Counting triangulations of balanced subdivisions of convex polygons [1]

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Abstract

We compute the number of triangulations of a convex k-gon each of whose sides is subdivided by $r-1$ points. We find explicit formulas and generating functions, and we determine the asymptotic behaviour of these numbers as k and/or r tend to infinity. We connect these results with the question of finding the planar set of n points in general position that has the minimum possible number of triangulations.

Keywords: Triangulations, generating functions, asymptotic analysis.

1 Introduction

Let k and r be two natural numbers, $k \geq 3$, $r \geq 1$. Let $\mathrm{SC}(k,r)$ denote a convex k-gon in the plane each of whose sides is subdivided by $r-1$ points. (Thus, the whole configuration consists of $n := kr$ points.)

A *triangulation* of a planar point set S is a dissection of its convex hull by non-crossing diagonals into triangles. We denote the number of triangulations of $\mathrm{SC}(k,r)$ by $\mathsf{tr}(k,r)$. Triangulations of subdivided convex polygons were studied to some extent by Hurtado and Noy [4] [7] and by Bacher and Mouton [2]. We find enumeration formulas and precise asymptotic results for the numbers $\mathsf{tr}(k,r)$. Some of our results extend those from earlier papers, and answer questions and conjectures stated there and in the OEIS [5].

2 Formulas

The first step is developing an inclusion-exclusion formula for $\mathsf{tr}(k,r)$. [8]

Theorem 2.1 *We have*

$$\mathsf{tr}(k,r) = \sum_{m=0}^{\lfloor r/2 \rfloor k} (-1)^m \, a_{k,r,m} \, C_{kr-m-2}, \tag{1}$$

where C_n is the nth Catalan number, and

$$a_{k,r,m} := [x^m] \left(\sum_{\ell=0}^{\lfloor r/2 \rfloor} \binom{r-\ell}{\ell} x^\ell \right)^k .$$

Proof (Sketch) We construct a bijection between triangulations of $\mathrm{SC}(k,r)$ and a subset of triangulations of the convex (kr)-gon, determined by certain

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[7] Notice the difference in notation: our k is their r, and our r is their $k+1$.

[8] An equivalent formula was found in the earlier work [4] [2].

"forbidden" diagonals. The expression $a_{k,r,m}$ is the number of triangulations of the convex (kr)-gon that use at least m forbidden diagonals. We apply the inclusion-exclusion principle and obtain formula (1). □

Next, we observe that $\sum_{\ell=0}^{\lfloor r/2\rfloor} \binom{r-\ell}{\ell}(-x)^\ell = x^{r/2}\, U_r\left(\frac{1}{2\sqrt{x}}\right)$, where $U_r(x)$ is the rth Chebyshev polynomial of the second kind. We use explicit expressions for these polynomials and for the generating function of Catalan numbers, and apply Cauchy's integral formula. This yields

$$\mathsf{tr}(k,r) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\left(\left(1+\sqrt{1-4x}\right)^{r+1} - \left(1-\sqrt{1-4x}\right)^{r+1}\right)^k \left(1-\sqrt{1-4x}\right)}{2^{(r+1)k+1}x^{rk}(1-4x)^{k/2}}dx, \tag{2}$$

where $\mathcal{C}$ is a small positively oriented circle around the origin.

Next we obtain the following expressions for $\mathsf{tr}(k,r)$.

