

Ramanujan's Life And His Contributions In The Field Of Mathematics

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Abstract: *Srinivasa Ramanujan, one of greatest mathematicians in history, made path-breaking contributions without any formal education, and undeterred by poverty and hardships. Ramanujan's accomplishments are compared with those of certain mathematical luminaries who in their own way faced tremendous difficulties in life, yet made revolutionary contributions.*

Srinivasa Ramanujan's spectacular discoveries revealed surprising connections between apparently unrelated topics, and provided food for thought for mathematicians of subsequent generations. This article describes early life and career and also describes how Ramanujan's results and ideas will continue to influence research in the century following his centenary in areas such as mock theta functions, congruences for partition functions and coefficients of modular forms, hypergeometric identities, special functions, mathematical physics, and computer algebra.

I. INTRODUCTION

Srinivasa Ramanujan (22 December 1887 – 26 April 1920) was an Indian mathematician and autodidact who, with almost no formal training in pure mathematics, made extraordinary contributions to mathematical analysis, number theory, infinite series, and continued fractions. Living in India with no access to the larger mathematical community, which was centred in Europe at the time, Ramanujan developed his own mathematical research in isolation. As a result, he rediscovered known theorems in addition to producing new work. Ramanujan was said to be a natural genius by the English mathematician G. H. Hardy, in the same league as mathematicians such as Euler and Gauss.^[1]

Ramanujan was born at Erode, Madras Presidency (now Tamil Nadu) in a Tamil family.^{[2][3]} His introduction to formal mathematics began at age of ten. He demonstrated a natural ability, and was given books on advanced trigonometry written by S. L. Loney that he mastered by the age of twelve; he even discovered theorems of his own, and re-discovered Euler's identity independently.^[4] He demonstrated unusual mathematical skills at school, winning accolades and awards. By the age of seventeen, Ramanujan had conducted his own

mathematical research on Bernoulli numbers and the Euler–Mascheroni constant.

Rightly regarded as 'natural genius' by the English mathematician G.H. Hardy, Srinivasa Ramanujan displayed an amazing talent in mathematics, even though he did not receive formal training in that subject. He contributed to several areas of mathematics such as the number theory, mathematical analysis, infinite series and continued fractions. This great mathematician of the 20th century added much to the field of advance mathematics with his fascinating theories and proofs, which are in use even today. Also, in 1997, 'The Ramanujan Journal' was published by an American mathematician Bruce .C. Berndt, which showed Ramanujan's areas of study. He formulated many formulas to solve problems, but his untimely death put an end to his great exploration to the unseen beauty and enormity of this subject. Within a short-life, he independently compiled about 3900 results involving identities and equations. Ramanujan used to jot down some of the proofs and theorems in his notebooks that had been studied by many mathematicians, after his death. Scroll further and read more about the profile, life, career and timeline of Srinivasa Ramanujan

Ramanujan received a scholarship to study at Government College in Kumbakonam, which was later rescinded when he failed his non-mathematical coursework. He joined another college to pursue independent mathematical research. During the same time, Diwan Bahadur R. Ramachandra Rao, ICS; who was a keen mathematician and served as President of the Indian Mathematical Society himself; assisted Ramanujan in getting a clerical job in the Accountant-General's office at the Madras Port Trust Office to support himself.^[5] In 1912–1913, he sent samples of his theorems to three academics at the University of Cambridge. G. H. Hardy, recognising the brilliance of his work, invited Ramanujan to visit and work with him at Cambridge. He became a Fellow of the Royal Society and a Fellow of Trinity College, Cambridge. Ramanujan died of tuberculosis caused by illness and malnutrition in 1920 at the age of 32.

During his short life, Ramanujan independently compiled nearly 3900 results (mostly identities and equations).^[6] Nearly all his claims have now been proven correct, although a small number of these results were actually false and some were already known.^[7] He stated results that were both original and

highly unconventional, such as the Ramanujan prime and the Ramanujan theta function, and these have inspired a vast amount of further research.^[8] The *Ramanujan Journal*, an international publication, was launched to publish work in all areas of mathematics influenced by his work.

