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# Convex Optimization-Based Beamforming

[From receive to transmit and network designs]

In this article, an overview of advanced convex optimization approaches to multisensor beamforming is presented, and connections are drawn between different types of optimization-based beamformers that apply to a broad class of receive, transmit, and network beamformer design problems. It is demonstrated that convex optimization provides an indispensable set of tools for beamforming, enabling rigorous formulation and effective solution of both long-standing and emerging design problems.

INTRODUCTION

Beamforming is a versatile and powerful approach to receive, transmit, or relay signals of interest in a spatially selective way in the presence of interference and noise. *Receive beamforming* is a classic yet continuously developing field that has a rich history of theoretical research and practical applications to radar, sonar, communications, microphone array speech/audio processing, biomedicine, radio astronomy, seismology, and other areas [1]. In the last decade, there has been renewed interest in beamforming driven by applications in wireless communications, where multiantenna techniques have emerged as one of the key technologies to accommodate the explosive growth of the number of users and rapidly increasing demands for new high data-rate services.

Recently, there has been significant progress in the field of receive beamforming facilitated by convex optimization. Motivated by the fact that the traditional adaptive beamforming techniques such as minimum variance beamforming lack robustness against even small mismatches in the desired signal steering vector [2], [3], several authors proposed robust techniques that are based on the concept of worst-case performance optimization; see [4]–[11] and references therein. One distinguishing feature of this line of work is that, using convex optimization theory, seemingly complex robust
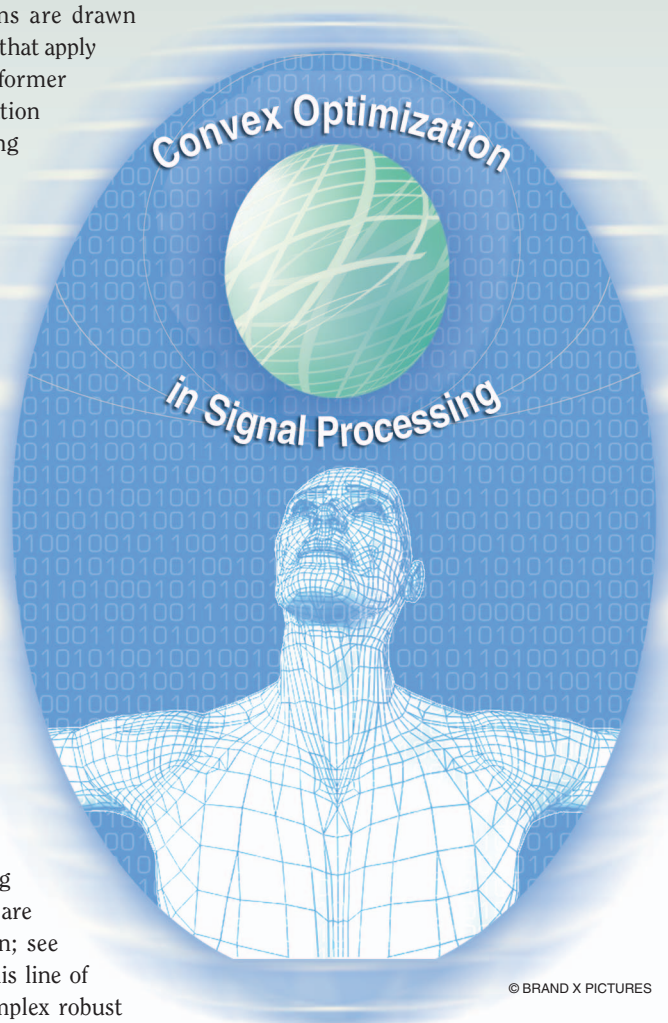
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design problems formulated in [4]–[11] have been recast into tractable convex forms and efficiently solved using interior point algorithms or other appropriate numerical techniques. Beyond the deterministic worst-case robust beamformer designs of [4]–[11], there has been a recent trend to alternatively use less conservative probabilistically constrained designs [12] that employ convex optimization to solve the resulting chance programming problems. Moreover, both the worst-case and probabilistically constrained beamforming approaches have been extended to the case of designing multiuser receivers for space-time coded multiple-input multiple-output (MIMO) communication systems [13], [14].

Transmit beamforming is a relatively young and dynamically developing research field. Classical beamforming is matched to a single steering vector of interest (or, in the case of robust beamforming, a "ball" of steering vectors around the "nominal" one) and its goal is to ensure that the inner product of the beamforming weight vector and the steering vector of interest is large, while the inner product of the beamforming weight vector and all other steering vectors is small (to mitigate interference). This paradigm applies to both receive beamforming and unicast transmit beamforming towards a single receiver. A related but different case is that of multiuser transmit beamforming, which arises in the cellular multiuser downlink when the transmitter is equipped with multiple transmit antennas. In this case, multiple transmit beamforming weight vectors are used to carry different cochannel unicast transmissions, each meant to reach the receiver of one user. These vectors are then jointly designed to balance the interference between different transmissions. The weight vector designed for a given user should have a large inner product with the steering vector of this user, and small inner products with the steering vectors of all other users. This concept was pioneered in [15] and [16] where several early downlink beamforming techniques have been developed in the context of voice services in a cellular mobile radio network where, from the operator's perspective, the system should provide an acceptable quality-of-service (QoS) to each user and serve as many users as possible, while radiating as low power as possible. An important step forward followed in [17], where convex optimization methods were used to solve the problems of [15] and [16] and their robust worst-case optimization based extensions. As the robust designs of [17] are based on several approximations and can be shown to be overly conservative, recent follow-up work has pursued less conservative robust designs based on convex optimization [18], [19]. To provide more flexibility than that of worst-case designs, outage probability-constrained downlink beamformers based on chance programming have also been recently developed [20]–[22].

What if we wish to transmit common information to many users? The traditional way of doing this is (semi) blind, in the

> **BEAMFORMING IS A VERSATILE AND POWERFUL APPROACH TO RECEIVE, TRANSMIT, OR RELAY SIGNALS-OF-INTEREST IN A SPATIALLY SELECTIVE WAY IN THE PRESENCE OF INTERFERENCE AND NOISE.**

sense that it assumes little if anything regarding the steering vectors or even the spatial distribution of users listening to the transmission at any given time. In traditional radio and TV broadcasting, for example, the signal is emitted either isotropically or with a fixed beampattern to cover a service area. There are many reasons for this, including the fact that analog receivers were passive devices incapable of providing feedback to the transmitting station. In modern digital wireless networks, particularly those based on subscription or offering location-aware services, we often have some level of channel state information (CSI) at the transmitter. This can be exploited to boost network reach, coverage, quality of service, and spectral efficiency and to minimize interference to other systems (thus facilitating cohabitation, as in cognitive radio). This is the premise of a recent line of work (starting with [23] and [24]) on multicast beamforming using convex optimization tools. *Multicast beamforming* is now part of the current universal mobile telecommunications system long-term evolution/enhanced multimedia broadcast/multicast service (UMTS-LTE/EMBMS) draft for next-generation cellular wireless services [25], [66]. Similar ideas are currently making their way through fixed wireless and local distribution standardization committees and are likely to influence media distribution to wireless hand-held devices.

Information-theoretic analysis of the relay channel [39] and multiple-relay networks [40] has paved the way for more practical network cooperation schemes. *Network beamforming* is a rapidly emerging area that belongs to the general field of cooperative communications [27]. The key idea of network beamforming is to use a "virtual array" of relay nodes that retransmit properly weighted signals from the source to the destination [28], thereby exploiting cooperation diversity. In the simplest setting, a distributed network beamformer uses an adaptive complex-valued weighting of the received signal, similar to the so-called amplify-and-forward protocol. More advanced types of relay processing (e.g., based on the decode-and-forward strategy) are also possible. An interesting feature of network beamforming is that it can be interpreted as a certain combination of receive and transmit strategies. However, the main difference between the concept of network beamforming and the more traditional concepts of receive and transmit beamforming is that the relays can hardly exchange information about their received signals, so that beamforming is performed in a *distributed* fashion. There has been a rapidly growing activity in this area over the last two years. Following [28], a number of new concepts and methods have been proposed, see [29]–[38] and references therein. These include multiuser and bidirectional extensions of the original approach of [28] and new beamforming strategies such as a filter-and-forward approach [33], [34]. Convex optimization techniques have been extensively used in these works to obtain computationally attractive (exact or approximate) solutions to originally difficult design problems.

