Help on SOS

In this issue of *IEEE Control Systems Magazine*, Andy Packard and friends respond to a query on determining the region of attraction using sum-of-squares methods.

Q. In my course on linear control, we learned that asymptotic convergence is global, but in nonlinear control we learned about the "domain of attraction." The instructor mentioned that it can be hard to figure out what the region of attraction (ROA) is, but that there is something called "SOS" that can be used. I know that SOS stands for sum of squares, but other than that I don't know anything about it. Is there anyone at *IEEE Control Systems Magazine* who can explain SOS?

Andy: I'm happy to try to help, with the assistance of my colleagues Ufuk Topcu, Pete Seiler, and Gary Balas. It's important to note that we're users of SOS methods, not experts, but I think we can answer your question or at least point you in the right direction. Your question leads with "what is SOS?" so let's begin there. Once that's out of the way, just a few steps lead to optimizations whose feasible solutions yield certified, quantitative inner estimates of the region of attraction.

In its basic form, SOS applies to polynomials in several real variables. A polynomial is a finite linear combination of monomials. For example, the polynomial

$$q(x_1, x_2) := x_1^2 + 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4 \qquad (1)$$

is a linear combination of five monomials in two variables. Quadratic polynomials, such as x^TQx, where Q is a symmetric matrix, appear frequently in control theory. This form can be generalized to polynomials of higher degree, namely, if $p(x)$ is a polynomial of degree less than or equal to $2d$, then a *Gram matrix representation* is $p(x) = z^T(x)Qz(x)$, where $z(x)$ is a vector of monomials of degree less than or equal to d, and Q is a symmetric matrix. For example, the polynomial $q(x_1, x_2)$ can be represented as $z^T(x)Qz(x)$, where

$$z(x) := \begin{bmatrix} x_1 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix},$$

$$Q := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & -0.5 \\ 0 & 1 & 0 & 0 \\ 0 & -0.5 & 0 & 5 \end{bmatrix}.$$

The Gram matrix Q is not unique due to the dependencies among the monomials in z. In this example, $x_1^2x_2^2$ can be expressed as either $(x_1x_2)(x_1x_2)$ or $(x_1^2)(x_2^2)$. Therefore, if

$$N := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.5 \\ 0 & 0 & 1 & 0 \\ 0 & -0.5 & 0 & 0 \end{bmatrix}$$

then $z^T(x)Nz(x) = 0$ for all x, and thus $Q + \lambda N$ also gives a Gram matrix representation of q for every $\lambda \in \mathbb{R}$.

A polynomial p is an SOS if there exist polynomials $g_1, \ldots, g_N$ such that $p = \sum_{i=1}^N g_i^2$. The set of SOS polynomials in the vector variable x is denoted by $\Sigma[x]$. One trivial, but important, fact is that every SOS polynomial is nonnegative everywhere.

The polynomial $q(x_1, x_2)$ given by (1) is an SOS since it can be expressed as

$$q(x_1, x_2) = x_1^2 + \frac{1}{2}(2x_1^2 - 3x_2^2 + x_1x_2)^2 + \frac{1}{2}(x_2^2 + 3x_1x_2)^2,$$

which is easy to verify by multiplying it out, but it is not so obvious how it was obtained. The question, then, is how to automate the search for such a decomposition. To answer this question, we use the following key result, which relates SOS polynomials to positive-semidefinite matrices. A polynomial p of degree $2d$, that is, a polynomial with monomials up to degree $2d$, is an SOS if and only if there exists $Q \succcurlyeq 0$ such that $p(x) = z^T(x)Qz(x)$ for all x, where $z(x)$ is the vector of all monomials of degree up to d. Here, $Q \succcurlyeq 0$ and $Q \succ 0$ mean that Q is positive semidefinite and positive definite, respectively. This result, which is proved in [1]–[3], follows from the following equivalent statements for a polynomial p of degree $2d$ and the vector z of all monomials of degree less than or equal to d:

1) p is SOS.
2) There exist row vectors $L_1, \ldots, L_N \in \mathbb{R}^{1\times l_z}$ such that $p(x) = \sum_{i=1}^N (L_i z(x))^2$ for all $x \in \mathbb{R}^n$.
3) There exists a matrix $L \in \mathbb{R}^{N\times l_z}$ such that $p(x) = z^T(x)L^TLz(x)$ for all $x \in \mathbb{R}^n$.
4) There exists a positive-semidefinite matrix Q such that $p(x) = z^T(x)Qz(x)$ for all $x \in \mathbb{R}^n$.

