NORTH-HOLLAND

Block Triangularization of Skew-Symmetric Matrices

Satoru Iwata
Division of Systems Science
Graduate School of Engineering Science
Osaka University
Toyonaka, Osaka 560, Japan

Submitted by Richard A. Brualdi

ABSTRACT

This paper addresses the finest block triangularization of nonsingular skew-symmetric matrices by simultaneous permutations of rows and columns. Hierarchical relations among components are represented in terms of signed posets. The finest block-triangular form can be computed efficiently with the aid of the strongly connected component decomposition of bidirected graphs. © 1998 Elsevier Science Inc.

I. INTRODUCTION

The *Dulmage-Mendelsohn* (DM) *Decomposition* [4] of bipartite graphs yields the finest block triangularization of matrices by independent permutations of rows and columns. It can be computed very efficiently with the aid of the bipartite matching algorithm and the strongly connected component decomposition. Thus the DM decomposition is now not only a fundamental result in combinatorial theory [3, 9] but also a useful tool in numerical computations and systems analysis [10], where efficient manipulation of sparse matrices is of practical significance. It should also be noted that the DM decomposition is characterized by a distributive lattice consisting of the minimizers of a certain submodular function [7], which fact serves as a prototype of the theory of principal partitions [8].

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As an extension of the DM decomposition, this paper discusses the finest block triangularization of nonsingular skew-symmetric matrices by simultaneous permutations of rows and columns. The hierarchical structure among components is represented in terms of signed posets introduced by Reiner [11] as hyperoctahedral (or signed) analogues of posets. We also present how to compute the finest block-triangular form efficiently. The algorithm makes use of the strongly connected component decomposition of bidirected graphs recently developed by Ando, Fujishige, and Nemoto [2].

The outline of this paper is as follows Section 2 is devoted to preliminaries on signed posets. In Section 3, we introduce a canonical block-triangular form and show that it gives the finest one. Finally, Section 4 presents an efficient algorithm for computing the canonical block-triangular form.

2. SIGNED POSETS

This section provides preliminaries on signed posets based on the results obtained by Ando and Fujishige [1].

Let V be a finite set. We denote by 3^V the set of all the ordered pairs of disjoint subsets, i.e., $3^V = \{(X, Y) \mid X, Y \subseteq V,\ X \cap Y = \emptyset\}$, while 2^V means the set of all the subsets as usual. For $(X_h, Y_h) \in 3^V$ $(h = 1, 2)$, we write $(X_1, Y_1) \sqsubseteq (X_2, Y_2)$ if $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$. Then the relation $\sqsubseteq$ is a partial order on 3^V. We write $(X_1, Y_1) \sqsubset (X_2, Y_2)$ if $(X_1, Y_1) \sqsubseteq (X_2, Y_2)$ but $(X_1, Y_1) \neq (X_2, Y_2)$. The characteristic vector $\chi_{(X,Y)} \in \mathbf{Z}^V$ is defined by

$$\chi_{(X,Y)}(v) = \begin{cases} 1 & \text{if } v \in X, \\ -1 & \text{if } v \in Y, \\ 0 & \text{otherwise.} \end{cases}$$

For $(X_h, Y_h) \in 3^V$ $(h = 1, 2)$, we denote

$$(X_1, Y_1) \sqcup (X_2, Y_2) = (X_1 \cap X_2, Y_1 \cap Y_2)$$

$$(X_1, Y_1) \sqcup (X_2, Y_2) = ((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)).$$

These operations $\sqcap$ and $\sqcup$ are called *intersection* and *reduced union*, respectively. A family $\mathscr{F} \subseteq 3^V$ that is closed with respect to the reduced union and the intersection is called a *signed ring family* (or $\sqcup$, $\sqcap$-closed family [1]). The partial order $\sqsubseteq$ is naturally induced on a signed ring family. Although the minimal element of $\mathscr{F}$ with respect to $\sqsubseteq$ is unique, there may be a number of maximal elements. It has been shown [1], however, that every maximal $(X, Y) \in \mathscr{F}$ has the same $X \cup Y$, which is called the *support* of $\mathscr{F}$ and denoted by $\mathrm{Supp}(\mathscr{F})$.