The main goal of this article is to present a systematic overview of the current state of the art of advanced optimization-based beamforming and to explore interrelationships between different types of beamformers that apply to a broad class of practically important receive, transmit, and network beamforming problems. While the focus of this article is on applications in wireless communications, several designs considered here are also applicable in quite different application contexts, such as MIMO radar.

[NETWORK BEAMFORMING IS A RAPIDLY EMERGING AREA THAT BELONGS TO THE GENERAL FIELD OF COOPERATIVE COMMUNICATIONS.]

$$\mathbf{y}(t) = \mathbf{s}(t) + \mathbf{n}(t),$$

where $\mathbf{s}(t)$ and $\mathbf{n}(t)$ are the desired signal and the interference-plus-noise components of $\mathbf{y}(t)$, respectively. In the point signal source case, $\mathbf{s}(t) = s(t)\mathbf{a}_s$ where $s(t)$ and $\mathbf{a}_s$ are the desired signal waveform and its steering vector (spatial signature), respectively.

If $\mathbf{a}_s$ and the true array covariance matrix $\mathbf{R} \triangleq \mathrm{E}\{\mathbf{y}(t)\mathbf{y}^H(t)\}$ are perfectly known, then the optimal weight vector can be straightforwardly obtained by means of maximizing the signal-to-interference-plus-noise ratio (SINR) [1]; see Figure 1. In the finite sample case, the true array covariance matrix is unavailable and, therefore, its sample estimate $\hat{\mathbf{R}} = (1/J)\sum_{t=1}^{J}\mathbf{y}(t)\mathbf{y}^H(t) = (1/J)\,\mathbf{YY}^H$ is used instead of $\mathbf{R}$ where $\mathbf{Y} \triangleq [\mathbf{y}(1), \dots, \mathbf{y}(J)]$ is the beamformer training data matrix and J is the number of snapshots available. Then, the optimal weight vector can be approximately computed by solving the following convex problem [1]–[3]

$$\min_{\mathbf{w}} \mathbf{w}^H\hat{\mathbf{R}}\mathbf{w} \quad \text{s.t.} \quad \mathbf{w}^H\mathbf{a}_s = 1. \quad (1)$$

The solution to (1) can be expressed in the following familiar closed form [1]:

$$\mathbf{w} = \beta\hat{\mathbf{R}}^{-1}\mathbf{a}_s, \quad (2)$$

where the scalar $\beta = (\mathbf{a}_s^H\hat{\mathbf{R}}^{-1}\mathbf{a}_s)^{-1}$ does not affect the beamformer output SINR. The beamformer in (2) is usually referred to as the sample matrix inversion (SMI)-based minimum variance (MV) technique.

The fact that the sample array covariance matrix $\hat{\mathbf{R}}$ is used instead of $\mathbf{R}$ in (1) is known to dramatically affect the performance of (2) as compared to the optimal beamformer in the case when the desired signal component is present in the training samples [2], [3]. Note that the latter case is typical for multi-antenna wireless communications and passive source localization. Such a performance degradation caused by signal cancellation is commonly termed as signal self-nulling. It becomes especially strong in practical scenarios, when the knowledge of $\mathbf{a}_s$ is imperfect as well [3].

One of the most popular ad hoc approaches to improve the robustness of the SMI-based MV technique and to avoid signal self-nulling is the diagonal loading (DL) method [2], [41] whose key idea is to regularize the solution of (1) by adding the quadratic penalty term $\gamma\mathbf{w}^H\mathbf{w}$ to the objective function, where γ is a preselected DL factor. The resulting loaded SMI (LSMI) beamformer amounts to replacing the sample covariance matrix $\hat{\mathbf{R}}$ in (2) by its diagonally loaded counterpart, $\gamma\mathbf{I} + \hat{\mathbf{R}}$.

The main shortcoming of the traditional DL approach is that there is no easy and reliable way of choosing the DL factor γ. Note that any fixed choice of γ can be only suboptimal, because the optimal choice of γ is known to be scenario-dependent [2],

NOTATION

Uppercase and lowercase bold letters denote matrices and vectors, respectively. $\mathrm{E}\{\cdot\}$, $\mathrm{Tr}(\cdot)$, $(\cdot)^T$, and $(\cdot)^H$ stand for the statistical expectation, trace of a matrix, transpose, and Hermitian transpose, respectively. $\mathbf{I}$ is the identity matrix. $\|\cdot\|$ denotes the Euclidean norm of a vector or the Frobenius norm of a matrix. $\odot$ denotes the Schur-Hadamard (element-wise) matrix or vector product, $\mathrm{diag}(\mathbf{a})$ stands for a diagonal matrix whose diagonal entries are the elements of vector $\mathbf{a}$, and $\lambda_{\max}(\cdot)$ stands for the principal eigenvalue of a matrix.
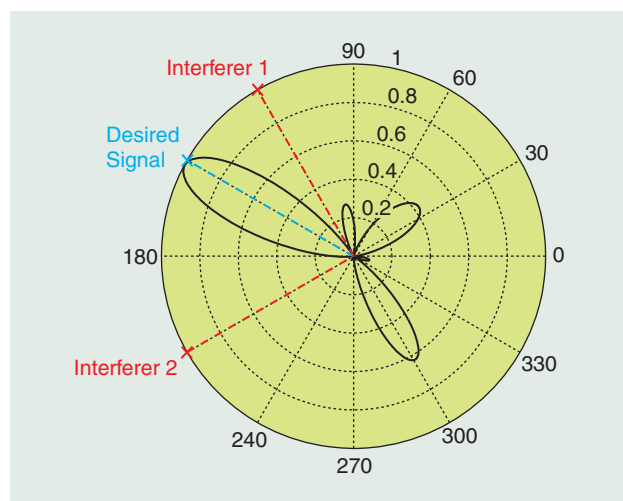
RECEIVE BEAMFORMING

The output signal of a narrowband receive beamformer can be written as

$$x(t) = \mathbf{w}^H\mathbf{y}(t),$$

where $\mathbf{w}$ is the $N \times 1$ vector of beamformer complex weight coefficients, $\mathbf{y}(t)$ is the $N \times 1$ complex snapshot vector of array observations, and N is the number of antenna array sensors.

The array observation vector can be modeled as

[FIG1] Illustration of receive adaptive beamforming: Polar plot of adapted beampattern. The beamformer output SINR is maximized by means of enhancing the desired signal by the beampattern mainlobe and rejecting the interferers by beampattern nulls.

[4]. To avoid the aforementioned drawbacks of the standard DL technique, more theoretically rigorous robust MV beamforming algorithms have been recently proposed in [4]–[8] based on worst-case designs.

The key idea of the beamformer developed in [4] and, independently, in [7], is to explicitly model the steering vector uncertainty as $\boldsymbol{\delta} \triangleq \widetilde{\mathbf{a}}_s - \mathbf{a}_s$ where $\widetilde{\mathbf{a}}_s$ and $\mathbf{a}_s$ are the actual and presumed signal steering vectors, respectively; and to assume that the Euclidean norm of $\boldsymbol{\delta}$ is upper-bounded by a known constant ε. This corresponds to the case of spherical uncertainty; a more general ellipsoidal uncertainty model has been considered in [7] and [8].

