problem.

The paper is organized as follows: In Section I-A, we briefly review interpolatory model reduction and revisit the optimal $\mathcal{H}_2$ method of Gugercin *et al* [16], a main tool in our methodology. Section III presents the proposed method for $\mathcal{H}_\infty$ model reduction. Section IV presents the properties of

This work was supported in parts by the NSF Grants DMS-0505971 and DMS-0645347

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the proposed method for the special case of symmetric state-space systems followed by a potential theoretic approach to $\mathcal{H}_\infty$ model reduction in Section V. Conclusions and future directions are presented in Section VII.

A. Interpolatory model reduction

Given a full-order model $H(s)$ and the interpolation points $\{s_i\}_{i=1}^q$, the goal of interpolatory model reduction is to construct a ROM, $H_r(s)$, that interpolates $H(s)$ at s_i, for $i = 1, \ldots, q$; i.e. $H(s_i) = H_r(s_i)$. The next theorem, due to Grimme [12], shows how to solve this problem in a projection setting.

Theorem 1: Given $H(s) = \boldsymbol{c}^T(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}$ and $2r$ interpolation points $s_1, \ldots, s_{2r} \in \mathbb{C}$, let

$$\boldsymbol{V}_r = [(s_1\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} \ldots (s_r\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}] \tag{4}$$

$$\boldsymbol{W}_r = [(\bar{s}_1\boldsymbol{I} - \boldsymbol{A}^T)\boldsymbol{c}^T \ldots (\bar{s}_r\boldsymbol{I} - \boldsymbol{A}^T)\boldsymbol{c}^T] \tag{5}$$

Assume $\boldsymbol{W}_r^T\boldsymbol{V}_r$ is nonsingular. Define the ROM $H_r(s) = \boldsymbol{c}_r^T(s\boldsymbol{I}_r - \boldsymbol{A}_r)^{-1}\boldsymbol{b}_r + d_r$ by:

$$\begin{aligned} \boldsymbol{A}_r &= (\boldsymbol{W}_r^T\boldsymbol{V}_r)^{-1}\boldsymbol{W}_r^T\boldsymbol{A}\boldsymbol{V}_r \quad \boldsymbol{c}_r = \boldsymbol{V}_r^T\boldsymbol{c} \\ \boldsymbol{b}_r &= (\boldsymbol{W}_r^T\boldsymbol{V}_r)^{-1}\boldsymbol{W}_r^T\boldsymbol{b}, \quad \text{and} \quad d_r = 0. \end{aligned} \tag{6}$$

Then $H(s_i) = H_r(s_i)$, for $i = 1, \ldots, 2r$. Moreover, if $s_i = s_{i+r}$ for $i = 1, \ldots, r$, then $H'(s_i) = H_r'(s_i)$ for $i = 1, \ldots, r$.

B. $\mathcal{H}_2$ -Optimal Interpolation Conditions

Even though Theorem 1 shows how to enforce interpolation, it does not specify a good/optimal strategy for choosing the interpolation points. This problem has been resolved recently for the special case of the $\mathcal{H}_2$ norm. For a SISO system $H(s)$, the $\mathcal{H}_2$ norm is defined as

$$\|H\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |\, H(\jmath\omega)\,|^2\, d\omega\right)^{1/2}. \tag{7}$$

Then, for the full-order model $H(s)$, and a given reduced order $r < n$, the optimal $\mathcal{H}_2$ approximation problem is to find a reduced-model $H_r(s)$ that solves

$$\min_{\hat{H}_r \in \mathcal{H}_\infty^r(\jmath\mathbb{R})} \left\| H - \hat{H}_r \right\|_{\mathcal{H}_2}. \tag{8}$$

The optimization problem (8) has been studied in the literature extensively, see, for example, [20], [28], [16], [25], [27], [14], [5], [7], [29] and the references there in. In this note, we only present its solution in the interpolation framework.

