ELSEVIER Discrete Applied Mathematics 53 (1994) 171-182

DISCRETE APPLIED MATHEMATICS

Broadcasting on recursively decomposable Cayley graphs†

Chandra GowriSankaran[a, b]

[a] *Department of Computer Science, Concordia University, Montreal, Que., Canada H3G 1M8*
[b] *Dawson College, Montreal, Que., Canada*

Received 14 August 1991; revised received 5 May 1992

Abstract

We present here a near-optimal broadcasting algorithm for a family of Cayley graphs, the pancake graphs. We also show that this algorithm can be applied to a wider class of recursively decomposable Cayley graphs.

1. Introduction

We use [4] for basic concepts in graph theory. Basic results in group theory can be found in [10]. All the logarithms used here have base 2. The graphs and networks considered here are finite. Let G be a finite group and let ϕ be a set of generators for G which is closed under inverses and which does not contain the identity element. The *Cayley graph* G associated with the group G and generator set ϕ is the graph with vertex set G and there is an edge joining vertex u to vertex v *iff* there is a generator g in ϕ such that $ug = v$. This resulting graph is simple and since ϕ is closed under inverse group operation, it can be considered as an undirected graph. Also, since ϕ generates G, the graph is connected. In [1] the Cayley graph model is used to define alternative models to the existing topologies of interconnection networks. One of the families of Cayley graphs presented in [1], which is a particularly attractive model for interconnection networks is the *pancake graphs.* We describe pancake graphs and list some of their properties in Section 2.

In this paper we consider the problem of broadcasting in a pancake graph. In an interconnection network each vertex may represent a processor and each edge a two-way communication link. Broadcasting is the communication of a message or data, originated at a single vertex, to all the vertices of the network. In each unit of time a vertex can communicate with only one adjacent vertex and receive or send the

† This work was supported by a grant from FCAR of Quebec.

SSDI 0166-218X(93)E0046-2

message. A survey of published results in the field of broadcasting in networks can be found in [9]. Let u be a message originator in a connected graph $\mathscr{G}$ of order N. The *broadcast time* $b(u)$ of u is the minimum number of time units needed to broadcast a message originating at u. The *broadcast time of the network* $\mathscr{G}$ is defined as $b(\mathscr{G}) = \max\{b(u) \mid u \in \mathscr{G}\}$. Since in any broadcasting process the number of vertices which have received the message at most doubles in each time unit, we have $b(u) \geqslant \lceil \log N \rceil$. Consequently, we have the following well-known fundamental result [9].

Theorem 1.1. *Let $\mathscr{G}$ be a connected graph of order N. Then $b(\mathscr{G}) \geqslant \lceil \log \mathscr{N} \rceil$.*

For most graphs, however, the lower bound for the broadcast time cannot be attained. A broadcasting algorithm describes the communication route from the message originator to all the nodes in the network. It aims to complete the broadcasting process in minimum possible time. A broadcasting algorithm is a prerequisite for many fundamental computational algorithms in multiprocessor topologies. In [1] an $O(n \log n)$ algorithm for finding the maximum of $n!$ elements distributed among $n!$ processors placed at the vertices of an n-pancake graph is presented. For the architecture of an interconnection network to be of practical use, it is necessary to have an efficient broadcasting algorithm. In [8], the authors present a recursive broadcasting algorithm on a class of Cayley graphs, the *star graphs*. We show here that their ideas can be generalized to obtain a near-optimal broadcasting algorithm for a wider class of recursively decomposable Cayley graphs (as defined in Section 3), which includes in particular the pancake graphs.

Algebraic treatment of interconnection networks can also be found in [5, 7] and in more recent works [3, 6]. These references show how group theory can be effectively employed in highly parallel computing. [6] presents construction of some Cayley graphs along with algebraic broadcast algorithms. A unified approach based on algebraic structures of the networks is used in dealing with communication in parallel networks in [7, 3].