Proposition 2.2 *We have the following formulas:*

$$\mathsf{tr}(k,r) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{(1-2t)\,dt}{t^{rk-1}(1-t)^{rk}(1-2t)^k}\left((1-t)^{r+1}-t^{r+1}\right)^k \tag{3}$$

$$= -\frac{1}{4\pi i}\int_{\mathcal{C}} \frac{dt}{t^{rk}(1-t)^{rk}(1-2t)^{k-2}}\left((1-t)^{r+1}-t^{r+1}\right)^k \tag{4}$$

$$= \sum_{j=0}^{k}\sum_{l=0}^{rk-(r+1)j-2}(-1)^j\,2^l\binom{k}{j}\binom{k-2+l}{l}\binom{(r-1)k-l-3}{rk-(r+1)j-l-2}. \tag{5}$$

Proof (Sketch) The formula (3) is obtained from (2) by the substitution $x = t(1-t)$. In order to obtain (4) (the advantage of which is in the symmetry in t and $1-t$), we take (3) and blow up the contour $\mathcal{C}$ so that it is sent to infinity. Thus we have to take the residue at $t = 1$ into account. In this way, another expression for $\mathsf{tr}(k,r)$ is obtained. Its arithmetic mean with (3) yields (4).

The formula (5) is obtained from (3) by using Cauchy's integral formula and interpreting it as a coefficient extraction formula. □

3 Generating Functions

Here we present formulas for "horizontal" and "vertical" generating functions for the numbers $\mathsf{tr}(k,r)$. In particular, we show that these generating functions are algebraic.

Theorem 3.1 *For fixed $r \geq 2$, we have*

$$\sum_{k\geq 1} \mathsf{tr}(k,r)x^k = -\frac{1}{2}\sum_{i=1}^{r} \frac{t_i(x)^r(1-t_i(x))^r(1-2t_i(x))^2}{(\frac{d}{dt}P_r)(x;t_i(x))}, \tag{6}$$

where the $t_i(x)$, $i = 1, 2, \ldots, r$, are the "small" zeroes of the polynomial

$$P_r(x;t) = t^r(1-t)^r - x\frac{(1-t)^{r+1} - t^{r+1}}{1-2t},$$

that is, the zeroes $t(x)$ for which $\lim_{x\to 0} t(x) = 0$.

Proof (Sketch) We sum up $\sum_{k\geq 1} \mathsf{tr}(k,r)x^k$, using formula (4) for $\mathsf{tr}(k,r)$ and the summation formula for geometric series. Thus, we obtain

$$\sum_{k\geq 1} \mathsf{tr}(k,r)x^k = -\frac{1}{4\pi i}\int_{\mathcal{C}} \frac{t^r(1-t)^r(1-2t)^2}{P_r(x;t)}\,dt, \tag{7}$$

where, as before, $\mathcal{C}$ is a small positively oriented circle around the origin. By the residue theorem, this integral equals the sum of the residues at poles of the integrand inside $\mathcal{C}$. The poles are the "small" zeroes of the denominator polynomial $P_r(x;t)$. We show that it has r "small" and r "not small" zeroes. We apply the residue theorem to (7) and use the formula for the residue of a quotient of two functions. This yields the claim. □

A similar reasoning yields a formula for "horizontal" generating functions.

4 Asymptotics

We prove the following asymptotic results for the numbers $\mathsf{tr}(k,r)$.

Theorem 4.1 *1. For fixed $k \geq 3$, we have*

$$\mathsf{tr}(k,r) = \frac{2^{(r-1)k}r^{k-3}}{\pi}\left(\int_{-\infty}^{\infty}\frac{du}{u^{k-2}}\sin^k(2u)\right)\Big(1+o(1)\Big), \qquad \textit{as } r\to\infty. \tag{8}$$

2. We have

$$\mathsf{tr}(k,r) = \frac{\big(2^r(r+1)\big)^k}{16\sqrt{\pi}(r(r+5)/6)^{3/2}k^{3/2}}\big(1+o(1)\big), \qquad \textit{as } k\to\infty. \tag{9}$$

Proof (Sketch) Starting from the integral representation (4), we deform the contour $\mathcal{C}$ into a shape that consists of a segment that connects the points

