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A finiteness theorem for primal extensions

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Abstract

A set $W \subseteq V(G)$ is called *homogeneous* in a graph G if $2 \leqslant |W| \leqslant |V(G)| - 1$, and $N(x)\backslash W = N(y)\backslash W$ for each $x, y \in W$. A graph without homogeneous sets is called *prime*. A graph H is called a (*primal*) *extension* of a graph G if G is an induced subgraph of H, and H is a prime graph. An extension H of G is *minimal* if there are no extensions of G in the set $\mathrm{ISub}(H)\backslash\{H\}$. We denote by $\mathrm{Ext}(G)$ the set of all minimal extensions of a graph G.

We investigate the following problem: find conditions under which $\mathrm{Ext}(G)$ is a finite set. The main result of Giakoumakis (Discrete Math. 177 (1997) 83–97) is the following sufficient condition.

Theorem. *If every homogeneous set of G has exactly two vertices then* $\mathrm{Ext}(G)$ *is a finite set.*

We extend this result to a wider class of graphs. A graph is *simple* if it is isomorphic to an induced subgraph of the path P_4.

Theorem. *If every homogeneous set of G induces a simple graph then* $\mathrm{Ext}(G)$ *is a finite set.*

We show that our result is best possible in the following sense. Specifically, we prove that for every non-simple graph F there exist a graph G and a homogeneous set W of G such that W induces a subgraph isomorphic to F and $\mathrm{Ext}(G)$ is infinite.

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1. Introduction

The *neighborhood* of a vertex $x \in V(G)$ is the set $N_G(x) = N(x)$ of all vertices in G that adjacent to x.

Definition 1. Let G and H be graphs. A *substitution of H in G for a vertex* $v \in V(G)$ is the graph $G(v \to H)$ consisting of disjoint union of H and $G - v$ with the additional edge-set $\{xy : x \in V(H), y \in N_G(v)\}$.

Definition 2. For a class $\mathscr{P}$ of graphs, its *substitutional closure* $\mathscr{P}^*$ consists of all graphs that can be obtained from $\mathscr{P}$ by repeated substitutions, i.e., $\mathscr{P}^*$ is generated by the following rules:

(S1): $\mathscr{P} \subseteq \mathscr{P}^*$, and
(S2): if $G, H \in \mathscr{P}^*$ and $v \in V(G)$, then $G(v \to H) \in \mathscr{P}^*$.

Let $\mathrm{ISub}(G)$ be the set of all induced subgraphs of a graph G [considered up to isomorphism]. A class of graphs $\mathscr{P}$ is called *hereditary* if $\mathrm{ISub}(G) \subseteq \mathscr{P}$ for every $G \in \mathscr{P}$. For a set of graphs Z, the class of *Z-free* graphs consists of all graphs G such that $\mathrm{ISub}(G) \cap Z = \emptyset$.

Proposition 1. *If $\mathscr{P}$ is a hereditary class then $\mathscr{P}^*$ is also a hereditary class.*

Problem 1. *For a hereditary class $\mathscr{P}$ given by a set Z of forbidden induced subgraphs, find a forbidden induced subgraph characterization of the substitutional closure $\mathscr{P}^*$.*

De Simone [3] and Bertolazzi et al. [1] noted that Problem 1 is especially interesting in the case where $\mathscr{P}$ is a good class for the vertex packing problem, i.e., the weighted stability number can be found in polynomial time for all graphs in $\mathscr{P}$. Also, it is useful for the domination problem (Zverovich [10]) and for perfect graphs (Zverovich and Zverovich [14]).

We discuss the Reducing Pseudopath Method proposed by Zverovich [9] for solving Problem 1 for an arbitrary hereditary class. Note that implementation of this method is not always straightforward.

Definition 3. A set $W \subseteq V(G)$ is called *homogeneous* in a graph G if

(H1): $2 \leqslant |W| \leqslant |V(G)| - 1$, and
(H2): $N(x) \backslash W = N(y) \backslash W$ for all $x, y \in W$.

According to (H2), a homogeneous set W defines a partition $W \cup W^+ \cup W^- = V(G)$ such that

- every vertex of W is adjacent to every vertex of W^+ [notation $W \sim W^+$], and
- every vertex of W is non-adjacent to every vertex of W^- [notation $W \not\sim W^-$].

By (H1), $W^+ \cup W^- \neq \emptyset$ for every homogeneous set W.

Definition 4. A graph without homogeneous sets is called *prime*. A graph H is called a (*primal*) *extension* of a graph G if

(E1): G is an induced subgraph of H, and
(E2): H is a prime graph.

Definition 5. An extension H of G is *minimal* if there are no extensions of G in the set $\mathrm{ISub}(H)\backslash\{H\}$. We denote by $\mathrm{Ext}(G)$ the set of all minimal extensions of a graph G.

For a set of graphs Z we put

$$\mathrm{Ext}(Z) = \bigcup_{G \in Z} \mathrm{Ext}(G),$$

and we define Z^o as the set of all minimal graphs in $\mathrm{Ext}(Z)$ with respect to the partial order 'to be an induced subgraph'. The following result is straightforward.