Theorem 2 ([20]): For a given $H(s)$, let $H_r(s)$ solve the optimal $\mathcal{H}_2$ problem (8). Also, let $\hat{\lambda}_1, \ldots, \hat{\lambda}_r$ denote the poles of $H_r(s)$ for $i = 1, \ldots, r$. Then, for $i = 1, \ldots, r$,

$$H(-\hat{\lambda}_i) = H_r(-\hat{\lambda}_i), \quad \text{and} \quad H'(-\hat{\lambda}_i) = H_r'(-\hat{\lambda}_i). \tag{9}$$

Theorem 2 states that the solution of the optimal $\mathcal{H}_2$ model reduction problem is a Hermite interpolant. However, constructing this interpolant requires interpolation conditions to be enforced at the mirror images of the poles of the reduced-order system. Clearly, the poles of $H_r(s)$ are not known *a priori*. This problem has been recently overcome by the Iterative Rational Krylov Algorithm (IRKA) of [16]. Starting from an initial selection of interpolation points, IRKA successively corrects these points until the first-order necessary conditions (9) for $\mathcal{H}_2$ optimality are satisfied. A brief sketch of IRKA is outlined below:

*Algorithm 1 (**Iterative Rational Krylov Algorithm**):*

1) Make an initial selection of interpolation points σ_i, for $i = 1, \ldots, r$ that is closed under complex conjugation and fix a convergence tolerance tol.
2) Choose $\boldsymbol{V}_r$ and $\boldsymbol{W}_r$ as in Theorem 1.
3) while (relative change in $\{\sigma_i\} < tol$) a.)
   a) Make a basis change so that $\boldsymbol{W}_r^T\boldsymbol{V}_r = \boldsymbol{I}_r$.
   b) $\boldsymbol{A}_r$=$\boldsymbol{W}_r^T\boldsymbol{A}\boldsymbol{V}_r$
   c) Assign $s_i \leftarrow -\lambda_i(\boldsymbol{A}_r)$ for $i = 1, \ldots, r$.
   d) Update $\boldsymbol{V}_r$ and $\boldsymbol{W}_r$ to satisfy hypotheses of Theorem 1 with new s_i's.
4) $\boldsymbol{A}_r$=$\boldsymbol{W}_r^T\boldsymbol{A}\boldsymbol{V}_r$, $\boldsymbol{b}_r$=$\boldsymbol{W}_r^T\boldsymbol{b}$, and $\boldsymbol{c}_r = \boldsymbol{V}_r^T\boldsymbol{c}_r$.

It has been observed that IRKA has fast convergence behavior, and tends to converge to at least a local minimum of the $\mathcal{H}_2$ optimal model reduction problem [16], producing high-fidelity reduced-order models in a numerically efficient way. Note that the only cost of IRKA is solving a sequence of linear systems. For details on IRKA, we refer to the original source [16].

II. The d_r-term in the interpolatory optimal $\mathcal{H}_\infty$ problem

The transfer function of an rth-order ROM has $2r+1$ free parameters. Theorem 1 determines all of these parameters by forcing $2r$ interpolation conditions and setting $d_r = 0$. Setting $d_r = 0$ forces $H(s) - H_r(s) = 0$ at $s = \infty$. This last interpolation condition is nonoptimal for $\mathcal{H}_\infty$ approximation, which requires control over precisely $2r + 1$ interpolation points in the *interior* of the right half plane. This fact is stated in the following theorem, a slight variation of a theorem due to Trefethen [26].

Theorem 3: Given the full-order model $H(s)$, let $H_r(s)$ be an r^{th} order approximation to $H(s)$. If $H(s) - H_r(s)$ has at least $2r + 1$ zeros in the right half plane, and the image of the imaginary axis under $H(s) - H_r(s)$ is a perfect circle about the origin, then $H_r(s)$ is an optimal $\mathcal{H}_\infty$ approximation to $H(s)$.

As [26] discusses, while a perfectly circular error curve with winding number at least $2r + 1$ indicates an optimal approximation, a nearly circular error curve that does not pass through origin with at least $2r + 1$ zeros indicates a nearly optimal approximation as noted in the next result, once more a slight variation of the result from [26].

Theorem 4: Given the full-order model $H(s)$, let $H_r^{\text{opt}}(s)$ denote an r^{th} order optimal $\mathcal{H}_\infty$ approximation to $H(s)$. Let $H_r(s)$ be an r^{th} order approximation such that $H(s) - H_r(s)$ has at least $2r + 1$ zeros in the right half plane, and the image of the imaginary axis under $H(s) - H_r(s)$ does not pass through the origin. Then

$$\min_{\omega\in\mathbb{R}} |H(\jmath\omega) - H_r(\jmath\omega)| \le \|H - H_r^{\text{opt}}\|_{\mathcal{H}_\infty} \le \|H - H_r\|_{\mathcal{H}_\infty}$$

Inspired by Theorem 3, we aim to place $2r + 1$ interpolation points in the desired location (right-half plane in

this case) as opposed to the $2r$ interpolation points used in Theorem 1. Therefore we need to exploit the freedom in choosing d_r to get the $(2r+1)^{\text{th}}$ interpolation point. So, we need a modification of Theorem 1 that will allow interpolation at the desired locations $s_1, \ldots, s_{2r}$ while keeping d_r a free parameter. The next theorem achieves precisely this goal. For a proof of this result, we refer to [19] and [6].