We've already seen that the Gram matrix representation for a polynomial p might not be unique. We

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now parameterize all Gram matrix representations of a given polynomial p of degree $2d$. To this end, define a linear operator $\mathcal{L}$ that maps each symmetric matrix Q to the polynomial $z^T(x)Qz(x)$, where $z(x)$ is a vector of monomials of degree up to d. Each Gram matrix representation for p is a solution of $\mathcal{L}(Q) = p$. Let Q_0 be a particular Gram matrix representation, that is, $\mathcal{L}(Q_0) = p$. Let the matrices $N_1, \ldots, N_M \in \mathbb{R}^{\nu \times \nu}$, where ν is the length of $z(x)$, span the null space of $\mathcal{L}$, that is, $\mathcal{L}$ maps each N_i to the zero polynomial and every matrix in the null space of $\mathcal{L}$ is a linear combination of $N_1, \ldots, N_M$. Then, for every value of $\lambda_i \in \mathbb{R}$, $Q = Q_0 + \sum_{i=1}^{M} \lambda_i N_i$, is a solution to $\mathcal{L}(Q) = p$. Consequently, p is an SOS if and only if there exist $\lambda_1, \ldots, \lambda_M$ such that

$$Q_0 + \sum_{i=1}^{M} \lambda_i N_i \succcurlyeq 0, \tag{2}$$

which is a linear matrix inequality (LMI) feasibility problem. A matrix representation of $\mathcal{L}$ can be computed since both the domain and range spaces of $\mathcal{L}$ are finite dimensional. Solving $\mathcal{L}(Q) = p$ for a particular solution and $\mathcal{L}(Q) = 0$ for all homogenous solutions, that is, for $N_1, \ldots, N_M$, reduces to standard matrix operations. Software tools, such as those given in [4]–[6], automate these procedures by determining whether a given polynomial is SOS and, if so, producing a polynomial SOS decomposition.

Thus far, SOS refers to a computationally viable sufficient condition for a polynomial in several real variables to be globally nonnegative by expressing the polynomial as a sum of squares. But how is SOS used in stability analysis? As motivation, recall that global stability of an equilibrium point can be ensured with a Lyapunov function, which is globally nonnegative and radially unbounded, along with a linear transformation of that function, namely, the derivative along the ordinary differential equation flow, which is globally nonpositive. Since SOS decompositions guarantee global nonnegativity, they can be used to verify these conditions.

To discuss ROA questions, we consider the autonomous nonlinear dynamical system

$$\dot{x}(t) = f(x(t)), \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and the locally Lipschitz function $f: \mathbb{R}^n \to \mathbb{R}^n$ determines the system dynamics. Assume that $f(0) = 0$, that is, the origin is an equilibrium point of (3). Let $\phi(\xi, t)$ denote the solution to (3) at time t with the initial condition $\phi(\xi, 0) = \xi$. The ROA for the equilibrium point $x = 0$ of the system (3) is $\{\xi \in \mathbb{R}^n : \lim_{t\to\infty} \phi(\xi, t) = 0\}$. A set $\mathcal{M}$ is called *invariant* under the flow of (3) if $\phi(\xi, t) \in \mathcal{M}$ for all $t \geq 0$ and $\xi \in \mathcal{M}$.

As an example, consider the time-reversed version of the Van der Pol dynamics

$$\dot{x}_1 = -x_2, \tag{4}$$

$$\dot{x}_2 = x_1 + (x_1^2 - 1)x_2. \tag{5}$$

This system has the unique equilibrium $x = 0$, and both eigenvalues of the linearization have negative real parts. Therefore, $x = 0$ is a locally asymptotically stable equilibrium point. However, $x = 0$ is not globally asymptotically stable. The phase plane plot in Figure 1 shows convergent and divergent trajectories of this system. The unstable limit cycle forms the boundary between the convergent and divergent trajectories. The region of attraction for this system consists of all points in the interior of the unstable limit cycle.