2. The pancake graphs

Since the underlying group in the definition of a Cayley graph is finite, one can consider it to be a permutation group. In particular, let the underlying group be S_n, the symmetric group of permutations on n symbols $1, 2, \ldots, n$. An element g of S_n will generally be denoted as a sequence of n distinct symbols (integers) in a row as $g = a_1 a_2 \ldots a_n$, where for all i, $1 \leqslant a_i \leqslant n$. $g(k)$ will denote the symbol in the kth position in g. For every permutation g in S_n the multiplication on right by g gives a transformation of the group $x \to xg$, which maps permutation x into permutation xg. Thus, considered as an operator a permutation will also be described by its effect on group elements and denoted as a product of independent cycles. In particular, the

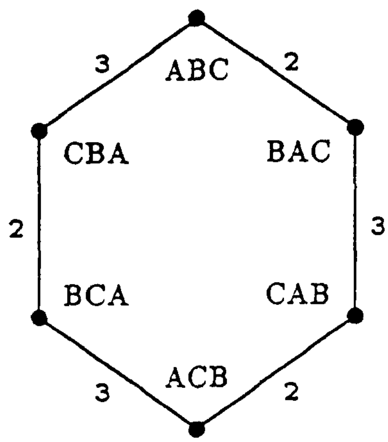

Fig. 1. The 3-pancake graph.

transposition interchanging the kth and jth symbols will be denoted by the cycle (jk). Now, for $2 \leqslant k \leqslant n$ consider the permutation g_k which *flips* the first (leftmost) k symbols, similar to the *flipping* with a spatula the top k pancakes from a stack of n pancakes. We have $g_k = k[k-1][k-2]\ldots 21[k+1][k+2]\ldots n$. Let $\phi_n = \{g_k \mid 2 \leqslant k \leqslant n\}$. Clearly, for all k, $g_k = g_k^{-1}$. Therefore, ϕ_n is closed for inverses. Also, since the composition $g_k g_2 g_k$ results in the transposition $([k-1]k)$, it follows that ϕ_n generates S_n.

The *n-pancake graph* (or the pancake graph on n symbols) $\mathscr{G}_n$ is the Cayley graph associated with the permutation group S_n and the generator set ϕ_n. Fig. 1 shows the 3-pancake graph $\mathscr{G}_3$ and Fig. 2 shows the 4-pancake graph $\mathscr{G}_4$. The symbols used in permutations here are A, B, C, D instead of 1, 2, 3, 4. Note that if $x = yg_k$ for some k, $2 \leqslant k \leqslant n$, then $y = xg_k$ and the edge joining vertices x and y can be labeled g_k, or for the sake of convenience, simply k.

2.1. Properties of pancake graphs

The pancake graphs are introduced in [1] and it is shown there that they present a good alternative to the widely used interconnection network models of n-cube. Clearly, the order of an n-pancake graph is $n!$ and it is regular of degree $n-1$. It is proved in [1] that its diameter is not more than $2n-3$. Now for given degree of $n-1$ the order of the multidimensional cube is 2^{n-1} and for given diameter of $2n-3$ the order of the cube is 2^{2n-3}. Therefore, comparing graphs of the same order, the pancake graphs have smaller diameter and degree than the n-cubes. Two properties of pancake graphs particularly of use in this paper are: the vertex symmetry and the hierarchical nature of the pancake graph.

A graph $\mathscr{G}$ is said to be *vertex symmetric* if for every pair of vertices u and v in $\mathscr{G}$ there is an automorphism of $\mathscr{G}$ that maps u into v. A vertex symmetric graph looks