Theorem 5: Given $H(s) = \boldsymbol{c}^T(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}$ and the $2r$ interpolation points $s_1, \ldots, s_{2r} \in \mathbb{C}$, let $\boldsymbol{V}_r$, $\boldsymbol{W}_r$, $\boldsymbol{A}_r$, $\boldsymbol{b}_r$, and $\boldsymbol{c}_r$ be as defined in Theorem 1. For any given $d_r \in \mathbb{R}$, define the new ROM

$$H_r^{d_r}(s) = \tilde{\boldsymbol{c}}_r^T(s\boldsymbol{I} - \tilde{\boldsymbol{A}}_r)^{-1}\tilde{\boldsymbol{b}}_r + d_r \tag{10}$$

with

$$\begin{aligned} \tilde{\boldsymbol{A}}_r &= \boldsymbol{A}_r + d_r(\boldsymbol{W}_r^T\boldsymbol{V}_r)^{-1}\boldsymbol{e}\boldsymbol{e}^T, \\ \tilde{\boldsymbol{b}}_r &= \boldsymbol{b}_r - d_r(\boldsymbol{W}_r^T\boldsymbol{V}_r)^{-1}\boldsymbol{e}, \quad \text{and} \quad \tilde{\boldsymbol{c}}_r = \boldsymbol{c}_r - d_r\boldsymbol{e}, \end{aligned} \tag{11}$$

where $\boldsymbol{e}$ denotes the vector of ones. Then $H(s_i) = H_r^{d_r}(s_i)$, for $i = 1, \ldots, 2r$. Moreover, if $s_i = s_{i+r}$ for $i = 1, \ldots, r$, then $H'(s_i) = (H_r^{d_r})'(s_i)$ as well, for $i = 1, \ldots, r$.

The new ROM of Theorem 5 is denoted by $H_r^{d_r}(s)$ to emphasize the parametric dependency on d_r. By $R(H_r, d_r)$ we will denote the set of all $H_r^{d_r}(s)$. The freedom in the d_r parameter is significant for at least two reasons. First, $R(H_r, d_r)$ is a parameterization of the set of all rational functions of degree r satisfying the same interpolation constraints as $H_r(s)$. A proof of this can be found in [19]. Second, it is now possible to construct reduced-order models of order r satsifying $2r + 1$ interpolation conditions. Since $H_r^{d_r}(s)$ interpolates $H(s)$ at $s_1, \ldots, s_r$ for any d_r, one can find a d_r that satisfies $H_r^{d_r}(s_{2r+1}) = H(s_{2r+1})$, which is a rootfinding problem. However, we will choose d_r in a special way to lower the $\mathcal{H}_\infty$ error which will in turn automatically choose the $(2r + 1)^{\text{th}}$ interpolation point.

III. An interpolatory approach for $\mathcal{H}_\infty$ approximation

Now that we have a tool at hand to enforce interpolation at $2r+1$ points we aim to satisfy the conditions of Theorem 4, by carefully choosing $2r$ interpolation points known to perform robustly in the $\mathcal{H}_\infty$ norm, and then use the d_r-term to center the error about the origin, driving down the $\mathcal{H}_\infty$-error over the set $R(H_r, d_r)$. As shown in [16], [3], the $\mathcal{H}_2$ optimal interpolation points due to IRKA not only produce optimal $\mathcal{H}_2$ behavior, but also a high fidelity $\mathcal{H}_\infty$ reduced model whose $\mathcal{H}_\infty$ error norm is less than or at least comparable to that of balanced truncation. Therefore, the $\mathcal{H}_2$ optimal interpolation points of Section I-B, i.e. the interpolation points resulting from IRKA, provide us with a good initial choice to start the search for the d_r term. Based on this observation, we propose the following heuristic method:

Algorithm 2: $\mathcal{H}_\infty$ approximation using IRKA points:

1) Run IRKA, i.e. Algorihm 1, to compute the $\mathcal{H}_2$ optimal interpolation points.
2) Use the resulting interpolation points and find the best d_r term by solving
$$d_r^* = \arg\min_{d_r \in \mathbb{R}} \left\| H(s) - H_r^{d_r}(s) \right\|_{\mathcal{H}_\infty}$$
where $H_r^{d_r}(s)$ is constructed as in Theorem 5.
3) Construct the final $\mathcal{H}_\infty$ approximant $H_r^{d_r^*}$ using d_r^* in Theorem 5.