Let $\mathscr{T}$ denote the family of $(X, Y) \in 3^V$ such that $|X| + |Y|$ is one or two. Then $\tilde{G} = (V, A; \tilde{\partial})$ with $\tilde{\partial} : A \to \mathscr{T}$ is called a *bidirected graph*. For a subset $U \subseteq V$, we denote $(X, Y): U = ((X - U) \cup (Y \cap U), (Y - U) \cup (X \cap U))$. A bidirected graph $\tilde{G} : U = (V, A; \tilde{\partial}')$ defined by $\tilde{\partial}'a = \tilde{\partial}a : U$ is called a *reflection* of $\tilde{G}$ by U.

A *signed poset* $\mathscr{P} = (V, A; \tilde{\partial})$ is a bidirected graph with the following properties:

Antisymmetry. There are no two arcs a_1, $a_2 \in A$ such that $\tilde{\partial}a_1 \sqcup \tilde{\partial}a_2 = (\varnothing, \varnothing)$.

Transitivity. If $\tilde{\partial}a_1 \sqcup \tilde{\partial}a_2 \in \mathscr{T}$ for two arcs a_1, $a_2 \in A$, then there exists an arc $a_3 \in A$ such that $\tilde{\partial}a_1 \sqcup \tilde{\partial}a_2 = \tilde{\partial}a_3$.

Two signed posets $\mathscr{P} = (V, A; \tilde{\partial})$ and $\mathscr{P}' = (V, A; \tilde{\partial}')$ are said to be *isomorphic* if $\mathscr{P}'$ is a reflection of $\mathscr{P}$ by some $U \subseteq V$.

An ideal of a signed poset $\mathscr{P} = (V, A; \tilde{\partial})$ is defined to be $(X, Y) \in 3^V$ that satisfies

$$\left\langle \tilde{\partial}a, (X, Y) \right\rangle \leqslant 0 \qquad (a \in A), \tag{1}$$

where $\langle \cdot, \cdot \rangle$ designates the inner product of a pair of characteristic vectors. It is not difficult to see that the set of all the ideals of a signed poset forms a signed ring family. Conversely, a signed ring family can be represented by a partition of V with a signed poset structure among components as follows [1].

Given a signed ring family $\mathscr{F}$, let (X_0, Y_0) be its minimum element, and put $V_* = \mathrm{Supp}(\mathscr{F}) - V_0$, where $V_0 = X_0 \cup Y_0$. Consider a maximal chain

$$(X_0, Y_0) \sqsubset (X_1, Y_1) \sqsubset \cdots \sqsubset (X_b, Y_b), \tag{2}$$

where (X_b, Y_b) is a maximal element of $\mathscr{F}$, and put $V_k = (X_k \cup Y_k) - (X_{k-1} \cup Y_{k-1})$ for $k = 1, \dots, b$. An equivalence relation $\sim$ on V_* is defined by

$$v \sim w \quad \Leftrightarrow \quad \forall (X, Y) \in \mathscr{F}: \quad v, w \in X \cup Y \text{ or } v, w \notin X \cup Y.$$

Then the partition $\Pi_*(\mathscr{F})$ of V_* into the equivalence classes agrees with $(V_1, \dots, V_b)$.

Choose a representative v_k from each component V_k, and put $\hat{V} = \{v_k | V_k \in \Pi_*(\mathscr{F})\}$. Construct a bidirected graph $\hat{G} = (\hat{V}, \hat{A}; \hat{\partial})$ by performing the following 1–2 for each $v \in \hat{V}$ and 3–9 for each distinct $v, w \in \hat{V}$:

1. Add a with $\hat{\partial}a = (\{v\}, \emptyset)$ to $\hat{A}$ if $v \notin X$ for every $(X, Y) \in \mathscr{F}$.
2. Add a with $\hat{\partial}a = (\emptyset, \{v\})$ to $\hat{A}$ if $v \notin Y$ for every $(X, Y) \in \mathscr{F}$.
3. Add a with $\hat{\partial}a = (\{v\}, \{w\})$ to $\hat{A}$ if $w \in X$ for every $(X, Y) \in \mathscr{F}$ with $v \in X$.
4. Add a with $\hat{\partial}a = (\{v, w\}, \emptyset)$ to $\hat{A}$ if $w \in Y$ for every $(X, Y) \in \mathscr{F}$ with $v \in X$.
5. Add a with $\hat{\partial}a = (\emptyset, \{v, w\})$ to $\hat{A}$ if $w \in X$ for every $(X, Y) \in \mathscr{F}$ with $v \in Y$.
6. Add a with $\hat{\partial}a = (\{w\}, \{v\})$ to $\hat{A}$ if $w \in Y$ for every $(X, Y) \in \mathscr{F}$ with $v \in Y$.
7. Add a with $\hat{\partial}a = (\{v\}, \{w\})$ to $\hat{A}$ if $v \notin X$ and $w \notin Y$ for every $(X, Y) \in \mathscr{F}$.
8. Add a with $\hat{\partial}a = (\{v, w\}, \emptyset)$ to $\hat{A}$ if $v, w \notin X$ for every $(X, Y) \in \mathscr{F}$.
9. Add a with $\hat{\partial}a = (\emptyset, \{v, w\})$ to $\hat{A}$ if $v, w \notin Y$ for every $(X, Y) \in \mathscr{F}$.

Then the bidirected graph $\hat{G}$ becomes a signed poset whose set of ideals coincides with $\mathscr{F}$. Although the bidirected graph $\hat{G}$ might depend on the choice of representatives, the signed poset structure is invariant up to isomorphisms.

Thus $\hat{G}$ naturally induces a signed poset structure among the components of $\Pi_*(\mathscr{F}) = (V_1, \dots, V_b)$. The partition $\Pi(\mathscr{F}) = (V_0; V_1, \dots, V_b; V_\infty)$, where $V_\infty = V - \mathrm{Supp}(\mathscr{F})$, together with this signed poset structure is refereed to as the decomposition induced by $\mathscr{F}$.

For an arbitrary bidirected graph $\tilde{G} = (V, A; \tilde{\partial})$, a pair $(X, Y) \in 3^V$ satisfying (1) is also called an ideal of $\tilde{G}$. The set of all ideals forms a signed ring family $\mathscr{F}$ with $(\emptyset, \emptyset) \in \mathscr{F}$. The strongly connected component decomposition of bidirected graphs established in [2] is nothing but the decomposition induced by this signed ring family. In this case, $V_0 = \emptyset$ and V_∞ corresponds to the set of inconsistent vertices of $\tilde{G}$.

Example 1. Let V be a set of cardinality six, say $V = \{1, 2, 3, 4, 5, 6\}$, and consider a signed ring family

$$\mathscr{F} = \{(\emptyset, \emptyset), (\{1\}, \{2\}), (\{4\}, \{5\}), (\{1, 3\}, \{2, 6\}), (\{1, 4\}, \{2, 5\}),$$

$$(\{1, 3, 5\}, \{2, 4, 6\}), (\{1, 3, 4\}, \{2, 5, 6\}), (\{1, 4, 6\}, \{2, 3, 5\})\},$$

whose Hasse diagram according to the partial order $\sqsubseteq$ is illustrated in Figure 1. Since $(\emptyset, \emptyset) \in \mathscr{F}$ and $\mathrm{Supp}(\mathscr{F}) = V$, both V_0 and V_∞ are empty.

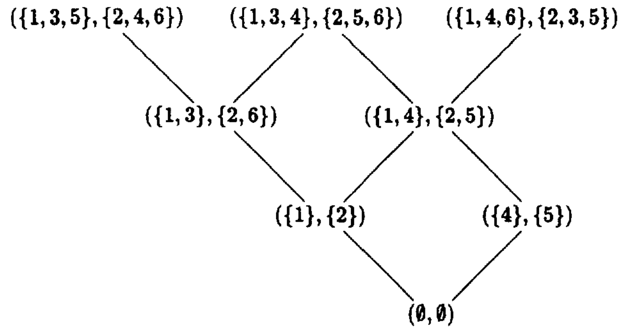

FIG. 1. Hasse diagram of a signed ring family $\mathscr{F}$ in Example 1 according to the partial order $\sqsubseteq$.

