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A Class of Unbounded Operator Algebras, IV

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1. INTRODUCTION

In this paper we continue our study of unbounded operator algebras begun in the previous papers [4–6]. The primary purpose of this paper is to investigate the direct sum, l_2^{ω}-direct sum and tensor product of unbounded Hilbert algebras and their left $EW^{\#}$-algebras. The second purpose is to study classifications of unbounded Hilbert algebras and of left $EW^{\#}$-algebras.

In this section let $\{\mathscr{D}_\lambda\}_{\lambda\in\Lambda}$ be a family of unbounded Hilbert algebras $\mathscr{D}_\lambda$ over $(\mathscr{D}_\lambda)_0$. We define the direct sum $\sum_{\lambda\in\Lambda}^{\oplus}\mathscr{D}_\lambda$ of $\{\mathscr{D}_\lambda\}_{\lambda\in\Lambda}$. Then $\sum_{\lambda\in\Lambda}^{\oplus}\mathscr{D}_\lambda$ is an unbounded Hilbert algebra over the direct sum $\sum_{\lambda\in\Lambda}^{\oplus}(\mathscr{D}_\lambda)_0$ of the Hilbert algebras $\{(\mathscr{D}_\lambda)_0\}_{\lambda\in\Lambda}$. Furthermore, we shall define a new direct sum $\bigoplus_{\lambda\in\Lambda}^{\omega}\mathscr{D}_\lambda$ called l_2^{ω}-direct sum. We find that $\bigoplus_{\lambda\in\Lambda}^{\omega}\mathscr{D}_\lambda$ is an unbounded Hilbert algebra containing $\sum_{\lambda\in\Lambda}^{\oplus}\mathscr{D}_\lambda$ and $L_2^{\omega}(\sum_{\lambda\in\Lambda}^{\oplus}(\mathscr{D}_\lambda)_0) = \bigoplus_{\lambda\in\Lambda}^{\omega}L_2^{\omega}((\mathscr{D}_\lambda)_0)$. Even if $\mathscr{D}_\lambda$ is a Hilbert algebra for every $\lambda\in\Lambda$, $\bigoplus_{\lambda\in\Lambda}^{\omega}\mathscr{D}_\lambda$ is not always a Hilbert algebra. There exist examples of such unbounded Hilbert algebras (Examples (1), (2) in Sect. 3).

An unbounded Hilbert algebra $\mathscr{D}$ over $\mathscr{D}_0$ is called weakly unbounded if there exists a family $\{(\mathscr{D}_0)_\lambda\}_{\lambda\in\Lambda}$ of Hilbert algebras $(\mathscr{D}_0)_\lambda$ such that $\mathscr{D}$ is a dense *-subalgebra of $\bigoplus_{\lambda\in\Lambda}^{\omega}(\mathscr{D}_0)_\lambda$. If $E\mathscr{D}$ is a pure unbounded Hilbert algebra for every non-zero projection E in $\mathscr{U}_0(\mathscr{D}_0)\cap\mathscr{V}_0(\mathscr{D}_0)$, then $\mathscr{D}$ is called strictly unbounded, where $\mathscr{U}_0(\mathscr{D}_0)$ (resp. $\mathscr{V}_0(\mathscr{D}_0)$) denotes the left (resp. right) von Neumann algebra of $\mathscr{D}_0$. Then there exists a projection E in $\mathscr{U}_0(\mathscr{D}_0)\cap\mathscr{V}_0(\mathscr{D}_0)$ such that $E\mathscr{D}$ is weakly unbounded, $(I-E)\mathscr{D}$ is strictly unbounded and $\mathscr{D}$ is a dense *-subalgebra of the direct sum $E\mathscr{D}\oplus(I-E)\mathscr{D}$ of $E\mathscr{D}$ and $(I-E)\mathscr{D}$.

We shall investigate the relation between the left $EW^{\#}$-algebras $\mathscr{U}(\sum_{\lambda\in\Lambda}^{\oplus}\mathscr{D}_\lambda)$, $\mathscr{U}(\bigoplus_{\lambda\in\Lambda}^{\omega}\mathscr{D}_\lambda)$ and the product $\prod_{\lambda\in\Lambda}\mathscr{U}(\mathscr{D}_\lambda)$ of the left $EW^{\#}$-algebras $\{\mathscr{U}(\mathscr{D}_\lambda)\}_{\lambda\in\Lambda}$. Let $\mathfrak{A}$ be a family of closable operators on a Hilbert space. Then we denote by $\bar{A}$ the closure of $A\in\mathfrak{A}$ and put $\overline{\mathfrak{A}} = \{\bar{A}; A\in\mathfrak{A}\}$. Let $\bigoplus_{\lambda\in\Lambda}\mathfrak{B}_\lambda$ be the direct sum of von Neumann algebras $\mathfrak{B}_\lambda$. Then $\overline{\mathscr{U}(\sum_{\lambda\in\Lambda}^{\oplus}\mathscr{D}_\lambda)}$ and $\overline{\mathscr{U}(\bigoplus_{\lambda\in\Lambda}^{\omega}\mathscr{D}_\lambda)}$ are EW^{*}-subalgebras of the EW^{*}-algebra $\overline{\prod_{\lambda\in\Lambda}\mathscr{U}(\mathscr{D}_\lambda)}$ under the operations of strong sum, strong product, adjoint and strong scalar multiplication and

$$\overline{\mathscr{U}\left(\sum_{\lambda\in\Lambda}^{\oplus}\mathscr{D}_\lambda\right)_b} = \overline{\mathscr{U}\left(\bigoplus_{\lambda\in\Lambda}^{\omega}\mathscr{D}_\lambda\right)_b} = \bigoplus_{\lambda\in\Lambda}\mathscr{U}_0((\mathscr{D}_\lambda)_0).$$

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We shall study the left $EW^{\#}$-algebra $\mathscr{U}(\mathscr{D})$ of a weakly or strictly unbounded Hilbert algebra $\mathscr{D}$. An $EW^{\#}$-algebra $\mathfrak{A}$ is called weakly unbounded if there exists a family $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda}$ of von Neumann algebras $\mathfrak{A}_\lambda$ such that $\overline{\mathfrak{A}}$ is a *-subalgebra of $\overline{\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda}$ and $\overline{\mathfrak{A}_b} = \bigoplus_{\lambda\in\Lambda}\mathfrak{A}_\lambda$. If there is not any non-zero projection E in $\overline{\mathfrak{A}_b} \cap \overline{\mathfrak{A}_b}'$ such that $\overline{\mathfrak{A}_E}$ is a von Neumann algebra, then $\mathfrak{A}$ is called strictly unbounded (, where $\mathfrak{A}_E$ denotes the reduced $EW^{\#}$-algebra of $\mathfrak{A}$). We can show that $\mathscr{D}$ is a weakly (resp. strictly) unbounded Hilbert algebra if and only if $\mathscr{U}(\mathscr{D})$ is a weakly (resp. strictly) $EW^{\#}$-algebra. Furthermore, there exists a projection E in $\mathscr{U}_0(\mathscr{D}_0) \cap \mathscr{V}_0(\mathscr{D}_0)$ such that $\mathscr{U}(\mathscr{D})_E$ is a weakly unbounded $EW^{\#}$-algebra, $\mathscr{U}(\mathscr{D})_{I-E}$ is a strictly unbounded $EW^{\#}$-algebra and $\mathscr{U}(\mathscr{D})$ equals the product of $\mathscr{U}(\mathscr{D})_E$ and $\mathscr{U}(\mathscr{D})_{I-E}$.

Finally we shall consider the tensor product of unbounded Hilbert algebras. Let $\mathscr{D}_1$ (resp. $\mathscr{D}_2$) be an unbounded Hilbert algebra over $(\mathscr{D}_1)_0$ (resp. $(\mathscr{D}_2)_0$). Then the algebraic tensor product $\mathscr{D}_1 \otimes \mathscr{D}_2$ of $\mathscr{D}_1$ and $\mathscr{D}_2$ is an unbounded Hilbert algebra over $(\mathscr{D}_1)_0 \otimes (\mathscr{D}_2)_0$. We shall investigate the left $EW^{\#}$-algebra $\mathscr{U}(\mathscr{D}_1 \otimes \mathscr{D}_2)$ of $\mathscr{D}_1 \otimes \mathscr{D}_2$. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be $EW^{\#}$-algebras on pre-Hilbert spaces $\mathfrak{D}_1$ and $\mathfrak{D}_2$ respectively. For each $T_1 \in \mathfrak{A}_1$ and $T_2 \in \mathfrak{A}_2$ we denote by $T_1 \otimes T_2$ the smallest linear extension of the map $\xi_1 \otimes \xi_2 \to T_1\xi_1 \otimes T_2\xi_2$ where $\xi_1 \in \mathfrak{D}_1$ and $\xi_2 \in \mathfrak{D}_2$ and set $\mathfrak{A}_1 \otimes \mathfrak{A}_2 = \{T_1 \otimes T_2;\ T_1 \in \mathfrak{A}_1,\ T_2 \in \mathfrak{A}_2\}$. Then $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ is a $\#$-algebra on $\mathfrak{D}_1 \otimes \mathfrak{D}_2$, but $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ is not generally an $EW^{\#}$-algebra. An $EW^{\#}$-algebra is called the tensor product of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ if it is minimal among $EW^{\#}$-algebras $\mathfrak{A}$ such that $\overline{\mathfrak{A}_b} = \overline{(\mathfrak{A}_1)_b} \mathbin{\overline{\otimes}} \overline{(\mathfrak{A}_2)_b}$ and $\overline{\mathfrak{A}} \supset \overline{\mathfrak{A}_1 \otimes \mathfrak{A}_2}$ (, where $\mathfrak{B}_1 \mathbin{\overline{\otimes}} \mathfrak{B}_2$ denotes the tensor product of von Neumann algebras $\mathfrak{B}_1$ and $\mathfrak{B}_2$) and is denoted by $\mathfrak{A}_1 \mathbin{\overline{\otimes}} \mathfrak{A}_2$. Does there exist the tensor product of the $EW^{\#}$-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$? If $\mathfrak{A}_1 = \mathscr{U}(\mathscr{D}_1)$ and $\mathfrak{A}_2 = \mathscr{U}(\mathscr{D}_2)$, then $\mathscr{U}(\mathscr{D}_1) \mathbin{\overline{\otimes}} \mathscr{U}(\mathscr{D}_2)$ exists and equals $\mathscr{U}(\mathscr{D}_1 \otimes \mathscr{D}_2)$.