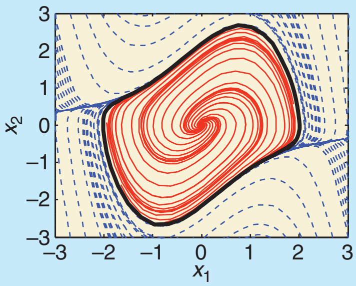

FIGURE 1 Phase plane for the time-reversed Van der Pol dynamics (4), (5). The unstable limit cycle (black, thick curve) forms the boundary between the convergent (red, solid curves) and divergent trajectories (blue, dashed curves). The region of attraction for this system consists of all points in the interior of the unstable limit cycle.

Computing the exact ROA or even an estimate of the ROA is, in general, a difficult task. For systems with two or three states, the ROA can be visualized by simulating the system from many initial conditions and plotting the trajectories in a phase plane plot. However, an analytical approach is desirable for higher-dimensional systems. The following slight modification of a result in [7] characterizes some invariant subsets of the ROA.

LEMMA 1

Let $\gamma > 0$ and assume that there exists a continuously differentiable function $V: \mathbb{R}^n \to \mathbb{R}$ such that

$$\Omega_{V,\gamma} := \{x \in \mathbb{R}^n : V(x) \leq \gamma\} \text{ is bounded}, \tag{6}$$

$$V(0) = 0, V(x) > 0 \text{ for all nonzero } x \in \mathbb{R}^n, \tag{7}$$

$$\Omega_{V,\gamma} \backslash \{0\} \subset \{x \in \mathbb{R}^n : \nabla V(x) f(x) < 0\}. \tag{8}$$

Then, for all $\xi \in \Omega_{V,\gamma}$, the solution $\phi(\xi, \cdot)$ of (3) exists on $[0, \infty)$, satisfies $\phi(\xi, t) \in \Omega_{V,\gamma}$ for all $t \geq 0$, and $\lim_{t\to\infty} \phi(\xi, t) = 0$.

Lemma 1 shows that $\Omega_{V,\gamma}$ is an invariant subset of the ROA for the equilibrium $x = 0$. Given a positive-definite function V, condition (8) must be verified. Note that both sets in (8) are defined in terms of inequalities, and generalizations of the S-procedure [8] (see "Generalized S-Procedure") can be used to verify containment. For example, if $l: \mathbb{R}^n \to \mathbb{R}$ is positive definite, $s: \mathbb{R}^n \to \mathbb{R}$ is positive semidefinite, and

$$-(l(x) + \nabla V(x) f(x)) + s(x)(V(x) - \gamma) \geq 0 \text{ for all } x, \tag{9}$$

then (8) holds. To prove this statement, let x be nonzero and satisfy $V(x) \leq \gamma$. Since $s(x) \geq 0$, it follows from (9) that $\nabla V(x) f(x) \leq -l(x) < 0$. This sufficient condition leads to the following optimization, which can enlarge the value of γ such that $\Omega_{V,\gamma}$ is an invariant subset of the ROA by the choice of positive-semidefinite function s

$$\max_{\gamma, s} \quad \gamma \tag{10}$$

subject to

$$s(x) \geq 0 \quad \text{for all } x, \tag{11}$$

$$-(l(x) + \nabla V(x) f(x)) + s(x)(V(x) - \gamma) \geq 0 \quad \text{for all } x, \tag{12}$$

where V, l are given, and the scalar γ and function s are decision variables. To solve (10)–(12), we must find a positive-semidefinite function s so that a specific affine map, namely, the left-hand side of the inequality in (12), of s is also globally nonnegative. SOS decompositions and SOS programs play a key role in this computation. Toward that end, if the functions V, f, l, and s are polynomial, then the polynomial nonnegativity constraints can be enforced with more restrictive SOS constraints, and the optimization (10)–(12) can be recast as the following optimization problem:

$$\max_{\gamma \in \mathbb{R}, s \in \mathcal{S}} \quad \gamma \tag{13}$$

subject to

$$s(x) \in \Sigma[x], \tag{14}$$

$$-(l(x) + \nabla V(x) f(x)) + s(x)(V(x) - \gamma) \in \Sigma[x], \tag{15}$$

where $\mathcal{S}$ is a given finite-dimensional subspace of polynomials, for example, all quadratic or quartic polynomials. This optimization problem involves two SOS conditions and motivates the definition of the following *SOS Program*.