Then the partition is given by $\Pi(\mathscr{F}) = (V_1, V_2, V_3)$, where $V_1 = \{1, 2\}$, $V_2 = \{3, 6\}$, and $V_3 = \{4, 5\}$. The signed poset structure among components is shown in Figure 2, where the set of representatives is $\hat{V} = \{1, 3, 5\}$.

3. CANONICAL BLOCK TRIANGULARIZATION

Let T be a skew-symmetric matrix whose row set Row(T) and column set Col(T) are identified with a finite set V. We denote by $T[I, J]$ the submatrix of T with row set I and the column set J. The zero-nonzero pattern of T can

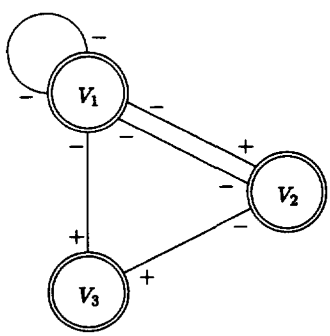

FIG. 2. The signed poset in Example 1. For an arc a, $\hat{\partial} a$ is represented by $\hat{\partial} a = (\hat{\partial}^+ a, \hat{\partial}^- a)$, where $\hat{\partial}^+ a$ is the set of its end vertices with $+$, and $\hat{\partial}^- a$ with the set of those with $-$.

be represented by a graph $G(T) = (V, E; \partial)$ with $E = \{(i,j) \mid T_{ij} \neq 0\}$ and $\partial e = \{i, j\}$ for $e = (i,j) \in E$. If there are no algebraic relations among the nonzero entries except for the skew symmetry, i.e., $T_{ij} + T_{ji} = 0$, then T is called a *generic skew-symmetric matrix*. The rank of a generic skew-symmetric matrix T is equal to twice the maximum cardinality of a matching in $G(T)$. Hence a generic skew-symmetric matrix T is nonsingular iff $G(T)$ has a perfect matching.

A nonsingular generic skew-symmetric matrix T is said to be in a *block-triangular form* if V is split into a certain number of disjoint blocks $(V_1^+, \ldots, V_b^+; V_\infty; V_b^-, \ldots, V_1^-)$ with $|V_k^+| = |V_k^-|$ for $k = 1, \ldots, b$ in such a way that

$$\begin{array}{lll} T[V_k^+, V_l^+] = O & & \text{for} \quad k = 1, \ldots, b, \quad l = 1, \ldots, b, \\ T[V_k^+, V_\infty] = O, & T[V_\infty, V_k^+] = O & \text{for} \quad k = 1, \ldots, b, \\ T[V_k^+, V_l^-] = O, & T[V_l^-, V_k^+] = O & \text{if} \quad 1 \leqslant k < l \leqslant b. \end{array}$$

Then T can be put into an explicitly upper block-triangular form by an appropriate simultaneous permutation of rows and columns (see Figure 3). The nonsingularity of T implies that $T[V_k^+, V_k^-]$ and $T[V_k^-, V_k^+]$ for $k = 1, \ldots, b$ as well as $T[V_\infty, V_\infty]$ are nonsingular.

We now construct a signed ring family that gives a canonical block triangularization of a nonsingular generic skew-symmetric matrix T. Let $\mathscr{F}(T)$

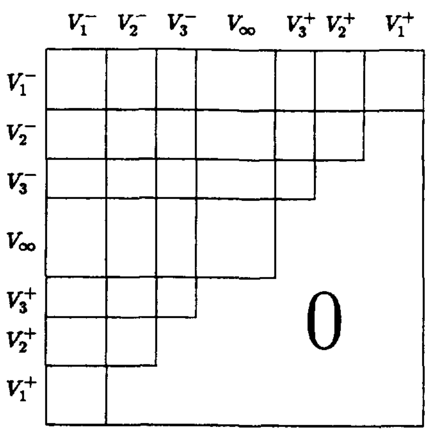

FIG. 3. A block-triangular form with $b = 3$.

denote the subfamily of 3^V defined by

$$\mathscr{F}(T) = \{(X,Y) | (X,Y) \in 3^V, T[X,\bar{Y}] = O, |X| = |Y|\},$$

where $\bar{Y} = V - Y$. Note that $\mathscr{F}(T)$ is nonempty, because $(\varnothing, \varnothing) \in \mathscr{F}(T)$. In order to show that $\mathscr{F}(T)$ is a signed ring family, we also consider

$$\mathscr{F}_*(T) = \{(X,Y) | (X,Y) \in 3^V, T[X,\bar{Y}] = O\}.$$

Then we have the following lemmas.