Theorem 1. *If Z is the set of all minimal forbidden induced subgraphs for a hereditary class $\mathscr{P}$ then Z^o is the set of all minimal forbidden induced subgraphs for $\mathscr{P}^*$.*

2. Reducing pseudopaths

The notation $x \sim y$ (respectively, $x \nsim y$) means that x and y are adjacent (respectively, non-adjacent). For disjoint sets X and Y, the notation $X \sim Y$ (respectively, $X \nsim Y$) means that every vertex of X is adjacent to (respectively, non-adjacent) to every vertex of Y. In case of $X = \{x\}$ we also write $x \sim Y$ and $x \nsim Y$ instead of $\{x\} \sim Y$ and $\{x\} \nsim Y$, respectively.

Here is the main definition of the Reducing Pseudopath Method.

Definition 6. Let G be an induced subgraph of a graph H, and let W be a homogeneous set of G. We define a *reducing W-pseudopath* [*with respect to G*] in H as a sequence

$$R = (u_1, u_2, \dots, u_t), \quad t \geqslant 1,$$

of pairwise distinct vertices of $V(H)\backslash V(G)$ satisfying the following conditions:

(R1): there exist vertices $w_1, w_2 \in W$ such that
  (R1a): $u_1 \sim w_1$, and
  (R1b): $u_1 \nsim w_2$,
(R2): for each $i = 2, 3, \dots, t$, either
  (R2a): $u_i \sim u_{i-1}$ and $u_i \nsim W \cup \{u_1, u_2, \dots, u_{i-2}\}$, or
  (R2b): $u_i \nsim u_{i-1}$ and $u_i \sim W \cup \{u_1, u_2, \dots, u_{i-2}\}$
  [when $i = 2$, $\{u_1, u_2, \dots, u_{i-2}\} = \emptyset$],
(R3): for every $i = 1, 2, \dots, t-1$, both
  (R3a): $u_i \sim W^+$, and
  (R3b): $u_i \nsim W^-$,

(R4): either
(R4a): $u_t \not\sim x$ for a vertex $x \in W^+$, or
(R4b): $u_t \sim y$ for a vertex $y \in W^-$.

We shall use the following result.

Theorem 2 (*Zverovich [9]*). *Let H be an extension of its induced subgraph G, and let W be a homogeneous set of G. Then there exists a reducing W-pseudopath with respect to any induced copy of G in H.*

Definition 7. We denote by $\mathscr{H}(G, W)$ the set of all graphs that are obtained from a graph G and a homogeneous set W of G by adding a reducing W-pseudopath.

A homogeneous set is called *maximal* if it is not contained in any other homogeneous set. We denote by $\mathrm{Hom}(G)$ the set of all maximal homogeneous sets in a graph G.

Algorithm 1 (*Graph Extension*).

Input: a graph G.
Output: a set $\mathrm{Ext} = \mathrm{Ext}(G)$.
Step 0. Set $S_0 = \{G\}$, $\mathrm{Ext} = \emptyset$, and $i = 0$.
Step i ($\mathbf{i} \geqslant 1$).

- If $S_i = \emptyset$ then delete from Ext all graphs H such that there exists a graph $H' \in \mathrm{ISub}(H)\backslash\{H\}$ in Ext, return Ext and Stop.
- If $S_i \neq \emptyset$ then for every graph $F \in S_i$ proceed as follows:
  - if $\mathrm{Hom}(F) = \emptyset$ then include F into Ext,
  - if $\mathrm{Hom}(F) \neq \emptyset$ then choose a set $W \in \mathrm{Hom}(F)$ and put into S_{i+1} all graphs of $\mathscr{H}(F, W)$,
  - set $i = i + 1$ and go to Step (i + 1).

Theorem 3 (*Zverovich [9]*). *If the set* Ext *is finite, then* Graph Extension Algorithm *constructs it in a finite number of steps.*

3. Some examples

Here we construct extensions for some graphs that are implicitly or explicitly involved into the proof of our main result (Theorem 5). They also illustrate Definition 6, Theorem 2, and Fact 1 (it will be proved later).

First we consider graphs Chair and P shown in Fig. 1.

Corollary 1 (*Zverovich [9]*). (i) $\mathrm{Ext(Chair)} = \mathrm{FIS}(G_1, G_2, \ldots, G_7)$ (Fig. 2).
(ii) $\mathrm{Ext}(P) = \mathrm{FIS}(H_1, H_2, \ldots, H_7)$ (Fig. 3).

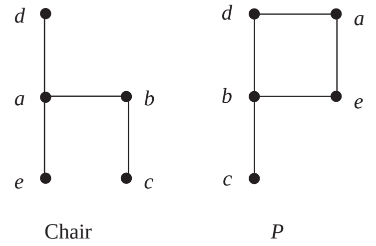

Fig. 1. Chair and P.

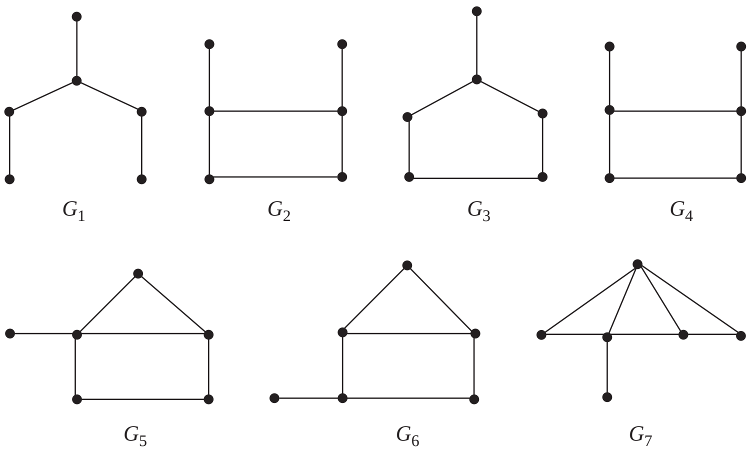

Fig. 2. $\mathrm{Ext(Chair)} = \mathrm{FIS}(G_1, G_2, \ldots, G_7)$.