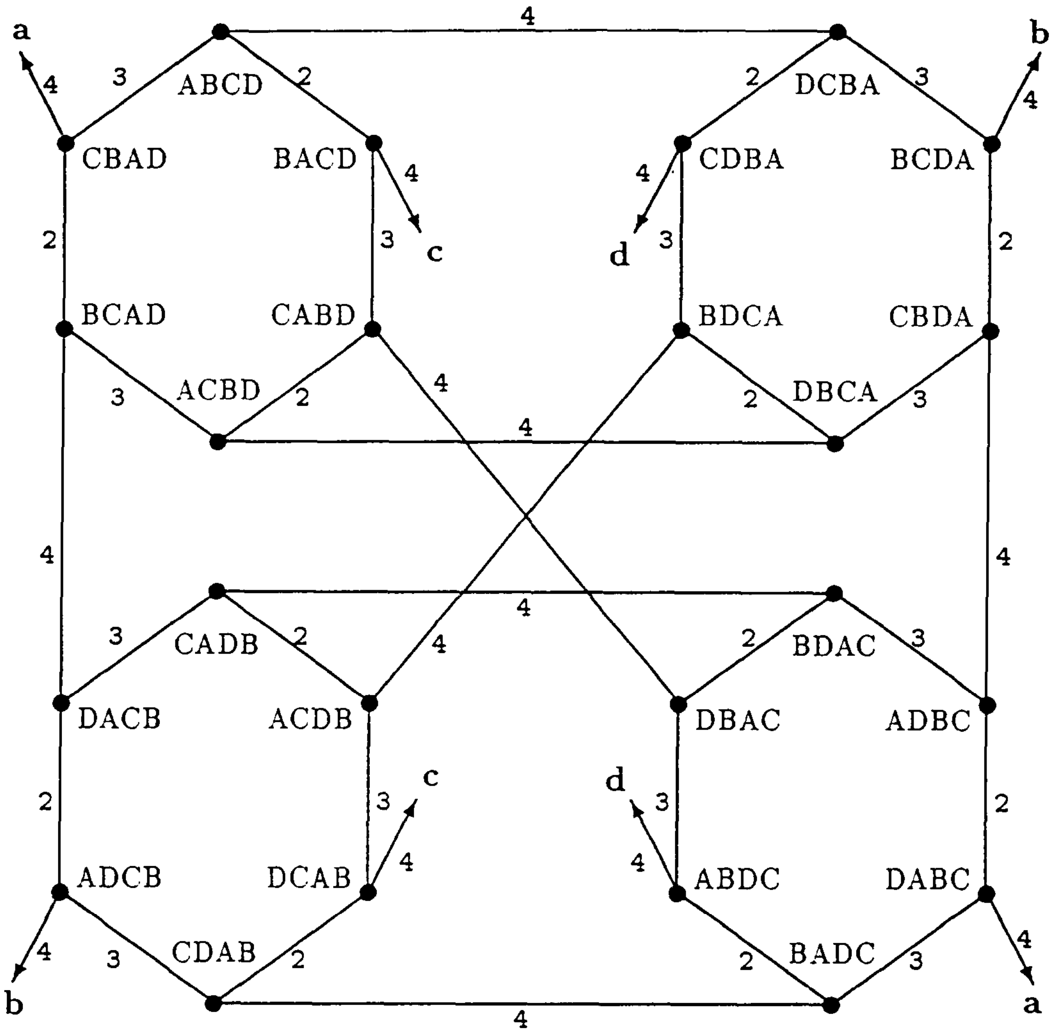

Fig. 2. The 4-pancake graph.

the same from each vertex. By using algebraic properties of the underlying group structure, it is shown in [1] that every Cayley graph is vertex symmetric.

The hierarchical nature of the pancake graph makes it possible to decompose an n-pancake graph $\mathscr{G}_n$ into $(n-1)$-pancake graphs. Let $n \geqslant 3$. First we consider the set F of all permutations in S_n with n in the nth position. This forms a subgroup of S_n of order $(n-1)!$ and is isomorphic to S_{n-1}, the symmetric group of all permutations on $n-1$ symbols $1, 2, \ldots, n-1$. Considered as vertices of $\mathscr{G}_n$, elements of F are joined only by edges corresponding to the generators $\phi_{n-1} = \{g_k \mid 2 \leqslant k \leqslant n-1\}$. Also, if u is in F and is joined to v in $\mathscr{G}_n$ by an edge corresponding to any generator in ϕ_{n-1}, then v is also in F. Thus the symbol n may be ignored and one can see that the subgraph induced on F is the $(n-1)$-pancake graph $\mathscr{G}_{n-1}$ with vertex set S_{n-1} and generator set ϕ_{n-1}.

Now let a be one of the $n-1$ symbols: $1, 2, \ldots, n-1$. Then as in the preceding paragraph it can be seen that the set of permutations in S_n with a in the nth position induces a subgraph isomorphic to the pancake graph $\mathscr{G}_{n-1}$ if the symbols n and a are interchanged. Thus one can express $\mathscr{G}_n$ as a vertex disjoint union of n pancake graphs of order $(n-1)!$ and the only edges connecting any two of these subgraphs are those corresponding to the generator g_n. Clearly each one of these n copies of $\mathscr{G}_{n-1}$, being an $(n-1)$-pancake graph, can itself be decomposed as a disjoint union of $n-1$ pancake graphs each of order $(n-2)!$, interconnected by edges corresponding to the generator g_{n-1}. This decomposition can be continued recursively down to pancake graphs of size 2. Fig. 2 shows the 4-pancake graph as a disjoint union of 4 isomorphic copies of 3-pancake graphs interconnected by edges labeled 4.