The essence of the approach of [4] is to add robustness to the standard MV beamforming problem (1) by using a distortionless response constraint that must be satisfied for all mismatched signal steering vectors in the given spherical uncertainty set. With such a constraint, robust MV beamformer design has been formulated in [4] as the following optimization problem:

$$\min_{\mathbf{w}} \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \quad \text{s.t.} \quad |\mathbf{w}^H(\mathbf{a}_s + \boldsymbol{\delta})| \geq 1 \ \forall \ \|\boldsymbol{\delta}\| \leq \varepsilon. \quad (3)$$

Note that the constraint in (3) warrants that the distortionless response will be maintained in the worst case, i.e., for the particular choice of $\boldsymbol{\delta}$ that corresponds to the smallest value of $|\mathbf{w}^H(\mathbf{a}_s + \boldsymbol{\delta})|$ provided that $\|\boldsymbol{\delta}\| \leq \varepsilon$. Towards converting (3) to convex form, it has been shown in [4] that, for reasonably small size of the uncertainty region, $\varepsilon \leq |\mathbf{w}^H \mathbf{a}_s|/\|\mathbf{w}\|$,

$$\min_{\|\boldsymbol{\delta}\| \leq \varepsilon} |\mathbf{w}^H(\mathbf{a}_s + \boldsymbol{\delta})| = |\mathbf{w}^H \mathbf{a}_s| - \varepsilon \|\mathbf{w}\|. \quad (4)$$

Using (4) and taking into account that the objective function in (3) remains unchanged when $\mathbf{w}$ undergoes an arbitrary phase rotation, it has been shown in [4] that (3) can be converted to the following convex form:

$$\min_{\mathbf{w}} \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \quad \text{s.t.} \quad \mathbf{w}^H \mathbf{a}_s \geq \varepsilon \|\mathbf{w}\| + 1, \quad (5)$$

where the constraint in (5) also implicitly constrains $\mathbf{w}^H \mathbf{a}_s$ to be real valued and positive. The problem in (5) belongs to the class of second-order cone programming (SOCP) problems that can be easily solved (with complexity comparable to that of the SMI-based MV beamformer) using standard and highly efficient convex optimization software [42], [43] or, alternatively, using Newton-type algorithms [7], [8], [44]. It can be proved [4] that the constraint in (5) is active, i.e., it is satisfied with equality. Interestingly, the robust design in (5) admits an adaptive DL interpretation; see "Adaptive DL Interpretation of the Robust Beamformer (5)" to appreciate this link.

EXTENSIONS OF WORST-CASE BEAMFORMER DESIGNS

One useful extension of the robust beamformer (5) has been developed in [6]. In this work, a more general case is

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ADAPTIVE DL INTERPRETATION OF THE ROBUST BEAMFORMER (5)

Taking into account that the constraint in (5) is active and using the Lagrange multiplier method, the solution of (5) can be found by minimizing the Lagrangian function [8]

$$L(\mathbf{w}, \lambda) = \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} - \lambda(\mathbf{w}^H \mathbf{a}_s - \varepsilon \|\mathbf{w}\| - 1),$$

where λ is the Lagrange multiplier. Computing the gradient of $L(\mathbf{w}, \lambda)$ and equating it to zero yields

$$\hat{\mathbf{R}} \mathbf{w} + \lambda \varepsilon \frac{\mathbf{w}}{\|\mathbf{w}\|} = \lambda \mathbf{a}_s. \quad (6)$$

Since rescaling the weight vector by an arbitrary constant does not affect the output SINR, (6) can be transformed to

$$\left(\hat{\mathbf{R}} + \frac{\varepsilon}{\|\mathbf{w}\|} \mathbf{I} \right) \mathbf{w} = \mathbf{a}_s, \quad (7)$$

where, for the sake of simplicity, the same notation $\mathbf{w}$ used for the rescaled weight vector as for the original one in (6).

From (7), it can be seen that the robust beamformer (5) belongs to the class of adaptive DL techniques because its DL factor $\varepsilon/\|\mathbf{w}\|$ depends on $\|\mathbf{w}\|$ and, therefore, is scenario-dependent. In contrast to the fixed DL approach, such adaptive diagonal loading optimally matches the DL factor to the radius ε of the uncertainty sphere.

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considered where, apart from the steering vector mismatch, there is a nonstationarity of the beamformer training data. This nonstationarity is characterized in [6] by the matrix $\boldsymbol{\Delta}$ that models the mismatch in the data matrix $\mathbf{Y}$, and it is proposed to combine the robustness against interference nonstationarity and steering vector errors using the ideas similar to that of [4] and [7]. Correspondingly, the objective function in (3) can be modified as

$$\max_{\|\boldsymbol{\Delta}\| \leq \eta} \|(\mathbf{Y} + \boldsymbol{\Delta})^H \mathbf{w}\|^2,$$

where η is some known upper bound on the norm of the matrix $\boldsymbol{\Delta}$.

It has been shown in [6] that the resulting modified problem can be converted to the following convex form:

$$\min_{\mathbf{w}} \|\mathbf{Y}^H \mathbf{w}\| + \eta \|\mathbf{w}\| \quad \text{s.t.} \quad \mathbf{w}^H \mathbf{a}_s \geq \varepsilon \|\mathbf{w}\| + 1. \quad (8)$$

Note that, similar to (5), the problem in (8) belongs to the class of SOCP problems and, hence, it can be easily solved.

Another useful extension of the approach of [4] and [7] has been developed in [12]. The authors of [12] argue that, although the worst-case beamformer designs are known to result in quite robust techniques, they might be overly conservative because the actual worst operational conditions may occur in practice with a very low (or even zero) probability. This motivated the authors of [12] to develop an alternative approach to robust beamforming that provides the robustness only against "likely" spatial signature errors.

Using this philosophy, the probabilistically constrained counterpart of the problem (3) can be written as [12]

$$\min_{\mathbf{w}} \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \quad \text{s.t.} \quad \Pr\{|\mathbf{w}^H(\mathbf{a}_s + \boldsymbol{\delta})| \geq 1\} > p, \qquad (9)$$

where $\boldsymbol{\delta}$ is assumed to be a random mismatch vector drawn from some known distribution, $\Pr\{\cdot\}$ is the probability operator whose explicit form can be obtained from the statistical assumptions on the steering vector errors, and p is some preselected probability threshold. In contrast to the deterministic constraint used in (3) (that requires the distortionless response to be maintained for all norm-bounded mismatch vectors in the uncertainty sphere), the soft (probabilistic) constraint in (9) maintains the distortionless response only for the mismatch vectors $\boldsymbol{\delta}$ whose probability is sufficiently high, while ignoring the values of $\boldsymbol{\delta}$ that are unlikely to occur. Therefore, the constraint in (9) can be interpreted as an outage probability constraint maintaining this probability ($p_{\text{out}} = 1 - p$) low.

It has been shown in [12] that for both circularly symmetric Gaussian and for the worst-case distribution of the steering vector mismatch, the chance programming problem in (9) can be tightly approximated by means of deterministic SOCP problems. The worst-case distribution (for given covariance) of the mismatch vector turns out to be discrete. This result entails an intermediate restriction (strengthening) of the outage constraint; see [12] for details. Interestingly, the resulting problems are mathematically quite similar to (5). However, an important advantage of the probability-constrained beamformers of [12] with respect to their worst-case counterparts of [4] and [7] is that the former beamformers enable to explicitly quantify the parameters of the uncertainty region in terms of the beamformer outage probability. This streamlines the choice of ε.

Further convex optimization-based extensions of the robust beamformers discussed in this section have been recently proposed. In [9], a semidefinite programming (SDP) approach has been developed to extend the beamformers of [4] and [7] to a more general (than the spherical and ellipsoidal) class of uncertainty models. Several broadband generalizations of [4] have been proposed in [10] and [11]. Extensions of the approaches of [4] and [12] to the problem of designing robust multiuser MIMO receivers have been developed in [13] and [14], respectively.

Before moving on, it is instructive to summarize three different approaches towards beamforming under uncertainty, using the relatively simple case of receive beamforming as an example: see "Beamforming Under Uncertainty."

TRANSMIT BEAMFORMING

DOWNLINK BEAMFORMING

For notational convenience, we consider a single base station equipped with N antennas, transmitting individual narrowband data streams to a set of M users, each having a single antenna. Note that all results in this section generalize straightforwardly to scenarios with multiple cells, see [17]. The signal transmitted at the base station is given by

$$\mathbf{y}(t) = \sum_{m=1}^{M} s_m(t) \mathbf{w}_m, \qquad (10)$$

where $s_m(t)$ and $\mathbf{w}_m$ are the transmitted signal intended for user m and the beamforming vector for this user, respectively. The signal received at user m is given by

$$x_m(t) = \mathbf{h}_m^H \mathbf{y}(t) + n_m(t), \qquad (11)$$

where $\mathbf{h}_m$ is the downlink channel vector of user m and $n_m(t)$ is additive noise with power σ_m^2.