Proof. (i) Chair has exactly one homogeneous set, namely $W = \{d, e\}$ shown Fig. 1. It will be shown in Fact 1(i) that each extension H of Chair contains a set $Y = \{a, b, c, d, e\}$ inducing Chair and a reducing $\{d, e\}$-pseudopath (u_1) with respect to $H(Y)$.

By (R1) and symmetry, we may assume that u_1 is adjacent to d and u_1 is non-adjacent to e. Since $t = 1$, (R4) implies that either u_1 is non-adjacent to a or u_1 is adjacent to at least one of b, c. As a result, we obtain seven graphs of Fig. 2.

(ii) P has exactly one homogeneous set, namely $W = \{d, e\}$ shown in see Fig. 1. Thus, we may use the same arguments as in (i). □

We denote by $K_1 \cup P_3$ a disjoint union of K_1 and the path P_3.

Corollary 2 (*Zverovich [9]*). $\mathrm{Ext}(K_1 \cup P_3) = \{P_5, \mathrm{Bull}, A\}$ (see Fig. 4).

Proof. We apply Fact 1(i) to the unique maximal homogeneous set of $K_1 \cup P_3$. As a result, we obtain graphs P_5, Bull, Chair and P. Corollary 1 implies that each extension of

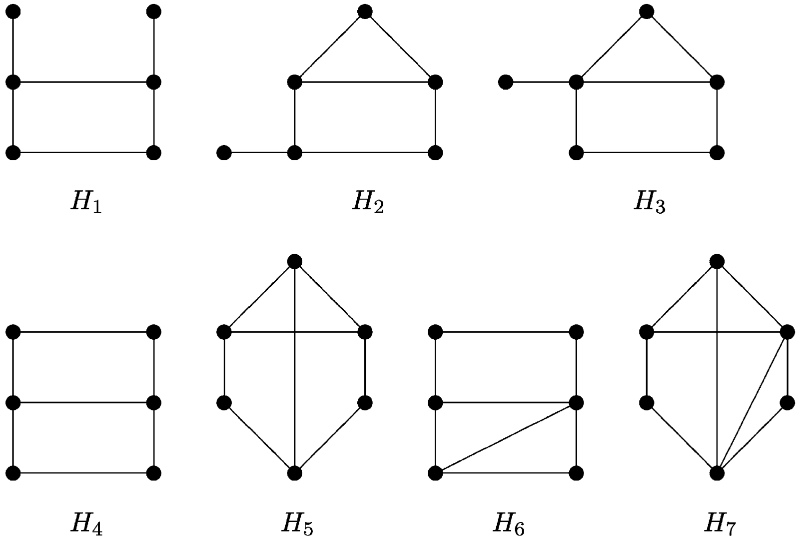

Fig. 3. $\mathrm{Ext}(P) = \mathrm{FIS}(H_1, H_2, \ldots, H_7)$.

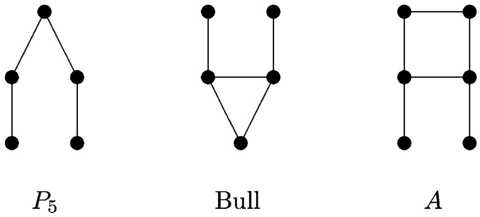

Fig. 4. $\mathrm{Ext}(K_1 \cup P_3) = \mathrm{Ext}(O_3) = \{P_5, \mathrm{Bull}, A\}$.

Chair or P either

- is isomorphic to A, or
- contains P_5 or Bull as an induced subgraph,

see Figs. 2 and 3. □

As usual, O_n is the edgeless graph of order n.

Corollary 3 (*Olariu [8]*). $\mathrm{Ext}(O_3) = \{P_5, \mathrm{Bull}, A\}$ (see Fig. 4).

Proof. Applying Fact 1(i) to any homogeneous set of O_3 produces $K_1 \cup P_3$. Now the result follows from Corollary 2 (Fig. 5). □

Corollary 4 (*Brandstädt et al. [2]*). $\mathrm{Ext}(K_1 \cup P_4) = \{L_1, L_2, \ldots, L_9\}$ (see Fig. 6).

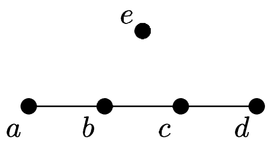

Fig. 5. $K_1 \cup P_4$.