2. Preliminaries

We give here only the basic definitions and facts needed. For a more complete discussion of the basic properties of unbounded Hilbert algebras and $EW^{\#}$-algebras the reader is referred to [4–7].

In this section let $\mathscr{D}$ be an unbounded Hilbert algebra over $\mathscr{D}_0$ in a Hilbert space $\mathfrak{H}$. Then $\mathscr{D}_0$ is a Hilbert algebra and the completion of $\mathscr{D}_0$ is the Hilbert space $\mathfrak{H}$. Let π (resp. π') be the left (resp. right) regular representation of $\mathscr{D}$ and let π_0 (resp. π_0') be the left (resp. right) regular representation of $\mathscr{D}_0$. For each $x \in \mathfrak{H}$ we define $\pi_0(x)$ and $\pi_0'(x)$ by:

$$\pi_0(x)\,\xi = \overline{\pi_0'(\xi)}\,x, \qquad \pi_0'(x)\,\xi = \overline{\pi_0(\xi)}\,x \qquad (\xi \in \mathscr{D}_0).$$

Then $\pi_0(x)$ and $\pi_0'(x)$ are linear operators on $\mathfrak{H}$ with the domain $\mathscr{D}_0$. The involution on $\mathscr{D}$ is extended to an involution on $\mathfrak{H}$, which is also denoted by *.

Then we have

$$\overline{\pi_0(x^*)} = \overline{\pi_0(x)}^*, \qquad \overline{\pi_0'(x^*)} = \overline{\pi_0'(x)}^* \qquad (x \in \mathfrak{H}),$$

$$\overline{\pi(\xi)} = \overline{\pi_0(\xi)}, \qquad \overline{\pi'(\xi)} = \overline{\pi_0'(\xi)}, \qquad \overline{\pi(\xi^*)} = \overline{\pi(\xi)}^*,$$

$$\overline{\pi'(\xi^*)} = \overline{\pi'(\xi)}^* \qquad (\xi \in \mathscr{D})$$

and for each $\lambda \in \mathbb{C}$ (the field of complex numbers) and $\xi, \eta \in \mathscr{D}$

$$\overline{\pi(\xi)} \mathbin{\boldsymbol{+}} \overline{\pi(\eta)} := \overline{\overline{\pi(\xi)} + \overline{\pi(\eta)}} = \overline{\pi(\xi + \eta)},$$

$$\overline{\pi(\xi)} \cdot \overline{\pi(\eta)} := \overline{\overline{\pi(\xi)}\, \overline{\pi(\eta)}} = \overline{\pi(\xi\eta)},$$

$$\lambda \cdot \overline{\pi(\xi)} := \begin{cases} \lambda\overline{\pi(\xi)}, & \text{if} \quad \lambda \neq 0 \\ 0, & \text{if} \quad \lambda = 0 \end{cases} = \overline{\pi(\lambda\xi)}.$$

Therefore $\overline{\pi(\mathscr{D})}$ is a *-algebra of closed operators on $\mathfrak{H}$ under the operations of strong sum, strong product, adjoint and strong scalar multiplication. Similarly $\overline{\pi'(\mathscr{D})}$ is a *-algebra of closed operators on $\mathfrak{H}$.

Let ϕ_0 be the natural trace on $\mathscr{U}_0(\mathscr{D}_0)^+$ and let $\mathfrak{B}(\mathfrak{H})$ be the set of all bounded linear operators on $\mathfrak{H}$. Putting $(\mathscr{D}_0)_b = \{x \in \mathfrak{H}; \overline{\pi_0(x)} \in \mathfrak{B}(\mathfrak{H})\}$, $(\mathscr{D}_0)_b$ is a Hilbert algebra containing $\mathscr{D}_0$. If $\mathscr{D}_0 = (\mathscr{D}_0)_b$, then $\mathscr{D}_0$ is called a maximal Hilbert algebra in $\mathfrak{H}$. Let $\mathfrak{M}$ (resp. $\mathfrak{M}^+$) be the set of all measurable (resp. positive measurable) operators on $\mathfrak{H}$ with respect to $\mathscr{U}_0(\mathscr{D}_0)$. For every $T \in \mathfrak{M}^+$ we put

$$\mu_0(T) = \sup[\phi_0(\overline{\pi_0(\xi)}); 0 \leqslant \overline{\pi_0(\xi)} \leqslant T, \xi \in (\mathscr{D}_0)_b^2]$$

and

$$L^p(\phi_0) = \{T \in \mathfrak{M}; \| T \|_p := \mu_0(| T |^p)^{1/p} < \infty\}, \qquad 1 \leqslant p < \infty.$$

Then $\| T \|_p$ is called the L^p-norm of $T \in L^p(\phi_0)$ and μ_0 is called the integral on $L^1(\phi_0)$. If $p = \infty$, we shall identify $\mathscr{U}_0(\mathscr{D}_0)$ with $L^\infty(\phi_0)$ and denote by $\| T \|_\infty$ the operator norm of $T \in \mathscr{U}_0(\mathscr{D}_0)$. We define L_2^ω-spaces with respect to ϕ_0 and $\mathscr{D}_0$ as follows;

$$L_2^\omega(\phi_0) = \bigcap_{2 \leqslant p < \infty} L^p(\phi_0) \qquad \text{and} \qquad L_2^\omega(\mathscr{D}_0) = \{x \in \mathfrak{H}; \overline{\pi_0(x)} \in L_2^\omega(\phi_0)\}$$

respectively. Then $L_2^\omega(\mathscr{D}_0)$ is an unbounded Hilbert algebra over $(\mathscr{D}_0)_b$ and $\mathscr{D}$ is a *-subalgebra of $L_2^\omega(\mathscr{D}_0)$. Hence $L_2^\omega(\mathscr{D}_0)$ is maximal among unbounded Hilbert algebras containing $\mathscr{D}_0$ ([5] Theorem 3.9), and so it is called a maximal unbounded Hilbert algebra of $\mathscr{D}_0$. If $(\mathscr{D}_0)_b \neq \mathfrak{H}$, i.e., $\mathfrak{H}$ is not a Hilbert algebra, then $L_2^\omega(\mathscr{D}_0)$ is pure [6, Theorem 3.4]. For $2 \leqslant p \leqslant \infty$ we set

$$L_2^p(\mathscr{D}_0) = \{x \in \mathfrak{H}; \overline{\pi_0(x)} \in L^p(\phi_0)\}, \qquad \| x \|_p = \| \overline{\pi_0(x)} \|_p \qquad (x \in L_2^p(\mathscr{D}_0)).$$

Then, for $2 \leqslant p < q < \infty$

$$L_2{}^2(\mathscr{D}_0) = \mathfrak{H} \supset L_2{}^p(\mathscr{D}_0) \supset L_2{}^q(\mathscr{D}_0) \supset L_2{}^\omega(\mathscr{D}_0) \supset L_2{}^\infty(\mathscr{D}_0) = (\mathscr{D}_0)_b\,,$$

and so $L_2{}^\omega(\mathscr{D}_0) = \bigcap_{2\leqslant n<\infty} L_2{}^n(\mathscr{D}_0)$ (n; integer).

Let $\pi_2{}^\omega$ be the left regular representation of $L_2{}^\omega(\mathscr{D}_0)$. Then $\pi_2{}^\omega(\mathscr{D})$ is a $\#$-algebra on $L_2{}^\omega(\mathscr{D}_0)$ under the involution $\pi_2{}^\omega(\xi)^\# = \pi_2{}^\omega(\xi^*)$ and since $\mathscr{U}_0(\mathscr{D}_0)\, L_2{}^\omega(\mathscr{D}_0) \subset L_2{}^\omega(\mathscr{D}_0)$, $\mathscr{U}_0(\mathscr{D}_0)/L_2{}^\omega(\mathscr{D}_0) := \{T/L_2{}^\omega(\mathscr{D}_0);\ T \in \mathscr{U}_0(\mathscr{D}_0)\}$ is a $\#$-algebra on $L_2{}^\omega(\mathscr{D}_0)$ under the involution $(T/L_2{}^\omega(\mathscr{D}_0))^\# = T^*/L_2{}^\omega(\mathscr{D}_0)$, where $T/L_2{}^\omega(\mathscr{D}_0)$ denotes the restriction of T onto $L_2{}^\omega(\mathscr{D}_0)$. We denote by $\mathscr{U}(\mathscr{D})$ the $\#$-algebra on $L_2{}^\omega(\mathscr{D}_0)$ generated by $\pi_2{}^\omega(\mathscr{D})$ and $\mathscr{U}_0(\mathscr{D}_0)/L_2{}^\omega(\mathscr{D}_0)$. Then $\mathscr{U}(\mathscr{D})$ and $\mathscr{U}(L_2{}^\omega(\mathscr{D}_0))$ are $EW^\#$-algebras on $L_2{}^\omega(\mathscr{D}_0)$ over $\mathscr{U}_0(\mathscr{D}_0)$. $\mathscr{U}(\mathscr{D})$ is called the left $EW^\#$-algebra of $\mathscr{D}$. In particular, if $(\mathscr{D}_0)_b \neq \mathfrak{H}$ then $\mathscr{U}(L_2{}^\omega(\mathscr{D}_0))$ is a pure $EW^\#$-algebra [6, Theorem 4.4].

