Chinese Journal of Aeronautics, (2015), 28(5): 1510–1518

Chinese Society of Aeronautics and Astronautics
& Beihang University

Chinese Journal of Aeronautics

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A new iterative algorithm for geolocating a known altitude target using TDOA and FDOA measurements in the presence of satellite location uncertainty

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Received 18 July 2014; revised 12 June 2015; accepted 16 July 2015
Available online 1 September 2015

KEYWORDS

Constrained Cramér-Rao lower bound (CCRLB);
Frequency difference of arrival (FDOA);
Generalized trust region sub-problem (GTRS);
Performance analysis;
Time difference of arrival (TDOA)

Abstract This paper considers the problem of geolocating a target on the Earth surface whose altitude is known previously using the target signal time difference of arrival (TDOA) and frequency difference of arrival (FDOA) measurements obtained at satellites. The number of satellites available for the geolocation task is more than sufficient and their locations are subject to random errors. This paper derives the constrained Cramér-Rao lower bound (CCRLB) of the target position, and on the basis of the CCRLB analysis, an approximately efficient constrained maximum likelihood estimator (CMLE) for geolocating the target is established. A new iterative algorithm for solving the CMLE is then proposed, where the updated target position estimate is shown to be the globally optimal solution to a generalized trust region sub-problem (GTRS) which can be found via a simple bisection search. First-order mean square error (MSE) analysis is conducted to quantify the performance degradation when the known target altitude is assumed to be precise but indeed has an unknown but deterministic error. Computer simulations are used to compare the performance of the proposed iterative geolocation technique with those of two benchmark algorithms. They verify the approximate efficiency of the proposed algorithm and the validity of the MSE analysis.

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Peer review under responsibility of Editorial Committee of CJA.

Production and hosting by Elsevier

1. Introduction

Determining the position of a target in a passive manner, usually referred to as passive localization, has been a central problem in many applications such as radar, sonar, navigation, tracking and wireless sensor networks.[1–5] For a passive localization task, the angle of arrival (AOA) of a target signal intercepted at

http://dx.doi.org/10.1016/j.cja.2015.08.015
1000-9361

receivers can be explored.[6] An alternative approach is to utilize the changing rate of phase difference (CRPD) obtained by an array of long baseline interferometers (LBIs) mounted on multiple receivers to jointly identify the position and velocity of a moving target.[7] Other commonly used positioning parameters include the time difference of arrival (TDOA) and the frequency difference of arrival (FDOA) of a target signal received at spatially distributed receivers.[8,9] As the positioning parameters mentioned above are all nonlinearly related to the target position, iterative numerical methods such as those developed in Refs.[10,11] may be used for target localization.

We shall consider in this work locating a target passively using TDOA and FDOA measurements, a problem that has been extensively investigated. When no prior information on the target position is available, besides the Taylor-series (TS)-based localization techniques,[10,11] algorithms that utilize the multidimensional scaling (MDS) and the semidefinite relaxation (SDR) have also been developed.[12,13] In Ref.[14], a constrained total least squares (CTLS) method was proposed to identify the target position from TDOA and FDOA measurements when the known sensor positions had errors. In Ref.[15], a closed-form localization algorithm was developed and it transforms the nonlinear TDOA and FDOA measurement equations into pseudo-linear ones via introducing nuisance variables as in Ref.[16]. It then eliminates the nuisance parameters using an orthogonal projection matrix and finds a least squares (LS) estimate of the target position. More recently, a two-step approach for the TDOA and FDOA-based target localization was proposed in Ref.[17], where the target position is first identified from TDOA asymptotes and then refined using the FDOA measurements.

When geometric constraints on the target position are available, they can be explored together with the target signal TDOA and FDOA measurements to improve the passive localization accuracy. In this paper, we shall focus on the passive localization of a target on the Earth surface with known altitude information, a problem referred to as target geolocation in literature.[18–20] This problem is of practical importance due to the large coverage of satellites, which makes it possible to monitor a large area of interest by using only a small number of receivers.

With the knowledge on the target altitude, a pair of satellites is sufficient for a geolocation task. This is because in this case, two satellites can produce one TDOA and one FDOA measurements, which, when combined with the altitude information, would be sufficient to geolocate a target in three-dimensional (3-D). In literature, there exist algebraic geolocation techniques for such dual-satellite systems.[18] The impacts of TDOA and FDOA measurement noises and the uncertainty in satellite positions and velocities on the performances of dual-satellite systems were analyzed in details in Refs.[21,22]. In Ref.[23], the authors considered an interesting dual-satellite geolocation scenario where two satellites had almost the same velocity. Recently, a particle swarm optimization (PSO)-based technique was developed for target geolocation in a dual-satellite system.[24] By restraining the particles on the Earth surface, it converts the constrained localization problem into an unconstrained one and finds the target position estimate iteratively.

When the number of satellites available for TDOA and FDOA-based target geolocation is greater than two, we have an over-determined case, which has been relatively less studied in literature. In Ref.[18], an algorithm was proposed for the over-determined case, where the iterative Newton's method was employed to jointly estimate the Lagrange multiplier and a nuisance parameter in order to find the target position estimate. A linear-correction least squares (LCLS) method was proposed for the case where only FDOA measurements and target altitude information were explored.[25]

The above works on target geolocation all assumed that satellite locations were known accurately. However, in practice, known satellite locations may be imprecise, which may degrade the target geolocation accuracy significantly if the satellite location uncertainty is simply ignored.[26,27] Recently, on the basis of the sequential quadratic programming (SQP) technique, an LSSQP algorithm was proposed in Ref.[28]. It takes into account the presence of satellite location uncertainty and solves a series of SQP sub-problems to geolocate a target with known altitude using TDOA and FDOA measurements.

This work aims at developing a new iterative algorithm for target geolocation that does not require the use of the computationally intensive SQP technique. Specifically, we shall consider the over-determined but realistic scenario where there are more than two satellites and their location information is subject to random errors. Target signal TDOA and FDOA measurements are explored for the geolocation task.

The study in this paper begins with formulating an equality constrained maximum likelihood estimator (CMLE) for the considered target geolocation problem, where the equality constraint comes from the altitude information. The constrained Cramer-Rao lower bound (CCRLB)[29,30] for the target position is derived. Inspired by the CCRLB analytical result, we transform the obtained CMLE which requires jointly identifying the target position and the satellite locations into a new CMLE that needs to estimate the target position only. The newly formulated CMLE has reduced complexity and the satellite location uncertainty is taken into account in its cost function. We show analytically that under a small satellite location error, the new CMLE is approximately efficient so that it can provide geolocation accuracy approximately equal to the target position CCRLB.

This paper proceeds to propose an iterative method to solve the reduced-complexity CMLE to estimate the target position. The development of the new algorithm follows the framework of Ref.[20]. Different from the previous work, we consider in this paper the presence of the satellite location error. Moreover, the newly proposed iterative technique formulates the problem of updating the target position estimate as a generalized trust region sub-problem (GTRS)[31,32] and finds the globally optimal solution via a simple bisection search. This is in contrast to the method developed in Ref.[20], where it needs to locate the smallest root of a 6th-order polynomial and computing all the six roots could be cumbersome. This paper also conducts a first-order mean square error (MSE) analysis to quantify the loss in the estimation accuracy of the newly proposed iterative geolocation algorithm when the known target altitude is assumed to be accurate but in fact it has unknown but deterministic deviation from the true value. In this case, the obtained target position estimate is biased. We adopt the re-parameterization technique[33] in the MSE analysis to gain more insights. Computer simulations are used to illustrate the good performance of the new geolocation technique and verify its approximate efficiency in attaining the CCRLB. The MSE analysis results are validated using simulations as well.