A basic (and meaningful from the network operator's perspective) formulation of the downlink beamforming problem is to impose a constraint on the received SINR of each user and minimize the total transmitted power subject to these constraints; this is often referred to as SINR balancing. The formulations considered below can be extended to systems using dirty paper coding, see e.g., [54] and references therein, which also provide connections to information-theoretic results on the rate region. When the channel vectors $\mathbf{h}_m$ are known, the resulting optimization problem can be written as

$$
\begin{aligned}
&\min_{\{\mathbf{w}_m\}_{m=1}^M} \sum_{m=1}^{M} \|\mathbf{w}_m\|^2 \\
&\text{s.t.} \quad \frac{|\mathbf{w}_k^H \mathbf{h}_k|^2}{\sum_{l \neq k}^{M} |\mathbf{w}_l^H \mathbf{h}_k|^2 + \sigma_k^2} \geq \gamma_k \; \forall k = 1, \dots, M,
\end{aligned}
\qquad (12)
$$

where γ_k denotes the desired minimum SINR for user k. As neither the objective function nor the constraints change if the beamforming vectors undergo a phase rotation [17], this problem can be formulated as a SOCP program

$$
\begin{aligned}
&\min_{\{\mathbf{w}_m\}_{m=1}^M} \sum_{m=1}^{M} \|\mathbf{w}_m\|^2 \\
&\text{s.t.} \; (\mathbf{w}_k^H \mathbf{h}_k)^2 \geq \gamma_k \sum_{l \neq k}^{M} |\mathbf{w}_l^H \mathbf{h}_k|^2 + \gamma_k \sigma_k^2 \; \forall k = 1, \dots, M,
\end{aligned}
\qquad (13)
$$

where $\mathbf{w}_k^H \mathbf{h}_k$ is real valued and positive.

In practical cases, instantaneous CSI is often unavailable. If the channels instead are assumed to be randomly fading with known second-order statistics $\mathbf{R}_m = \mathbb{E}[\mathbf{h}_m \mathbf{h}_m^H]$, the SINR balancing problem can be written as

$$
\begin{aligned}
&\min_{\{\mathbf{w}_m\}_{m=1}^M} \sum_{m=1}^{M} \|\mathbf{w}_m\|^2 \\
&\text{s.t.} \quad \frac{\mathbf{w}_k^H \mathbf{R}_k \mathbf{w}_k}{\sum_{l \neq k}^{M} \mathbf{w}_l^H \mathbf{R}_k \mathbf{w}_l + \sigma_k^2} \geq \gamma_k \; \forall k = 1, \dots, M.
\end{aligned}
\qquad (14)
$$

Note that the previous case of known instantaneous channels corresponds to setting $\mathbf{R}_m = \mathbf{h}_m \mathbf{h}_m^H$. Note also that here the quality of service (QoS) constraints are expressed in terms of the

ratio between the average signal power and the average interference plus noise power. For receive beamforming, where the signal of interest and the interference pass through independently fading channels, such a constraint provides a (tight) lower bound on the average SINR. This follows from Jensen's inequality, since $1/x$ is a convex function. See also [45] and [46] for outage bounds and other results related to this measure of SINR. For downlink beamforming, the physical channel for the intracell interference is the same as for the desired signal, but the above results still hold approximately, since the effective channels (including beamforming) will be approximately uncorrelated in the numerator and denominator for any good choice of beamformers—at least if the number of antennas is sufficiently large.

The problem in (14) is a nonconvex quadratic program; in general, such problems can be NP-hard. However, the specific problem formulation (14) exhibits much extra structure that allows it to be solved efficiently. One possibility is to use the idea of semidefinite relaxation [47]. Defining the matrices $\mathbf{W}_m \triangleq \mathbf{w}_m \mathbf{w}_m^H$ and noting that $\mathbf{w}_l^H \mathbf{R}_k \mathbf{w}_l = \text{Tr}(\mathbf{R}_k \mathbf{W}_l)$, the problem (14) can be transformed to

$$\min_{\{\mathbf{W}_m\}_{m=1}^M} \sum_{m=1}^{M} \text{Tr}(\mathbf{W}_m)$$

$$\text{s.t. } \text{Tr}(\mathbf{R}_k \mathbf{W}_k) - \gamma_k \sum_{l \neq k}^{M} \text{Tr}(\mathbf{R}_k \mathbf{W}_l) \geq \gamma_k \sigma_k^2,$$

$$\mathbf{W}_k \succcurlyeq 0, \text{rank}(\mathbf{W}_k) = 1, k = 1, \dots, M, \quad (15)$$

where $\mathbf{A} \succcurlyeq 0$ means that $\mathbf{A}$ is positive semidefinite. The key in this transformation is that it explicitly reveals and isolates the nonconvex part of the problem. Except for the rank-one constraints $\text{rank}(\mathbf{W}_k) = 1$ ($k = 1, \dots, M$), the remaining problem is convex. Dropping these constraints (thus generally enlarging the feasible set), one obtains a relaxed SDP problem, which is convex and far easier to solve.

There are at least three possible interpretations of this semidefinite relaxation

▪ With the rank constraints $\text{rank}(\mathbf{W}_k) = 1$, (15) is completely equivalent to (14). This problem can become a relaxed version of (14) only when the matrices $\mathbf{W}_k$ are allowed to have any rank.

▪ It can be shown that the semidefinite relaxation of (15) is the Lagrange dual of the Lagrange dual of (14).

▪ If we allow the beamforming vectors to be randomly time-varying with covariance matrix $\mathbf{W}_k = \text{E}[\mathbf{w}_k \mathbf{w}_k^H]$, the optimal choice of $\mathbf{W}_k$ is given by the semidefinite relaxation of (15). One possible practical implementation of such a scheme is to use a space-time code with the corresponding transmit covariance matrix.

For general nonconvex quadratic programs, semidefinite relaxation can only be used to obtain a lower bound on the optimal objective function and possibly determine an approximate solution to the original problem, such as in the multicast beamforming problem described below. However, in the specific

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BEAMFORMING UNDER UNCERTAINTY

▪ Worst-case design:

$$\min_{\mathbf{w}} \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \quad \text{s.t.} \quad |\mathbf{w}^H(\mathbf{a_s} + \boldsymbol{\delta})| \geq 1 \ \forall \ \|\boldsymbol{\delta}\| \leq \varepsilon.$$

▪ Probabilistic design:

$$\min_{\mathbf{w}} \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \quad \text{s.t.} \quad \text{Pr}\{|\mathbf{w}^H(\mathbf{a_s} + \boldsymbol{\delta})| \geq 1\} > p.$$

Interpretation: In a long sequence of "trials" (i.i.d. draws of $\boldsymbol{\delta}$), acceptable performance is guaranteed in $p \times 100\%$ of cases; signal outage happens with probability (at the rate of) less than $1 - p$.

▪ Expectation-based design:

$$\min_{\mathbf{w}} \mathbf{w}^H \hat{\mathbf{R}} \mathbf{w} \quad \text{s.t.} \quad \text{E}\{|\mathbf{w}^H(\mathbf{a_s} + \boldsymbol{\delta})|\} \geq t.$$

The latter only requires knowledge of the second-order statistics (instead of the distribution) of $\mathbf{h} := \mathbf{a_s} + \boldsymbol{\delta}$, which is convenient. The flip side is that this formulation offers no outage performance guarantee in general. For this reason, expectation-based design is the "last resort" way of dealing with uncertainty.

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case of (15) with dropped rank constraints, it turns out that the ostensible relaxation is not a relaxation, i.e., the "relaxed" problem is exactly equivalent to the original problem. In other words, it can be shown that the solution to (15) with dropped rank constraints always yields rank-one matrices $\mathbf{W}_k$, which directly provides the solution to (14) using $\mathbf{W}_k = \mathbf{w}_k \mathbf{w}_k^H$. In optimization terminology, this result shows that strong duality holds for problem (14), i.e., that the dual of (14) has the same optimal objective as the primal problem. This result is not so surprising, considering that there are also several other algorithms available to solve problem (14); see [15] and [48]. These algorithms are not based on the convex reformulation of (15) but rather on rewriting the problem into an equivalent *virtual uplink* problem, that can be solved using fixed point iterations [49] (see "Connection Between Lagrange Duality and Virtual Uplink"). From practical experience, the algorithm in [48] is preferable compared to the semidefinite reformulation in terms of computational speed, at least so long as (15) with dropped rank constraints is solved using general purpose SDP software like [42].