The most expensive part of Algorithm 2 is Step 2, a scalar non-linear optimization problem. The main cost in this step is the computation of the cost function, the $\mathcal{H}_\infty$ norm, during the optimization. However, this is not specific to our approach since any optimization algorithm will require function evaluations. On the other hand, to make the algorithm cheaper, we also performed an approximant $\mathcal{H}_\infty$ norm evaluation in Step 2 by sampling the transfer function over the imaginary axis and used a discrete set for the $\mathcal{H}_\infty$ norm computation. We have observed that the sampling of the transfer function yielded almost the same accuracy as using the exact $\mathcal{H}_\infty$ norm computation; an encouraging result in making the algorithm faster.

A. Numerical Results for Algorithm 2

We demonstrate Algorithm 2 on the CD player model of order $n = 120$. For details on this model, see [9]. We reduced the order to $r = 4, 8, 12$. We stopped at $r = 12$ since the relative error fell below 10^{-3}. IRKA in Step 1 of the algorithm was randomly initialized. For comparison purposes, in Step 2 of the algorithm, we used both the exact $\mathcal{H}_\infty$ computation and the sampling-based $\mathcal{H}_\infty$ computation at 75 sampling points logarithmically spaces between 10^{-2} and 10^6. We compare Algorithm 2 to balanced truncation and Hankel norm approximation. Note that the generic balanced truncation will yield $d_r = d$. Therefore, to present a fair comparison to balanced truncation, we have also varied the d_r term in balanced truncation and found the best d_r-term yielding the minimum $\mathcal{H}_\infty$ norm. *Hence, through out the paper, whenever we refer to balanced truncation, we mean the balanced truncation with the optimally chosen d_r term.* The results are illustrated in Table I where $H_r^{d_r^*}$-sample and $H_r^{d_r^*}$-exact denotes the reduced models of Algorithm 2 where the sampling-based and exact $\mathcal{H}_\infty$ norm computations are used, respectively; H_r^{BAL} denotes the reduced-model due to balanced truncation and H_r^{HNA} the one due to Hankel norm approximation. The first observation is that approximating the $\mathcal{H}_\infty$-norm by sampling is essentially equivalent to computing the $\mathcal{H}_\infty$-norm of the error exactly at each step of the minimization. Moreover, by an interpolation-based method of reduced order modeling, for this example we are consistently able to produce smaller $\mathcal{H}_\infty$ error norms than balanced truncation and Hankel norm approximation.

IV. A special case: $\mathcal{H}_\infty$ model reduction of state-space-symmetric systems

In this section, we examine the special case of state-space-symmetric (*SSS*) systems. $H(s) = \boldsymbol{c}^T(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}$ is called *SSS* if $\boldsymbol{A} = \boldsymbol{A}^T$ and $\boldsymbol{c} = \boldsymbol{b}^T$.

TABLE I

RELATIVE $\mathcal{H}_\infty$-NORM ERROR NORMS FOR THE CD PLAYER MODEL

r	$H_r^{d_r^*}$-sample	$H_r^{d_r^*}$-exact	H_r^{BAL}	H_r^{HNA}
4	2.2×10^{-2}	2.1×10^{-2}	2.3×10^{-2}	2.0×10^{-2}
8	5.2×10^{-3}	4.9×10^{-3}	6.4×10^{-3}	6.0×10^{-3}
12	6.8×10^{-4}	6.5×10^{-4}	9.7×10^{-4}	7.4×10^{-4}