3. The recursively decomposable Cayley graphs

Let us note that the recursive decomposition of the pancake graph $\mathscr{G}_n$ as seen before is possible due to the following property of the generator set ϕ_n. For each r with $2 < r \leqslant n$, the generators $\{g_t \mid 2 \leqslant t < r\}$, when applied to any permutation in S_n, leave all but the leftmost $r-1$ symbols unchanged. Therefore, we find it natural to consider a more general class of Cayley graphs, namely, Cayley graphs with the property of being *recursively decomposable* (or with property RD for short) as defined below.

Definition 3.1 (*Property RD*). A Cayley graph $\mathscr{G}$ has property RD if, for some $n \geqslant 2$, its vertex set is S_n, the group of all permutations on n symbols, and its generator set ϕ satisfies the following conditions:

(1) ϕ contains exactly $n-1$ elements and for each $g \in \phi$, $g = g^{-1}$.

(2) There exists an ordering $g_2, g_3, \ldots, g_n$ of the elements of ϕ such that

(a) for all r, with $2 \leqslant r \leqslant n-1$, $g_r(k) = k$ for $r < k \leqslant n$,

(b) for all r, with $2 \leqslant r \leqslant n$, $g_r(1) = r$ and $g_r(r) = 1$.

Note that S_n is the vertex set of $\mathscr{G}$ if and only if order of $\mathscr{G}$ is $n!$. Also, the condition (2) on ϕ in Definition 3.1 can alternatively be stated as

(2a) for all r, with $2 \leqslant r \leqslant n-1$, the permutation g_r leaves the symbols in positions $r+1, r+2, \ldots, n$ unchanged and

(2b) for all r, with $2 \leqslant r \leqslant n$, g_r interchanges the symbols in the first and the rth positions.

We will refer to the graphs satisfying the conditions in Definition 3.1 as Cayley graphs of order $n!$ (or Cayley graphs on n symbols) with property RD. Clearly the pancake graphs, taken with the natural ordering of the generators as $g_2, g_3, \ldots, g_n$ possess the RD property. Another family of Cayley graphs, the *star graphs*, is studied in [1, 2]. The vertex set of a star graph of order $n!$, or an *n-star graph*, is the group S_n and the generator set consists of $n-1$ transpositions $(1k)$ for $k \leqslant 2 \leqslant n$. It is easy to see that with natural ordering of these generators star graphs possess property RD. More generally, star graphs can be decomposed recursively using an arbitrary ordering of the generators.

The following property of the generators of a Cayley graph satisfying the conditions of Definition 3.1 can easily be proved by induction on r. We omit the proof.

Lemma 3.2. *Let $n \geqslant 2$ and let $\mathscr{G}$ be a Cayley graph on n symbols with property RD. Then for every r, with $2 \leqslant r \leqslant n$, the set $\{g_t \mid 2 \leqslant t \leqslant r\}$ generates S_r, the group of all permutations of r symbols in the leftmost r positions.*

Proposition 3.3. *Every Cayley graph of order $n!$, $n \geqslant 3$, with property RD can be expressed as a finite vertex disjoint union of n copies of a Cayley graph of order $(n-1)!$ with property RD.*

Proof. Let $n \geqslant 3$ and let $\mathscr{G}$ be a Cayley graph on n symbols with property RD. Then the vertex set of $\mathscr{G}$ is S_n and the set ϕ of generators of $\mathscr{G}$ has $n-1$ elements. Let the generators be ordered as $g_2, g_3, \ldots, g_n$. Since for every $g \in \phi$, $g = g^{-1}$, every edge in the graph can be labeled unambiguously with one of the generators in ϕ. For each integer b with $1 \leqslant b \leqslant n$, consider the set of permutations $F_b = \{g \in S_n \mid g(n) = b\}$ and let $\mathscr{H}_b$ be the subgraph induced by F_b. Then $\mathscr{G}$ is a vertex disjoint union of the n subgraphs $\mathscr{H}_b$ for $1 \leqslant b \leqslant n$. Note that due to condition (2a) of Definition 3.1, for each b with $1 \leqslant b \leqslant n$, if $x \in F_b$, then for all k, $2 \leqslant k \leqslant n-1$, xg_k is also in F_b whereas, by condition (2b), xg_n is not in F_b. Let $\psi = \{g_k \mid 2 \leqslant k \leqslant n-1\}$. Then any edge of $\mathscr{G}$ is labeled with a generator in ψ if and only if it is within $\mathscr{H}_b$ for some b and it is labeled with g_n if and only if it interconnects $\mathscr{H}_i$ and $\mathscr{H}_j$ for some i and j, $1 \leqslant i, j \leqslant n$, $i \neq j$.