FURTHER MODIFICATIONS AND EXTENSIONS

Several modifications and extensions have been proposed to the SINR balancing problem (14). For multicell scenarios, the problem can be extended to not only find the jointly optimal set of transmit beamformers but also to determine to which base station each user should be assigned to. See Figure 2 for an illustration. Surprisingly enough, this mixed combinatorial and nonconvex quadratic problem can be solved easily; see [50] and [51]. The trick is to conceptually view all the base stations as a single virtual base station that jointly transmits to all users, and solve the corresponding beamforming problem. By making the channel covariance matrices for the virtual base station block

[FIG2] Illustration of downlink beamforming, where one base station transmits to two mobiles and another base station transmits to a third mobile, using the strategy (14).

diagonal, the solution will not benefit from using coherent transmission from several base stations (in contrast to so-called coherent coordinated multipoint transmission schemes that recently have been proposed for use in IMT-advanced), and it can be shown that the optimal beamforming vectors will only be nonzero in a subvector corresponding to one of the base stations. One proof is based on the simple observation that the corresponding optimal matrices $\mathbf{W}_k$ in the semidefinite reformulation will be both block diagonal and rank one, which is only possible if only one of the blocks is nonzero.

An alternative is to exploit the virtual uplink formulation and use general results from the theory of standard interference functions [49]. Algorithmically, this conceptual idea can be implemented with a computational complexity that is only K times larger than in the case with a given base station assignment, where K is the number of base stations. Unfortunately, to our knowledge, there is no convex formulation of the resulting optimization problem, which however can still be solved very efficiently using the virtual uplink formulation and a fixed-point iteration, see [48]. An alternative solution using quasi convexity is described in [53].

An advantage of the semidefinite relaxation technique is that it is very easy to add more constraints to the problem formulation. For example, the transmit power of the individual antenna elements can be constrained; see [17]. As long as the corresponding constraints on the matrices $\mathbf{W}_k$ are linear (or convex), the problem can be solved efficiently. However, there are in general

no guarantees that strong duality holds (i.e., that the optimal matrices $\mathbf{W}_k$ are rank one) when additional convex constraints are included. Strong duality has been proven only for certain special cases. For example, individual power constraints per antenna are considered in [54], where an efficient algorithm based on a generalization of the virtual uplink duality is developed. In [55] it is shown that an indefinite constraint of the form $\mathbf{w}_k^H \mathbf{C}_k \mathbf{w}_k = 0$ or $\mathbf{w}_k^H \mathbf{C}_k \mathbf{w}_k \geq 0$ can be added for each user, where the matrices $\mathbf{C}_k$ are indefinite Hermitian matrices, and strong duality still holds for the resulting optimization problem. Such constraints may, for example, be used to enforce solutions with increased path diversity in code-division multiple access (CDMA) scenarios, or to constrain the relative power transmitted to surrounding cells. This result has recently been generalized in [56], where both aforementioned types of indefinite constraints and so-called soft-shaping interference constraints of the form $\sum_{m=1}^{M} \mathbf{w}_m^H \mathbf{S}_{km} \mathbf{w}_m \leq \tau_k$ with positive semidefinite matrices $\mathbf{S}_{km}$ are considered. It is shown that the semidefinite relaxation of the corresponding optimization problems has a rank-one solution in a certain number of cases, for example when there are up to two indefinite constraints for each user, or when there are up to two soft-shaping constraints for the system. The proofs are constructive, showing how a given high-rank solution can be converted into a solution with lower rank, under certain conditions.

Just as for the receive beamforming problem, many papers have been devoted to robust extensions that can cope with uncertainties in the channel knowledge, due to estimation errors, feedback quantization or delays between channel estimation and the actual transmission, for example. Most references consider worst-case strategies given bounds on the errors. Errors in the channel covariance matrices are considered in [17] and [19] where the true covariance matrix is modeled as $\mathbf{R}_m + \boldsymbol{\Delta}_m$ with a known bound $\|\boldsymbol{\Delta}_m\| \leq \varepsilon_m$ on the error term. Replacing the numerator and denominator in (14) by a lower and upper bound, respectively, as proposed in [17], results in a kind of diagonal loading of the matrix $\mathbf{R}_k$ in the numerator and denominator of the constraints in (14). Unfortunately, this approach is very conservative and the resulting optimization problem often does not even have a feasible solution. A less conservative approach is proposed in [19], wherein the worst-case matrix $\boldsymbol{\Delta}_m$ for each constraint is found by solving the dual problem, and the resulting beamforming problem is solved using the semidefinite relaxation technique. Numerical experiments in [19] indicate that the obtained solution is always rank one, but no proof of this empirical observation is available. An alternative outage probability based approach is proposed in [20], where the matrices $\boldsymbol{\Delta}_m$ are assumed to be Gaussian distributed and the SINR constraints are required to hold with a certain probability. Again, semidefinite relaxation is used to solve the problem and the numerical results in [20] indicate that the solution always is rank one.

> ONE OF THE MOST POPULAR AD HOC APPROACHES TO IMPROVE THE ROBUSTNESS OF THE SMI-BASED MV TECHNIQUE AND TO AVOID SIGNAL SELF-NULLING IS THE DIAGONAL LOADING METHOD.

Robustness against uncertainties in the channel vectors $\mathbf{h}_m$ has been considered in [18], [57], and [58] using a QoS constraint expressed in terms of the MSE instead of SINR. For exactly known channel vectors, such an MSE constraint is equivalent to an SINR constraint. For partial CSI, on the other hand, MSE constraints provide a conservative lower bound on the SINR, as shown in [18]. Stochastic error models have also been considered where the channel vector uncertainties are assumed to be Gaussian distributed. In [57], the average MSE is optimized subject to a constraint on the total transmitted power, whereas in [21], a prespecified outage level on the MSE is used.

MULTICAST BEAMFORMING

Consider a base station or wireless access point that uses N antennas to transmit common information to a pool of M users, each equipped with a single receive antenna. This problem statement corresponds to the case of single-group multicast (or, equivalently, broadcast) beamforming. Let $\mathbf{h}_m$ be the $N \times 1$ complex downlink channel vector of user m and $\mathbf{w}$ be the $N \times 1$ complex beamforming vector. The objective is to design $\mathbf{w}$ in such a way that the inner product of $\mathbf{w}$ and each $\mathbf{h}_m$ $(m = 1, \ldots, M)$ is large, while the norm of $\mathbf{w}$ is small. This philosophy is rather different from the robust receive or unicast transmit beamforming paradigms, because the different $\mathbf{h}_m$s need not be clustered in a small neighborhood and the
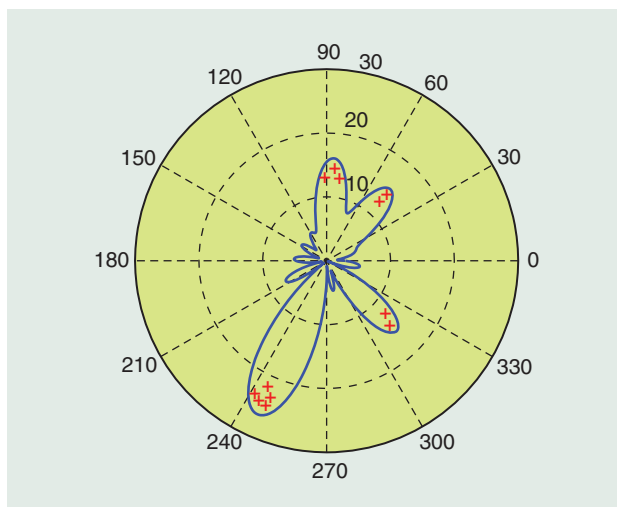
[FIG3] Illustration of multicast beamforming: Polar plot of a multicast transmit beampattern. Multicast subscriber locations are denoted by red crosses. Unlike receive beamforming or single-user transmit beamforming, the multicast beampattern comprises multiple lobes, each generally serving a group of users. Unlike multiuser transmit beamforming that uses multiple beampatterns, each carrying a different unicast stream, here a single beampattern that carries a common information stream is used. Notice that the lobe serving the lower-left group has higher peak magnitude, to compensate for a higher path loss experienced by distant users.

resulting adapted transmit beampattern has multiple main lobes; see Figure 3.