2. Geolocation scenario

We consider the geolocation of a target at an unknown position $\boldsymbol{u}^o = [x, y, z]^T$ using M satellites, where $M \geqslant 3$. Denote the sum of the target altitude and the local Earth radius at the target position as R. We have $\boldsymbol{u}^o$ must satisfy

$$||\boldsymbol{u}^o|| = R \tag{1}$$

where $||\cdot||$ denotes the 2-norm.

The satellite positions and velocities available for the geolocation task are subject to random errors and they are equal to $\boldsymbol{s}_i = \boldsymbol{s}_i^o + \Delta\boldsymbol{s}_i$ and $\dot{\boldsymbol{s}}_i = \dot{\boldsymbol{s}}_i^o + \Delta\dot{\boldsymbol{s}}_i$ $(i = 1,\ 2\ ,\ldots,\ M)$, where $\boldsymbol{s}_i^o = [x_i^o, y_i^o, z_i^o]^T$ and $\dot{\boldsymbol{s}}_i^o = [\dot{x}_i^o, \dot{y}_i^o, \dot{z}_i^o]^T$ are the true but unknown position and velocity of satellite i while $\Delta\boldsymbol{s}_i$ and $\Delta\dot{\boldsymbol{s}}_i$ are corresponding random errors. For notation simplicity, we collect $\boldsymbol{s}_i$ and $\dot{\boldsymbol{s}}_i$ in the column vector $\boldsymbol{\beta} = [\boldsymbol{s}^T, \dot{\boldsymbol{s}}^T]^T$, where $\boldsymbol{s} = [\boldsymbol{s}_1^T, \boldsymbol{s}_2^T, \ldots, \boldsymbol{s}_M^T]^T$ and $\dot{\boldsymbol{s}} = [\dot{\boldsymbol{s}}_1^T, \dot{\boldsymbol{s}}_2^T, \ldots, \dot{\boldsymbol{s}}_M^T]^T$. Therefore, the satellite location error vector is $\Delta\boldsymbol{\beta} = \boldsymbol{\beta} - \boldsymbol{\beta}^o$, where $\boldsymbol{\beta}^o = [\boldsymbol{s}^{oT}, \dot{\boldsymbol{s}}^{oT}]^T$, $\boldsymbol{s}^o = [\boldsymbol{s}_1^{oT}, \boldsymbol{s}_2^{oT}, \ldots, \boldsymbol{s}_M^{oT}]^T$ and $\dot{\boldsymbol{s}}^o = [\dot{\boldsymbol{s}}_1^{oT}, \dot{\boldsymbol{s}}_2^{oT}, \ldots, \dot{\boldsymbol{s}}_M^{oT}]^T$. We shall model $\Delta\boldsymbol{\beta}$ as a zero-mean Gaussian random vector with the covariance matrix $\boldsymbol{Q}_{\boldsymbol{\beta}}$.[26,27]

The target signal TDOAs and FDOAs are measured with satellite 1 as the reference satellite. Let c be the signal propagation speed that is known. The measured target signal TDOA between satellite pair i and 1, denoted by d_{i1}, is

$$d_{i1} = \frac{r_{i1}}{c} = \frac{r_{i1}^o + n_{i1}}{c} = \frac{r_i^o - r_1^o + n_{i1}}{c} \tag{2}$$

where $r_{i1} = cd_{i1}$ is the range difference of arrival (RDOA); r_{i1}^o is the true RDOA; $\frac{n_{i1}}{c}$ is the TDOA measurement noise; r_1^o is the true distance between the target $\boldsymbol{u}^o$ and satellite 1; and r_i^o is the true distance between the target and satellite i, which is equal to

$$r_i^o = ||\boldsymbol{u}^o - \boldsymbol{s}_i^o|| \tag{3}$$

Let f be the known target carrier frequency. The measured target signal FDOA between satellite pair i and 1, denoted by $\dot{d}_{i1}$, is

$$\dot{d}_{i1} = \frac{\dot{r}_{i1}}{c/f} = \frac{\dot{r}_{i1}^o + \dot{n}_{i1}}{c/f} = \frac{\dot{r}_i^o - \dot{r}_1^o + \dot{n}_{i1}}{c/f} \tag{4}$$

where $\dot{r}_{i1} = \frac{c}{f}\dot{d}_{i1}$ is the range rate difference; $\dot{r}_{i1}^o$ is the true range rate difference; $\frac{\dot{n}_{i1}}{c/f}$ is the FDOA measurement noise; $\dot{r}_1^o$ is the true range rate between the target and satellite 1; and $\dot{r}_i^o$ is the true range rate between the target and satellite i. $\dot{r}_i^o$ can be obtained by taking the time derivative of Eq. (3) and is equal to

$$\dot{r}_i^o = \frac{-\dot{\boldsymbol{s}}_i^{oT}\left(\boldsymbol{u}^o - \boldsymbol{s}_i^o\right)}{r_i^o} \tag{5}$$

To facilitate the presentation of the new iterative geolocation algorithm, we multiply the obtained TDOAs with the signal propagation speed c and the measured FDOAs with c/f, the wavelength of the target signal. Collecting the results yields the measurement vector $\boldsymbol{m} = [\boldsymbol{r}^T, \dot{\boldsymbol{r}}^T]^T$, where $\boldsymbol{r} = [r_{21}, r_{31}, \ldots, r_{M1}]^T$ and $\dot{\boldsymbol{r}} = [\dot{r}_{21}, \dot{r}_{31}, \ldots, \dot{r}_{M1}]^T$. The measurement error in $\boldsymbol{m}$ is $\Delta\boldsymbol{m} = \boldsymbol{m} - \boldsymbol{m}^o$, where $\boldsymbol{m}^o = [\boldsymbol{r}^{oT}, \dot{\boldsymbol{r}}^{oT}]^T$, $\boldsymbol{r}^o = [r_{21}^o, r_{31}^o, \ldots, r_{M1}^o]^T$ and $\dot{\boldsymbol{r}}^o = [\dot{r}_{21}^o, \dot{r}_{31}^o, \ldots, \dot{r}_{M1}^o]^T$. We shall model $\Delta\boldsymbol{m}$ as a zero-mean Gaussian distributed vector with the covariance matrix $\boldsymbol{Q}_{\boldsymbol{m}}$. It is further assumed that $\Delta\boldsymbol{m}$ is independent to the satellite location error $\Delta\boldsymbol{\beta}$.[26,27]

We are interested in determining the target position $\boldsymbol{u}^o$ using the noisy TDOA and FDOA measurements in $\boldsymbol{m}$, the erroneous satellite locations in $\boldsymbol{\beta}$, and the geometric equality constraint on $\boldsymbol{u}^o$ (see Eq. (1)).