3. $l_2{}^\omega$-Direct Sums of Unbounded Hilbert Algebras

In this section let Λ be an infinite set and let $\{\mathscr{D}_\lambda\}_{\lambda\in\Lambda}$ be a family of unbounded Hilbert algebras $\mathscr{D}_\lambda$ over $(\mathscr{D}_\lambda)_0$. Let $\mathfrak{H}_\lambda$ be the completion of $\mathscr{D}_\lambda$ for every $\lambda \in \Lambda$ and let $\mathsf{X}_{\lambda\in\Lambda}\, \mathscr{D}_\lambda$ be the Cartesian product of $\{\mathscr{D}_\lambda\}_{\lambda\in\Lambda}$. Under the operations: $\{\xi_\lambda\} + \{\eta_\lambda\} = \{\xi_\lambda + \eta_\lambda\}$, $\alpha\{\xi_\lambda\} = \{\alpha\xi_\lambda\}$, $\{\xi_\lambda\}\{\eta_\lambda\} = \{\xi_\lambda\eta_\lambda\}$ and $\{\xi_\lambda\}^* = \{\xi_\lambda{}^*\}$ ($\{\xi_\lambda\}, \{\eta_\lambda\} \in \mathsf{X}_{\lambda\in\Lambda}\, \mathscr{D}_\lambda$, $\alpha \in \mathbb{C}$), $\mathsf{X}_{\lambda\in\Lambda}\, \mathscr{D}_\lambda$ is a $*$-algebra.

We denote by $\sum_{\lambda\in\Lambda}^{\oplus} \mathscr{D}_\lambda$ the set of all elements of $\mathsf{X}_{\lambda\in\Lambda}\, \mathscr{D}_\lambda$ with only a finite number of non-zero coordinates. Then we can easily show that $\sum_{\lambda\in\Lambda}^{\oplus} \mathscr{D}_\lambda$ is an unbounded Hilbert algebra in the direct sum $\bigoplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda$ of the Hilbert spaces $\mathfrak{H}_\lambda$. We call $\sum_{\lambda\in\Lambda}^{\oplus} \mathscr{D}_\lambda$ the direct sum of the unbounded Hilbert algebras $\{\mathscr{D}_\lambda\}_{\lambda\in\Lambda}$. If $\mathscr{D}_\lambda$ is a Hilbert algebra for every $\lambda \in \Lambda$, then $\sum_{\lambda\in\Lambda}^{\oplus} \mathscr{D}_\lambda$ is a Hilbert algebra and

$$\mathscr{U}_0\Big(\sum_{\lambda\in\Lambda}^{\oplus} \mathscr{D}_\lambda\Big) = \bigoplus_{\lambda\in\Lambda} \mathscr{U}_0(\mathscr{D}_\lambda), \qquad \mathscr{V}_0\Big(\sum_{\lambda\in\Lambda}^{\oplus} \mathscr{D}_\lambda\Big) = \bigoplus_{\lambda\in\Lambda} \mathscr{V}_0(\mathscr{D}_\lambda).$$

Now we shall define a new direct sum (called $l_2{}^\omega$-direct sum) of unbounded Hilbert algebras.

Proposition 3.1. *We set*

$$\overset{\omega}{\bigoplus_{\lambda\in\Lambda}} \mathscr{D}_\lambda = \Big\{\{\xi_\lambda\} \in \mathsf{X}_{\lambda\in\Lambda}\, \mathscr{D}_\lambda;\ \sum_{\lambda\in\Lambda} \|\, \xi_\lambda \|_p^p < \infty \textit{ for all } p \geqslant 2\Big\}.$$

Then $\bigoplus_{\lambda\in\Lambda}^{\omega} \mathscr{D}_\lambda$ *is an unbounded Hilbert algebra in* $\bigoplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda$ *containing* $\sum_{\lambda\in\Lambda}^{\oplus} \mathscr{D}_\lambda$.

Proof. Let $\{\xi_\lambda\}, \{\eta_\lambda\} \in \bigoplus^{\omega}_{\lambda\in\Lambda} \mathscr{D}_\lambda$ and $\alpha \in \mathfrak{C}$. Then we have

$$\left[\sum_{\lambda\in\Lambda} \| \xi_\lambda + \eta_\lambda \|_p^p\right]^{1/p} \leqslant \left[\sum_{\lambda\in\Lambda} (\| \xi_\lambda \| + \| \eta_\lambda \|_p)^p\right]^{1/p}$$

$$\leqslant \left[\sum_{\lambda\in\Lambda} \| \xi_\lambda \|_p^p\right]^{1/p} + \left[\sum_{\lambda\in\Lambda} \| \eta_\lambda \|_p^p\right]^{1/p},$$

$$\left[\sum_{\lambda\in\Lambda} \| \alpha\xi_\lambda \|_p^p\right]^{1/p} = | \alpha | \left[\sum_{\lambda\in\Lambda} \| \xi_\lambda \|_p^p\right]^{1/p},$$

$$\sum_{\lambda\in\Lambda} \| \xi_\lambda\eta_\lambda \|_p^p \leqslant \sum_{\lambda\in\Lambda} \| \xi_\lambda \|_{2p}^p \| \eta_\lambda \|_{2p}^p \leqslant \tfrac{1}{2}\left[\sum_{\lambda\in\Lambda} \| \xi_\lambda \|_{2p}^{2p} + \sum_{\lambda\in\Lambda} \| \eta_\lambda \|_{2p}^{2p}\right]$$

and

$$\sum_{\lambda\in\Lambda} \| \xi_\lambda^* \|_p^p = \sum_{\lambda\in\Lambda} \| \xi_\lambda \|_p^p .$$

Hence, $\{\xi_\lambda\} + \{\eta_\lambda\}, \alpha\{\xi_\lambda\}, \{\xi_\lambda\}\{\eta_\lambda\}, \{\xi_\lambda\}^* \in \bigoplus^{\omega}_{\lambda\in\Lambda} \mathscr{D}_\lambda$ and $\|\{\xi_\lambda\}\|_p := [\sum_{\lambda\in\Lambda} \| \xi_\lambda \|_p^p]^{1/p}$ is a norm on $\bigoplus^{\omega}_{\lambda\in\Lambda} \mathscr{D}_\lambda$. Furthermore, it is easily showed that $(\{\xi_\lambda\} \mid \{\eta_\lambda\}) = (\{\eta_\lambda\}^* \mid \{\xi_\lambda\}^*)$, $(\{\xi_\lambda\}\{\eta_\lambda\} \mid \{\zeta_\lambda\}) = \{\{\eta_\lambda\} \mid \{\xi_\lambda\}^*\{\zeta_\lambda\}\}$ and $\sum^{\oplus}_{\lambda\in\Lambda} (\mathscr{D}_\lambda)_0 \subset (\bigoplus^{\omega}_{\lambda\in\Lambda} \mathscr{D}_\lambda)_0$. Thus $\bigoplus^{\omega}_{\lambda\in\Lambda} \mathscr{D}_\lambda$ is an unbounded Hilbert algebra.

DEFINITION 3.2. $\bigoplus^{\omega}_{\lambda\in\Lambda} \mathscr{D}_\lambda$ is called the l_2^{ω}-direct sum of the unbounded Hilbert algebras $\{\mathscr{D}_\lambda\}_{\lambda\in\Lambda}$ and is also denoted by $l_2^{\omega}(\{\mathscr{D}_\lambda\})$.

Even if $\mathscr{D}_\lambda$ is a Hilbert algebra for every $\lambda \in \Lambda$, $\bigoplus^{\omega}_{\lambda\in\Lambda} \mathscr{D}_\lambda$ is not always a Hilbert algebra. That is, there are examples such that $\bigoplus^{\omega}_{\lambda\in\Lambda} \mathscr{D}_\lambda$ becomes a pure unbounded Hilbert algebra.

EXAMPLES. (1) For every positive integer n we denote by $\mathfrak{C}_n$ the complex field $\mathfrak{C}$ with the inner product $(\alpha \mid \beta)_n := \alpha\bar{\beta}/n^2$. We can easily show that $\mathfrak{C}_n$ is a Hilbert algebra under the usual multiplication $\alpha\beta$ and involution $\alpha^* = \bar{\alpha}$. Then,

$$\bigoplus_n^{\omega} \mathfrak{C}_n = l_2^{\omega}(\{1/n^2\})$$

$$:= \left\{\{\alpha_n\};\ \alpha_n \in \mathfrak{C} \text{ for every } n \text{ and } \sum_{n=1}^{\infty} | \alpha_n |^p/n^2 < \infty \text{ for all } p \geqslant 2\right\}.$$

From [8, Example 3.5] $\bigoplus^{\omega}_n \mathfrak{C}_n$ is a pure unbounded Hilbert algebra.

(2) We set $(L^\infty[0, 1])_n = L^\infty[0, 1]$ $(n = 1, 2, \ldots)$. Then $L^\infty[0, 1]$ is a maximal

Hilbert algebra under the usual operations and inner product $(f \mid g) = \int_0^1 f(x)\overline{g(x)}\,dx$. We put

$$f_n(x) = |\log x|, \qquad 1/(n+1) \leqslant x \leqslant 1/n,$$
$$= 0, \qquad \text{otherwise},$$

and

$$f = (f_1, f_2, \ldots, f_n, \ldots).$$

Then, $f_n \in L^\infty[0, 1]$ $(n = 1, 2, \ldots)$ and for $k \geqslant 2$

$$\sum_{n=1}^{\infty} \| f_n \|_k^k = \sum_{n=1}^{\infty} \left[\int_{1/(n+1)}^{1/n} |\log x|^k\, dx \right] = \int_0^1 |\log x|^k\, dx = k!.$$

Hence, $f \in \oplus_n^\omega (L^\infty[0, 1])_n$. Since $\sup_n \| f_n \|_\infty = \sup_n[\log(n+1)] = \infty$, $f \notin (\oplus_n^\omega (L^\infty[0, 1])_n)_0$. Thus $\oplus_n^\omega (L^\infty[0, 1])_n$ is a pure unbounded Hilbert algebra.