LEMMA 1. *The family $\mathscr{F}_*(T)$ is a signed ring family.*

Proof. Suppose $(X_h, Y_h) \in \mathscr{F}_*(T)$ for $h = 1, 2$. Then we have $T[X_1 \cap X_2, \overline{Y_1 \cap Y_2}] = T[X_1 \cap X_2, \bar{Y}_1 \cup \bar{Y}_2] = O$, because $T[X_1 \cap X_2, \bar{Y}_1] = O$ and $T[X_1 \cap X_2, \bar{Y}_2] = O$. This means $(X_1, Y_1) \sqcap (X_2, Y_2) \in \mathscr{F}_*(T)$. It follows from $T[X_1, \bar{Y}_1 \cap \bar{Y}_2] = O$ and $T[X_2, \bar{Y}_1 \cap \bar{Y}_2] = O$ that $T[X_1 \cup X_2, \overline{Y_1 \cup Y_2}] = T[X_1 \cup X_2, \bar{Y}_1 \cap \bar{Y}_2] = O$. By the skew symmetry, we also have $T[\overline{Y_1 \cup Y_2}, X_1 \cup X_2] = O$, which together with $(X_1 \cup X_2 - Y_1 \cup Y_2) \subseteq \overline{Y_1 \cup Y_2}$ implies $T[X_1 \cup X_2 - Y_1 \cup Y_2, X_1 \cup X_2] = O$. Therefore

$$\begin{aligned} &T\left[X_1 \cup X_2 - Y_1 \cup Y_2, \overline{Y_1 \cup Y_2 - X_1 \cup X_2}\right] \\ &\quad = T\left[X_1 \cup X_2 - Y_1 \cup Y_2, \overline{Y_1 \cup Y_2} \cup X_1 \cup X_2\right] = O, \end{aligned}$$

which means $(X_1, Y_1) \sqcup (X_2, Y_2) \in \mathscr{F}_*(T)$. Thus $\mathscr{F}_*(T)$ is closed with respect to the intersection and the reduced union. ∎

LEMMA 2. *If $(X, Y) \in \mathscr{F}_*(T)$, then $|X| \leqslant |Y|$.*

Proof. Since $T[X, \bar{Y}] = O$, we have $\operatorname{rank} T \leqslant \operatorname{rank} T[X, Y] + \operatorname{rank} T[\bar{X}, V] \leqslant |Y| + |\bar{X}|$, which together with the nonsingularity of T implies $|X| \leqslant |Y|$. ∎

We now turn to the family $\mathscr{F}(T)$ and show that it is closed with respect to the intersection and the reduced union.

THEOREM 3. *The family $\mathscr{F}(T)$ is a signed ring family.*

Proof. Suppose $(X_h, Y_h) \in \mathscr{F}(T)$ for $h = 1, 2$. Then it follows from $|X_1| = |Y_1|$ and $|X_2| = |Y_2|$ that $|X_1 \cap X_2| + |X_1 \cup X_2 - Y_1 \cup Y_2| = |Y_1 \cap Y_2| + |Y_1 \cup Y_2 - X_1 \cup X_2|$. On the other hand, Lemmas 1 and 2 imply $|X_1 \cup X_2 - Y_1 \cup Y_2| \leqslant |Y_1 \cup Y_2 - X_1 \cup X_2|$ and $|X_1 \cap X_2| \leqslant |Y_1 \cap Y_2|$. Therefore both of these inequalities hold with equalities, which means $(X_1, Y_1) \sqcup (X_2, Y_2) \in \mathscr{F}(T)$ and $(X_1, Y_1) \sqcap (X_2, Y_2) \in \mathscr{F}(T)$. ∎

As is described in Section 2, the signed ring family $\mathscr{F}(T)$ induces a decomposition. With reference to the maximal chain (2) of $\mathscr{F}(T)$, where $(X_0, Y_0) = (\emptyset, \emptyset)$, put $V_k^+ = X_k - X_{k-1}$ and $V_k^- = Y_k - Y_{k-1}$ for $k = 1, \ldots, b$. Then T is in a block-triangular form with the partition $(V_1^+, \ldots, V_b^+; V_\infty; V_b^-, \ldots, V_1^-)$. We call this block triangularization *canonical*.