3. CMLE and CCRLB

From Section 2, we know that the target position $\boldsymbol{u}^o$ and the true satellite location vector $\boldsymbol{\beta}^o$ are both unknown. Besides, due to the availability of the target altitude R, the MLE for $\boldsymbol{u}^o$ and $\boldsymbol{\beta}^o$ would become equality-constrained. Applying the fact that the TDOA and FDOA measurement noise $\Delta\boldsymbol{m}$ and the satellite location error $\Delta\boldsymbol{\beta}$ are independent zero-mean Gaussian random vectors, we can express the CMLE of $\boldsymbol{u}^o$ and $\boldsymbol{\beta}^o$ as

$$\begin{aligned} &\min_{\boldsymbol{u}^o,\boldsymbol{\beta}^o}\Big\{(\boldsymbol{m} - \boldsymbol{m}^o)^T\boldsymbol{Q}_{\boldsymbol{m}}^{-1}(\boldsymbol{m} - \boldsymbol{m}^o) \\ &\quad + (\boldsymbol{\beta} - \boldsymbol{\beta}^o)^T\boldsymbol{Q}_{\boldsymbol{\beta}}^{-1}(\boldsymbol{\beta} - \boldsymbol{\beta}^o)\Big\} \\ &\text{s.t.}\quad ||\boldsymbol{u}^o|| = R \end{aligned} \tag{6}$$

where the dependence of the true TDOA and FDOA vector $\boldsymbol{m}^o$ on $\boldsymbol{u}^o$ and $\boldsymbol{\beta}^o$ is given in Eqs. (2)–(5). The CMLE is asymptotically unbiased and can reach the CCRLB for the target position $\boldsymbol{u}^o$, which is[29,30]

$$\mathrm{CCRLB}(\boldsymbol{u}^o) = \boldsymbol{J}^{-1} - \boldsymbol{J}^{-1}\boldsymbol{F}(\boldsymbol{F}^T\boldsymbol{J}^{-1}\boldsymbol{F})^{-1}\boldsymbol{F}^T\boldsymbol{J}^{-1} \tag{7}$$

where $\boldsymbol{J}^{-1}$ is the Cramér-Rao lower bound (CRLB) of $\boldsymbol{u}^o$ without the knowledge on its altitude and $\boldsymbol{J}$ is the associated Fisher information matrix (FIM). As shown in Ref.[27], $\boldsymbol{J}$ is equal to

$$\boldsymbol{J} = \boldsymbol{X} - \boldsymbol{Y}\boldsymbol{Z}^{-1}\boldsymbol{Y}^T \tag{8}$$

where

$$\begin{cases} \boldsymbol{X} = \left(\dfrac{\partial\boldsymbol{m}^o}{\partial\boldsymbol{u}^o}\right)^T\boldsymbol{Q}_{\boldsymbol{m}}^{-1}\dfrac{\partial\boldsymbol{m}^o}{\partial\boldsymbol{u}^o} \\ \boldsymbol{Y} = \left(\frac{\partial\boldsymbol{m}^o}{\partial\boldsymbol{u}^o}\right)^T\boldsymbol{Q}_{\boldsymbol{m}}^{-1}\frac{\partial\boldsymbol{m}^o}{\partial\boldsymbol{\beta}^o} \\ \boldsymbol{Z} = \boldsymbol{Q}_{\boldsymbol{\beta}}^{-1} + \left(\frac{\partial\boldsymbol{m}^o}{\partial\boldsymbol{\beta}^o}\right)^T\boldsymbol{Q}_{\boldsymbol{m}}^{-1}\frac{\partial\boldsymbol{m}^o}{\partial\boldsymbol{\beta}^o} \end{cases} \tag{9}$$

The derivation of the partial derivatives in Eq. (9) can also be found in Ref.[27] and as a result, it is omitted here for brevity. The matrix $\boldsymbol{F}$ is the partial derivative of the equality constraint in Eq. (6) with respect to the target position $\boldsymbol{u}^o$, which is equal to

$$\boldsymbol{F} = \frac{\partial(||\boldsymbol{u}^o|| - R)}{\partial\boldsymbol{u}^o} = \frac{\boldsymbol{u}^o}{||\boldsymbol{u}^o||} \tag{10}$$

Solving Eq. (6) for the target position $\boldsymbol{u}^o$ can be computationally demanding, because we need to jointly estimate $\boldsymbol{u}^o$ and the true satellite location vector $\boldsymbol{\beta}^o$. To address this problem, we shall propose an alternative CMLE for $\boldsymbol{u}^o$ that has approximately the same accuracy as the one in Eq. (6). For this purpose, we put Eq. (9) into Eq. (8) and re-arrange the result to arrive at

$$\boldsymbol{J}=\left(\frac{\partial\boldsymbol{m}^{\mathrm{o}}}{\partial\boldsymbol{u}^{\mathrm{o}}}\right)^{\mathrm{T}}\left[\boldsymbol{Q}_m^{-1}-\boldsymbol{Q}_m^{-1}\frac{\partial\boldsymbol{m}^{\mathrm{o}}}{\partial\boldsymbol{\beta}^{\mathrm{o}}}\boldsymbol{Z}^{-1}\left(\frac{\partial\boldsymbol{m}^{\mathrm{o}}}{\partial\boldsymbol{\beta}^{\mathrm{o}}}\right)^{\mathrm{T}}\boldsymbol{Q}_m^{-1}\right]\frac{\partial\boldsymbol{m}^{\mathrm{o}}}{\partial\boldsymbol{u}^{\mathrm{o}}} \tag{11}$$

Applying the matrix inversion Lemma[34] to the matrix in the middle yields

$$\boldsymbol{J}=\left(\frac{\partial\boldsymbol{m}^{\mathrm{o}}}{\partial\boldsymbol{u}^{\mathrm{o}}}\right)^{\mathrm{T}}\left[\boldsymbol{Q}_m+\frac{\partial\boldsymbol{m}^{\mathrm{o}}}{\partial\boldsymbol{\beta}^{\mathrm{o}}}\boldsymbol{Q}_\beta\left(\frac{\partial\boldsymbol{m}^{\mathrm{o}}}{\partial\boldsymbol{\beta}^{\mathrm{o}}}\right)^{\mathrm{T}}\right]^{-1}\frac{\partial\boldsymbol{m}^{\mathrm{o}}}{\partial\boldsymbol{u}^{\mathrm{o}}} \tag{12}$$

This indicates that without the target altitude information, the FIM of the target position $\boldsymbol{u}^{\mathrm{o}}$ in the presence of satellite location uncertainty would be identical to that of the case where the satellite locations are known precisely but the TDOA and FDOA measurements have an increased covariance matrix $\boldsymbol{Q}_m+\frac{\partial\boldsymbol{m}^{\mathrm{o}}}{\partial\boldsymbol{\beta}^{\mathrm{o}}}\boldsymbol{Q}_\beta\left(\frac{\partial\boldsymbol{m}^{\mathrm{o}}}{\partial\boldsymbol{\beta}^{\mathrm{o}}}\right)^{\mathrm{T}}$.

With the above observations in mind, we expand, using the first-order Taylor Series expansion, the true TDOA and FDOA measurement vector $\boldsymbol{m}^{\mathrm{o}}$ at the known satellite location vector $\boldsymbol{\beta}$. From Eqs. (2)–(5), we have, after proper manipulations,

$$\boldsymbol{m}^{\mathrm{o}}\approx\tilde{\boldsymbol{m}}^{\mathrm{o}}-\boldsymbol{D}^{\mathrm{o}}\Delta\boldsymbol{\beta} \tag{13}$$

where $\tilde{\boldsymbol{m}}^{\mathrm{o}}=[\tilde{\boldsymbol{r}}^{\mathrm{oT}},\tilde{\dot{\boldsymbol{r}}}^{\mathrm{oT}}]^{\mathrm{T}}$, with $\tilde{\boldsymbol{r}}^{\mathrm{o}}=[\tilde{r}^{\mathrm{o}}_{21},\tilde{r}^{\mathrm{o}}_{31},\ldots,\tilde{r}^{\mathrm{o}}_{M1}]^{\mathrm{T}}$ and $\tilde{\dot{\boldsymbol{r}}}^{\mathrm{o}}=[\tilde{\dot{r}}^{\mathrm{o}}_{21},\tilde{\dot{r}}^{\mathrm{o}}_{31},\ldots,\tilde{\dot{r}}^{\mathrm{o}}_{M1}]^{\mathrm{T}}$; $\boldsymbol{D}^{\mathrm{o}}$ has the same functional form as $\frac{\partial\boldsymbol{m}^{\mathrm{o}}}{\partial\boldsymbol{\beta}^{\mathrm{o}}}$, except that the true satellite positions and velocities are replaced by their known but erroneous versions. The (i-1)th (i = 2, 3, …, M) elements of $\tilde{\boldsymbol{r}}^{\mathrm{o}}$ and $\tilde{\dot{\boldsymbol{r}}}^{\mathrm{o}}$ are