Now F_n is a subgroup of S_n of order $(n-1)!$ and is isomorphic to S_{n-1} the symmetric group of permutations of $n-1$ symbols. By Lemma 3.2, ψ is a set of generators for F_n. The induced subgraph is thus a Cayley graph $\mathscr{H}_n$ with vertex set F_n and generator set ψ. Since $\psi \subset \phi$, all the conditions of Definition 3.1 are satisfied for ψ and $\mathscr{H}_n$ clearly has property RD. Furthermore for each b, $1 \leqslant b \leqslant n-1$, the induced subgraph on F_b, namely $\mathscr{H}_b$, is isomorphic to $\mathscr{H}_n$ under interchange of symbols n and b which associates $x \in F_b$ to $x(bn) \in F_n$. Thus $\mathscr{G}$ is a vertex disjoint union of n copies of $\mathscr{H}_n$, each with generator set ψ and with property RD. We note that these n subgraphs are interconnected by edges labeled g_n. □

4. A near-optimal broadcasting algorithm

Algorithm broadcast_message: In this section we present a near-optimal broadcasting algorithm on Cayley graphs on n symbols with RD property. This algorithm works recursively, exploiting the hierarchical structure of RD graphs. It generalizes the recursive broadcasting algorithm on star graphs given in [8], to the class of Cayley graphs with RD property. In particular, it gives a broadcasting algorithm on pancake graphs.

Let $\mathscr{G}$ be a Cayley graph of order $n!$ with RD property. The algorithm is presented below as *procedure broadcast_message* with two subprocedures: *procedure phase_one* and *procedure phase_two.* The parameters are: n, where $n!$ is the order of the Cayley graph and v, the message originator. Since $\mathscr{G}$ is vertex symmetric, the same algorithm works for every message originator v. A copy of the algorithm is to be executed concurrently and synchronously at all the nodes of the network. We say the message is sent along *dimension* k if it is sent along edge g_k. At each time unit the algorithm determines at each node the dimension through which it should communicate with an adjacent node. The notations g_k, $\mathscr{H}_k$ and ϕ_k are used as in preceding sections. The two main steps in the algorithm are:

Step 1. Starting from v, send the message to one vertex (say v_k) in $\mathscr{H}_k$, for each k such that $1 \leqslant k \leqslant n$.

Step 2. For each k with $1 \leqslant k \leqslant n$, apply the algorithm recursively (and concurrently) to the n Cayley graphs $\mathscr{H}_k$ of order $(n-1)!$ with message originator v_k.

Since the graph $\mathscr{H}_k$ itself is an RD graph, it admits of a decomposition into $n-1$ RD subgraphs each of order $(n-2)!$ consisting of all the permutations in S_n which have same two symbols in the last two positions, the symbol in the nth position being k. At the second recursive step the algorithm sends the message to one message originator vertex in each one of these $n(n-1)$ subgraphs. Thus continued recursive application of the algorithm results in recursive decomposition of $\mathscr{G}$ into Cayley graphs of order 2. Each one of these graphs has 2 permutations which differ only in the first (leftmost) symbol and one of the 2 nodes becomes a message originator. At this stage all message originators send the message through dimension 2. This completes the broadcasting process.