We shall show that $L_2^\omega(\sum_{\lambda\in\Lambda}^{\oplus} (\mathscr{D}_\lambda)_0) = \oplus_{\lambda\in\Lambda}^{\omega} L_2^\omega((\mathscr{D}_\lambda)_0)$. Let X_λ be a linear operator on $\mathfrak{H}_\lambda$ with the domain $\mathscr{D}(X_\lambda)$ for every $\lambda \in \Lambda$. We define the linear operator $\{X_\lambda\}$ on $\mathfrak{H} := \oplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda$ with the domain $\mathscr{D}(\{X_\lambda\})$ as follows:

$$\mathscr{D}(\{X_\lambda\}) = \left\{ \{x_\lambda\} \in \mathfrak{H};\ x_\lambda \in \mathscr{D}(X_\lambda) \text{ for all } \lambda \in \Lambda \text{ and } \sum_{\lambda\in\Lambda} \| X_\lambda x_\lambda \|_2^2 < \infty \right\},$$

$$\{X_\lambda\}\{x_\lambda\} = \{X_\lambda x_\lambda\}, \qquad \{x_\lambda\} \in \mathscr{D}(\{X_\lambda\}).$$

Let X_λ be a densely-defined closable operator on $\mathfrak{H}_\lambda$ and let $\overline{X_\lambda} = U_\lambda \mid \overline{X_\lambda} \mid$ be the polar decomposition of $\overline{X_\lambda}$ for every $\lambda \in \Lambda$. We set $X = \{X_\lambda\}$ and $U = \{U_\lambda\}$. Then we can easily show that: $\bar{X} = \{\overline{X_\lambda}\}$, $X^* = \{X_\lambda{}^*\}$, $|\bar{X}| = \{|\overline{X_\lambda}|\}$ and $\bar{X} = U \mid \bar{X} \mid$ is the polar decomposition of $\bar{X}$. From the above facts we obtain the following lemma.

Lemma 3.3. *Let* π_0 (resp. $\pi_0{}^\lambda$) *be the left regular representation of the Hilbert algebra* $\sum_{\lambda\in\Lambda}^{\oplus} (\mathscr{D}_\lambda)_0$ (resp. $(\mathscr{D}_\lambda)_0$). *Suppose that* $x = \{x_\lambda\} \in \mathfrak{H}$. *Let* $\overline{\pi_0{}^\lambda(x_\lambda)} = U_\lambda \mid \overline{\pi_0{}^\lambda(x_\lambda)}\mid$ *be the polar decomposition of* $\overline{\pi_0{}^\lambda(x_\lambda)}$ *and let* $U = \{U_\lambda\}$. *Then:*

(1) $\overline{\pi_0(x)} = \{\overline{\pi_0{}^\lambda(x_\lambda)}\}$,

(2) $|\overline{\pi_0(x)}| = \{|\overline{\pi_0{}^\lambda(x_\lambda)}|\}$,

(3) $\overline{\pi_0(x)} = U \mid \overline{\pi_0(x)}\mid$ *is the polar decomposition of* $\pi_0(x)$.

Theorem 3.4. *The maximal unbounded Hilbert algebra* $L_2^\omega(\sum_{\lambda\in\Lambda}^{\oplus} (\mathscr{D}_\lambda)_0)$ *of* $\sum_{\lambda\in\Lambda}^{\oplus} (\mathscr{D}_\lambda)_0$ *equals the* l_2^ω*-direct sum* $\oplus_{\lambda\in\Lambda}^{\omega} L_2^\omega((\mathscr{D}_\lambda)_0)$ *of the maximal unbounded Hilbert algebras* $L_2^\omega((\mathscr{D}_\lambda)_0)$ *of* $(\mathscr{D}_\lambda)_0$.

Proof. Suppose that $x = \{x_\lambda\} \in L_2^\omega(\sum_{\lambda\in\Lambda}^{\oplus} (\mathscr{D}_\lambda)_0)$. Let π_0 (resp. $\pi_0{}^\lambda$) be the left regular representation of the Hilbert algebra $\sum_{\lambda\in\Lambda}^{\oplus}(\mathscr{D}_\lambda)_0$ (resp. $(\mathscr{D}_\lambda)_0$). Let $\overline{\pi_0(x)} = U\,|\,\overline{\pi_0(x)}|$ be the polar decomposition of $\overline{\pi_0(x)}$ and let $U = \{U_\lambda\}$. Then, by Lemma 3.3, $\overline{\pi_0{}^\lambda(x_\lambda)} = U_\lambda\,|\,\overline{\pi_0{}^\lambda(x_\lambda)}|$ is the polar decomposition of $\overline{\pi_0{}^\lambda(x_\lambda)}$. We have $|\,\overline{\pi_0(x)}| = U^*\overline{\pi_0(x)} = \overline{\pi_0(U^*x)}$ and $U^*x \in L_2{}^\omega(\sum_{\lambda\in\Lambda}^{\oplus}(\mathscr{D}_\lambda)_0)$. Let ϕ_0 (resp. $\phi_0{}^\lambda$) be the natural trace on $\mathscr{U}_0(\sum_{\lambda\in\Lambda}^{\oplus}(\mathscr{D}_\lambda)_0)^+$ (resp. $\mathscr{U}_0((\mathscr{D}_\lambda)_0)^+$) and let μ_0 (resp. $\mu_0{}^\lambda$) be the integral on $L^1(\phi_0)$ (resp. $L^1(\phi_0{}^\lambda)$). For each integer $n \geqslant 2$ we have

$$\begin{aligned}\| x \|_n^n &= \mu_0(|\,\overline{\pi_0(x)}|^n) = \mu_0(\overline{\pi_0((U^*x)^n)})\\ &= ((U^*x)^{n-1}\,|\,U^*x) = \sum_{\lambda\in\Lambda}((U_\lambda{}^*x_\lambda)^{n-1}\,|\,U_\lambda{}^*x_\lambda)\\ &= \sum_{\lambda\in\Lambda}\mu_0{}^\lambda(\overline{\pi_0{}^\lambda(U_\lambda{}^*x_\lambda)^n}) = \sum_{\lambda\in\Lambda}\mu_0{}^\lambda(|\,\overline{\pi_0{}^\lambda(x_\lambda)}|^n)\\ &= \sum_{\lambda\in\Lambda}\| x_\lambda \|_n^n .\end{aligned}$$

Hence, $L_2{}^\omega(\sum_{\lambda\in\Lambda}^{\oplus}(\mathscr{D}_\lambda)_0) \subset \bigoplus_{\lambda\in\Lambda}^{\omega} L_2{}^\omega((\mathscr{D}_\lambda)_0)$. On the other hand, $\bigoplus_{\lambda\in\Lambda}^{\omega} L_2{}^\omega((\mathscr{D}_\lambda)_0)$ is an unbounded Hilbert algebra containing $\sum_{\lambda\in\Lambda}^{\oplus}(\mathscr{D}_\lambda)_0$ (Prop. 3.1) and $L_2{}^\omega(\sum_{\lambda\in\Lambda}^{\oplus}(\mathscr{D}_\lambda)_0)$ is maximal among unbounded Hilbert algebras containing $\sum_{\lambda\in\Lambda}^{\oplus}(\mathscr{D}_\lambda)_0$ ([5] Theorem 3.9), and so the reverse inclusion is satisfied.

Thus, $\sum_{\lambda\in\Lambda}^{\oplus}(\mathscr{D}_\lambda)_0$ is a Hilbert algebra and $\bigoplus_{\lambda\in\Lambda}^{\omega}(\mathscr{D}_\lambda)_0$, $\sum_{\lambda\in\Lambda}^{\oplus}\mathscr{D}_\lambda$, $\bigoplus_{\lambda\in\Lambda}^{\omega}\mathscr{D}_\lambda$ and $\bigoplus_{\lambda\in\Lambda}^{\omega} L_2{}^\omega((\mathscr{D}_\lambda)_0)$ are unbounded Hilbert algebras. Furthermore, they have the following inclusions:

$$\sum_{\lambda\in\Lambda}^{\oplus}(\mathscr{D}_\lambda)_0 \subset \begin{matrix}\bigoplus_{\lambda\in\Lambda}^{\omega}(\mathscr{D}_\lambda)_0\\ \sum_{\lambda\in\Lambda}^{\oplus}\mathscr{D}_\lambda\end{matrix} \subset \bigoplus_{\lambda\in\Lambda}^{\omega}\mathscr{D}_\lambda \subset \bigoplus_{\lambda\in\Lambda}^{\omega} L_2{}^\omega((\mathscr{D}_\lambda)_0) = L_2{}^\omega\Big(\sum_{\lambda\in\Lambda}^{\oplus}(\mathscr{D}_\lambda)_0\Big).$$

If $\mathscr{D}_\lambda$ is pure for some $\lambda \in \Lambda$, then $\sum_{\lambda\in\Lambda}^{\oplus}\mathscr{D}_\lambda$, $\bigoplus_{\lambda\in\Lambda}^{\omega}\mathscr{D}_\lambda$, $\bigoplus_{\lambda\in\Lambda}^{\omega} L_2{}^\omega((\mathscr{D}_\lambda)_0)$ are clearly pure. We shall consider the problem: "Is $\bigoplus_{\lambda\in\Lambda}^{\omega} L_2{}^\omega((\mathscr{D}_\lambda)_0)$ is pure?" From [6, Theorem 3.4], if $\mathfrak{H}_\lambda$ is not a Hilbert algebra then $L_2{}^\omega((\mathscr{D}_\lambda)_0)$ is pure, and so $\bigoplus_{\lambda\in\Lambda}^{\omega} L_2{}^\omega((\mathscr{D}_\lambda)_0)$ is pure. Hence we have only to consider the above problem when the Hilbert space $\mathfrak{H}_\lambda$ is a Hilbert algebra for every $\lambda \in \Lambda$.