$$\tilde{r}^{\mathrm{o}}_{i1}=\tilde{r}^{\mathrm{o}}_{i}-\tilde{r}^{\mathrm{o}}_{1}=||\boldsymbol{u}^{\mathrm{o}}-\boldsymbol{s}_i||-||\boldsymbol{u}^{\mathrm{o}}-\boldsymbol{s}_1|| \tag{14}$$

$$\tilde{\dot{r}}^{\mathrm{o}}_{i1}=\tilde{\dot{r}}^{\mathrm{o}}_{i}-\tilde{\dot{r}}^{\mathrm{o}}_{1}=\frac{-\dot{\boldsymbol{s}}_i^{\mathrm{T}}(\boldsymbol{u}^{\mathrm{o}}-\boldsymbol{s}_i)}{||\boldsymbol{u}^{\mathrm{o}}-\boldsymbol{s}_i||}-\frac{-\dot{\boldsymbol{s}}_1^{\mathrm{T}}(\boldsymbol{u}^{\mathrm{o}}-\boldsymbol{s}_1)}{||\boldsymbol{u}^{\mathrm{o}}-\boldsymbol{s}_1||} \tag{15}$$

Putting Eq. (13) into $\boldsymbol{m}-\boldsymbol{m}^{\mathrm{o}}$ yields

$$\boldsymbol{m}-\boldsymbol{m}^{\mathrm{o}}\approx(\boldsymbol{m}-\tilde{\boldsymbol{m}}^{\mathrm{o}})+\boldsymbol{D}^{\mathrm{o}}\Delta\boldsymbol{\beta}+\Delta\boldsymbol{m} \tag{16}$$

The sum of the last terms in Eq. (16) can be considered the new TDOA and FDOA measurement error, which is zero-mean Gaussian distributed because $\Delta\boldsymbol{m}$ and $\Delta\boldsymbol{\beta}$ are both zero-mean Gaussian random vectors. It is straightforward to show that $\boldsymbol{D}^{\mathrm{o}}\Delta\boldsymbol{\beta}+\Delta\boldsymbol{m}$ has a covariance matrix

$$\boldsymbol{W}=\boldsymbol{Q}_m+\boldsymbol{D}^{\mathrm{o}}\boldsymbol{Q}_\beta\boldsymbol{D}^{\mathrm{oT}} \tag{17}$$

which is approximately equal to $\boldsymbol{Q}_m+\frac{\partial\boldsymbol{m}^{\mathrm{o}}}{\partial\boldsymbol{\beta}^{\mathrm{o}}}\boldsymbol{Q}_\beta\left(\frac{\partial\boldsymbol{m}^{\mathrm{o}}}{\partial\boldsymbol{\beta}^{\mathrm{o}}}\right)^{\mathrm{T}}$ under the condition of small satellite location errors.

On the basis of the above theoretical developments, we propose to use the following CMLE for determining the target position $\boldsymbol{u}^{\mathrm{o}}$

$$\begin{aligned}&\min_{\boldsymbol{u}^{\mathrm{o}}}(\boldsymbol{m}-\tilde{\boldsymbol{m}}^{\mathrm{o}})^{\mathrm{T}}\boldsymbol{W}^{-1}(\boldsymbol{m}-\tilde{\boldsymbol{m}}^{\mathrm{o}})\\&\text{s.t.}\quad ||\boldsymbol{u}^{\mathrm{o}}||=R\end{aligned} \tag{18}$$

Compared with the original CMLE in Eq. (6), the new target position estimator no longer requires jointly identifying the true satellite location vector $\boldsymbol{\beta}^{\mathrm{o}}$. Hence, it has reduced complexity. The presence of satellite location errors has been taken into account in Eq. (18) in the weighting matrix $\boldsymbol{W}$. More importantly, the new CMLE has a CCRLB approximately equal to the one given in Eq. (7), which is the CCRLB of the original CMLE in Eq. (6). To verify this, we can firstly follow the approach in Ref.[30] and obtain that the CCRLB of the estimator in Eq. (18) is $\tilde{\boldsymbol{J}}^{-1}-\tilde{\boldsymbol{J}}^{-1}\boldsymbol{F}(\boldsymbol{F}^{\mathrm{T}}\tilde{\boldsymbol{J}}^{-1}\boldsymbol{F})^{-1}\boldsymbol{F}^{\mathrm{T}}\tilde{\boldsymbol{J}}^{-1}$, which has the same functional form as the one in Eq. (7). Secondly, we note that

$$\begin{aligned}\tilde{\boldsymbol{J}}&=\left(\frac{\partial\boldsymbol{m}^{\mathrm{o}}}{\partial\boldsymbol{u}^{\mathrm{o}}}\right)^{\mathrm{T}}\boldsymbol{W}^{-1}\frac{\partial\boldsymbol{m}^{\mathrm{o}}}{\partial\boldsymbol{u}^{\mathrm{o}}}\\&=\left(\frac{\partial\boldsymbol{m}^{\mathrm{o}}}{\partial\boldsymbol{u}^{\mathrm{o}}}\right)^{\mathrm{T}}(\boldsymbol{Q}_m+\boldsymbol{D}^{\mathrm{o}}\boldsymbol{Q}_\beta\boldsymbol{D}^{\mathrm{oT}})^{-1}\frac{\partial\boldsymbol{m}^{\mathrm{o}}}{\partial\boldsymbol{u}^{\mathrm{o}}}\end{aligned} \tag{19}$$

Comparing Eq. (19) with Eq. (12) and utilizing the small satellite location error condition yield $\boldsymbol{J}\approx\tilde{\boldsymbol{J}}$. This completes the proof that as the original CMLE given in Eq. (6), the newly proposed CMLE for the target position $\boldsymbol{u}^{\mathrm{o}}$ can attain the CCRLB in Eq. (7) approximately.

4. New iterative geolocation algorithm

4.1. Algorithm development

We shall propose in this section a new iterative algorithm for solving the CMLE in Eq. (18) to estimate the target position $\boldsymbol{u}^{\mathrm{o}}$ in the presence of satellite position uncertainty. The development of the new algorithm follows the similar procedure as in Ref.[20]. Specifically, it starts with an initial solution guess $\hat{\boldsymbol{u}}$ and expands $\tilde{\boldsymbol{m}}^{\mathrm{o}}$ around $\hat{\boldsymbol{u}}$ up to the first-order term as

$$\tilde{\boldsymbol{m}}^{\mathrm{o}}\approx\hat{\boldsymbol{m}}+\left.\frac{\partial\tilde{\boldsymbol{m}}^{\mathrm{o}}}{\partial\boldsymbol{u}^{\mathrm{o}}}\right|_{\boldsymbol{u}^{\mathrm{o}}=\hat{\boldsymbol{u}}}\times(\boldsymbol{u}^{\mathrm{o}}-\hat{\boldsymbol{u}}) \tag{20}$$

where $\hat{\boldsymbol{m}}$ has the same functional form as $\tilde{\boldsymbol{m}}^{\mathrm{o}}$ but its elements are computed by replacing $\boldsymbol{u}^{\mathrm{o}}$ in Eqs. (14) and (15) with $\hat{\boldsymbol{u}}$. The gradient matrix $\left.\frac{\partial\tilde{\boldsymbol{m}}^{\mathrm{o}}}{\partial\boldsymbol{u}^{\mathrm{o}}}\right|_{\boldsymbol{u}^{\mathrm{o}}=\hat{\boldsymbol{u}}}$ has the same functional form as $\frac{\partial\boldsymbol{m}^{\mathrm{o}}}{\partial\boldsymbol{u}^{\mathrm{o}}}$, except that the partial derivatives in $\left.\frac{\partial\tilde{\boldsymbol{m}}^{\mathrm{o}}}{\partial\boldsymbol{u}^{\mathrm{o}}}\right|_{\boldsymbol{u}^{\mathrm{o}}=\hat{\boldsymbol{u}}}$ are evaluated at $\hat{\boldsymbol{u}}$ and the erroneous satellite locations.