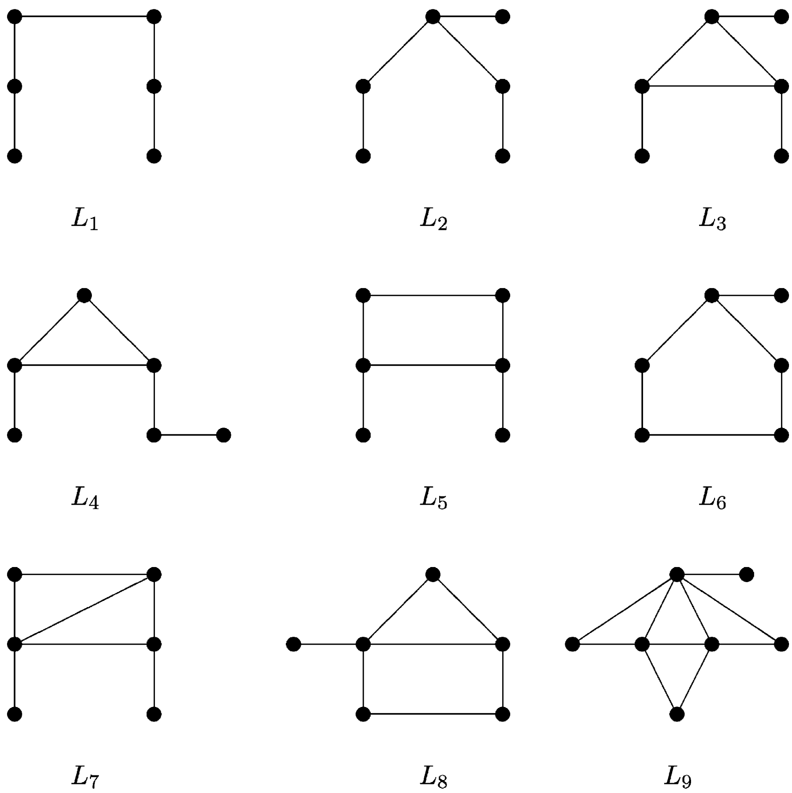

Fig. 6. $\mathrm{Ext}(K_1 \cup P_4) = \{L_1, L_2, \ldots, L_9\}$.

Fig. 7. $O_2 \cup K_2$.

Proof. We apply Fact 1 to the homogeneous set $\{a, b, c, d\}$. The statement (i) of Fact 1 produces graphs $L_1, L_2, \ldots, L_8$. The statement (ii) of Fact 1 produces L_9 [if u_2 satisfies (R2b)] and a redundant graph [if u_2 satisfies (R2a)]. □

Now we consider graph $O_2 \cup K_2$ shown in Fig. 7.

Corollary 5. $\mathrm{Ext}(O_2 \cup K_2) = \{G_1, G_2, \ldots, G_7, L_1, L_3, L_4\}$ (Figs. 2 and 6).

Proof. We apply Fact 1(i) to the homogeneous set $\{c, d\}$. It gives Chair and $K_1 \cup P_4$. Now the result follows from Corollary 1(i) and Corollary 4. □

4. Main results

We investigate the following problem: find conditions under which $\mathrm{Ext}(G)$ is a finite set. In view of Theorem 3 it is a key problem in finding forbidden induced subgraph characterization of the substitutional closure of hereditary classes. For a graph G, let $\mathrm{HomInd}(G) = \{G(W) : W$ is a homogeneous set of $G\}$. We solve the following problem.

Problem 2. *Characterize lists L of graphs such that* $\mathrm{Ext}(G)$ *is finite for each graph G with* $\mathrm{HomInd}(G) = L$.

The main result of Giakoumakis [5] is the following sufficient condition.

Theorem 4 (*Giakoumakis [5]*). *If every homogeneous set of G has exactly two vertices, then* $\mathrm{Ext}(G)$ *is a finite set.*

A graph is *simple* if it is isomorphic to an induced subgraph [not necessarily proper] of the path P_4. We generalize Theorem 4 as follows.

Theorem 5. *If every homogeneous set of G induces a simple graph, then* $\mathrm{Ext}(G)$ *is a finite set.*

Proof. We choose a maximal homogeneous set W of G. We use notation $P_4 = (a, b, c, d)$ to indicate that a and d are end-vertices of the P_4, and b and c are mid-vertices of the P_4. □

Fact 1. *Let W be a homogeneous set in G, and let H be an extension of G. If W induces* $P_2, P_3, \overline{P}_2, \overline{P}_3$, *or* $P_4 = (a, b, c, d)$, *then either*

(i) *there exists a set* $Y \subseteq V(H)$ *that induces G, and H contains a reducing W-pseudopath* (u_1) *with respect to* $H(Y)$, *or*

(ii) $W = \{a, b, c, d\}$ *and there exists a set* $Y \subseteq V(H)$ *that induces G, and H contains a reducing W-pseudopath* (u_1, u_2) *with respect to* $H(Y)$; *moreover,* $N(u_1) \cap W = \{b, c\}$.

Proof. Let $X \subseteq V(H)$ be a set that induces G in H. By Theorem 2, there exists a reducing W-pseudopath $R = (u_1, u_2, \ldots, u_t)$ with respect to $G = H(X)$ in H. We may assume that t has the minimum value taken over all induced copies of G in H and all corresponding reducing pseudopaths.

By (R1), $u_1 \sim w_1$ and $u_1 \not\sim w_2$ for some $w_1, w_2 \in W$.

Case 1: $W = \{a, b, c, d\}$ and $N(u_1) \cap W = \{b, c\}$.

If $t = 2$, we have nothing to prove. Let $t \geqslant 3$. If u_2 satisfies (R2a) then the set $W' = \{a, b, u_1, u_2\}$ induces P_4. If u_2 satisfies (R2b) then the set $W' = \{a, u_2, c, u_1\}$ induces P_4. As it follows from (R3), the set $Y = (X \backslash W) \cup W'$ induces G.

The condition (R2) implies that $R' = (u_3, u_4, \ldots, u_t)$ is a reducing W-pseudopath with respect to $G = H(Y)$ in H. Since R' is shorter that R, we obtain a contradiction to minimality of R.