A. SSS interpolants satisfying the $\mathcal{H}_2$ optimality conditions

While reducing *SSS* systems using Theorem 1, it is important to preserve the symmetric structure in the resulting reduced-order model. Indeed, like balanced truncation, IRKA applied to an *SSS* system preserves the symmetry as well. One obtains $\boldsymbol{V}_r = \boldsymbol{W}_r$ throughout Algorithm 1 and the reduced-model can be obtained by

$$\boldsymbol{A}_r = \boldsymbol{Q}_r^T \boldsymbol{A}_r \boldsymbol{Q}_r, \quad \boldsymbol{b}_r = \boldsymbol{c}_r = \boldsymbol{Q}_r^T \boldsymbol{b}, \tag{12}$$

where $\boldsymbol{Q}_r$ is an orthonormal basis for $\boldsymbol{V}_r$; hence, preserving the *SSS* structure. However, even more can be said about the model reduction of *SSS* systems using (12). In the SISO case, IRKA, i.e. Algorithm 1, is a fixed point iteration. Even though it has been shown to converge rapidly for numerous numerical examples even for very bad initialization strategies [16], [14], [3], convergence could not be guaranteed except in the Newton formulation of IRKA. However, a much stronger statement can be made for IRKA in the case of *SSS* systems.

Theorem 6: Let IRKA be applied to a stable *SSS* system $H(s)$. Then every fixed point of IRKA which is a local minimizer is locally attractive, and every fixed point which is a local max or a saddle point is repellent.

Proof: The proof makes use of Appendix I in [18], in addition to several technical lemmas; and is left out due to the page limitations. It will be included in the full paper. ■

Due to Theorem 6, we expect IRKA to converge even faster for *SSS* systems than the general case. This is crucial since this will make Algorithm 2 even faster in this special case. Moreover, we are guaranteed that IRKA will never converge to a local max or to a saddle point.

B. Main results for the SSS case

In this section, we will present some (local) optimality results for the approximant once we initiate the algorithm with an IRKA interpolant. These results make use of the properties of *SSS* IRKA interpolants. We state two theorems without proofs. The proofs are omitted due to the page limitation and will be included in the full paper. We refer the reader to [10] for details.

Let $E_{d_r}(s)$ denote the error system $H(s) - H_r^{d_r}(s)$. Theorem 7 presents a necessary condition for $\mathcal{H}_\infty$ optimality over $R(H_r, d_r)$ similar to the necessary condition given by Antoulas and Astolfi for $\mathcal{H}_\infty$ optimality over $\mathcal{H}_\infty^r(\jmath\mathbb{R})$ [2].

Theorem 7: Let $H(s)$ be a stable SSS system and $H_r(s)$ a reduced-order model computed from IRKA. Suppose $d_r^* = \arg\min_{d_r \in \mathbb{R}} \|E_{d_r}(s)\|_{\mathcal{H}_\infty}$, and $E_{d_r^*}(0) = \left\|E_{d_r^*}\right\|_{\mathcal{H}_\infty}$. Then there exists at least one point $\jmath\omega^*$ such that $\omega^* \in \mathbb{R} \cup \{\infty\}$ and $E_{d_r^*}(0) = |E_{d_r^*}(\jmath\omega^*)|$.

Theorem 8: Suppose $H_r(s)$ is a reduced order model of $H(s)$ computed by IRKA, and let

$$\mathcal{D} = \{d_r \geq 0 : E_{d_r}(0) = \|E_{d_r}\|_{\mathcal{H}_\infty} \text{ and } H_r^{d_r}(s) \text{ stable}\}$$

and assume $\mathcal{D} \neq \emptyset$. If $d_r^* = \sup \mathcal{D}$ then $d_r^* \in \mathcal{D}$, and there exists some point $\jmath\omega^* \in \mathbb{R} \cup \{\infty\}$ such that $E_{d_r^*}(0) = E_{d_r^*}(i\omega^*) = \left\|E_{d_r^*}\right\|_{\mathcal{H}_\infty}$. Furthermore, if $-d_r^* < d_r < d_r^*$, then $\|E_{d_r}\|_{\mathcal{H}_\infty} > \left\|E_{d_r^*}\right\|_{\mathcal{H}_\infty}$. If $\jmath\omega^* = \infty$, then $d_r^* = \arg\min_{d_r \in \mathbb{R}} \left\|H(s) - H_r^{d_r}(s)\right\|_{\mathcal{H}_\infty}$.

Theorem 8 is a sufficient condition for optimality over $R(H_r, d_r)$, which is computationally inexpensive to enforce, requiring at most one computation of the $\mathcal{H}_\infty$ norm. Algorithm 3 shows how we may use Theorems 7, and 8 to compute the optimal value d_r^*.