Given $c \in \mathbb{R}^m$ and polynomials $f_{j,k}$, for $1 \leq j \leq N_s$ and $0 \leq k \leq m$, solve

$$\max_{\alpha \in \mathbb{R}^m} \quad c^T \alpha$$

subject to

$$\begin{aligned} f_{1,0}(x) + \alpha_1 f_{1,1}(x) &+ \cdots \\ + \alpha_m f_{1,m}(x) &\in \Sigma[x], \\ &\vdots \\ f_{N_s,0}(x) + \alpha_1 f_{N_s,1}(x) &+ \cdots \\ + \alpha_m f_{N_s,m}(x) &\in \Sigma[x]. \end{aligned}$$

Each SOS constraint leads to an LMI feasibility constraint. Therefore, an SOS program is transformed to a linear semidefinite program (SDP), where α and the homogeneous terms in the Gram matrices, that is, $\lambda_1, \ldots, \lambda_M$ in (2), constitute the decision variables.

Returning to the constraint (15), if basis functions are chosen to parameterize the search space for s, for example, all quadratic functions, then the optimization is nearly an SOS program. This program has SOS constraints and an objective function that is a linear function of the decision variables. However, the constraint (15) involves the term $-\gamma s(x)$ and hence is bilinear in the decision variables. More specifically, (15) is quasi-convex [9], that is, for each fixed value of γ, (15) is convex in the remaining decision variables. Therefore, the optimization (13)–(15) can be solved using bisection on γ, and a computational strategy for the ROA estimation is given by the following procedure:

1) Let $A = \partial f / \partial x |_{x=0}$ be the linearization of (3). If A is Hurwitz then, for each $Q > 0$, there exists $P > 0$ that satisfies the Lyapunov equation $A^T P + PA = -Q$.
2) $V(x) := x^T P x$ satisfies the condition in (7) and the constraints in (6) and (8), respectively, for all and sufficiently small values of $\gamma > 0$.
3) With this V, maximize γ subject to condition (15).

For the Van der Pol example, the choice

$$Q = I \text{ leads to } P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}.$$

Optimization (13)–(15) is solved with $l(x) = 10^{-6} x^T x$ and s restricted to be quadratic. The largest inner estimate of the ROA in this manner is $\Omega_{V,2.3} = \{x \in \mathbb{R}^n : V(x) \leq 2.3\}$, so that $\gamma = 2.3$. Different choices of Q lead to different inner estimates, as shown in Figure 2. While these estimates are similar, it can be seen that each estimate is better than the remaining estimates along some direction of the state space. This difference motivates the use of a "shape" function h to further optimize the estimate by choice of V as well.

The shape function h is a fixed positive-definite polynomial in x whose

Generalized S-Procedure

In robust control theory, we often encounter problems with constraints of the form

$$g_0(x) \geq 0 \tag{S1}$$

for all x satisfying

$$g_1(x) \geq 0, \ldots, g_m(x) \geq 0, \tag{S2}$$

where $g_0, g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$. Note that (S1) and (S2) can equivalently be written as the set-containment constraint

$$\{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0\} \subseteq \{x \in \mathbb{R}^n : g_0(x) \geq 0\}.$$

A potentially conservative but useful algebraic sufficient condition for (S1)–(S2) is the existence of positive-semidefinite functions $s_1, \ldots, s_m : \mathbb{R}^n \to \mathbb{R}$ such that

$$g_0(x) - \sum_{i=1}^{m} s_i(x) g_i(x) \geq 0 \text{ for all } x \in \mathbb{R}^n. \tag{S3}$$

To verify that (S3) implies the set containment condition in (S1)–(S2), take an arbitrary point x such that $g_1(x) \geq 0, \ldots, g_m(x) \geq 0$. Then, $g_i(x) s_i(x) \geq 0$ for all $i = 1, \ldots, m$. Consequently, $g_0(x) \geq 0$ is satisfied due to (S3), and the constraint in (S1) and (S2) holds. For the case in which $g_0, g_1, \ldots, g_m$ are quadratic functions, the sufficient condition in (S3) is known as the *S-procedure relaxation* for (S1) and (S2), which can equivalently be written as a linear matrix inequality [8].