Case 2: The condition of Case 1 does not take place.

Suppose that $t \geqslant 2$. It is easy to see that there exists a vertex $w \in W$ such that the set $Y = (X \backslash \{w_2\}) \cup \{u_1\}$ induces G. Recall that according to (R3), $u_1 \sim W^+$ and $u_1 \not\sim W^-$, since $t \geqslant 2$.

The condition (R2) implies that $R' = (u_2, u_3, \ldots, u_t)$ is a reducing W-pseudopath with respect to $G = H(Y)$ in H. Since R' is shorter that R, we obtain a contradiction to minimality of R. Thus, $t = 1$. $\square$

We denote by $\mathrm{Comp}(G)$ the number of connected components of a graph G. We put

$$c(G) = \max\{\mathrm{Comp}(G), \mathrm{Comp}(\overline{G})\},$$

where $\overline{G}$ is the complement of G.

Fact 2. *If $c(G) \geqslant 3$ then $\mathrm{Ext}(G)$ is a finite set.*

Proof. Without loss of generality we may assume that G is disconnected. Specifically, G has $c \geqslant 3$ components $G_1, G_2, \ldots, G_c$. If $c \geqslant 4$ then G contains a homogeneous set $V(G_1) \cup V(G_2) \cup V(G_3)$ which induces a non-simple graph, a contradiction. Thus, $c = 3$.

The homogeneous set $V(G_i) \cup V(G_j)$, $1 \leqslant i < j \leqslant 3$, must induce a simple graph. It is clear that $G_i, G_j \in \{K_1, K_2\}$ and that G has at most one component K_2. Recall that K_n denotes the complete graph of order n. Thus, G is either O_3 or $O_2 \cup K_2$.

By Corollary 3, $\mathrm{Ext}(O_3)$ consists of three graphs. By Corollary 5, $\mathrm{Ext}(O_2 \cup K_2)$ consists of ten graphs. $\square$

In view of Fact 2, it remains to consider case where $c(G) \leqslant 2$.

Fact 3. *Let $c(G) \leqslant 2$, and let W be a maximal homogeneous set in G. If H is obtained from G by adding a reducing W-pseudopath $R = (u_1, u_2, \ldots, u_t)$, then $c(H) = 1$.*

Proof. First we suppose that G is a connected graph. By (R1a), $u_1 \sim w_1 \in W$. By (R2), each vertex u_i, $i = 2, 3, \ldots, t$, is adjacent to exactly one of u_{i-1}, w_1. This observation and the connectedness of G imply that H is also a connected graph.

Let now $\overline{G}$ be a connected graph. By (R1b), $u_1 \not\sim w_2 \in W$. By (R2), each vertex u_i, $i = 2, 3, \ldots, t$, is adjacent to exactly one of u_{i-1}, w_2. Hence $\overline{H}$ is also a connected graph. Thus, if $c(G) = 1$ then $c(H) = 1$.

Suppose that G has two connected components G_1 and G_2. Clearly, $W = V(G_i)$ for $i \in \{1, 2\}$, $W^+ = \emptyset$ and $W^- = V(G_j)$, where $G_j \in \{G_1, G_2\} \backslash \{G_i\}$.

According to (R1a), $u_1 \sim w_1 \in W = V(G_i)$. As before, (R2) implies that $R \cup V(G_i)$ is in the same component of H. Since $W^+ = \emptyset$, the condition (R4b) must hold, i.e., $u_t \sim y$ for some $y \in W^- = V(G_j)$. Therefore H is a connected graph.

In a similar way we can prove that if $\overline{H}$ is also a connected graph. □

Below $c(G) \leqslant 2$ and H is a graph obtained from G by adding a reducing W-pseudopath $R = (u_1, u_2, \ldots, u_t)$.

Fact 4. *If X is a maximal homogeneous set of H then either $X \cap R = \emptyset$ or $R \subseteq X$.*

Proof. If $X \cap R = \emptyset$ then the proof is complete. Otherwise we can choose the minimum $i \in \{1, 2, \ldots, t\}$ such that $u_i \in X$.

Case 1: $W \cap X \neq \emptyset$.

Let $w \in W \cap X$. We choose the maximum $j \in \{i, i+1, \ldots, t\}$ such that $u_i, u_{i+1}, \ldots, u_j \in X$. Suppose that $j \leqslant t-1$. Condition (R2) implies that u_{j+1} is adjacent to exactly one of u_j, w. Since $u_j, w \in X$ and X is a homogeneous set, we have $u_{j+1} \in X$, a contradiction to the choice of j. Thus $j = t$ and $u_i, u_{i+1}, \ldots, u_t \in X$.

If $i = 1$ then $R \subseteq X$ and the proof is complete. Let $i \geqslant 2$. By the choice of i, $u_1 \notin X$. The vertex u_t satisfies either (R4a) or (R4b). By symmetry, we may assume that (R4a) holds, i.e., $u_t \not\sim x$ for some $x \in W^+$.

By the definition of W^+, $w \sim x$. Since $u_t, w \in X$ and X is a homogeneous set, $x \in X$. According to (R3a), $u_1 \sim x$. Since $x \in X$, $u_1 \notin X$ and $u_1 \sim x$, we have $u_1 \in X^+$. By (R1b), $u_1 \not\sim w_2$ for some $w_2 \in W$. Since $u_1 \in X^+$ and $u_1 \not\sim w_2$, $w_2 \notin X$. It follows from $w_2 \in W$ and $x \in W^+$ that $w_2 \sim x$. Since $w_2 \notin X$, $x \in X$ and $w_2 \sim x$, we obtain $w_2 \in X^+$.