Algorithm 3: $\mathcal{H}_\infty$ approximation for *SSS* systems:

1) fix a tolerance tol
2) Compute $H_r(s)$=($\boldsymbol{A}_r$,$\boldsymbol{b}_r$,$\boldsymbol{c}_r$,0) from IRKA.
3) Solve $H(0) - H_r^{d_r}(0) - d_r = 0$
4) Sample $E_{d_r}(s)$ along the imaginary axis once at some points $\jmath\omega_1, \ldots, \jmath\omega_n$ and find $|E_{d_r}(\jmath\omega^*)| = max\{|E_{d_r}(\jmath\omega_i)|\}$.
5) If $|E_{d_r}(\jmath\omega^*)| \leq d_r$ end.
   else, Compute $d_r^* > d_r$ such that $|E_{d_r^*}(0) - E_{d_r^*}(\jmath\omega^*)| \leq tol$
   end

C. Numerical examples for the SSS case

In this section, we illustrate Algorithm 3 for the spiral inductor system PEEC model [9], a SSS systems. The system is of order $n = 1434$. We will use σ_i to denote the ith Hankel singular value of the full-order system. The order of approximation is $r = 4$. Here the optimal $H_r^{d_r^*}(s)$ satisfies $|E_{d_r^*}(0)| = |E_{d_r^*}(\jmath\omega^*)| = \left\|E_{d_r^*}(s)\right\|_{\mathcal{H}_\infty}$, $\jmath\omega^* < \infty$. The resulting $\mathcal{H}_\infty$ error norm is 1.35×10^{-4}. The $\mathcal{H}_\infty$ error norm for balanced truncation and the Hankel norm approximation were 2.78×10^{-4} and 1.24×10^{-4}. Note that the lowest achievable error is $\sigma_5 = 1.22 \times 10^{-4}$; hence the proposed method is very close to producing the true optimal $\mathcal{H}_\infty$ approximant. The reduced-model due to Algorithm 3 results in $2r + 1 = 9$ interpolation points in the right-half plane as desired and a nearly circular error curve as illustrated in Figure 1.

V. A POTENTIAL THEORETIC APPROACH TO $\mathcal{H}_\infty$ APPROXIMATION

Using the overwhelming numerical evidence that the optimal $\mathcal{H}_2$ interpolation points yields accurate $\mathcal{H}_\infty$ approximations as well, our approaches of the previous sections used the $2r$ interpolation points from IRKANote that of the $2r+1$ zeroes of the resulting error system there are r distinct zeroes with multiplicity 2. However, unlike the $\mathcal{H}_2$ approximation problem, the double zeroes are not a part of the sufficient conditions for $\mathcal{H}_\infty$ optimality. Hence, even though we were able to obtain high fidelity $\mathcal{H}_\infty$ approximants using IRKA points as initialization, in this section, we will look at a

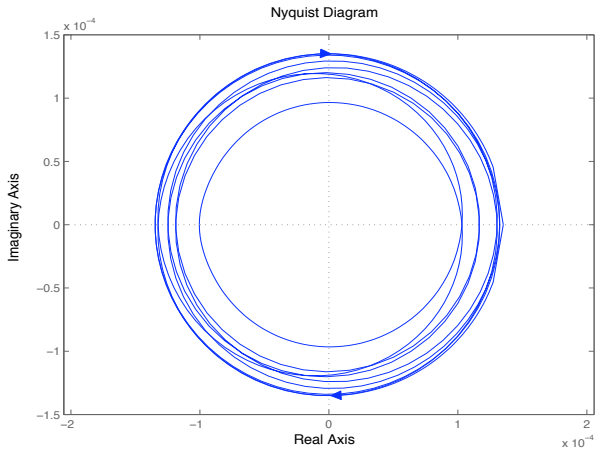

Fig. 1. Nyquist Plot of $E_{d_r^*}(s)$ for PEEC Model

slightly different interpolation-based approach and construct approximants with $2r+1$ distinct zeroes in the right-half plane. Here, r of the $2r+1$ zeroes will again be the mirror images of the poles of the reduced-system and the remaining $r+1$ zeroes will be chosen upon the basis of an analysis of the error system through the lens of potential theory.