Suppose that $\mathfrak{H}_\lambda$ is a Hilbert algebra for every $\lambda \in \Lambda$. $\bigoplus_{\lambda\in\Lambda}^{\omega}\mathfrak{H}_\lambda$ is not always pure. In fact, the complex field $\mathfrak{C}$ is a Hilbert space and a Hilbert algebra under the usual multiplication $\alpha\beta$, involution $\alpha^* = \bar{\alpha}$ and inner product $(\alpha\,|\,\beta) = \alpha\bar{\beta}$. Putting $\mathfrak{H}_n = \mathfrak{C}$ $(n = 1, 2, \ldots)$, $\bigoplus_n^\omega \mathfrak{H}_n = l^2$. Hence $\bigoplus_n^\omega \mathfrak{H}_n$ is a Hilbert algebra. We shall consider under what conditions $\bigoplus_{\lambda\in\Lambda}^{\omega}\mathfrak{H}_\lambda$ is pure. We set

$$l_2{}^p(\{\mathfrak{H}_\lambda\}) = \Big\{\{x_\lambda\} \in \bigoplus_{\lambda\in\Lambda}\mathfrak{H}_\lambda;\ \sum_{\lambda\in\Lambda}\| x_\lambda \|_p^p < \infty\Big\}, \qquad 2 \leqslant p < \infty,$$

$$l_2^{\infty}(\{\mathfrak{H}_\lambda\}) = \left\{\{x_\lambda\} \in \bigoplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda;\ \sup_{\lambda\in\Lambda} \| x_\lambda \|_\infty < \infty\right\},$$

$$l_2^{\omega}(\{\mathfrak{H}_\lambda\}) = \bigcap_{2\leqslant p<\infty} l_2^{p}(\{\mathfrak{H}_\lambda\}) = \overset{\omega}{\bigoplus_{\lambda\in\Lambda}} \mathfrak{H}_\lambda .$$

LEMMA 3.5. *$l_2^{\infty}(\{\mathfrak{H}_\lambda\})$ is a maximal Hilbert algebra and $l_2^{\omega}(\{\mathfrak{H}_\lambda\})$ is a maximal unbounded Hilbert algebra over $l_2^{\infty}(\{\mathfrak{H}_\lambda\})$ in $\oplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda$. For $q > p \geqslant 2$,*

$$\bigoplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda \supset l_2^{p}(\{\mathfrak{H}_\lambda\}) \supset l_2^{q}(\{\mathfrak{H}_\lambda\}) \supset \overset{\omega}{\bigoplus_{\lambda\in\Lambda}} \mathfrak{H}_\lambda \supset l_2^{\infty}(\{\mathfrak{H}_\lambda\}) \supset \sum_{\lambda\in\Lambda}^{\oplus} \mathfrak{H}_\lambda$$

and

$$\overset{\omega}{\bigoplus_{\lambda\in\Lambda}} \mathfrak{H}_\lambda = \bigcap_{2\leqslant n<\infty} l_2^{n}(\{\mathfrak{H}_\lambda\}) \qquad (n;\ \text{integer}).$$

Proof. From [6, Lemma 3.1], for $x_\lambda \in \mathfrak{H}_\lambda$ we have

$$\| x_\lambda \|_p^p \leqslant \| x_\lambda \|_2^2 + \| x_\lambda \|_q^q, \qquad q > p \geqslant 2.$$

Hence, $l_2^{q}(\{\mathfrak{H}_\lambda\}) \subset l_2^{p}(\{\mathfrak{H}_\lambda\})$. The other arguments are easily shown.

PROPOSITION 3.6. *The following conditions are equivalent.*

(1) $\oplus_{\lambda\in\Lambda}^{\omega} \mathfrak{H}_\lambda$ *is pure.*

(2) $\oplus_{\lambda\in\Lambda}^{\omega} \mathfrak{H}_\lambda \neq l_2^{\infty}(\{\mathfrak{H}_\lambda\})$.

(3) $\oplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda \neq l_2^{\infty}(\{\mathfrak{H}_\lambda\})$, *i.e.*, $\oplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda$ *is not a Hilbert algebra.*

(4) *There exists a sequence $\{e_n\}$ of non-zero projections e_n in $\mathfrak{H}_{\lambda_n}$ $(\lambda_n \in \Lambda)$ such that $\sum_{n=1}^{\infty} \| e_n \|_2^2 < \infty$.*

(5) $l_2^{p}(\{\mathfrak{H}_\lambda\}) \supsetneqq l_2^{q}(\{\mathfrak{H}_\lambda\})$ *for some $q > p \geqslant 2$.*

(6) $\oplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda \supsetneqq l_2^{p}(\{\mathfrak{H}_\lambda\})$ *for each $p > 2$.*

Proof. From Theorem 3.4, $\oplus_{\lambda\in\Lambda}^{\omega} \mathfrak{H}_\lambda = L_2^{\omega}(\sum_{\lambda\in\Lambda}^{\oplus} \mathfrak{H}_\lambda)$ and it is a maximal unbounded Hilbert algebra over $l_2^{\infty}(\{\mathfrak{H}_\lambda\})$ in $\oplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda$. Hence Proposition 3.6 follows from [6, Theorem 3.4].

We shall investigate the relation between the left $EW^{\#}$-algebras $\mathscr{U}(\sum_{\lambda\in\Lambda}^{\oplus} \mathscr{D}_\lambda)$, $\mathscr{U}(\oplus_{\lambda\in\Lambda}^{\omega} \mathscr{D}_\lambda)$ and the product $\prod_{\lambda\in\Lambda} \mathscr{U}(\mathscr{D}_\lambda)$ of the left $EW^{\#}$-algebras $\mathscr{U}(\mathscr{D}_\lambda)$. $\mathscr{U}(\sum_{\lambda\in\Lambda}^{\oplus} \mathscr{D}_\lambda)$ and $\mathscr{U}(\oplus_{\lambda\in\Lambda}^{\omega} \mathscr{D}_\lambda)$ are $EW^{\#}$-algebras on $\oplus_{\lambda\in\Lambda}^{\omega} L_2^{\omega}((\mathscr{D}_\lambda)_0) = L_2^{\omega}(\sum_{\lambda\in\Lambda}^{\oplus} (\mathscr{D}_\lambda)_0)$ over $\oplus_{\lambda\in\Lambda} \mathscr{U}_0((\mathscr{D}_\lambda)_0)$. Let $\mathfrak{A}_\lambda$ be an $EW^{\#}$-algebra on a pre-Hilbert space $\mathfrak{D}_\lambda$ and let $\mathfrak{H}_\lambda$ be the completion of $\mathfrak{D}_\lambda$. The product $\prod_{\lambda\in\Lambda} \mathfrak{A}_\lambda := \{(A_\lambda);\ A_\lambda \in \mathfrak{A}_\lambda\}$ is defined as follows:

$$\mathscr{D}\left(\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda\right) = \Big\{\{x_\lambda\}\in\mathfrak{H} := \bigoplus_{\lambda\in\Lambda}\mathfrak{H}_\lambda;\ x_\lambda\in\mathfrak{D}_\lambda \text{ for all } \lambda\in\Lambda \text{ and } \sum_{\lambda\in\Lambda}\|A_\lambda x_\lambda\|_2^2 < \infty$$

$$\text{for all } A_\lambda\in\mathfrak{A}_\lambda\Big\},\qquad (A_\lambda)\{x_\lambda\} = \{A_\lambda x_\lambda\},\qquad \{x_\lambda\}\in\mathscr{D}\left(\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda\right).$$

From [4, Theorem 3.7] $\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda$ is an $EW^{\#}$-algebra on $\mathscr{D}(\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda)$ over $\oplus_{\lambda\in\Lambda}\overline{(\mathfrak{A}_\lambda)_b}$. We previously defined a closed operator $\{\bar{A}_\lambda\}$ $(A_\lambda\in\mathfrak{A}_\lambda)$ on $\mathfrak{H}$ as follows:

$$\mathscr{D}(\{\overline{A_\lambda}\}) = \Big\{\{x_\lambda\}\in\mathfrak{H};\ x_\lambda\in\mathscr{D}(\overline{A_\lambda}) \text{ for all } \lambda\in\Lambda \text{ and } \sum_{\lambda\in\Lambda}\|\overline{A_\lambda}x_\lambda\|_2^2 < \infty\Big\},$$

$$\{\overline{A_\lambda}\}\{x_\lambda\} = \{\overline{A_\lambda}x_\lambda\},\qquad \{x_\lambda\}\in\mathscr{D}(\{\overline{A_\lambda}\}).$$

We denote the set $\{\{\bar{A}_\lambda\};\ A_\lambda\in\mathfrak{A}_\lambda\}$ of closed operators on $\mathfrak{H}$ by $\mathsf{X}^{\mathrm{op}}_{\lambda\in\Lambda}\overline{\mathfrak{A}_\lambda}$. Then it is easily proved that $\{\overline{A_\lambda}\} = \overline{(A_\lambda)}$ for every $(A_\lambda)\in\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda$, and so $\mathsf{X}^{\mathrm{op}}_{\lambda\in\Lambda}\overline{\mathfrak{A}_\lambda} = \overline{\prod_{\lambda\in\Lambda}\mathfrak{A}_\lambda}$. It follows that $\mathsf{X}^{\mathrm{op}}_{\lambda\in\Lambda}\overline{\mathfrak{A}_\lambda}$ is an $EW^{\#}$-algebra over $\oplus_{\lambda\in\Lambda}\overline{(\mathfrak{A}_\lambda)_b}$.