Now we choose the maximum $k \in \{1, 2, \ldots, i-1\}$ such that $u_1, u_2, \ldots, u_k \in X^+$. Condition (R2) implies that u_{k+1} is adjacent to exactly one of u_k, w_2. Since $w_2, u_k \in X^+$, $u_{k+1} \notin X$. By the choice of k, $u_{k+1} \notin X^+$. Hence $u_{k+1} \in X^-$.

It is clear that $k+1 < i \leqslant t$, i.e., $u_{k+1} \neq u_t$. Condition (R3a) implies that $u_{k+1} \sim W^+$. In particular, $u_{k+1} \sim x$. On the other hand, $x \in X$ and $u_{k+1} \in X^-$, so $u_{k+1} \not\sim x$, a contradiction.

Case 2: $W \cap X = \emptyset$.

Subcase 2(a). $|X \cap R| \geqslant 2$:

Let $u_i, u_j \in X \cap R$, where $1 \leqslant i < j \leqslant t$. We choose the maximum $k \in \{j, j+1, \ldots, t\}$ such that $u_j, u_{j+1}, \ldots, u_k \in X$. We show that $k = t$.

Suppose that $k \leqslant t-1$. Condition (R2) implies that u_{k+1} is adjacent to exactly one of u_k, u_i; recall that $i < j \leqslant k$. Since both u_k and u_i belong to a homogeneous set X, $u_{k+1} \in X$, a contradiction. Thus, $k = t$ and $u_t = u_k \in X$.

As before, we shall assume that (R4a) holds [the case where (R4b) holds is similar]. Then $u_t \not\sim x$ for some $x \in W^+$. Condition (R3a) and $x \in W^+$ imply that $u_i \sim x$. Since $u_i, u_t \in X$, $u_i \sim x$ and $u_t \not\sim x$, we have $x \in X$.

Further, $W \subseteq X^+$. Indeed, $x \sim W$, $W \cap X = \emptyset$ and $x \in X$. According to (R1b), $u_1 \not\sim w_2$ for some $w_2 \in W$. Since $w_2 \in W \subseteq X^+$ and $u_1 \not\sim w_2$, $u_1 \notin X$. In fact, $u_1 \in X^+$, since $u_1 \sim x$ and $x \in X$.

We choose the maximum $l \in \{1, 2, \ldots, i-1\}$ such that $u_1, u_2, \ldots, u_l \in X^+$. It follows from (R2) that u_{l+1} is adjacent to exactly one of u_l, w_2. Since $u_l, w_2 \in X^+$, $u_{l+1} \notin X$. Clearly, $l+1 \leqslant i < j \leqslant t$, i.e., $u_{l+1} \neq u_t$.

By (R3a), $u_{l+1} \sim x \in W^+$. It follows from $u_{l+1} \notin X$, $x \in X$, and $u_{l+1} \sim x$ that $u_{l+1} \in X^+$, a contradiction to the choice of l.

Subcase 2(b). $X \cap R = \{u_i\}$:

By the definition of homogeneous set, $|X| \geqslant 2$. Hence there exists a vertex $w \in X \backslash \{u_i\}$. According to the condition, $W \cap X = \emptyset$. Therefore $w \in W^+ \cup W^-$.

We shall assume that $w \in W^+$. The case where $w \in W^-$ is similar. Since $w \in W^+$, $w \sim W$. It follows from $w \in X$, $X \cap W = \emptyset$ and $w \sim W$ that $W \subseteq X^+$.

According to (R1b), $u_1 \not\sim w_2$ for some vertex $w_2 \in W$. Since $w_2 \in W \subseteq X^+$ and $u_1 \not\sim w_2$, we have $u_1 \notin X$. We show that $u_1 \in X^+$. By (R3a), $u_1 \sim w \in W^+$ [since $1 < i \leqslant t$]. But $w \in X$ and $u_1 \notin X$. Hence $u_1 \in X^+$.

Now we choose the maximum $k \in \{1, 2, \ldots, i-1\}$ such that $u_1, u_2, \ldots, u_k \in X^+$. According to (R2), the vertex u_{k+1} is adjacent to exactly one of u_k, w_2. Since both u_k and w_2 are in X^+, $u_{k+1} \notin X$.

By (R3a), $u_{k+1} \sim w \in W^+$; recall that $k+1 < i \leqslant t$. Since $w \in X$, $u_{k+1} \sim w$ and $u_{k+1} \notin X$, we have $u_{k+1} \notin X^+$. We obtain that $u_{k+1} \in X^+$, a contradiction to the choice of k. $\square$

Fact 5. *If X is a homogeneous set of H and $R \subseteq X$, then $Y = X \backslash R$ is a homogeneous set of G with $Y \cap W \neq \emptyset$ and $Y \cap (W^+ \cup W^-) \neq \emptyset$.*

Proof. Note that the set Y is a homogeneous set in G if and only if $|Y| \geqslant 2$. So it is sufficient to show that $Y \cap W \neq \emptyset$ and $Y \cap (W^+ \cup W^-) \neq \emptyset$.

First, let $t \geqslant 2$. By (R1), $u_1 \sim w_1$ and $u_1 \not\sim w_2$ for some vertices $w_1, w_2 \in W$. It follows from (R2) and $u_1 \neq u_t$ that either $u_t \sim \{w_1, w_2\}$ or $u_t \not\sim \{w_1, w_2\}$.