Given a full order system $H(s) = \boldsymbol{c}^T(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}$, and an approximation $H_r(s) = \boldsymbol{c}_r(s\boldsymbol{I}_r - \boldsymbol{A}_r)^{-1}\boldsymbol{b}_r + d_r$, the logarithm of the error system $E_r(s)$ can be written as

$$\log|E_r(s)| = \sum_{i=1}^{n+r} \log|s - z_i| - \sum_{i=1}^{n+r} \log|s - \lambda_i|, \qquad (13)$$

where the z_i, λ_i are the zeros and poles of $E_r(s)$ respectively. The summands of (13) are the expressions for the potential due to a point-charge of amplitude -1 or 1 placed at the points z_i or λ_i respectively, and therefore (13) yields an expression for the potential difference induced by the total charge configuration associated with the zeros and poles of $E_r(s)$. Viewing the error in this form, we will try to place $2r+1$ negative charges (the interpolation points) in the right half plane in such a way that the imaginary axis is an equipotential of the total charge configuration.

Note, however, that in (13) there are $n+r$ negative charges present. $2r+1$ of these negative charges will be controlled by the choice of interpolation points. We denote the remaining set of $n-r-1$ negative charges which are not explicitly controlled by the interpolation conditions by $\boldsymbol{\mu}$. In [10] we observed that, after normalizing each point in $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}(\boldsymbol{A})$, the spectrum of $\boldsymbol{A}$, by its distance to the imaginary axis, $\boldsymbol{\mu}$ was a very close approximation to a fixed set of full order poles, which we denote by $\boldsymbol{\lambda}_{\text{match}}$. We have observed in [10] that good/near optimal approximations in the $\mathcal{H}_\infty$-norm place interpolation points near the remaining points in the set $-\boldsymbol{\lambda}_{\text{unmatch}} = -(\boldsymbol{\lambda}(\boldsymbol{A}) \setminus \boldsymbol{\lambda}_{\text{match}})$ as well as points at or near $-\boldsymbol{\lambda}(\boldsymbol{A}_r)$. We have also observed that $\boldsymbol{\lambda}_{\text{unmatch}}$ corresponds strongly to the set of r dominant poles of $H(s)$, dominant in the sense of [23], and we will therefore identify $\boldsymbol{\lambda}_{\text{unmatch}}$ with the r most dominant poles. To induce an equipotential along the imaginary axis we first seek to replace the total charge distribution given by $\boldsymbol{\lambda}(\boldsymbol{A})$ and $\boldsymbol{\mu}$ with a set of $r+1$ "equivalent charges," which induce a nearly equivalent potential. The interpolation points will then be chosen as the reflection of these equivalent charges over the imaginary axis. Given that $\boldsymbol{\mu} \approx \boldsymbol{\lambda}_{\text{match}}$ but the charges associated with each set are opposite, we expect that the potential induced by $\boldsymbol{\lambda}(\boldsymbol{A}) \cup \boldsymbol{\mu}$ will be dominated by the charges associated with $\boldsymbol{\lambda}_{\text{unmatch}}$. Thus, to compute a set of equivalent charges we must account for the predominance of the poles in $\boldsymbol{\lambda}_{\text{unmatch}}$. One way of doing this is to use $-\boldsymbol{\lambda}_{\text{unmatch}}$ as shifts in the rational arnoldi iteration. For a hermitian matrix $\boldsymbol{M}$, Beckermann *et. al.* [8] have shown that the rational Ritz values computed from the rational Arnoldi iteration are asymptotically optimal approximations to the charge distribution which minimizes the potential difference between the distribution of the shifts, and the eigenvalues of $\boldsymbol{M}$. In general, we might therefore anticipate that $r+1$ rational Ritz values constitute a set of equivalent charges which "sees" the contribution made by all the charges in $\boldsymbol{\lambda}(\boldsymbol{A}) \cup \boldsymbol{\mu}$, but is weighted towards the contribution made by $\boldsymbol{\lambda}_{\text{unmatch}}$. The strategy for choosing equivalent charges as interpolation points is outlined in Algorithm 4. For further information on rational Arnoldi, see [24], [8].