Formally, such a design may be stated as the following optimization problem [23]

$$\min_{\mathbf{w}} \|\mathbf{w}\|^2 \quad \text{s.t.} \quad |\mathbf{w}^H \mathbf{h}_m|^2 \geq \sigma_m^2 \gamma_m \ \forall \, m = 1, \ldots, M, \quad (20)$$

where σ_m^2 is the additive noise power at user m, and γ_m is the desired signal-to-noise power ratio (SNR) at user m. The right-hand side of each inequality in (20) can be absorbed in $\mathbf{h}_m$, yielding $|\mathbf{w}^H \widetilde{\mathbf{h}}_m|^2 \geq 1$ with $\widetilde{\mathbf{h}}_m \triangleq \mathbf{h}_m / \sqrt{\sigma_m^2 \gamma_m}$. Dropping the tilde sign for brevity yields the following QoS formulation:

$$\min_{\mathbf{w}} \|\mathbf{w}\|^2 \quad \text{s.t.} \quad |\mathbf{w}^H \mathbf{h}_m|^2 \geq 1 \ \forall \, m = 1, \ldots, M. \quad (21)$$
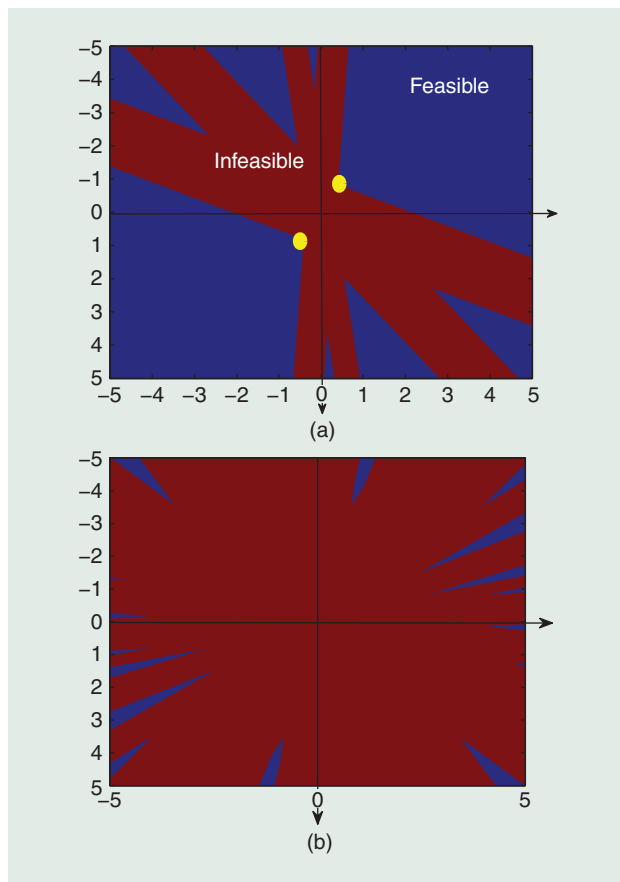
[FIG4] Peeking through shattered glass: Geometry of the feasible set in the real-valued two-dimensional case ($N = 2$ transmit antennas; w and $\{\mathbf{h}_i\}_{i=1}^M$ are real valued). (a) $M = 4$ and (b) $M = 40$. The axes correspond to the elements of w. Every constraint $(\mathbf{w}^T \mathbf{h}_i)^2 \geq 1$ excludes a strip perpendicular to $\mathbf{h}_i$ and passing through the origin. The nonconvex feasible set is shown in blue, while the infeasible set is shown in red. Notice that both are symmetric with respect to reflection (changing the sign of both elements of w). Since the norm of w is also invariant with respect to change of sign, there are two optimal solutions for w indicated by the yellow circles. As the number of users/constraints (M) increases, the glass tends to shatter in a larger number of smaller pieces that are further away from the origin, as illustrated in (b).

Note that this formulation has a certain similarity to (12) as in both cases the total transmit power is minimized subject to QoS constraints. However, contrary to (12), a single weight vector is used in (21).

An alternative formulation to the one in (21) arises from an information theory standpoint. Intuitively, if one transmits a single information-bearing signal to be decoded by a group of users, the attainable information rate that can be decoded by all interested users is determined by the weakest link, i.e., the user with the smallest SNR. This suggests the following "democratic" max-min-fair formulation:

$$\max_{\mathbf{w}} \ \min_{m \in \{1, \ldots, M\}} |\mathbf{w}^H \mathbf{h}_m|^2 \quad \text{s.t.} \quad \|\mathbf{w}\|^2 \leq P,$$

where P is an upper bound on the allowable transmission power. Without loss of generality, we may absorb P in the channel vectors and henceforth set $P = 1$. It is also easy to see that an optimal solution will use all available power, hence we may replace the power inequality with equality.

It is important to note that beamforming does not generally attain the multicast channel capacity—this may require a higher-rank transmit covariance. The capacity-attaining strategy, however, is often impractical for a number of reasons, including the complexity of multistream Shannon (de)coding, and incompatibility with existing and emerging standards. Beamforming, on the other hand, is simple to implement and often attains a significant fraction of multicast capacity [24].

Interestingly, it can be shown that the QoS and the max-min-fair formulations of multicast beamforming are equivalent up to scaling [24], and the scaling constant can be easily determined. Furthermore, multicast beamformer design naturally yields the optimal information-theoretic transmission strategy as a by-product, as we will see in the sequel.

How difficult is the multicast beamforming problem? It is certainly clean cut, and easy in the case of a single user: by virtue of the Cauchy-Schwartz inequality, an optimal solution is simply a scaled version of the user's steering vector. When we add more users, the situation becomes less clear. At this point, it is instructive to turn to the QoS formulation to gain insight. Notice that the quadratic constraints $|\mathbf{w}^H \mathbf{h}_m|^2 \geq 1$ are nonconvex. This implies that we are dealing with a nonconvex optimization problem, as the first clue. Going one step further, let us visualize the structure of the feasible set in the QoS formulation. Towards this end, we will consider the special case where all vectors are real valued and $N = 2$. Figure 4 illustrates the intricate structure of the "playing field" in this simplified scenario, and the picture is not pretty. Whereas we could perhaps characterize the potentially interesting vertices when M is small, this seems daunting for large M. It has been shown in [24] that the multicast beamforming problem is in fact NP-hard for $M \geq N$; it contains the *partition* problem as a special case. In plain words, this means that we have to give up hope of exactly solving an arbitrary problem instance at reasonable complexity. This motivates the pursuit of approximate solutions that can

approach optimal performance at moderate complexity. Towards this end, a convex approximation strategy based on semidefinite relaxation is considered next.

Using $|\mathbf{w}^H\mathbf{h}_m|^2 = \text{Tr}(\mathbf{w}\mathbf{w}^H\mathbf{h}_m\mathbf{h}_m^H)$ and defining $\mathbf{R}_m \triangleq \mathbf{h}_m\mathbf{h}_m^H$, we may recast the max-min fair problem as follows:

$$\max_{\mathbf{w}} \min_{m\in\{1,\ldots,M\}} \text{Tr}(\mathbf{w}\mathbf{w}^H\mathbf{R}_m) \quad \text{s.t.} \quad \text{Tr}(\mathbf{w}\mathbf{w}^H) = 1. \quad (22)$$

By change of variable $\mathbf{X} \triangleq \mathbf{w}\mathbf{w}^H$, we may further restate (22) as

$$\max_{\mathbf{X}} \min_{m\in\{1,\ldots,M\}} \text{Tr}(\mathbf{X}\mathbf{R}_m)$$

$$\text{s.t.} \quad \text{Tr}(\mathbf{X}) = 1, \mathbf{X} \succeq 0, \text{rank}(\mathbf{X}) = 1. \quad (23)$$

Following the idea of semidefinite relaxation, we can drop the nonconvex rank constraint $\text{rank}(\mathbf{X}) = 1$ to approximate (23) by an SDP problem. Notice that this relaxation "restores" up to full covariance rank, yielding the capacity-optimal transmit covariance [24], [59] [cf. first-principles definition of multicast capacity, using $\text{Tr}(\mathbf{X}\mathbf{R}_m) = \text{Tr}(\mathbf{X}\mathbf{h}_m\mathbf{h}_m^H) = \mathbf{h}_m^H\mathbf{X}\mathbf{h}_m$, and monotonicity of $\log(\cdot)$].