Suppose conversely that the nonsingular generic skew-symmetric matrix T is in a block-triangular form with a partition $(V_1^+, \ldots, V_b^+; V_\infty; V_b^-, \ldots, V_1^-)$. Put $X_k = \bigcup_{l=1}^k V_l^+$ and $Y_k = \bigcup_{l=1}^k V_l^-$ for $k = 1, \ldots, b$. Then it is obvious from the definition of $\mathscr{F}(T)$ that $(X_k, Y_k) \in \mathscr{F}(T)$ holds. This means that the canonical block-triangular form is finer than any other block triangularization of T.

The signed poset structure among components in the canonical block-triangular form reflects the zero-nonzero pattern of the upper block-triangular part of T as follows. Construct a bidirected graph $\hat{G}(T) = (W, \hat{E}; \hat{\partial})$ with $W = \{V_1, \ldots, V_b\}$ by applying the following procedure to each pair of components:

1. Add a with $\hat{\partial} a = (\{V_k\}, \{V_l\})$ to $\hat{E}$ if $T[V_k^+, V_l^-] \neq O$.
2. Add a with $\hat{\partial} a = (\emptyset, \{V_k, V_l\})$ to $\hat{E}$ if $T[V_k^-, V_l^-] \neq O$.
3. Add a with $\hat{\partial} a = (\emptyset, \{V_k\})$ to $\hat{E}$ if $T[V_k^-, V_\infty] \neq O$.

Then the signed poset induced by $\mathscr{F}(T)$ is nothing but the transitive closure of $\hat{G}(T)$.

4. AN ALGORITHM

In this section, we present an efficient algorithm for computing the canonical block-triangular form.

Let M be a perfect matching of the undirected graph $G(T) = (V, E; \partial)$. We construct an auxiliary bidirected graph $\tilde{G}_M = (V, \tilde{E}_M; \tilde{\partial})$ with $\tilde{E}_m = E \cup$

M', where M' is a copy of M and $\tilde{\partial} : \tilde{E}_M \to \mathscr{T}$ is defined by

$$\tilde{\partial} a = \begin{cases} (\partial a, \emptyset) & \text{for} \quad a \in E, \\ (\emptyset, \partial \bar{a}) & \text{for} \quad a \in M' \ \ (\text{copy of } \bar{a} \in M). \end{cases}$$

Then we have the following lemma.

LEMMA 4. *The family $\tilde{\mathscr{F}}_M$ of the ideals of the auxiliary bidirected graph $\tilde{G}_M$ coincides with the signed ring family $\mathscr{F}(T)$.*

Proof. Suppose $(X, Y) \in \mathscr{F}(T)$. Then an edge $a \in E$ with one end vertex in X has another end vertex in Y. Hence $\langle \tilde{\partial} a, (X, Y) \rangle \leqslant 0$ for an arbitrary $a \in E$. Since $|X| = |Y|$ and every edge a with an end vertex in X must have the other end vertex in Y, the perfect matching M must match Y completely with X, which means a matching edge $\bar{a} \in M$ has two end vertices in $\overline{X \cup Y}$ or it has one in X and the other in Y. In either case, $\langle \tilde{\partial} a, (X, Y) \rangle \leqslant 0$ holds. Therefore (X, Y) is an ideal of $\tilde{G}_M$.

Conversely, suppose $(X, Y) \in \tilde{\mathscr{F}}_M$. Note that all the vertices in V are covered by the perfect matching M. All the edges in E incident to X are also incident to Y, which implies $T[X, \overline{Y}] = O$, and all the matching edges incident to Y are incident to X. Hence $|X| = |Y|$ holds, and we may assert that $(X, Y) \in \mathscr{F}(T)$. ■

This lemma shows that the set of ideals, or the strongly connected component decomposition, of auxiliary bidirected graphs is independent of the choice of the perfect matching M. An efficient algorithm, which runs in time proportional to the number of arcs, for the strongly connected component decomposition of bidirected graphs has been developed in [2]. Thus we have an efficient algorithm for computing the canonical block-triangular form as follows.