Algorithm 4: $\mathcal{H}_\infty$ Approximation Using Equivalent Charges (EC) Method

1) select a convergence tolerance tol
2) Compute $\boldsymbol{\lambda}_{\text{unmatch}}$, the r most dominant poles of $H(s)$.
3) Compute a set of $r+1$ rational ritz values, $\{\tilde{\lambda}_i\}$, from rational Arnoldi applied to $\boldsymbol{A}$, initialized with -$\boldsymbol{\lambda}_{\text{unmatch}}$ and $\boldsymbol{b}$.
4) Select one real-valued point γ from $\{-\tilde{\lambda}_i\}$ and construct $\boldsymbol{W}_r$ and $\boldsymbol{V}_r$ using the r interpolation points $s_i = \{-\tilde{\lambda}_i\} \setminus \{-\gamma\}$
5) while (relative change in $\{s_i\} < tol$)
   a) Make a basis change so that $\boldsymbol{W}_r^T\boldsymbol{V}_r = \boldsymbol{I}_r$.
   b) $\boldsymbol{A}_r$=$\boldsymbol{W}_r^T\boldsymbol{A}\boldsymbol{V}_r$, $\boldsymbol{b}_r$=$\boldsymbol{W}_r\boldsymbol{b}$, $\boldsymbol{c}_r$=$\boldsymbol{c}\boldsymbol{V}_r$.
   c) Find d_r satisfying $H_r^{d_r}(\gamma) = H(\gamma)$
   d) Assign $s_i \leftarrow -\lambda_i(\tilde{\boldsymbol{A}}_r)$ for $i = 1, \ldots, r$, where $\tilde{\boldsymbol{A}}_r$ is defined as in Theorem 5.
   e) Update only $\boldsymbol{V}_r$ to satisfy hypotheses of Theorem 1 with new s_i's.
6) Define $H_r^{d_r}(s)$ as in Theorem 5 from $\boldsymbol{A}_r$, $\boldsymbol{b}_r$, $\boldsymbol{c}_r$, and d_r.

Overall, we obtain $2r+1$ interpolation points which are chosen based on viewing the error term in (13) as the potential difference and eliminating the dominant terms from this expression. We finally note that Step 2) can be achieved in a computationally effective way by making use of the dominant pole algorithm of [23]; hence there is no need for a full eigenvalue decomposition.

A. Numerical Results: EADY Model 4

The full-order model is the EADY model of order $n = 598$. For further details concerning this model see [9]. Here we compare the equivalent charge distribution method of

Algorithm 4 with balanced truncation, and the Hankel norm approximation for $r = 2, 4, 6$; stopping at $r = 6$ because the relative error fell below 10^{-3} for the Hankel norm Approximation. The resulting relative $\mathcal{H}_\infty$ error norms are tabulated in Table V-A where H_r^{EC} denotes the reduced model due to Algorithm 4. For all orders of approximation, Algorithm 4 performs consistently better than balanced truncation. Even though the Hankel norm approximation yields the smallest error, Algorithm 4 performs very close to Hankel norm approximation.

TABLE II

RELATIVE $\mathcal{H}_\infty$ ERROR NORMS FOR THE EADY MODEL

r	H_r^{BAL}	H_r^{HAN}	H_r^{EC}
2	1.11×10^{-1}	7.63×10^{-2}	9.70×10^{-2}
4	4.02×10^{-2}	2.44×10^{-2}	3.77×10^{-2}
6	1.05×10^{-2}	4.79×10^{-3}	1.03×10^{-2}

VI. ACKNOWLEDGEMENTS

The authors thank Dr. L.N. Trefethen for greatly useful comments regarding rational Chebyshev approximation on the unit disc and for the careful reading of the paper.

VII. CONCLUSIONS AND FUTURE DIRECTIONS

We introduced an interpolation framework for $\mathcal{H}_\infty$ model reduction of large-scale dynamical systems. The main tool is, for a given reduced-order r, the ability to enforce $2r + 1$ interpolation conditions using the freedom in choosing the d_r-term in interpolatory model reduction. We discussed two different approaches to choose the interpolation points, one inspired by the $\mathcal{H}_2$ optimal points and one by the potential theoretic inspection of the $\mathcal{H}_\infty$ error. Numerical experiments illustrated the effectiveness of the both approaches, in some cases almost achieving the acceptable lower bound.

Several future directions are possible. One is the investigation of different initialization technique for the potential theoretic approach. Can we analytically classify the set $\boldsymbol{\lambda}_{\text{unmatch}}$? Is there a better initialization than the dominant poles of $H(s)$? Also, the theory and computations are presented for the SISO case. How does the discussion extend to the MIMO case? Are the numerics modified drastically? All these questions will be the focus of future research.

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