If $u_t \sim \{w_1, w_2\}$ then $w_2 \in X$. Indeed, $w_2 \notin X^+$ [since $w_2 \not\sim u_1$ and $u_1 \in X$] and $w_2 \notin X^-$ [since $w_2 \sim u_t$ and $u_t \in X$]. Similarly, if $u_t \not\sim \{w_1, w_2\}$ then $w_1 \in X$. Thus, $|X \cap \{w_1, w_2\}| \geqslant 1$ and $|Y \cap W| \geqslant |Y \cap \{w_1, w_2\}| \geqslant 1$.

Further, we prove that $Y \cap (W^+ \cup W^-) \neq \emptyset$. If (R4a) holds then $u_t \not\sim x$ for some vertex $x \in W^+$. By (R3a), $u_1 \sim sx$. Since $u_t \not\sim x$ and $u_t \in X$, $x \notin X^+$. Since $u_1 \sim x$ and $u_1 \in X$, $x \notin X^-$. We have $x \in X$, or $x \in Y \cap W^+$.

Similarly, if (R4b) holds then $|Y \cap W^-| \geqslant 1$.

It remains to consider the case $t = 1$. By the definition of a homogeneous set, $|X| \geqslant 2$. Hence there is a vertex $w \in X \cap V(G)$.

Case 1: $w \in W$ and (R4a) holds.

By (R4a), $u_1 = u_t \not\sim x$ for some vertex $x \in W^+$. But $w \in W$ is adjacent to $x \in W^+$. Since $w, u_1 \in X$, we have $x \in X$. Thus, $|Y \cap W^+| \geqslant 1$ completing the proof,

Case 2: $w \in W$ and (R4b) holds.

By (R4b), $u_1 = u_t \sim y$ for some vertex $y \in W^+$. But $w \in W$ is non-adjacent to $y \in W^-$. Since $w, u_1 \in X$, we have $y \in X$. Thus, $|Y \cap W^-| \geqslant 1$ and the proof is complete.

Case 3: $w \in W^+$.

By (R1b), $u_1 \not\sim w_2$ for some vertex $w_2 \in W$. It follows from $w \in W^+$ and $w_2 \in W$ that $w \sim w_2$. Since $w, u_1 \in X$, $u_1 \not\sim w_2$ and $u \sim w_2$, we have $w_2 \in X$. Thus, $|Y \cap W| \geqslant 1$.

Case 4: $w \in W^-$.

By (R1a), $u_1 \not\sim w_1$ for some vertex $w_1 \in W$. It follows from $w \in W^-$ and $w_1 \in W$ that $w \not\sim w_1$. As before, $w_1 \in X$. Thus, $|Y \cap W| \geqslant 1$. □

Fact 6. *If X is a maximal homogeneous set in H, then* $X \cap R = \emptyset$ *and X is a homogeneous set of G.*

Proof. Suppose that $X \cap R \neq \emptyset$. By Fact 4, $R \subseteq X$. We denote $Y = X \backslash R$. By Fact 5, Y is a homogeneous set in G with $Y \cap W \neq \emptyset$ and $Y \cap (W^+ \cup W^-) \neq \emptyset$. Let Y' be a maximal homogeneous set in F that contains Y. Since $Y \cap W \neq \emptyset$, $Y' \cap W \neq \emptyset$. Since $Y \cap (W^+ \cup W^-) \neq \emptyset$, $Y' \neq W$. We arrive to a contradiction to a result of Gallai [4] (see also [7]) that if $c(G) \geqslant 2$ then the maximal homogeneous sets of G are disjoint. Note that Gallai's theorem is formulated for $c(G) = 1$, but the case $c(G) = 2$ is straightforward. □

According to Fact 6, all homogeneous sets in H are homogeneous sets of G. Hence they induce simple graphs.

In view of Theorem 5, it is not surprisingly that $\mathrm{Ext}(K_1 \cup P_4)$ is a finite set, see Corollary 4. Brandstädt et al. [2] proved that $\mathrm{Ext}(K_{1,3})$ is also a finite set [consisting of 12 graphs]. Note that the unique homogeneous set of $K_{1,3}$ induces O_3 which is not a simple graph. Nevertheless we show that Theorem 5 is best possible in the following sense.

Theorem 6. *For every non-simple graph F, there exist a graph G and a homogeneous set W of G such that W induces a subgraph isomorphic to F and* $\mathrm{Ext}(G)$ *is infinite.*

Proof. We start with some simple observations. □

Fact 7. *At least one of F or* $\overline{F}$ *has a cycle.*

Proof. If both F and $\overline{F}$ are acyclic, then F is a simple graph. Indeed, the class of simple graphs is characterized by C_3, $\overline{C}_3$ and C_5 as minimal forbidden induced subgraphs. □

Without loss of generality we may assume that F contains a cycle D. Let J be a graph in $\mathrm{Ext}(F)$ of minimum order. We construct a graph G as a disjoint union of F and a cycle C of order $|V(J)| + 1$.

Fact 8. *C has at least five vertices.*

Proof. By Fact 7, $|V(F)| \geqslant |V(D)| \geqslant 3$. Hence $|V(J)| \geqslant 4$ and C has $|V(J)| + 1 \geqslant 5$ vertices. □

We denote $W = V(F)$. Clearly, W is a homogeneous set of G and W induces F.