Once the relaxed problem is solved, the only direct claim one can make is that the resulting objective t_{opt} is no less than the optimal max-min value of the original NP-hard problem (since by dropping a constraint we expanded the feasible set). In many (but not all) cases, it turns out that the associated $\mathbf{X}_{\text{opt}}$ is rank one, which means that our relaxation was not a relaxation after all (see also the earlier discussion for downlink beamforming). If $\mathbf{X}_{\text{opt}}$ is rank one, then we can find $\mathbf{w}_{\text{opt}}$ for the original problem simply by taking the principal component of $\mathbf{X}_{\text{opt}}$. When $\mathbf{X}_{\text{opt}}$ has higher rank, there is more work to be done in "rounding" $\mathbf{X}_{\text{opt}}$ to a rank-one matrix—simply taking the principal component is not the best strategy. The prevailing rounding strategy is based on randomization: drawing i.i.d. random vectors from a zero-mean multivariate Gaussian of covariance $\mathbf{X}_{\text{opt}}$ and picking the best one. Details can be found in [24]. Note that randomization can be theoretically justified in this context—it is possible to bound the gap to the optimal solution of the original NP-hard problem. On this issue, see [47], [60], and [61].

At the end of the day, one is interested in how well the overall relaxation-randomization algorithm works in practice. The answer is that it works very well [24], although, for the case of a single-group multicast, there are now better and simpler algorithms available as a result of considerable follow-up work [62], [63]. The power of convex optimization/approximation lies in its generality; for example, it can handle the case of multiple (interfering) multicast groups [64]–[66] and additional convex constraints.

FURTHER EXTENSIONS

Robust multicast beamforming has been dealt with in [64]. Bridging the ground between multiuser downlink and multicast beamforming, the general case of multiple interfering multicast groups has been studied in [62], [65], and [66], and cross-layer multicast beamforming and admission control in

[62]. Instead of (exact or approximate) instantaneous CSI, it is possible to use long-term average CSI in the form of estimated channel correlation matrices $\hat{\mathbf{R}}_m$, albeit only average QoS guarantees can be offered in this case. Going one step further, [67] considered the case when the only information available for the channel vectors is their prior distribution. This is naturally modeled as a mixture distribution—e.g., a Gaussian mixture comprising components centered at different locations and with varying spread. Such a model can capture subscriber clustering in malls, campuses, or other urban "hotspots." In this case, similarly to the receive and downlink beamforming techniques of [12] and [20], the pertinent design criterion is the beamformer outage probability. While outage probability minimization also turns out to be NP-hard, an effective approximation is again possible [67]. It is worth noting that this last approach is particularly appealing in practice, because the mixture model can be built using historical data and/or field measurements around local points of interest.

RELAY NETWORK BEAMFORMING

Let us now consider a wireless network that consists of a source, a destination, and N relay nodes as shown in Figure 5. Assume that due to the poor quality of the channel between the source and destination, they cannot communicate directly with each other, but the destination cooperates with the N single-antenna relay nodes to receive the information transmitted by the source node and retransmitted by the relays. We use f_i and g_i to denote the channel complex coefficients between the source and the ith relay and between the ith relay and destination, respectively. In earlier network beamformer designs [28], is has been assumed that the instantaneous CSI is perfectly known at the destination or relays. However, this assumption is often violated in practical scenarios with randomly fading channels. To avoid the need to know instantaneous CSI, f_i and g_i can be modeled as random variables [29], and it can be assumed that their joint second-order statistics are known at the destination node that uses this knowledge to compute the relay complex weight coefficients and feed them back to the relay nodes. Alternatively, such second-order CSI may be available at the relay nodes rather than destination. In the latter case, each relay has to compute its own weight coefficient.
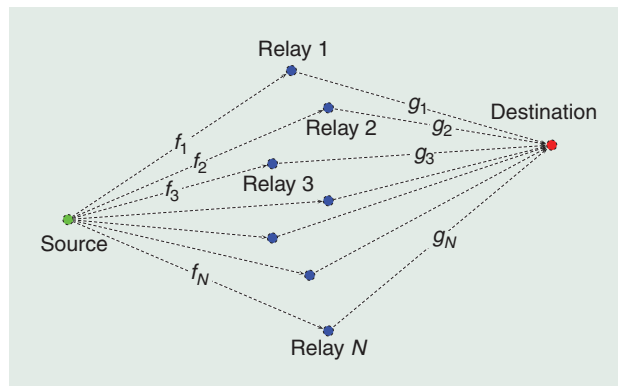
During the first step of a two-step amplify-and-forward protocol, the source transmits the signal $\sqrt{P_0}s$ to the relays, where s is the information symbol, P_0 is the source transmit power, and without loss of generality it is assumed that $\mathrm{E}\{|s|^2\} = 1$. The received signal at the ith relay is given by

$$x_i = \sqrt{P_0} f_i s + \nu_i, \tag{24}$$

where ν_i is the noise at the ith relay whose variance is known to be σ_ν^2. In the second step, the ith relay transmits y_i, which is an amplified and phase-steered version of its received signal and can be written as

$$y_i = w_i x_i. \tag{25}$$

Here, w_i is the complex relay beamforming weight that is used by the ith relay to adjust the phase and the amplitude of the corresponding signal.

Interestingly, network beamforming can be viewed as a certain combination of receive and transmit beamforming as the same weights are used for the signal reception and transmission. Moreover, network beamforming is distributed as each relay node knows only its own received signal, and does not know the signals received by the other relay nodes.

The signal received by the destination is given by

$$z = \sum_{i=1}^{r} g_i y_i + n, \tag{26}$$

where n is the receiver noise whose variance σ_n^2 is known. Using (24) and (25), we can rewrite (26) as

$$z = \underbrace{\sqrt{P_0} \sum_{i=1}^{r} w_i f_i g_i s}_{\text{signal component}} + \underbrace{\sum_{i=1}^{r} w_i g_i \nu_i + n}_{\text{total noise, } n_T}. \tag{27}$$

To optimally calculate the relay weight coefficients, the destination SNR has to be maximized subject to some power constraints. To illustrate the application of convex optimization to this problem, let us consider the individual relay power constraints. Then, the following optimization problem has to be solved:

$$\max_{\mathbf{w}} \text{SNR} \quad \text{s.t.} \quad P_i \leq \mathcal{P}_i \ \forall \, i = 1, \ldots, N, \tag{28}$$

SUMMARY OF THE NETWORK BEAMFORMING ALGORITHM
Step 1: Properly set the initial values of t_l and t_u.
Step 2: Set $t := (t_l + t_u)/2$ and solve (30).
Step 3: If (30) is feasible, then $t_l := t$; otherwise $t_u := t$.
Step 4: If $t_u - t_l < \delta$, then go to Step 5; otherwise go to Step 2.
Step 5: Find the weight vector from the principal eigenvector of the resulting matrix $\mathbf{X}_{\text{opt}}$.

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where P_i and $\mathcal{P}_i$ are, respectively, the actual and maximum allowable transmit powers of the ith relay. As in (14), we use the ratio of expected signal power to expected noise power as a measure of SNR. In [29], it has been shown that this is given by $(\mathbf{w}^H \mathbf{R} \mathbf{w})/(\sigma_n^2 + \mathbf{w}^H \mathbf{Q} \mathbf{w})$ and $P_i = D_{ii}|w_i|^2$ where $\mathbf{w} \triangleq [w_1, \ldots, w_N]^T$, $\mathbf{f} \triangleq [f_1, \ldots, f_N]^T$, $\mathbf{g} \triangleq [g_1, \ldots, g_N]^T$, $\mathbf{Q} \triangleq \sigma_\nu^2 \mathrm{E}\{\mathbf{g}\mathbf{g}^H\}$, $\mathbf{R} \triangleq P_0 \mathrm{E}\{(\mathbf{f} \odot \mathbf{g})(\mathbf{f} \odot \mathbf{g})^H\}$, $\mathbf{D} \triangleq P_0 \, \text{diag} \, ([\mathrm{E}\{|f_1|^2\}, \ldots, \mathrm{E}\{|f_N|^2\}]) + \sigma_\nu^2 \mathbf{I}$, and D_{ii} is the ith diagonal entry of $\mathbf{D}$.