$(1/2,-R)$ and $(1/2,R)$ and the left half-circle whose diameter is this very segment. It is easy to show that the integral over the half-circle tends to 0 as $R\to\infty$, and, thus, upon the substitution $t=\frac{1}{2}+iu$, we obtain

$$\mathsf{tr}(k,r)=-\frac{2^{(r-2)k}}{\pi}\int_{-\infty}^{\infty}\frac{du}{(1+4u^2)^{rk}(iu)^{k-2}}\left(\left(1+2iu\right)^{r+1}-\left(1-2iu\right)^{r+1}\right)^{k}.$$

Then the substitutions $u\to u/r$ and $u\to u\sqrt{kR}$, where $R=r(r+5)/6$, yield (8) and (9), respectively, after applying standard techniques. □

5 The case $k=3$

For $k=3$, we also deal with the non-balanced case, and obtain some compact formulas. Let $\Delta(a,b,c)$ denote a triangle whose sides are subdivided by a, b and c points, and let $\mathsf{tr}(\Delta(a,b,c))$ denote the number of its triangulations.

Theorem 5.1 *We have*

$$\mathsf{tr}(\Delta(a,b,c))=2^s-\sum_{\ell=0}^{a-2}\binom{s}{\ell}-\sum_{\ell=0}^{b-2}\binom{s}{\ell}-\sum_{\ell=0}^{c-2}\binom{s}{\ell},\tag{10}$$

where $s=a+b+c-1$; and

$$\mathsf{tr}(\Delta(a,b,c))=\sum_{i,j,m\geq 0}\binom{a}{i+j}\binom{b}{j+m}\binom{c}{m+i}.\tag{11}$$

Proof (Sketch) In order to prove (10), we notice that each triangulation either contains a diagonal that connects a corner of the basic triangle to an interior point of the opposite side, or it contains a triangle whose vertices are interior points of different sides of the basic triangle. Counting triangulations of the first kind is elementary; counting triangulations of the second kind boils down to determining

$$[x^a y^b z^c]\,\frac{xyz}{(1-x-y)(1-y-z)(1-z-x)},$$

which can be done by manipulations with binomial coefficients. Putting everything together, we obtain (10).

In order to prove (10) [9], we construct a bijection between triangulations of $\Delta(a,b,c)$ and the ways to choose $i+j$ out of a points, $j+m$ out of b points,

[9] This formula, restricted to the balanced case, was conjectured in OEIS/A087809 [5].

$m+i$ out of c points, that subdivide the corresponding sides, over all triples of non-negative integers (i, j, m). $\square$

6 Generalizations of the Double Circle and its triangulations

An *almost convex polygon* $\mathrm{ISC}(k, r)$ is obtained from $\mathrm{SC}(k, r)$ by "infinitesimal" pulling in all the interior points of the strings into the interior of the convex hull of the basic k-gon along circular arcs of sufficiently big radius [10]. Any triangulation of the convex hull of $\mathrm{ISC}(k, r)$ consists of a triangulation of $\mathrm{ISC}(k, r)$ and triangulations of k sets of r points in convex position. Therefore our results imply asymptotic estimates for the numbers of triangulations of $\mathrm{ISC}(k, r)$ and its convex hull. In particular, the exponential growth rate for the number of triangulations of $\mathrm{ISC}(k, r)$ is 8 for any fixed k and $r \to \infty$; and it is $2(r+1)^{1/r} C_{r-1}^{1/r}$ for $k \to \infty$ [11]. For $r = 2$, the convex hull of $\mathrm{ISC}(k, r)$ is called *Double Circle*. It was conjectured by Aichholzer, Hurtado and Noy [1] that the Double Circle of size n has the minimum number of triangulations over all planar sets of n points in general position. Our results support this conjecture showing that it is impossible to improve this example using balanced almost convex polygons of any kind and letting n to tend to ∞ *in whatever way*.

References

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[10] See [4] for details.

[11] The latter result is also stated in [3]; however, the argument given there is non-rigorous since it relies on [4, Theorem 3] which holds for *fixed* k rather than for $k \to \infty$.