Let $\boldsymbol{\theta}=\boldsymbol{u}^{\mathrm{o}}-\hat{\boldsymbol{u}}$ and rewrite the equality constraint in Eq. (18) as

$$\boldsymbol{u}^{\mathrm{oT}}\boldsymbol{D}\boldsymbol{u}^{\mathrm{o}}=R^2 \tag{21}$$

where

$$\boldsymbol{D}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \tag{22}$$

Applying the definition of $\boldsymbol{\theta}$, using the Cholesky decomposition $\boldsymbol{W}^{-1}=\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C}$, and putting Eq. (20) into the cost function of the CMLE in Eq. (18), after some manipulations, we can express the CMLE as

$$\begin{aligned}&\min_{\boldsymbol{\theta}\in\mathrm{R}^3}||\boldsymbol{A}\boldsymbol{\theta}-\boldsymbol{b}||^2\\&\text{s.t.}\quad\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{\theta}+2\boldsymbol{d}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{\theta}=h\end{aligned} \tag{23}$$

where $\boldsymbol{A}=\boldsymbol{C}\times\left.\frac{\partial\tilde{\boldsymbol{m}}^{\mathrm{o}}}{\partial\boldsymbol{u}^{\mathrm{o}}}\right|_{\boldsymbol{u}^{\mathrm{o}}=\hat{\boldsymbol{u}}}$, $\boldsymbol{b}=\boldsymbol{C}\times(\boldsymbol{m}-\hat{\boldsymbol{m}})$, $\boldsymbol{d}=\hat{\boldsymbol{u}}$ and $h=R^2-\hat{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{D}\hat{\boldsymbol{u}}$.

Eq. (23) is indeed a generalized trust region sub-problem (GTRS) and can be solved globally.[31,32] Denoting the globally optimal solution to Eq. (23) as $\hat{\boldsymbol{\theta}}$, we may obtain an improved estimate of the target position $\hat{\boldsymbol{u}}+\hat{\boldsymbol{\theta}}$. The newly proposed iterative geolocation algorithm would repeat the computations in Eqs. (20)–(23) with $\hat{\boldsymbol{u}}$ being replaced with $\hat{\boldsymbol{u}}+\hat{\boldsymbol{\theta}}$. The iteration will terminate to output the final geolocation result when the maximum allowable number of iterations has been reached and/or the update in the target position estimate $\hat{\boldsymbol{\theta}}$ is sufficiently small.

We next present the method for obtaining $\hat{\boldsymbol{\theta}}$ to complete the development of the new iterative geolocation algorithm. As shown in Refs.[31,32], for $\hat{\boldsymbol{\theta}}$ to be the globally optimal solution to Eq. (23), it must satisfy the following sufficient and necessary conditions

$$(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}+\lambda\boldsymbol{D})\hat{\boldsymbol{\theta}}=\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}-\lambda\boldsymbol{D}\boldsymbol{d} \tag{24}$$

$$\hat{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{D}\hat{\boldsymbol{\theta}}+2\boldsymbol{d}^{\mathrm{T}}\boldsymbol{D}\hat{\boldsymbol{\theta}}=h \tag{25}$$

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}+\lambda\boldsymbol{D}\geqslant\boldsymbol{0} \tag{26}$$

where λ is the Lagrange multiplier that makes $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}+\lambda\boldsymbol{D}$ positive semi-definite. Note that Eqs. (24) and (25) are actually the Karush–Kuhn–Tacker (KKT) conditions of the equality-constrained minimization problem given in Eq. (23).

To find $\hat{\boldsymbol{\theta}}$, we shall follow the approach in Ref.[32]. Specifically, from Eq. (26), we immediately have that the interval I consisting of all λ for which $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}+\lambda\boldsymbol{D}$ is positive semi-definite is

$$I=\left(-1/\lambda_1\left(\boldsymbol{D},\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\right),+\infty\right) \tag{27}$$

where $\lambda_1(\boldsymbol{D},\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})$ represents the biggest generalized eigenvalue of the matrix pair $(\boldsymbol{D},\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})$. Next, from Eq. (24), the globally optimal solution $\hat{\boldsymbol{\theta}}$ has the following functional form

$$\hat{\boldsymbol{\theta}}(\lambda)=(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}+\lambda\boldsymbol{D})^{-1}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}-\lambda\boldsymbol{D}\boldsymbol{d}) \tag{28}$$

Putting Eq. (28) into Eq. (25) yields the Eq. (29) for the Lagrange multiplier λ

$$\varphi(\lambda)=\hat{\boldsymbol{\theta}}^{\mathrm{T}}(\lambda)\boldsymbol{D}\hat{\boldsymbol{\theta}}(\lambda)+2\boldsymbol{d}^{\mathrm{T}}\boldsymbol{D}\hat{\boldsymbol{\theta}}(\lambda)=h \quad \lambda\in I \tag{29}$$

As shown in Theorem 5.2 of Ref.[31], $\varphi(\lambda)$ is strictly decreasing over I. Therefore, a simple bisection method can be used to find λ over the interval I.[32] Putting the obtained λ into Eq. (28) yields the desired globally optimal solution $\hat{\boldsymbol{\theta}}$ to Eq. (23).

It is worthwhile to point out that the newly proposed iterative geolocation algorithm has a computation procedure that is very similar to the one developed in Ref.[20] for geolocating a known altitude target using TDOA and FDOA measurements obtained at $M\geqslant 3$ satellites. However, this work differs from Ref.[20] in the following two important aspects. Firstly, in Ref.[20], the satellite positions and velocities are assumed to be known accurately, but in this paper, we consider a more realistic scenario where the satellite location uncertainty is present. Secondly, the iterative approach developed in Ref.[20] finds the update for the target position estimate via solving an equality-constrained minimization problem similar to Eq. (23), but the solution is found by the use of the method of the Lagrange multiplier. In particular, Ref.[20] resorts to solving a 6th-order polynomial in the Lagrange multiplier, which is cumbersome in the sense that all six roots need to be found in order to locate the smallest root and proper initial solution guesses have to be provided. On the contrary, via invoking the GTRS formulation and the bisection technique, we are able to guarantee obtaining the globally optimal solution as the update for the target position estimate.

4.2. Algorithm implementation

There are several aspects that need to be addressed in the realization of the newly proposed geolocation technique.

Firstly, the proposed algorithm requires an initial estimate of the target position $\boldsymbol{u}^{\mathrm{o}}$, denoted by $\hat{\boldsymbol{u}}$. In this work, we use the following method to obtain $\hat{\boldsymbol{u}}$. We choose three satellites out of the available M satellites that are not collinear and apply the algebraic TDOA-based geolocation algorithm developed in Ref.[18] to find $\hat{\boldsymbol{u}}$ from the known target altitude information (see Eq. (1)) and the two target signal TDOA measurements obtained at the three selected satellites. To reduce the initialization error, it is suggested that the two satellites with the longest inter-satellite range are selected first and a third satellite that is farthest to the line connecting the two selected satellites is then found to produce a 3-satellite subset. When the number of the available satellites M is small, as in many practical scenarios, an alternative method is to find all 3-satellite subsets, geolocate the target using each satellite subset, and output the one with the best theoretical geolocation accuracy, which can be computed by using e.g., Eq. (37) in Ref.[18].