Fact 9. *Every homogeneous set of G that does not contain* $V(C)$ *is a homogeneous set of F.*

Proof. By Fact 8, $V(C)$ has no homogeneous sets. Since C is a component of G, $V(C)$ cannot contain a vertex of a homogeneous set. Finally, $V(G)\backslash V(C) = V(F)$. □

To show that $\mathrm{Ext}(G)$ is infinite, we shall construct a sequence of graphs

$$(H_j : j = 1, 2, \ldots) \tag{1}$$

that will be shown to contain an infinite subsequence of pairwise distinct graphs from $\mathrm{Ext}(G)$. First, we define graphs L_i, $i = 1, 2, \ldots$, as follows:

- take disjoint copies of J, C and a path $P_i = (v_1, v_2, \ldots, v_i)$,
- choose vertices $x \in V(J)$ and $y \in V(C)$, and
- add edges xv_1 and $v_i y$.

By the construction and Fact 7, each L_j is a prime graph that contains G as an induced subgraph. Therefore L_j contains some graph $H_j \in \mathrm{Ext}(G)$ as an induced subgraph. We fix H_j and include it into (1).

Fact 10. *H_i contains C and a cycle D' of order $|V(D)|$ with $V(D') \subseteq V(J)$.*

Proof. The cycle C is the unique longest cycle in both G and H_i. Since G is as an induced subgraph of H_i, $V(C) \subseteq V(H_i)$.

Further, no vertex in $\{v_1, v_2, \ldots, v_i\} \cup V(C)$ belongs to a cycle of order $V(D)$ in L_i. Since G is as an induced subgraph of H_i and D is a cycle of G, H_i must contain a cycle D' of order $|V(D)|$ and $V(D') \subseteq V(J)$. $\square$

Since H_i is a prime graph, it must be connected. Every path in L_i that connects C and D' contains all vertices $v_1, v_2, \ldots, v_i$. We have

$$|V(H_i)| \geqslant |V(D')| + |V(C)| + i \tag{2}$$

and

$$|V(H_i)| \leqslant |V(L_i)| = |V(J)| + |V(C)| + i,$$

or

$$i \geqslant |V(H_i)| - |V(J)| - |V(C)|. \tag{3}$$

Inequalities (2) and (3) imply that

$$|V(H_{i+k})| \geqslant |V(H_i)| + k - (|V(J)| - |V(D)|). \tag{4}$$

As it follows from (4), $|V(H_{i+k})| \geqslant |V(H_i)| + 1$ if $k = |V(J)| - |V(D)| + 1$. Thus, we can define an infinite subsequence of the sequence (1) putting $j = (|V(J)| - |V(D)| + 1)m$ for $m = 1, 2, \ldots$. Since all graphs in this subsequence have different orders, they are pairwise distinct.

Theorems 5 and 6 solve Problem 2 completely.

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References

[1] P. Bertolazzi, C. De Simone, A. Galluccio, A nice class for the vertex packing problem, Discrete Appl. Math. 76 (1–3) (1997) 3–19.
[2] A. Brandstädt, C. Hoàng, I. Zverovich, Extension of claw-free graphs and $(K_1 \cup P_4)$-free graphs with substitutions, RUTCOR Research Report RRR 28-2001, Rutgers University, 2001, 16 pp.
[3] C. De Simone, On the vertex packing problem, Graphs and Combin. 9 (1) (1993) 19–30.
[4] T. Gallai, Transitiv orientierbare Graphen, Acta Math. Acad. Sci. Hungar. 18 (1967) 25–66 (in German).
[5] V. Giakoumakis, On the closure of graphs under substitution, Discrete Math. 177 (1–3) (1997) 83–97.
[6] C.T. Hoàng, B.A. Reed, Some classes of perfectly orderable graphs, J. Graph Theory 13 (4) (1989) 445–463.
[7] F. Maffray, M. Preissmann, A translation of T. Gallai's paper: "Transitiv orientierbare Graphen" in: Perfect Graphs, Wiley, Chichester, 2001, pp. 25–66 (Acta Math. Acad. Sci. Hungar. 18 (1967) 25–66).
[8] S. Olariu, On the closure of triangle-free graphs under substitution, Inform. Process. Lett. 34 (2) (1990) 97–101.
[9] I.E. Zverovich, Extension of hereditary classes with substitutions, Discrete Appl. Math. 128 (2–3) (2003) 487–509.
[10] I.E. Zverovich, A characterization of domination reducible graphs, Graphs Combin. 20 (2004) 281–289.
[14] I.E. Zverovich, V.E. Zverovich, Basic perfect graphs and their extensions, Discrete Math., accepted for publication.

Further reading

[11] I. Zverovich, Yu. Orlovich, Homogeneous sets in graphs and some algorithmic questions, in: A. Dolgui et al. (Eds.), Proceedings of the Ninth International Multi-Conference "Advanced Computer Systems" (ACS'02), Part II, Miedzyzdroje (Poland), October 23–25, 2002, pp. 565–572.
[12] I.E. Zverovich, Yu.L. Orlovich, Implementation of the reducing copath method for specific homogeneous sets, Vestsī Nats. Akad. Navuk Belarusī Ser. Fīz.-Mat. Navuk (2) (2004) 48–55 (in Russian).
[13] I.E. Zverovich, I.I. Zverovich, A characterization of superbipartite graphs, Graph Theory Notes of New York XLVI (2004) 31–35.