Proposition 3.7. *$\overline{\mathscr{U}(\sum^{\oplus}_{\lambda\in\Lambda}\mathscr{D}_\lambda)}$ and $\overline{\mathscr{U}(\oplus^{\omega}_{\lambda\in\Lambda}\mathscr{D}_\lambda)}$ are EW^*-subalgebras of the EW^*-algebra $\mathsf{X}^{\mathrm{op}}_{\lambda\in\Lambda}\overline{\mathscr{U}(\mathscr{D}_\lambda)}$ under the operations of strong sum, strong product, adjoint and strong scalar multiplication and $\overline{\mathscr{U}(\sum^{\oplus}_{\lambda\in\Lambda}\mathscr{D}_\lambda)}_b = \overline{\mathscr{U}(\oplus^{\omega}_{\lambda\in\Lambda}\mathscr{D}_\lambda)}_b = (\mathsf{X}^{\mathrm{op}}_{\lambda\in\Lambda}\overline{\mathscr{U}(\mathscr{D}_\lambda)})_b = \oplus_{\lambda\in\Lambda}\mathscr{U}_0((\mathscr{D}_\lambda)_0)$.*

4. Weakly Unbounded Hilbert Algebras

In this section let $\mathscr{D}$ be a pure unbounded Hilbert algebra over $\mathscr{D}_0$ in a Hilbert space $\mathfrak{H}$. If E is a projection in $\mathscr{U}_0(\mathscr{D}_0)\cap\mathscr{V}_0(\mathscr{D}_0)$, then we have

$$(E\xi)(E\eta) = E(\xi\eta),\qquad (E\xi)^* = E\xi^*,\qquad \xi,\eta\in\mathscr{D}.$$

Hence $E\mathscr{D}$ is an unbounded Hilbert algebra containing the Hilbert algebra $E\mathscr{D}_0$. From Examples in Section 3 even if $\mathscr{D}$ is pure, $E\mathscr{D}$ is not always pure. So, we shall consider a classification of unbounded Hilbert algebras.

Definition 4.1. $\mathscr{D}$ is called a weakly unbounded Hilbert algebra if there exists a family $\{\mathscr{D}_\lambda\}_{\lambda\in\Lambda}$ of Hilbert algebras such that $\mathscr{D}$ is a dense *-subalgebra of $\oplus^{\omega}_{\lambda\in\Lambda}\mathscr{D}_\lambda$. If $E\mathscr{D}$ is a pure unbounded Hilbert algebra for every non-zero projection E in $\mathscr{U}_0(\mathscr{D}_0)\cap\mathscr{V}_0(\mathscr{D}_0)$, then $\mathscr{D}$ is called a strictly unbounded Hilbert algebra.

Proposition 4.2. *The following conditions are equivalent.*

(1) *$\mathscr{D}$ is weakly unbounded.*

(2) *There exists a family $\{E_\lambda\}_{\lambda\in\Lambda}$ of mutually orthogonal projections in $\mathscr{U}_0(\mathscr{D}_0) \cap \mathscr{V}_0(\mathscr{D}_0)$ such that $\sum_{\lambda\in\Lambda} E_\lambda = I$ and $E_\lambda\mathscr{D}$ is a Hilbert algebra for every $\lambda \in \Lambda$.*

Proof. 1 ⇒ (2) Suppose that there exists a family $\{\mathscr{D}_\lambda\}_{\lambda\in\Lambda}$ of Hilbert algebras such that $\mathscr{D}$ is a dense *-subalgebra of $\oplus^{\omega}_{\lambda\in\Lambda} \mathscr{D}_\lambda$. Let $\mathfrak{H}_\lambda$ be the completion of $\mathscr{D}_\lambda$. Then, $\mathfrak{H} = \oplus_{\lambda\in\Lambda} \mathfrak{H}_\lambda$. Furthermore, we can easily show that the projection E_λ $(:= P\mathfrak{H}_\lambda)$ onto $\mathfrak{H}_\lambda$ belongs to $\mathscr{U}_0(\mathscr{D}_0) \cap \mathscr{V}_0(\mathscr{D}_0)$, $\sum_{\lambda\in\Lambda} E_\lambda = I$ and $E_\lambda\mathscr{D} \subset \mathscr{D}_\lambda$. Since $\mathscr{D}_\lambda$ is a Hilbert algebra, $E_\lambda\mathscr{D}$ is a Hilbert algebra.

(2) ⇒ (1) Putting $U\xi = \{E_\lambda\xi\}$ $(\xi \in \mathscr{D})$, U is an isometric isomorphism of $\mathscr{D}$ onto $\oplus^{\omega}_{\lambda\in\Lambda} E_\lambda\mathscr{D}$. Identifying $\mathscr{D}$ with $U\mathscr{D}$, $\mathscr{D}$ is clearly a dense *-subalgebra of $\oplus^{\omega}_{\lambda\in\Lambda} E_\lambda\mathscr{D}$.

PROPOSITION 4.3. *The following conditions are equivalent.*

(1) *$L_2^{\omega}(\mathscr{D}_0)$ is weakly unbounded.*

(2) *There exists a family $\{E_\lambda\}_{\lambda\in\Lambda}$ of mutually orthogonal projections in $\mathscr{U}_0(\mathscr{D}_0) \cap \mathscr{V}_0(\mathscr{D}_0)$ such that $\sum_{\lambda\in\Lambda} E_\lambda = I$, $E_\lambda\mathfrak{H}$ is a Hilbert algebra for every $\lambda \in \Lambda$ and $L_2^{\omega}(\mathscr{D}_0) = \oplus^{\omega}_{\lambda\in\Lambda} E_\lambda\mathfrak{H}$.*

Proof. (2) ⇒ (1) Obvious.

(1) ⇒ (2) From Proposition 4.2 there exists a family $\{E_\lambda\}_{\lambda\in\Lambda}$ of mutually orthogonal projections in $\mathscr{U}_0(\mathscr{D}_0) \cap \mathscr{V}_0(\mathscr{D}_0)$ such that $\sum_{\lambda\in\Lambda} E_\lambda = I$ and $E_\lambda L_2^{\omega}(\mathscr{D}_0)$ is a Hilbert algebra for every $\lambda \in \Lambda$. From Theorem 3.4, $L_2^{\omega}(\mathscr{D}_0) = \oplus^{\omega}_{\lambda\in\Lambda} L_2^{\omega}(E_\lambda\mathscr{D}_0)$. Furthermore, $E_\lambda L_2^{\omega}(\mathscr{D}_0) = L_2^{\omega}(E_\lambda\mathscr{D}_0)$, and so $L_2^{\omega}(E_\lambda\mathscr{D}_0)$ is a Hilbert algebra. From [6, Theorem 3.4], $(E_\lambda\mathscr{D}_0)_b = L_2^{\omega}(E_\lambda\mathscr{D}_0) = E_\lambda\mathfrak{H}$. Hence $E_\lambda\mathfrak{H}$ is a Hilbert algebra and $L_2^{\omega}(\mathscr{D}_0) = \oplus^{\omega}_{\lambda\in\Lambda} E_\lambda\mathfrak{H}$.

THEOREM 4.4. *There exists a projection E in $\mathscr{U}_0(\mathscr{D}_0) \cap \mathscr{V}_0(\mathscr{D}_0)$ such that:*

(1) *$E\mathscr{D}$ is weakly unbounded and $(I - E)\mathscr{D}$ is strictly unbounded;*

(2) *$\mathscr{D}$ is a dense *-subalgebra of the direct sum $E\mathscr{D} \oplus (I - E)\mathscr{D}$ of the unbounded Hilbert algebras $E\mathscr{D}$ and $(I - E)\mathscr{D}$;*

(3) $(\mathscr{D}_0)_b = E(\mathscr{D}_0)_b \oplus (I - E)(\mathscr{D}_0)_b$.

Proof. If there is not any non-zeio projection E in $\mathscr{U}_0(\mathscr{D}_0) \cap \mathscr{V}_0(\mathscr{D}_0)$ such that $E\mathscr{D}$ is a Hilbert algebra, then $\mathscr{D}$ is strictly unbounded. If there exists a projection E_1 such that $E_1\mathscr{D}$ is a Hilbert algebra, then $(I - E_1)\mathscr{D}$ is a pure unbounded Hilbert algebra. Then if there is not any non-zero projection E_2 such that $I - E_1 \geqslant E_2$ and $E_2\mathscr{D}$ is a Hilbert algebra, then $(I - E_1)\mathscr{D}$ is strictly unbounded. So, we have only to put $E = E_1$. If otherwise, $(I - E_2)\mathscr{D}$ is pure. Thus, by Zorn's lemma there exists a maximal family $\{E_\lambda\}_{\lambda\in\Lambda}$ of mutually orthogonal projections in $\mathscr{U}_0(\mathscr{D}_0) \cap \mathscr{V}_0(\mathscr{D}_0)$ such that $E_\lambda\mathscr{D}$ is a Hilbert algebra for every $\lambda \in \Lambda$. Putting $E = \sum_{\lambda \in \Lambda} E_\lambda$, $E\mathscr{D}$ is a weakly unbounded Hilbert algebra. Furthermore, by the maximality of $\{E_\lambda\}_{\lambda \in \Lambda}$, if $E \neq I$ then $(I - E)\mathscr{D}$ is a strictly unbounded Hilbert algebra. It is easy to show that E satisfies the conditions (2) and (3) of the theorem.

COROLLARY 4.5. *There exists a projection E in $\mathscr{U}_0(\mathscr{D}_0) \cap \mathscr{V}_0(\mathscr{D}_0)$ such that $L_2^\omega(E\mathscr{D}_0)$ is weakly unbounded, $L_2^\omega((I - E)\mathscr{D}_0)$ is strictly unbounded and $L_2^\omega(\mathscr{D}_0) = L_2^\omega(E\mathscr{D}_0) \oplus L_2^\omega((I - E)\mathscr{D}_0)$. Furthermore, there exists a family $\{E_\lambda\}_{\lambda \in \Lambda}$ of mutually orthogonal projections in $\mathscr{U}_0(\mathscr{D}_0) \cap \mathscr{V}_0(\mathscr{D}_0)$ such that $E = \sum_{\lambda \in \Lambda} E_\lambda$, $E_\lambda \mathfrak{H}$ is a Hilbert algebra for every $\lambda \in \Lambda$ and $L_2^\omega(E\mathscr{D}_0) = \oplus^\omega_{\lambda \in \Lambda} E_\lambda \mathfrak{H}$.*

Proof. This follows from Theorem 3.4, Proposition 4.3 and Theorem 4.4.

We shall consider a classification of the left $EW^\#$-algebra $\mathscr{U}(\mathscr{D})$ of the unbounded Hilbert algebra $\mathscr{D}$.