The second aspect in the geolocation algorithm realization is that in the formulation of the equality-constrained MLE for the target position $\boldsymbol{u}^{\mathrm{o}}$ (see Eq. (18)), the weighting matrix $\boldsymbol{W}$ needs to be generated. From its definition in Eq. (17), we note that the evaluation of $\boldsymbol{W}$ requires $\boldsymbol{u}^{\mathrm{o}}$, which is unknown. To address this difficulty, in this work, we shall replace $\boldsymbol{u}^{\mathrm{o}}$ with its initial estimate $\hat{\boldsymbol{u}}$ when producing $\boldsymbol{W}$. We may update $\boldsymbol{W}$ using the improved target position estimate obtained during the iterations. However, our simulations indicate that this does not lead to significantly enhanced geolocation accuracy. As a result, in the realization of the proposed geolocation algorithm, the weighting matrix $\boldsymbol{W}$ is not updated during the iterations.

Thirdly, when the target altitude R is not known but we have the a priori information that it is a ground target, we may apply the oblate spheroid Earth model so that the target position satisfies

$$\boldsymbol{u}^{\mathrm{oT}}\boldsymbol{D}_1\boldsymbol{u}^{\mathrm{o}}=r^2 \tag{30}$$

where

$$\boldsymbol{D}_1=\mathrm{diag}(1,1,1/(1-e^2)) \tag{31}$$

r = 6378.137 km and $e=0.0818191908426214957$ is the eccentricity. The proposed iterative geolocation algorithm can still be applied in this case after the substitution of $\boldsymbol{D}$ with $\boldsymbol{D}_1$ and R^2 with r^2 in Eq. (23).

5. Effect of target altitude error

In this section, we shall investigate the impact of error in the known target altitude R on the geolocation accuracy. In particular, a first-order MSE analysis is conducted to derive the geolocation MSE when the known target altitude is considered accurate but in fact has error. In practice, the ignored target altitude error should be considered deterministic but unknown,[18] which renders the geolocation result biased. Different from the approach adopted in Ref.[18], we shall employ in this section the re-parameterization technique[33] to accomplish the theoretical analysis.

The theoretical development starts with expressing the target position in terms of the known altitude R and its virtual latitude B and longitude L as

$$\boldsymbol{u}(\boldsymbol{\varphi}, R) = R\begin{bmatrix}\cos B\cos L\\ \cos B\sin L\\ \sin B\end{bmatrix} \tag{32}$$

where $\boldsymbol{\varphi} = [B, L]^{\mathrm{T}}$. With the above re-parameterization, we may re-write the CMLE for $\boldsymbol{u}^{\mathrm{o}}$ in Eq. (18) as

$$\min_{B,L}(\boldsymbol{m} - \tilde{\boldsymbol{m}}^{\mathrm{o}}(\boldsymbol{\varphi}, R))^{\mathrm{T}}\boldsymbol{W}^{-1}(\boldsymbol{m} - \tilde{\boldsymbol{m}}^{\mathrm{o}}(\boldsymbol{\varphi}, R)) \tag{33}$$

where the symbol $\tilde{\boldsymbol{m}}^{\mathrm{o}}(\boldsymbol{\varphi}, R)$ is introduced to reflect the dependence of the TDOA and FDOA measurements given in Eqs. (14) and (15) on the target latitude and longitude in $\boldsymbol{\varphi}$ as well as the target altitude. Note that with the re-parameterization and the known target altitude, the geolocation task reduces to estimating the target latitude and longitude, and the equality constraint in the original CMLE in Eq. (18) has been eliminated. Moreover, if R, the known target altitude, is precise, solving Eq. (33) for B and L and plugging the result into Eq. (32) would yield the same geolocation result as directly solving for $\boldsymbol{u}^{\mathrm{o}}$ using Eq. (18).

We proceed to derive the geolocation MSE using Eq. (33) under the condition that R is deviated from the true target latitude by ΔR, i.e., the true target position is indeed equal to $\boldsymbol{u}^{\mathrm{o}} = \boldsymbol{u}(\boldsymbol{\varphi}, R + \Delta R)$. Assume the availability of noisy versions of B and L, denoted by $\tilde{B}$ and $\tilde{L}$. For notation simplicity, let $\tilde{\boldsymbol{\varphi}} = [\tilde{B}, \tilde{L}]^{\mathrm{T}}$ and $\tilde{\boldsymbol{u}} = \boldsymbol{u}(\tilde{\boldsymbol{\varphi}}, R)$. Expanding $\tilde{\boldsymbol{m}}^{\mathrm{o}}(\boldsymbol{\varphi}, R)$ around $\tilde{\boldsymbol{\varphi}}$ up to the first-order terms, substituting the result back into Eq. (34), and going through the minimization process yield the weighted least squares (WLS) estimate of $\boldsymbol{\varphi}$, which is given by

$$\hat{\boldsymbol{\varphi}} = \tilde{\boldsymbol{\varphi}} + \underbrace{(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{W}^{-1}\boldsymbol{H})^{-1}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{W}^{-1}(\boldsymbol{m} - \tilde{\boldsymbol{m}}^{\mathrm{o}}(\tilde{\boldsymbol{\varphi}}, R))}_{\Delta\boldsymbol{\varphi}} \tag{34}$$

Here, matrix $\boldsymbol{H}$ is defined as

$$\boldsymbol{H} = \frac{\partial\tilde{\boldsymbol{m}}^{\mathrm{o}}(\boldsymbol{\varphi}, R)}{\partial\boldsymbol{u}(\boldsymbol{\varphi}, R)}\cdot\frac{\partial\boldsymbol{u}(\boldsymbol{\varphi}, R)}{\partial\boldsymbol{\varphi}}\bigg|_{\boldsymbol{\varphi}=\tilde{\boldsymbol{\varphi}}} \tag{35}$$

Putting $\hat{\boldsymbol{\varphi}}$ back into Eq. (32) gives the target position estimation. Subtracting from it the true target position $\boldsymbol{u}^{\mathrm{o}} = \boldsymbol{u}(\boldsymbol{\varphi}, R + \Delta R)$ yields the geolocation error under imprecise target altitude information, which is equal to

$$\Delta\boldsymbol{u} = \boldsymbol{u}(\hat{\boldsymbol{\varphi}}, R) - \boldsymbol{u}^{\mathrm{o}} = \boldsymbol{u}(\hat{\boldsymbol{\varphi}}, R) - \boldsymbol{u}(\boldsymbol{\varphi}, R + \Delta R) \tag{36}$$

When the initial solution guess $\tilde{\boldsymbol{\varphi}}$ is sufficiently close to its true value $\boldsymbol{\varphi}$, i.e., $\tilde{\boldsymbol{\varphi}} \approx \boldsymbol{\varphi}$, the geolocation error can be approximated up to the first-order error terms as

$$\Delta\boldsymbol{u} \approx \frac{\partial\boldsymbol{u}(\boldsymbol{\varphi}, R + \Delta R)}{\partial\boldsymbol{\varphi}}\Delta\boldsymbol{\varphi} - \frac{\partial\boldsymbol{u}(\boldsymbol{\varphi}, R + \Delta R)}{\partial R}\Delta R \tag{37}$$