Procedure phase_one (n, v); $\{v$ is the message originator in RD Cayley graph of order $n!$, for $n > 2$. The message is sent using each one of the dimensions $2, 3, \ldots, n-1$ exactly once. Total time taken is $\lceil \log(n-1) \rceil$. For each vertex u, which is not a message originator, the *source_dimension* is the dimension through which u receives the message. Initially, the value of source_dimension of the message originator v is 1 and for all other vertices it is 0. Every vertex sends the message through its *target_dimension* which is a function of its source_dimension and the current unit of time.$\}$

```
begin
  for time := 1 to ⌈log(n − 1)⌉ do
  {at each vertex do}
    begin
      if source_dim(u) > 0 then
```

```
            begin
               target_dim(u):= source_dim(u) + 2^(time-1);
               if target_dim(u) < max(3, n) then send message
                  along target_dimension(u);
            end;
         if u receives the message then
         source_dim(u):= dimension through which u receives the message;
      end
   end; {phase_one}
```

Procedure phase_two (n); {This procedure sends the message through dimension n to one node each in the subgraphs $\mathscr{H}_b$ of order $(n-1)!$. The source_dimension of such a node is set to 1 indicating that it will be the message originator in this subgraph. Total time taken by this step is 1 unit.}

```
begin
{at each vertex u do}
   if source_dim(u) > 0 then
      begin
         if source_dim(u) > 1 then source_dim(u):= 0;
         send message along dimension n;
      end;
   if u receives the message in this phase then source_dim(u):= 1;
end; {phase_two}
```

And now the main procedure:

Procedure broadcast_message (n, v); {broadcasts message in a RD Cayley graph of order $n!$. v is the message originator.}

```
begin
   if n > 2 then phase_one (n, v);
   phase_two (n);
   if n > 2 then
      begin
         {at each vertex u do}
         if source_dim(u) = 1 then broadcast_message (n - 1, u);
      end;
end;
```

4.1. Analysis of the algorithm

Theorem 4.1. *Let $\mathscr{G}_n$ be a Cayley graph of order $n!$ with RD property and let v be any vertex in $\mathscr{G}_n$. Then* **procedure broadcast_message** (n, v) *will broadcast a message* *from v to all vertices of $\mathscr{G}_n$ in time*

$$\beta(\mathscr{G}_n) = \sum_{p=2}^{n-1} \lceil \log p \rceil + n - 1.$$

Proof. Without loss of generality, we may assume that initially the message originator is $e = 1\ 2 \dots n$ and that **procedure phase_one** is called with parameters n and e. In **phase_one** every path along which the message is sent has the following property. Starting from e, the edges in the path are chosen in increasing order of dimension. Because of the conditions (2a) and (2b) of Definition 3.1, it follows that if a vertex u receives a message through edge g_k, then the first symbol in u is k. Moreover, each one of the edges $\{g_k \mid 2 \leqslant k \leqslant n-1\}$ is used exactly once in **phase_one**. Therefore, at the end of this phase, for each k with $1 \leqslant k \leqslant n-1$, there is one vertex with starting symbol k which has received the message. When **procedure phase_two** is called, each of these vertices sends the message along dimension n. Therefore, at the end of **phase_two**, for each k with $1 \leqslant k \leqslant n$, there is a vertex with last symbol k which has received the message. These n vertices are the message originators in the corresponding subgraphs of order $(n-1)!$ to which they belong. The algorithm continues recursively down to decomposition into subgroups of size 2. Thus clearly, a call to **procedure broadcast_message** with parameters n and v will transmit the message from v to all the vertices of $\mathscr{G}_n$.

Procedure phase_one will be called with message originators in subgraphs of successively diminishing order and thus the values taken by the first parameter will be $p = n, n-1, \dots, 2$. When $3 \leqslant p \leqslant n$, the message will be sent along dimensions $2, 3, \dots, p-1$ in **phase_one** and along dimension p in **phase_two**. Therefore, the total time taken will be $\lceil \log(p-1) \rceil + 1$. When $p = 2$ **phase_one** will not be called and **phase_two** will send the message along dimension 2 and thus will use 1 unit of time. Let $\beta(v)$ denote the time algorithm **broadcast_message** takes to broadcast a message from v and $\beta(\mathscr{G}_n)$ denote the maximum of $\beta(v)$ taken over v in $\mathscr{G}_n$. The vertex transitivity of $\mathscr{G}_n$ implies $\beta(v) = \beta(\mathscr{G}_n)$. Thus the broadcast time of algorithm **broadcast_message** is given by

$$\begin{aligned} \beta(\mathscr{G}_n) &= 1 + \sum_{p=3}^{n} \lceil \log(p-1) \rceil + n - 2 \\ &= \sum_{p=2}^{n-1} \lceil \log p \rceil + n - 1. \qquad \square \end{aligned}$$