DEFINITION 4.6. Let $\mathfrak{A}$ be an $EW^\#$-algebra on a pre-Hilbert space $\mathfrak{D}$. If there exists a family $\{\mathfrak{A}_\lambda\}_{\lambda \in \Lambda}$ of von Neumann algebras $\mathfrak{A}_\lambda$ such that $\overline{\mathfrak{A}}$ is a *-subalgebra of $\mathsf{X}^{\mathrm{op}}_{\lambda \in \Lambda} \mathfrak{A}_\lambda$ and $\overline{\mathfrak{A}}_b = \oplus_{\lambda \in \Lambda} \mathfrak{A}_\lambda$, then $\mathfrak{A}$ is called a weakly unbounded $EW^\#$-algebra.

Let E be a projection in $\overline{\mathfrak{A}}_b \cap \overline{\mathfrak{A}}_b'$, T_E the restriction of T onto $E\mathfrak{D}$ and let $\mathfrak{A}_E = \{T_E;\, T \in \mathfrak{A}\}$. From [4, Theorem 3.1] $\mathfrak{A}_E$ is an $EW^\#$-algebra on $E\mathfrak{D}$.

DEFINITION 4.7. An $EW^\#$-algebra $\mathfrak{A}$ is called a strictly unbounded $EW^\#$-algebra if there is not any non-zero projection E in $\overline{\mathfrak{A}}_b \cap \overline{\mathfrak{A}}_b'$ such that $\overline{\mathfrak{A}}_E$ is a von Neumann algebra.

THEOREM 4.8. *$\mathscr{D}$ is a weakly (resp. strictly) unbounded Hilbert algebra if and only if $\mathscr{U}(\mathscr{D})$ is a weakly (resp. strictly) unbounded $EW^\#$-algebra.*

Proof. Suppose that $\mathscr{D}$ is weakly unbounded, that is, there exists a family $\{\mathscr{D}_\lambda\}_{\lambda \in \Lambda}$ of Hilbert algebras $\mathscr{D}_\lambda$ such that $\mathscr{D}$ is a dense *-subalgebra of $\overline{\oplus^\omega_{\lambda \in \Lambda} \mathscr{D}_\lambda}$. Clearly $\mathscr{U}_0(\mathscr{D}_0) = \oplus_{\lambda \in \Lambda} \mathscr{U}_0(\mathscr{D}_\lambda)$ and $\overline{\mathscr{U}(\mathscr{D})}$ is a *-subalgebra of $\overline{\mathscr{U}(\oplus^\omega_{\lambda \in \Lambda} \mathscr{D}_\lambda)}$. From Proposition 3.7. $\overline{\mathscr{U}(\oplus^\omega_{\lambda \in \Lambda} \mathscr{D}_\lambda)}$ is a *-subalgebra of $\mathsf{X}^{\mathrm{op}}_{\lambda \in \Lambda} \mathscr{U}_0(\mathscr{D}_\lambda)$. Hence $\mathscr{U}(\mathscr{D})$ is weakly unbounded.

Conversely suppose that $\mathscr{U}(\mathscr{D})$ is weakly unbounded, that is, there exists a family $\{\mathfrak{A}_\lambda\}_{\lambda \in \Lambda}$ of von Neumann algebras $\mathfrak{A}_\lambda$ on Hilbert spaces $\mathfrak{H}_\lambda$ such that $\overline{\mathscr{U}(\mathscr{D})}$ is *-subalgebra of $\mathsf{X}^{\mathrm{op}}_{\lambda \in \Lambda} \mathfrak{A}_\lambda$ and $\mathscr{U}_0(\mathscr{D}_0) = \oplus_{\lambda \in \Lambda} \mathfrak{A}_\lambda$. We set $\mathfrak{H} = \oplus_{\lambda \in \Lambda} \mathfrak{H}_\lambda$ and $E_\lambda = P\mathfrak{H}_\lambda$ for all $\lambda \in \Lambda$. Then we can easily show that $E_\lambda \in \oplus_{\lambda \in \Lambda} \mathfrak{A}_\lambda \cap \oplus_{\lambda \in \Lambda} \mathfrak{A}_\lambda' = \mathscr{U}_0(\mathscr{D}_0) \cap \mathscr{V}_0(\mathscr{D}_0)$, $\sum_{\lambda \in \Lambda} E_\lambda = I$ and $\overline{\mathscr{U}(\mathscr{D})_{E_\lambda}} = \overline{\mathscr{U}(E_\lambda \mathscr{D})} = \mathfrak{A}_\lambda$. Since $\mathfrak{A}_\lambda$ is a von Neumann algebra for every $\lambda \in \Lambda$, $E_\lambda \mathscr{D}$ is a Hilbert algebra for every $\lambda \in \Lambda$. From Proposition 4.2, $\mathscr{D}$ is weakly unbounded. Similarly (2) is showed.

THEOREM 4.9. *There exists a projection E in $\mathscr{U}_0(\mathscr{D}_0) \cap \mathscr{V}_0(\mathscr{D}_0)$ such that:*

(1) *$\mathscr{U}(\mathscr{D})_E$ is a weakly unbounded $EW^{\#}$-algebra and $\mathscr{U}(\mathscr{D})_{I-E}$ is a strictly unbounded $EW^{\#}$-algebra;*

(2) *$\mathscr{U}(\mathscr{D})$ equals the product $\mathscr{U}(\mathscr{D})_E \times \mathscr{U}(\mathscr{D})_{I-E}$ of the $EW^{\#}$-algebras $\mathscr{U}(\mathscr{D})_E$ and $\mathscr{U}(\mathscr{D})_{I-E}$.*

Proof. From Theorem 4.4 there exists a projection E such that $E\mathscr{D}$ is weakly unbounded and $(I - E)\mathscr{D}$ is strictly unbounded. By Theorem 4.8 $\mathscr{U}(\mathscr{D})_E$ (resp. $\mathscr{U}(\mathscr{D})_{I-E}$) is a weakly (resp. strictly) unbounded $EW^{\#}$-algebra. Putting $\Phi(T) = \{T_E, T_{I-E}\}$, Φ is an isomorphism of $\mathscr{U}(\mathscr{D})$ onto $\mathscr{U}(\mathscr{D})_E \times \mathscr{U}(\mathscr{D})_{I-E}$ and $L_2^{\omega}(\mathscr{D}_0) = EL_2^{\omega}(\mathscr{D}_0) \oplus (I - E)L_2^{\omega}(\mathscr{D}_0)$. Hence we can identify $\mathscr{U}(\mathscr{D})$ with $\mathscr{U}(\mathscr{D})_E \times \mathscr{U}(\mathscr{D})_{I-E}$.

5. TENSOR PRODUCTS OF UNBOUNDED HILBERT ALGEBRAS

Let $\mathscr{D}_1$ (resp. $\mathscr{D}_2$) be an unbounded Hilbert algebra over $(\mathscr{D}_1)_0$ (resp. $(\mathscr{D}_2)_0$). Let $\mathscr{D}_1 \otimes \mathscr{D}_2$ be the algebraic tensor product of $\mathscr{D}_1$ and $\mathscr{D}_2$. We can easily show that $\mathscr{D}_1 \otimes \mathscr{D}_2$ is an unbounded Hilbert algebra over $(\mathscr{D}_1)_0 \otimes (\mathscr{D}_2)_0$ under the involution $(\xi_1 \otimes \xi_2)^* = \xi_1^* \otimes \xi_2^*$ and inner product $(\xi_1 \otimes \xi_2 \mid \eta_1 \otimes \eta_2) = (\xi_1 \mid \eta_1)(\xi_2 \mid \eta_2)$. We call it the tensor product of $\mathscr{D}_1$ and $\mathscr{D}_2$.

If X_1 and X_2 are linear operators on Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively, then we define their algebraic tensor product, denoted by $X_1 \otimes X_2$, to be the smallest linear extension of the map $x_1 \otimes x_2 \to X_1x_1 \otimes X_2x_2$ where $x_1 \in \mathscr{D}(X_1)$ and $x_2 \in \mathscr{D}(X_2)$. If X_1 and X_2 are closed operators, then we define $X_1 \boldsymbol{\otimes} X_2$ to be the closure of $X_1 \otimes X_2$ and call it the strong tensor product of X_1 and X_2. From [11, Theorem 8.1] if X_1 and X_2 are closed, densely-defined operators, then $(X_1 \boldsymbol{\otimes} X_2)^* = X_1^* \boldsymbol{\otimes} X_2^*$.

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be $EW^{\#}$-algebras on $\mathfrak{D}_1$ and $\mathfrak{D}_2$ respectively. Then we have, for each $S_1, T_1 \in \mathfrak{A}_1$ and $S_2, T_2 \in \mathfrak{A}_2$, $T_1 \otimes T_2$ is a bilinear map of T_1 and T_2; $(S_1 \otimes S_2)(T_1 \otimes T_2) = S_1T_1 \otimes S_2T_2$; $(T_1 \otimes T_2)^{\#} = T_1^{\#} \otimes T_2^{\#}$. We denote by $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ the $^{\#}$-algebra on $\mathfrak{D}_1 \otimes \mathfrak{D}_2$ generated by $\{T_1 \otimes T_2;\ T_1 \in \mathfrak{A}_1, T_2 \in \mathfrak{A}_2\}$.

DEFINITION 5.1. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be $EW^{\#}$-algebras on $\mathfrak{D}_1$ and $\mathfrak{D}_2$ respectively. An $EW^{\#}$-algebra is called the tensor product of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ if it is minimal among $EW^{\#}$-algebras $\mathfrak{A}$ such that $\overline{\mathfrak{A}_b} = \overline{(\mathfrak{A}_1)_b} \mathbin{\overline{\otimes}} \overline{(\mathfrak{A}_2)_b}$ and $\overline{\mathfrak{A}} \supset \overline{\mathfrak{A}_1 \otimes \mathfrak{A}_2}$, (where $\mathfrak{B}_1 \mathbin{\overline{\otimes}} \mathfrak{B}_2$ denotes the tensor product of von Neumann algebras $\mathfrak{B}_1$ and $\mathfrak{B}_2$) and is denoted by $\mathfrak{A}_1 \mathbin{\overline{\otimes}} \mathfrak{A}_2$.