We proceed to evaluate $\Delta\boldsymbol{\varphi}$. In particular, we have, from the definition of $\boldsymbol{m}$ given under Eqs. (14) and (15),

$$\boldsymbol{m} - \tilde{\boldsymbol{m}}^{\mathrm{o}}(\boldsymbol{\varphi}, R) \approx \Delta\boldsymbol{m} - \boldsymbol{D}^{\mathrm{o}}\Delta\boldsymbol{\beta} + \frac{\partial\tilde{\boldsymbol{m}}^{\mathrm{o}}(\boldsymbol{\varphi}, R)}{\partial\boldsymbol{u}(\boldsymbol{\varphi}, R)}\cdot\frac{\partial\boldsymbol{u}(\boldsymbol{\varphi}, R)}{\partial R}\Delta R \tag{38}$$

Putting Eq. (38) into Eq. (34) and substituting the result back into Eq. (37) yield the geolocation error given by

$$\begin{aligned}\Delta\boldsymbol{u} = {} & \frac{\partial\boldsymbol{u}(\boldsymbol{\varphi}, R+\Delta R)}{\partial\boldsymbol{\varphi}}(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{H})^{-1}\left(\frac{\partial\boldsymbol{u}(\boldsymbol{\varphi}, R)}{\partial\boldsymbol{\varphi}}\right)^{\mathrm{T}}\\ & \left(\frac{\partial\tilde{\boldsymbol{m}}^{\mathrm{o}}(\boldsymbol{\varphi}, R)}{\partial\boldsymbol{u}(\boldsymbol{\varphi}, R)}\right)^{\mathrm{T}}\boldsymbol{W}^{-1}(\Delta\boldsymbol{m} - \boldsymbol{D}^{\mathrm{o}}\Delta\boldsymbol{\beta})\\ & + \left(\frac{\partial\boldsymbol{u}(\boldsymbol{\varphi}, R+\Delta R)}{\partial\boldsymbol{\varphi}}(\boldsymbol{H}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{H})^{-1}\left(\frac{\partial\boldsymbol{u}(\boldsymbol{\varphi}, R)}{\partial\boldsymbol{\varphi}}\right)^{\mathrm{T}}\right.\\ & \left(\frac{\partial\tilde{\boldsymbol{m}}^{\mathrm{o}}(\boldsymbol{\varphi}, R)}{\partial\boldsymbol{u}(\boldsymbol{\varphi}, R)}\right)^{\mathrm{T}}\boldsymbol{W}^{-1}\frac{\partial\tilde{\boldsymbol{m}}^{\mathrm{o}}(\boldsymbol{\varphi}, R)}{\partial\boldsymbol{u}(\boldsymbol{\varphi}, R)}\cdot\frac{\partial\boldsymbol{u}(\boldsymbol{\varphi}, R)}{\partial R}\\ & \left. - \frac{\partial\boldsymbol{u}(\boldsymbol{\varphi}, R+\Delta R)}{\partial\boldsymbol{\varphi}}\right)\Delta R\end{aligned} \tag{39}$$

Multiplying $\Delta\boldsymbol{u}$ by its transpose and taking expectation would produce the desired geolocation MSE, which is also the geolocation MSE of the newly proposed iterative algorithm when the known target altitude has an error of ΔR.

Carefully examining Eq. (39) reveals that the geolocation error consists of two terms. The first term denotes the geolocation error due to the TDOA and FDOA measurement noise and the satellite location error that are random (see Section 2). The second term comes from the presence of the target altitude error ΔR, which is non-zero. In other words, if the known target altitude has an error, we would obtain a biased geolocation result.

6. Simulations

We shall contrast via simulations the estimation performance of the newly proposed iterative geolocation algorithm with those of two benchmark techniques. We shall also corroborate the theoretical development in Section 5 on the geolocation MSE when the known target altitude is subject to an unknown but deterministic error.

6.1. Benchmark algorithms

The first benchmark algorithm considered is the iterative geolocation algorithm developed in Ref.[20]. We extend it to solve Eq. (18) to find the position of a known altitude target in the presence of satellite location uncertainty. Different from the method proposed in this work, the algorithm from Ref.[20] computes the updated target position estimate in the current iteration (see Eq. (22)) via finding the smallest root of a 6th-order polynomial in the Lagrange multiplier. In this section, we adopt the commonly used Newton's method to compute the desired root.

The second benchmark algorithm comes from Ref.[18], where an algebraic method was developed for geolocating a known altitude target using TDOA and FDOA measurements in the over-determined case. This method has been widely used in practice (e.g., see Ref.[35]). We explore the theoretical developments in Ref.[27] and generalize the geolocation technique from Ref.[18] to take into account the presence of satellite position uncertainty. As in Ref.[18], the obtained algorithm estimates the target position via solving simultaneously for the range between the target and satellite 1, denoted by r_1 and a Lagrange multiplier. To improve the robustness of the generalized geolocation algorithm, we assume that r_1 lies in the interval $[r_{1\,\mathrm{min}}, r_{1\,\mathrm{max}}]$[1] and invoke a bisection search to estimate r_1 as well as the Lagrange multiplier jointly.

[1] This prior information is available if the target region of interest is known.

6.2. Simulation setup

We shall consider the same geolocation geometry used in Ref.[18]. Specifically, the target has a longitude and latitude of 75.9° W and 45.35° N. The local Earth radius is 6367.287 km and the target lies on the Earth surface with a zero altitude so that $R = 6367.287$ km. Four geosynchronous satellites are used to identify the target position. All of them are 42164 km from the Earth center and their positions under the geocentric coordinate system are $\boldsymbol{s}_1 = [50.0°\ \mathrm{W}, 2.0°\ \mathrm{N}]$, $\boldsymbol{s}_2 = [47.0°\ \mathrm{W}, 0°\ \mathrm{N}]$, $\boldsymbol{s}_3 = [53.0°\ \mathrm{W}, 0°\ \mathrm{N}]$ and $\boldsymbol{s}_4 = [51.5°\ \mathrm{W}, 3.0°\ \mathrm{N}]$. The velocities of the four geosynchronous satellites with respect to the Earth are equal to $\dot{\boldsymbol{s}}_1 = [-15.48, -13.0, -772.04]$ km/h, $\dot{\boldsymbol{s}}_2 = [-30.78, -28.70, 972.72]$ km/h, $\dot{\boldsymbol{s}}_3 = [-0.054, -0.041, -38.60]$ km/h and $\dot{\boldsymbol{s}}_4 = [-119.62, -95.15, 1920.34]$ km/h.

We generate the noisy TDOA and FDOA measurements by adding to the true values zero-mean Gaussian noises with the covariance matrix

$$\boldsymbol{Q}_m = \begin{bmatrix} (c\delta_\mathrm{t})^2\boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & \left(\frac{c\delta_\mathrm{f}}{f}\right)^2\boldsymbol{Q} \end{bmatrix} \tag{40}$$

Here, δ_t and δ_f represent respectively the standard deviations of the TDOA and FDOA noises; $c = 3 \times 10^8$ m/s; $f = 14$ GHz; $\boldsymbol{Q}$ is an $(M-1) \times (M-1)$ matrix with its diagonal elements being equal to 1 and its off-diagonal elements being equal to 0.5. The erroneous but known satellite location vectors are produced similarly with the covariance matrix

$$\boldsymbol{Q}_\beta = \begin{bmatrix} \delta_s^2\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \delta_{\dot{s}}^2\boldsymbol{I} \end{bmatrix} \tag{41}$$

where δ_s and $\delta_{\dot{s}}$ are the standard deviations of the satellite position and velocity errors.

To initialize the iterative geolocation algorithm proposed in this paper, we apply the algebraic TDOA-based geolocation algorithm developed in Ref.[18] and compute an initial target position estimate using the target altitude information and the two TDOA measurements obtained by the first three satellites. The geolocation technique from Ref.[20] is initialized in the same manner for a fair comparison. The estimation performance from simulation is quantified using the root mean square error (RMSE) and the number of ensemble runs is 5000.