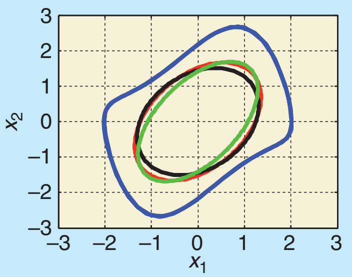

FIGURE 2 Various region of attraction (ROA) estimates for the time-reversed Van der Pol dynamics (4), (5) (red, black, and green curves) and the limit cycle (blue curve). The ROA estimates are computed using quadratic Lyapunov functions $V(x) = x^TPx$, where $P > 0$ satisfies the Lyapunov equation $A^TP + PA = -Q$ with $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (red), $Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ (black), $Q = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$ (green).

sublevel set $\Omega_{h,\beta}$ is contained in the ROA. The goal of optimization is to maximize the value of β for which this containment can be certified using Lyapunov methods. The choice of h is problem dependent, and the sublevel sets of h reflect the goals of the analyst, namely, easy-to-understand quantitative inner estimates of the ROA in high-dimensional problems. The choice of h can reflect dimensional scaling information as well as the importance of certain directions in the state space.

Given h, one method for enlarging the ROA estimate is illustrated in Figure 3, where V is adjusted to enlarge β. The corresponding optimization problem can be written as

$$\max_{\beta>0,\, V\in v} \quad \beta \tag{16}$$

subject to (6)–(8) and

$$\Omega_{h,\beta} \subseteq \Omega_{V,\gamma}. \tag{17}$$

Here, v denotes the set of candidate Lyapunov functions over which the maximum is defined, for example, polynomials in x of a fixed degree. A suboptimal value of β in (16)–(17) can be computed through the SOS optimization

$$\max_{V\in v,\, \gamma,\, \beta,\, s_i\in\mathcal{S}_i} \quad \beta \tag{18}$$

subject to

$$V(0)=0,\, s_i\in\Sigma[x], \beta>0, \tag{19}$$
$$V-l_1\in\Sigma[x], \tag{20}$$
$$-[(\beta-h)s_1+(V-\gamma)]\in\Sigma[x], \tag{21}$$
$$-(l_2+\nabla Vf)+s_2(V-\gamma)\in\Sigma[x]. \tag{22}$$

Here, l_1 and l_2 are fixed, positive-definite polynomials, and the sets $\mathcal{S}_i$ are given finite-dimensional subspaces of polynomials. The constraints (21) and (22) imply the set containments in (17) and (6), respectively, while (20) imposes the positive-definiteness of V and the boundedness of $\Omega_{V,\gamma}$.

Both V and the multipliers s_1 and s_2 are decision variables in (18)–(22), which is a critical difference between the optimization problems in (18)–(22) and (13)–(15), where V is fixed. Consequently, the problem in (18)–(22) is bilinear in the decision variables due mainly to the product term s_2V. Optimization problems with bilinear SOS constraints as in (18)–(22) result in bilinear SDPs, that is, SDPs with bilinear matrix inequality constraints, and are much more theoretically and pragmatically difficult to solve than those with only affine SOS constraints. Bilinear SDPs are nonconvex in general and are usually attacked by using local solvers. For example, PENBMI, a local solver for bilinear SDPs [10], is used to compute invariant subsets of the ROA [11]. Alternatively, observe that if V is fixed in (18)–(22), the problem becomes affine in the multipliers s_1 and s_2 and vice versa. This observation leads to the following coordinate-wise optimization approach: starting from feasible decision variables V, s_1, s_2, β, and γ, the solution can be improved by solving (18)–(22) for the multipliers as an affine SDP, holding V fixed, then solving (18)–(22) for V, holding the multipliers fixed, and repeating these steps until a stopping criterion is satisfied. Initial feasible solutions can be constructed in multiple ways either including the linear analysis as discussed above or by incorporating simulation data to restrict the set of

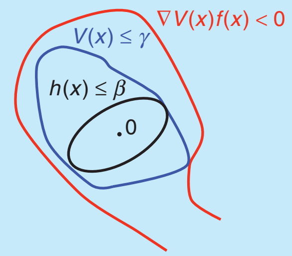

FIGURE 3 Enlarging the region of attraction estimate. Given the shape function h, the optimization problem in (16) and (17) maximizes β such that the set-containment conditions $\Omega_{h,\beta} \subseteq \Omega_{V,\gamma} \subset \{x:\nabla V(x)f(x) < 0\} \cup \{0\}$ hold.

candidate Vs and sampling a convex outer bound on the set of Vs that are feasible for (18)–(22) [21].