Since $\mathscr{G}_n$ is of order $n!$, using the lower bound for the optimal broadcast time $b(\mathscr{G}_n)$ given in Theorem 1.1, we have

$$b(\mathscr{G}_n) \geqslant \lceil \log n! \rceil.$$

Since $\mathscr{G}_n$ is regular of degree $n-1$ which is strictly less than $\log n!$ (with the exception of $n = 1, 2$), it is not possible for any broadcasting algorithm for $\mathscr{G}_n$ to attain

Table 1
Comparison of $BT(n)$ and $\log n!$

n	$BT(n)$	$\lceil \log n! \rceil$	n	$BT(n)$	$\lceil \log n! \rceil$
3	3	3	12	40	29
4	6	5	13	45	33
5	9	7	14	50	37
6	13	10	15	55	41
7	17	13	16	60	45
8	21	16	17	65	49
9	25	19	18	71	53
10	30	22	19	77	57
11	35	26	20	83	62

this lower bound. Moreover,

$$\begin{aligned}\lceil \log n! \rceil &= \left\lceil \sum_{p=2}^{n} \log p \right\rceil \\ &\geqslant \sum_{p=2}^{n} \lceil \log p \rceil - (n-2) \\ &= \sum_{p=2}^{n-1} \lceil \log p \rceil + \lceil \log n \rceil - (n-2).\end{aligned}$$

Combining this inequality with the expression for $\beta(\mathscr{G}_n)$ we get

$$\beta(\mathscr{G}_n) \leqslant \lceil \log n! \rceil + 2n - 3 - \lceil \log n \rceil.$$

Therefore, the broadcast time $\beta(\mathscr{G}_n)$ appears to be very close to the optimum value.

The proof of Theorem 4.1 does not use any other specific property of the graphs (or generators) involved except those stated in the Definition 3.1. Therefore, the broadcast time of our algorithm in all such graphs of order $n!$, including the pancake graphs and the star graphs, is $\beta(\mathscr{G}_n)$. We will denote this broadcast time by $BT(n)$ since it depends only on the size $n!$ of the Cayley graph $\mathscr{G}_n$ with RD property. Table 1 compares the two functions $\lceil \log n! \rceil$ and $BT(n) = \sum_{p=2}^{n-1} \lceil \log p \rceil + n - 1$ for some values of n.

5. Conclusion

We have presented in this paper a recursive broadcasting algorithm for interconnection networks which have the topology of Cayley graphs of order $n!$ whose generators satisfy the RD property. These topologies include the pancake graphs as well as the star graphs. Such graphs can be decomposed into smaller graphs with RD property. The broadcasting algorithm presented here exploits this hierarchical nature of these Cayley graphs and in that sense is a natural generalization of the broadcasting algorithm for the n-cube. In [8] the authors have shown that their recursive broadcasting algorithm for star graphs works in near-optimal time. We have shown here that algorithm **broadcast_message** will accomplish broadcasting on pancake graphs in near-optimal time.

As seen in Section 4 the broadcast time $BT(n)$ for algorithm **broadcast_message** holds for all Cayley graphs of order $n!$ with RD property. We can now define a new class of Cayley graphs which have RD property and (consequently broadcasting time $BT(n)$ by algorithm **broadcast_message**) as follows. Let ϕ be the set of generators $g_2, g_3, \ldots, g_n$ of the n-pancake graph and let ψ be the set of generators $f_2, f_3, \ldots, f_n$ of the n-star. Both the graphs have the same vertex set S_n. Let us recall that for all k with $2 \leqslant k \leqslant n$, g_k is the flip of first k symbols and f_k is the transposition $(1k)$. Note that $f_k = g_k$ for $k = 2, 3$. The property RD will hold even if we take certain combinations of the generators from the two sets. For example let $4 < M < n$. Consider the set χ of generators $h_2, h_3, \ldots, h_n$, where

$$h_k = g_k \quad \text{if} \quad k \leqslant M \quad \text{and} \quad h_k = f_k \quad \text{if} \quad k > M.$$