Hence, the problem in (28) can be rewritten as

$$\max_{\mathbf{w}} \frac{\mathbf{w}^H \mathbf{R} \mathbf{w}}{\sigma_n^2 + \mathbf{w}^H \mathbf{Q} \mathbf{w}} \quad \text{s.t.} \quad D_{ii}|w_i|^2 \leq \mathcal{P}_i \ \forall \, i = 1, \ldots, N.$$

Defining $\mathbf{X} \triangleq \mathbf{w}\mathbf{w}^H$, this optimization problem can be rewritten as

$$\max_{\mathbf{X}} \frac{\text{Tr}(\mathbf{R}\mathbf{X})}{\sigma_n^2 + \text{Tr}(\mathbf{Q}\mathbf{X})}$$

$$\text{s.t.} \ D_{ii}X_{ii} \leq \mathcal{P}_i \, \forall \, i = 1, \ldots, N; \text{rank}(\mathbf{X}) = 1, \mathbf{X} \succcurlyeq 0,$$

where X_{ii} is the ith diagonal entry of $\mathbf{X}$. Following the idea of semidefinite relaxation and dropping the nonconvex rank-one constraint, the latter problem can be relaxed as

$$\max_{\mathbf{X}, t} t$$
$$\text{s.t.} \quad \text{Tr}(\mathbf{X}(\mathbf{R} - t\mathbf{Q})) \geq \sigma_n^2 t,$$
$$X_{ii} \leq \mathcal{P}_i/D_{ii} \ \forall \, i = 1, \ldots, N; \ \mathbf{X} \succcurlyeq 0. \tag{29}$$

Note that, for any fixed value of t the set of feasible $\mathbf{X}$ in (29) is convex; it follows that the optimization problem in (29) is quasi convex.

Solving (29), one can obtain the maximum achievable SNR (which is the maximum value of t, denoted as $t_{\max}$). To solve (29), the following key observation [52] has been used in [29]. If, for some given SNR value t, the convex feasibility problem

$$\text{find} \, \mathbf{X}$$
$$\text{s.t.} \quad \text{Tr}(\mathbf{X}(\mathbf{R} - t\mathbf{Q})) \geq \sigma_n^2 t,$$
$$X_{ii} \leq \mathcal{P}_i/D_{ii} \ \forall \, i = 1, \ldots, N; \ \mathbf{X} \succcurlyeq 0 \tag{30}$$

is feasible, then $t_{\max} \geq t$. Conversely, if (30) is not feasible, then $t_{\max} < t$. Based on this observation, one can check whether the optimal value $t_{\max}$ of the quasi-convex problem (29) is smaller or greater than any given value t. In [29], it has been proposed to use a simple bisection algorithm for solving (29), where (30) has to be solved at each step of this algorithm. Let us start with some preselected interval $[t_l \, t_u]$ that is known to contain the optimal value $t_{\max}$, the problem (30) is then solved at the midpoint $t = (t_l + t_u)/2$. If (30) is feasible for this value of t, then $t_l = t$ is set; otherwise $t_u = t$ is chosen. This procedure is repeated until the difference between t_u and t_l is smaller than some preselected threshold δ. (See "Summary of the Network Beamforming

Algorithm'' for pseudocode of the overall solution.)

Numerical examples in [29] have shown that, similar to the case of downlink beamforming, the so-obtained solution $\mathbf{X}_{\text{opt}}$ is always rank one and, therefore, no randomization is needed to obtain the beamforming weight vector. However, no proof of this empirical observation is available in [29].

One suitable choice of the initial values of t_l and t_u is 0 and $\text{SNR}_{\max}(\mathcal{P}_{\max})$, respectively, where $\text{SNR}_{\max}(\mathcal{P}_{\max})$ is the maximum achievable SNR under the total relay power budget $\mathcal{P}_{\max} = \sum_{i=1}^{N} \mathcal{P}_i$. It has been shown in [29] that

$$\text{SNR}_{\max}(\mathcal{P}_{\max}) = \mathcal{P}_{\max}\lambda_{\max}(\mathbf{G}), \qquad (31)$$

where $\mathbf{G} \triangleq (\sigma_n^2\mathbf{I} + \mathcal{P}_{\max}\mathbf{D}^{-1/2}\mathbf{Q}\mathbf{D}^{-1/2})^{-1}\mathbf{D}^{-1/2}\mathbf{R}\mathbf{D}^{-1/2}$.

The results of [28] and [29] are applicable only when the relays are fully synchronized at the symbol level and when the source-to-relay and relay-to-destination channels are frequency flat. When these channels are frequency selective or the time synchronization between the relays is poor, the signal replicas passed through different relays and/or channel paths will arrive to the destination node with different delays. This will result in inter-symbol-interference (ISI).

To combat such ISI, two different approaches have been presented in the literature. In [33] and [34], a filter-and-forward protocol has been introduced for frequency selective relay networks, and several related network beamforming techniques have been developed. In these techniques, the relays deploy finite impulse response (FIR) filters to compensate for the effect of source-to-relay and relay-to-destination channels; that is, the burden of mitigating ISI is put on the shoulders of the relay nodes. One of these techniques can be viewed as an extension of (29) because it is based on maximizing the destination QoS (measured in terms of SINR) subject to the individual relay power constraints. The latter technique is also based on a combination of bisection search and convex feasibility problem-solving.

Another beamforming approach developed in [35] for asynchronous but flat-fading relay networks, suggests the relay processing to be simple (i.e., to follow the amplify-and-forward protocol), while the source and destination nodes carry the main burden of mitigating ISI. Viewing an asynchronous flat-fading relay network as an artificial multipath channel (where each channel path corresponds to one particular relay), the authors of [35] use the orthogonal frequency division multiplexing (OFDM) scheme at the source and destination nodes to deal with this artificial multipath channel.

Convex optimization has also found its application to multiuser (i.e., multiple-source, multiple-destination) network beamforming techniques. In [30], a network of relays is used to establish communication between multiple source destination

> **NETWORK BEAMFORMING CAN BE VIEWED AS A CERTAIN COMBINATION OF RECEIVE AND TRANSMIT BEAMFORMING AS THE SAME WEIGHTS ARE USED FOR THE SIGNAL RECEPTION AND TRANSMISSION.**

pairs. The relays amplify and phase adjust the signal they receive from all transmitting sources by multiplying it with a complex beamforming weight. To obtain the optimal value of beamforming weights, the total relay transmit power is minimized subject to QoS constraints on the received SINRs at the destinations. It is then shown that using semidefinite relaxation, this power minimization problem can be turned into a convex SDP problem. In light of the results of [68], when the number source-destination pairs is less than or equal to three, the semidefinite relaxation approach is always guaranteed to have a rank-one solution, and therefore, in this case it is not a relaxation but exact transformation of the original problem (note here some similarity to the downlink beamforming case, where the resulting solution after semidefinite relaxation yields rank-one matrices as well).

Considering the same problem as in [30], the authors of [36] use additional constraints to enforce the signals received by the destinations be all in-phase. This will turn the aforementioned constrained total relay power minimization problem into an SOCP problem. As SOCP problems can be solved with much lower computational complexity than SDP problems, the approach of [36] to peer-to-peer network beamforming is computationally less expensive than that of [30]. The price for this computational complexity improvement is a small increase in the relay transmitted power.

Convex optimization has also proven instrumental in application to the design of beamformers for two-way (bidirectional) relay networks. Such beamformers have been developed in [32] for three-node two-way networks with one multiantenna relay node and two single-antenna transceivers, and in [31] and [38] for multiple-node two-way networks with all single-antenna nodes involving two transceivers and multiple relays.

CONCLUSIONS AND FUTURE DIRECTIONS

In this article, we have presented an overview of the current state of the art of advanced optimization-based beamforming with application to the receive, transmit and network beamformer design problems. Connections have been drawn between different types of optimization-based beamformers, and it has been demonstrated that convex optimization is an indispensable toolbox for beamformer designs.

Promising future research directions include beamformer designs for frequency-selective scenarios; incorporating practical communications engineering aspects, such as synchronization, modulation, and coding; and real-time beamformer weight optimization to account for time-selective fading and other sources of temporal variation in the operational environment. Robustness issues will likely remain high in the research agenda, in light of erroneous/delayed/quantized CSI encountered

in practical systems. This is especially true for network beamforming that is still in its infancy. Computationally efficient implementations of beamforming techniques are critical for applications of beamforming in practical systems, and it can be foreseen that this field will keep benefiting from advances in convex optimization theory—including relevant work towards real-time convex optimization [69].

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