To illustrate the V-s iteration for the time-reversed Van der Pol dynamics (4)–(5), we set $h(x) = x^Tx$ and initialize the search over polynomials of degree 6 with a quadratic Lyapunov function obtained from the linearized system with $Q = I$. The resulting ROA estimate and maximal level set of h are shown in Figure 4. For this particular example, the ROA estimate from the iteration covers almost the entirety of the ROA.

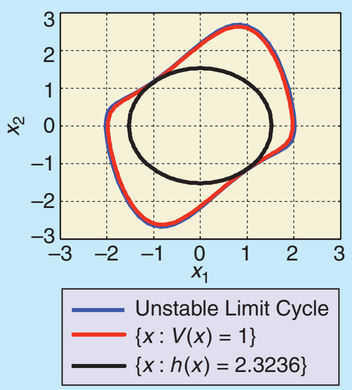

FIGURE 4 A region of attraction (ROA) estimate for the time-reversed Van der Pol dynamics (4)–(5) using V-s iterations. The blue curve is the unstable limit cycle (boundary of the actual ROA), while the red and black curves show the boundaries of $\Omega_{V,1}$ and $\Omega_{h,\beta}$ after 30 iterations, where the shape function h is $h(x) = x^Tx$ and a degree-6 polynomial Lyapunov function is used. The degrees of the multipliers s_1 and s_2 in the optimization problem (18)–(22) are two and four, respectively.

In conclusion, we have described how the ROA can be estimated using SOS methods. SOS techniques can also be used to perform other nonlinear analyses, including computation of input/output gains, estimation of reachable sets, and computation of robustness margins. A similar procedure is used to solve each of these problems: i) formulate the systems question in terms of set-containment conditions, ii) use the generalized S-procedure to convert set-containment conditions to global nonnegativity constraints, iii) relax global nonnegativity constraints to SOS constraints, and iv) solve the resulting bilinear SOS problem using coordinate-wise affine iterations or other heuristics.

We emphasize two caveats that apply to SOS methods. First, the computational requirements grow rapidly in the number of variables and polynomial degree, which roughly limits SOS-based analysis to systems with at most eight to ten states, one to two inputs, and polynomial vector fields of degree 3. Second, numerical issues can arise in solving the SDPs that result from SOS programs. For example, it is possible to inadvertently formulate SOS programs where one SOS constraint forces a decision variable c to satisfy $c \geq 0$ and another constraint forces the same decision variable to satisfy $c \leq 0$. See the appendix of [12] for an example of how such constraints can arise. The implicit constraint $c = 0$ can cause SDP solvers to have difficulty detecting feasibility of the constraints. These implicit constraints can be automatically detected and removed leading to improvements in the numerical reliability of the SDP solvers [6]. Additional research is needed to fully understand the numerical reliability of SOS methods.

Finally, detailed notes, working software, and demonstration examples can be found in [6]. This reference includes software for creating and manipulating polynomials, a solver for SOS programs, and code to solve various nonlinear analysis problems including region of attraction estimation. A good starting point for additional details on ROA estimation using SOS methods is [13] and [14]. There is also a rich theory surrounding SOS methods that we have only briefly mentioned in this note. In particular, there are connections to algebraic geometry and dual interpretations involving statistical moments. A few good starting references for a deeper understanding of the theory are [1]–[3], [15], [16], and [22].

In conclusion, we have described how ROA can be estimated using SOS methods.

AUTHOR INFORMATION

Andrew Packard joined the University of California at Berkeley (UCB) Mechanical Engineering Department in 1990. His technical interests include quantitative nonlinear systems analysis and optimization and data structure issues associated with large-scale collaborative research for predictive modeling of complex physical processes. He is an author of the Robust Control Toolbox distributed by Mathworks. The Meyersound X-10 loudspeaker utilizes feedback control circuitry developed by his UCB research group. He received the campus Distinguished Teaching Award, the 1995 Eckman Award, and the 2005 IEEE Control System Technology Award. He is an IEEE Fellow.

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