6.3. Results and discussions

In Fig. 1, we plot the geolocation RMSE results of the newly proposed iterative geolocation algorithm as we vary the standard deviations of the TDOA noise, the FDOA noise and the satellite position error. In Fig. 1(a), we set $\delta_\mathrm{f} = 20$ Hz, $\delta_s = 1$ km and $\delta_{\dot{s}} = 0.05$ m/s. In Fig. 1(b), we set $\delta_\mathrm{t} = 0.1$ μs, $\delta_s = 1$ km and $\delta_{\dot{s}} = 0.05$ m/s. In Fig. 1(c), we set $\delta_\mathrm{t} = 0.1$ μs, $\delta_\mathrm{f} = 20$ Hz and $\delta_{\dot{s}} = 0.05$ m/s. To demonstrate the effect of iteration on improving the geolocation result, the RMSE of the initial target position estimate S is shown in Fig. 1. Also included in Fig. 1 for the purpose of comparison are the geolocation RMSEs of the two benchmark algorithms and the associated geolocation CCRLBs.

It can be seen from Fig. 1(a) that under small noises, the three algorithms in consideration can all attain the CCRLB accuracy, as expected. As the noise levels increase, especially when the FDOA error becomes larger than 25 Hz (see Fig. 1(b)) and the satellite position error is bigger than 1.4 km (see Fig. 1(c)), the estimation accuracy of the iterative method from Ref.[20] deviates significantly from the CCRLB, probably due to the increased initialization error and the local convergence of the Newton's method in locating the smallest root for the Lagrange multiplier. On the other hand, the newly proposed iterative method exhibits better robustness to increased noise levels and provides greatly improved estimation performance over the algorithm from Ref.[20], thanks to the use of the GTRS formulation (see Eqs. (22) and (23)) and the guaranteed global convergence in solving for the updated target position estimate during iterations. The method from Ref.[18] can offer similar estimation performance to that of the new iterative geolocation algorithm but it is much more computationally intensive (see Section 6.4).

Fig. 2 investigates the effect of errors in the known target altitude on the target geolocation accuracy. We plot in Fig. 2 the geolocation RMSE of the proposed iterative geolocation algorithm and the theoretical value derived in Section 5. In this simulation, we fix the standard deviations of the TDOA and FDOA measurement noises at $\delta_\mathrm{t} = 0.1$ μs and $\delta_\mathrm{f} = 46.7$ Hz. The standard deviations of the satellite position and velocity errors are fixed at 1 km and 0.05 m/s. We can observe that when the target altitude error is less than 10 km, the simulation RMSE matches the theoretical value very well, which verifies the validity of the theoretical derivations in Section 5. Besides, in this simulation, the target

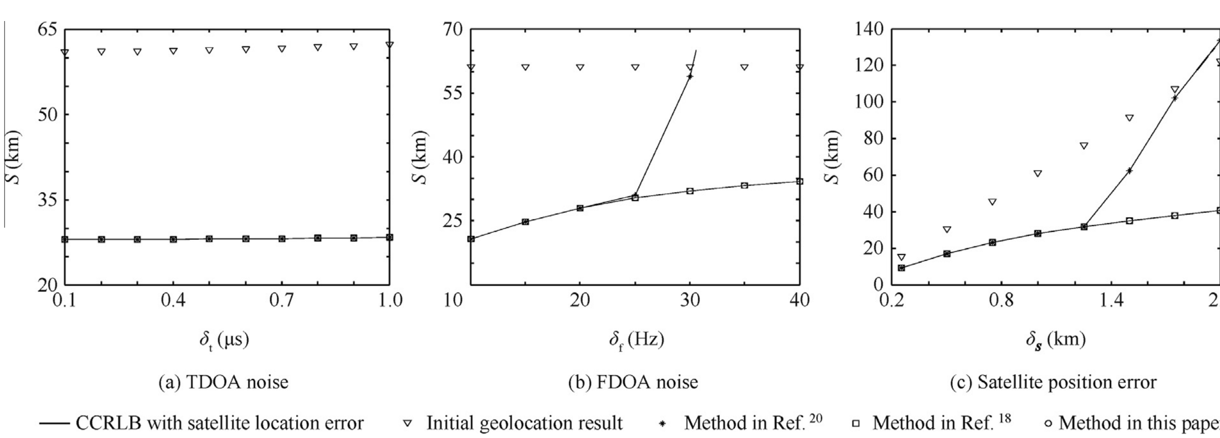

Fig. 1 Comparisons of geolocation accuracy as a function of TDOA noise, FDOA noise and satellite position error.

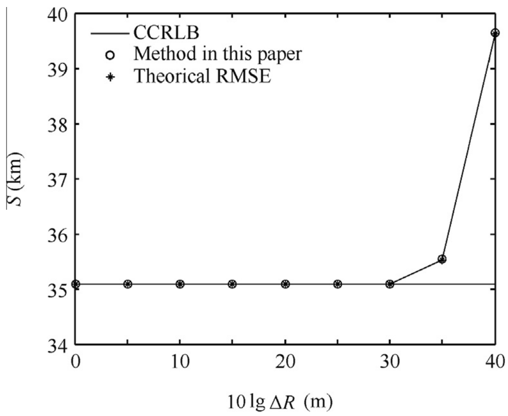

Fig. 2 Effect of errors in the known target altitude on the target geolocation accuracy.

Table 1 Normalized computation times of three geolocation methods in consideration.

Algorithm	Normalized computation time
Proposed in this paper	1
Ref.[20]	0.8
Ref.[18]	40

geolocation accuracy is degraded greatly only when the target altitude error is larger than 1 km. This could be attributed to the long target-satellite distance, which renders the target geolocation accuracy relatively insensitive to small errors in target altitude information.

6.4. Computational complexity

Table 1 is the normalized computation times of the three geolocation methods in consideration. In Table 1, we provide the computation times of the three methods in consideration. The results are normalized against the computation time of the new iterative geolocation technique for ease of comparison. It can be observed that the newly developed iterative approach and the algorithm from Ref.[20] exhibit similar computational complexity, which is somewhat expected as they differ mainly in their methods for updating the target position estimate during iterations. The method from Ref.[18], however, is much more computationally complex. Specifically, the algorithm in Ref.[18] involves a tedious process of applying the bisection technique to jointly identify the range between the target and satellite 1 as well as the Lagrange multiplier.

7. Conclusions

(1) A new iterative algorithm for geolocating a target on the Earth surface from the target signal TDOA and FDOA measurements obtained at more than two satellites was developed in this paper. The target altitude is assumed known and the satellite locations are subject to random errors.

(2) Under the Gaussian noise model, we formulated the geolocation problem as a CMLE and derived the CCRLB of the target position. It was shown analytically that by properly taking the presence of satellite location uncertainty into account in the formulation of the CMLE, estimating the target only, instead of identifying it together with the satellite locations, could still offer a geolocation accuracy approximately equal to the CCRLB.

(3) A new iterative algorithm for solving the CMLE was established. It does not need the cumbersome root-finding process commonly adopted in existing geolocation techniques. On the other hand, during iterations, it computes the updated target position estimate through solving a GTRS sub-problem globally via a simple bisection search. The estimation MSE of the newly developed geolocation algorithm under imprecisely known target altitude was then derived. The geolocation result in this case was shown to be biased if the target altitude error was deterministic.

(4) Computer simulations illustrated the good performance of the new iterative geolocation algorithm and verified its approximate efficiency as well as the validity of the theoretical MSE analysis.

Acknowledgments

This study was co-supported by the National Natural Science Foundation of China (Nos. 61304264 and 61305017) and the Innovation Foundation of Industry, Education and Research of Jiangsu Province (No. BY2014023-25).

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