ALGORITHM (The canonical block triangularization).

Step 1. Find a perfect matching M in $G(T)$.
Step 2. Construct the auxiliary bidirected graph $\tilde{G}_M$.
Step 3. Compute the strongly connected component decomposition of $\tilde{G}_M$.

The most time-consuming part of this algorithm is step 1, which is performed by the blossom algorithm [6]. In other words, we can obtain the

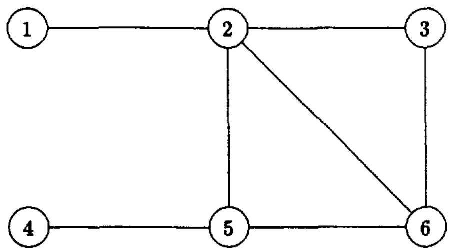

FIG. 4. An undirected graph G in Example 2.

decomposition at fairly small expense once a perfect matching is computed. One can easily see how analogous this algorithm is to the standard algorithm for the DM decomposition of bipartite graphs [5] (cf. [10, §3]).

EXAMPLE 2. Consider a generic skew-symmetric matrix

$$T = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & -t_1 & 0 & 0 & 0 & 0 \\ t_1 & 0 & -t_2 & 0 & -t_3 & -t_4 \\ 0 & t_2 & 0 & 0 & 0 & -t_5 \\ 0 & 0 & 0 & 0 & -t_6 & 0 \\ 0 & t_3 & 0 & t_6 & 0 & -t_7 \\ 0 & t_4 & t_5 & 0 & t_7 & 0 \end{pmatrix} \end{array}$$

with row column set $V = \{1, 2, 3, 4, 5, 6\}$. The undirected graph $G(T)$ is illustrated in Figure 4. Then $M = \{(1, 2), (3, 6), (4, 5)\}$ is the unique perfect

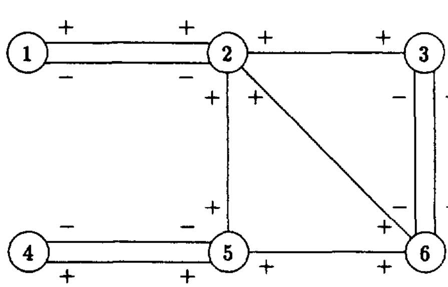

FIG. 5. The auxiliary bidirected graph $\tilde{G}_M$ in Example 2.

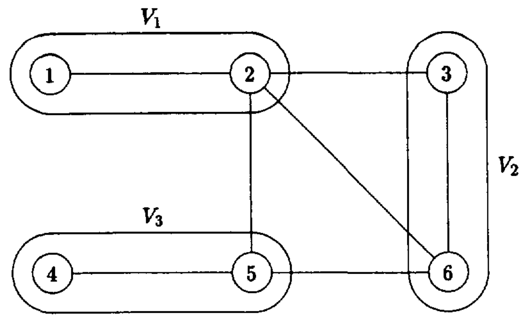

FIG. 6. The decomposition of the undirected graph $G(T)$ in Example 2.

matching, and hence T is nonsingular. The auxiliary bidirected graph $\tilde{G}_M$ is presented in Figure 5. The strongly connected component decomposition of $\tilde{G}_M$ gives a partition (V_1, V_2, V_3) shown in Figure 6. The signed poset structure among the three components is actually the same as that in Example 1. See Figure 2. In fact, the signed ring family $\mathcal{F}$ defined from $G(T)$ coincides with that in Example 1. We can permute the row-column set simultaneously so that T is in an explicitly block-triangular form as follows:

$$T = \begin{array}{c} \begin{array}{cccccc} 2 & 6 & 4 & 5 & 3 & 1 \end{array} \\ \begin{pmatrix} 0 & -t_4 & 0 & -t_3 & -t_2 & t_1 \\ t_4 & 0 & 0 & t_7 & t_5 & 0 \\ 0 & 0 & 0 & -t_6 & 0 & 0 \\ t_3 & -t_7 & t_6 & 0 & 0 & 0 \\ t_2 & -t_5 & 0 & 0 & 0 & 0 \\ -t_1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

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