It is not difficult to see that χ generates S_n and the Cayley graph A_n associated with S_n and χ satisfies the RD property. A_n will be decomposed into smaller Cayley graphs which do not all necessarily look like A_n, however they all satisfy the RD property. For any r with $2 \leqslant r \leqslant n$, all the subgraphs at the rth stage of decomposition will be isomorphic. Algorithm **broadcast_message** will still be applicable and the broadcasting process will be completed successfully in time $BT(n)$. One can also consider the set of generators $w_2, w_3, \ldots, w_n$, where

$$w_k = f_k \quad \text{if} \quad k \leqslant M \quad \text{and} \quad w_k = g_k \quad \text{if} \quad k > M$$

and the same observation is true. It remains to be seen, however, what attractive properties such mixture of topologies may have.

Another family of Cayley graphs with RD family, which may be considered to be in between the star graphs and the pancake graphs, may be defined making use of the following observation. The generator f_k of star graphs consists of a single transposition $(1k)$ whereas the generator g_k of pancake graphs is a composition of $\lfloor k/2 \rfloor$ transpositions.

$$g_k = (1k)(2[k-1])(3[k-2])\ldots(\lfloor k/2 \rfloor \lceil (k+2)/2 \rceil).$$

We can choose an integer $C \geqslant 2$ and define the $n-1$ generators of S_n using the compositions of only C of these transpositions from left. For example, let $n > 5$ and $C = 2$. Then the set θ of generators $s_2, s_3, \ldots, s_n$ will be given by

$$s_k = g_k \quad \text{if} \quad k \leqslant 5 \quad \text{and} \quad s_k = (1k)(2[k-1]) \quad \text{if} \quad 5 < k \leqslant n.$$

Clearly, θ generates S_n and satisfies RD property.

In algorithm **broadcast_message** once a vertex becomes a message originator in a subgraph at a certain level of recursion, it remains from then on a message originator in subgraphs of smaller sizes to which it belongs, created in succeeding recursive calls. In **phase_one** of every recursive call a message originator in a subgraph of order $m!$ sends messages along dimensions $2, 3, 5, \ldots, 1 + 2^r$ where $1 + 2^r < m$. Thus, for example, the initial message originator $e = 1\,2 \ldots n$ sends in **phase_one** messages along the same $\lceil \log n \rceil$ dimensions even though its degree is $n - 1$. e sends the message to $u = 2\,1\,3 \ldots n$ along dimension 2 in **phase_one** of every recursive call, that is $n - 1$ times. In all but the last of these calls, that is $n - 2$ times, u sends the message to $v = n(n - 1) \ldots 3\,1\,2$ along dimension n. This redundancy however does not result in overall delay since meanwhile the message is being sent, in parallel, to newer nodes in other subgraphs entered for the first time.

References

[1] S.B. Akers and B. Krishnamurthy, A group-theoretic model for symmetric interconnection networks, IEEE Trans. Comput. 38 (1989) 555–566.

[2] S.B. Akers, D. Harel and B. Krishnamurthy, The star graph: an attractive alternative to the n-cube, in: Proceeding 16th International Conference Parallel Processing (1987) 393–400.

[3] F. Annexstein and M. Baumslag, A unified approach to off-line permutation routing on parallel networks, Proceedings 2nd Annual ACM Symposium on Parallel Algorithms and Architectures, Crete, Greece, July (1990).

[4] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, New York, 1976).

[5] G. Carlsson, J. Cruthirds, H. Sexton and C. Wright, Interconnection networks based on a generalization of cube-connected cycles, IEEE Trans. Comput. 34 (1985) 769–772.

[6] M. Dineen, V. Faber and M. Fellows, Algebraic constructions of efficient broadcast networks, DCS-153-IR. University of Victoria, B.C. (1991).

[7] V. Faber, Global communication algorithms for hypercubes and other cayley coset graphs, LA-UR-87-3136, Los Alamos National Laboratory (1987).

[8] V.E. Mendia and D. Sarkar, Optimal broadcasting on the star graph, Tech. Rept., Department of Mathematics and Computer Science, University of Florida, Gainesville, FL (1990).

[9] S.M. Hedetniemi, S.T. Hedetniemi and A.L. Liestman, A survey of gossiping and broadcasting in communications networks, Networks 18 (1989) 319–349.

[10] I.N. Herstein, Topics in Algebra (Blaisdell, New York, 1964).