We shall consider the problem: "Does there exist the tensor product of each $EW^{\#}$-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$?" When $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are the left $EW^{\#}$-algebras of unbounded Hilbert algebras $\mathscr{D}_1$ and $\mathscr{D}_2$ respectively, we shall find that the answer is affirmative.

LEMMA 5.2. *Let $\mathscr{D}_i$ be an unbounded Hilbert algebra over $(\mathscr{D}_i)_0$ and let $\mathfrak{H}_i$ be the completion of $\mathscr{D}_i$ $(i = 1, 2)$. Let π_0^1 (resp. π_0^2, π_0, π^1, π^2, π) be the left regular representation of $(\mathscr{D}_1)_0$ (resp. $(\mathscr{D}_2)_0$, $(\mathscr{D}_1)_0 \otimes (\mathscr{D}_2)_0$, $\mathscr{D}_1$, $\mathscr{D}_2$, $\mathscr{D}_1 \otimes \mathscr{D}_2$). Then:*

(1) $\overline{\pi_0(x_1 \otimes x_2)} = \overline{\pi_0^1(x_1) \otimes \pi_0^2(x_2)} = \overline{\pi_0^1(x_1)} \mathbin{\boldsymbol{\otimes}} \overline{\pi_0^2(x_2)}$ $(x_1 \in \mathfrak{H}_1, x_2 \in \mathfrak{H}_2)$;

(2) $\overline{\pi(\xi_1 \otimes \xi_2)} = \overline{\pi^1(\xi_1) \otimes \pi^2(\xi_2)} = \overline{\pi^1(\xi_1)} \mathbin{\boldsymbol{\otimes}} \overline{\pi^2(\xi_2)}$ $(\xi_1 \in \mathscr{D}_1, \xi_2 \in \mathscr{D}_2)$.

Proof. For every $\eta_1 \in (\mathscr{D}_1)_0$ and $\eta_2 \in (\mathscr{D}_2)_0$, $(\pi_0^1(x_1) \otimes \pi_0^2(x_2))(\eta_1 \otimes \eta_2) = \pi_0(x_1 \otimes x_2)(\eta_1 \otimes \eta_2)$. Hence, $\overline{\pi_0^1(x_1) \otimes \pi_0^2(x_2)} = \overline{\pi_0(x_1 \otimes x_2)}$. Clearly, $\overline{\pi_0^1(x_1) \otimes \pi_0^2(x_2)} \subset \overline{\pi_0^1(x_1)} \mathbin{\boldsymbol{\otimes}} \overline{\pi_0^2(x_2)}$. On the other hand, $\overline{\pi_0(x_1^* \otimes x_2^*)} = \overline{\pi_0((x_1 \otimes x_2)^*)} = \pi_0(x_1 \otimes x_2)^* \supset (\overline{\pi_0^1(x_1)} \mathbin{\boldsymbol{\otimes}} \overline{\pi_0^2(x_2)})^* = \overline{\pi_0^1(x_1)}^* \mathbin{\boldsymbol{\otimes}} \overline{\pi_0^2(x_2)}^* = \overline{\pi_0^1(x_1^*)} \mathbin{\boldsymbol{\otimes}} \overline{\pi_0^2(x_2^*)}$. Thus, $\overline{\pi_0^1(x_1) \otimes \pi_0^2(x_2)} = \overline{\pi_0^1(x_1)} \mathbin{\boldsymbol{\otimes}} \overline{\pi_0^2(x_2)}$. Similarly (2) is shown.

THEOREM 5.3. *Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be unbounded Hilbert algebras over $(\mathscr{D}_1)_0$ and $(\mathscr{D}_2)_0$ respectively. Then $\mathscr{U}(\mathscr{D}_1) \overline{\otimes} \mathscr{U}(\mathscr{D}_2)$ exists and equals $\mathscr{U}(\mathscr{D}_1 \otimes \mathscr{D}_2)$.*

Proof. Since $\mathscr{D}_1 \otimes \mathscr{D}_2$ is an unbounded Hilbert algebra over $(\mathscr{D}_1)_0 \otimes (\mathscr{D}_2)_0$, $\mathscr{U}(\mathscr{D}_1 \otimes \mathscr{D}_2)$ is an $EW^\#$-algebra on $L_2^\omega((\mathscr{D}_1)_0 \otimes (\mathscr{D}_2)_0)$ over $\mathscr{U}_0((\mathscr{D}_1)_0 \otimes (\mathscr{D}_2)_0)$. Hence,

$$\begin{aligned} \mathscr{U}(\mathscr{D}_1 \otimes \mathscr{D}_2)_b &= \mathscr{U}_0((\mathscr{D}_1)_0 \otimes (\mathscr{D}_2)_0) \\ &= \mathscr{U}_0((\mathscr{D}_1)_0) \overline{\otimes} \mathscr{U}_0((\mathscr{D}_2)_0) \\ &= \overline{\mathscr{U}(\mathscr{D}_1)_b} \overline{\otimes} \overline{\mathscr{U}(\mathscr{D}_2)_b}\,. \end{aligned}$$

Next we shall show that $\overline{\mathscr{U}(\mathscr{D}_1 \otimes \mathscr{D}_2)} \supset \overline{\mathscr{U}(\mathscr{D}_1) \otimes \mathscr{U}(\mathscr{D}_2)}$. Let π_0^1 (resp. π_0^2, π_0, $(\pi_2^\omega)^1$, $(\pi_2^\omega)^2$, π_2^ω) be the left regular representation of $(\mathscr{D}_1)_0$ (resp. $(\mathscr{D}_2)_0$, $(\mathscr{D}_1)_0 \otimes (\mathscr{D}_2)_0$, $L_2^\omega((\mathscr{D}_1)_0)$, $L_2^\omega((\mathscr{D}_2)_0)$, $L_2^\omega((\mathscr{D}_1)_0 \otimes (\mathscr{D}_2)_0)$). From Lemma 5.2, for every $x_1 \in \mathscr{D}_1$ and $x_2 \in \mathscr{D}_2$ we have

$$\begin{aligned} \overline{(\pi_2^\omega)^1(x_1) \otimes (\pi_2^\omega)^2(x_2)} &= \overline{(\pi_2^\omega)^1(x_1)} \mathbin{\boldsymbol{\otimes}} \overline{(\pi_2^\omega)^2(x_2)} \\ &= \overline{\pi_0^1(x_1)} \mathbin{\boldsymbol{\otimes}} \overline{\pi_0^2(x_2)} \\ &= \overline{\pi_0(x_1 \otimes x_2)} = \overline{\pi_2^\omega(x_1 \otimes x_2)}. \end{aligned}$$

Furthermore, $\overline{\mathscr{U}(\mathscr{D}_1 \otimes \mathscr{D}_2)_b} = \overline{\mathscr{U}(\mathscr{D}_1)_b} \overline{\otimes} \overline{\mathscr{U}(\mathscr{D}_2)_b} \supset \overline{\mathscr{U}(\mathscr{D}_1)_b \otimes \mathscr{U}(\mathscr{D}_2)_b}$. Hence we have $\overline{\mathscr{U}(\mathscr{D}_1) \otimes \mathscr{U}(\mathscr{D}_2)} \subset \overline{\mathscr{U}(\mathscr{D}_1 \otimes \mathscr{D}_2)}$. Finally we shall show that $\overline{\mathscr{U}(\mathscr{D}_1 \otimes \mathscr{D}_2)}$ is minimal among $EW^\#$-algebras $\mathfrak{A}$ such that $\mathfrak{A}_b = \overline{\mathscr{U}(\mathscr{D}_1)_b} \overline{\otimes} \overline{\mathscr{U}(\mathscr{D}_2)_b}$ and $\mathfrak{A} \supset \overline{\mathscr{U}(\mathscr{D}_1) \otimes \mathscr{U}(\mathscr{D}_2)}$. Suppose that $\mathfrak{A}$ is such an $EW^\#$-algebra. For every $x_1 \in \mathscr{D}_1$ and $x_2 \in \mathscr{D}_2$, $\overline{\pi_2^\omega(x_1 \otimes x_2)} = \overline{(\pi_2^\omega)^1(x_1) \otimes (\pi_2^\omega)^2(x_2)}$. Hence, $\overline{\pi_2^\omega(x_1 \otimes x_2)} \in \overline{\mathscr{U}(\mathscr{D}_1) \otimes \mathscr{U}(\mathscr{D}_2)} \subset \mathfrak{A}$. Furthermore, $\overline{\mathscr{U}(\mathscr{D}_1 \otimes \mathscr{D}_2)_b} = \overline{\mathscr{U}(\mathscr{D}_1)_b} \overline{\otimes} \overline{\mathscr{U}(\mathscr{D}_2)_b} = \mathfrak{A}_b$. Since $\mathscr{U}(\mathscr{D}_1 \otimes \mathscr{D}_2)$ is a *-algebra generated by $\mathscr{U}(\mathscr{D}_1 \otimes \mathscr{D}_2)_b$ and $\pi_2^\omega(\mathscr{D}_1 \otimes \mathscr{D}_2)$, we get $\overline{\mathscr{U}(\mathscr{D}_1 \otimes \mathscr{D}_2)} \subset \mathfrak{A}$.

COROLLARY 5.4. *If* $\mathscr{D}_0^1$ *and* $\mathscr{D}_0^2$ *are Hilbert algebras, then* $\mathscr{U}(L_2^\omega(\mathscr{D}_0^1)) \mathbin{\overline{\otimes}} \mathscr{U}(L_2^\omega(\mathscr{D}_0^2)) = \mathscr{U}(L_2^\omega(\mathscr{D}_0^1) \otimes L_2^\omega(\mathscr{D}_0^2))$ *and they are* $EW^\#$*-subalgebras of* $\mathscr{U}(L_2^\omega(\mathscr{D}_0^1 \otimes \mathscr{D}_0^2))$.

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