In December 2011, in recognition of his contribution to mathematics, the Government of India declared that Ramanujan's birth date (22 December) would be celebrated every year as National Mathematics Day and declared 2012 the National Mathematics Year.

II. EARLY LIFE

Srinivasa Ramanujan was born at his grandmother's house in Erode, a small town located about 400 km towards southwest Madras. His father was a clerk in a textile shop in Kumbakonam. Young Ramanujan contracted small pox in 1889 December. However, unlike many other people in that town, Ramanujan overcame the epidemic invasion even though his family his father's income was barely sufficient to meet extra medical expenses. When he was five, he was sent to a primary school in Kumbakonam. Before he entered the Town High school in Kumbakonam in 1898 January, he went to several other private schools. While in school, he excelled in all the subjects and was considered as an all-rounder. Towards 1900, he began to work towards developing his mathematical ability, dealing with geometrics and arithmetic series. His talent was exposed to the world very early in 1902, when he showed how to solve cubic equations and also sought a method to solve quartic. While in Town High School, he read a book 'Synopsis of elementary results in mathematics', which was very concise that he could teach himself without taking help from any tutor. In this book, various theorems were mentioned in the book, along with shortcuts and formulas to solve them. During this time, Ramanujan engaged himself in deep research in 1904 and during this time he investigated the series 'sigma 1/n' and also extended Euler's constant to 15 decimal points. Because of his great work in school studies, he was awarded a scholarship to attend Government College in Kumbakonam, in 1904. Due to his lack of interest in other subjects, he could not utilize this opportunity properly. He kept up his mathematical works and studied in depth about hyper geometric series and the relationship between series and integrals.

III. CAREER

Ramanujan was focused to pass the First Arts examination, which would be his ticket to the University of Madras. Hence, he went to Pachaiyappa's College in Madras in 1906 and put all his efforts in studying and attended all the lectures. Unfortunately, after three months of his dedicated study, he became ill. He appeared for the Fine Arts examination and cleared in mathematics, but failed in all the other subjects. This stopped him from pursuing his dream of getting into the University of Madras. He left college without a degree and pursued independent research in Mathematics. In

1908, he studied fractions and divergent series. His health deteriorated and this time, it became worse and he had to undergo an operation in 1909. It took considerable time for him to recover.

IV. ATTENTION TOWARDS MATHEMATICS

Ramanujan met deputy collector V. Ramaswamy Aiyer, who had recently founded the Indian Mathematical Society. Ramanujan, wishing for a job at the revenue department where Ramaswamy Aiyer worked, showed him his mathematics notebooks. Ramanujan spent more time and effort in developing his mathematical ability and solved problems in the Journal of the 'Indian Mathematical Society', developing relations between elliptic modular equations.

Ramaswamy Aiyer sent Ramanujan, with letters of introduction, to his mathematician friends in Madras. Some of these friends looked at his work and gave him letters of introduction to R. Ramachandra Rao, the district collector for Nellore and the secretary of the Indian Mathematical Society. Ramachandra Rao was impressed by Ramanujan's research but doubted that it was actually his own work. Ramanujan mentioned a correspondence he had with Professor Saldhana, a notable Bombay mathematician, in which Saldhana expressed a lack of understanding of his work but concluded that he was not a phoney. Ramanujan's friend, C. V. Rajagopalachari, persisted with Ramachandra Rao and tried to quell any doubts over Ramanujan's academic integrity. Rao agreed to give him another chance, and he listened as Ramanujan discussed elliptic integrals, hypergeometric series, and his theory of divergent series, which Rao said ultimately "converted" him to a belief in Ramanujan's mathematical brilliance. When Rao asked him what he wanted, Ramanujan replied that he needed some work and financial support. Rao consented and sent him to Madras. He continued his mathematical research with Rao's financial aid taking care of his daily needs. Ramanujan, with the help of Ramaswamy Aiyer, had his work published in the *Journal of the Indian Mathematical Society*.

One of the first problems he posed in the journal was:

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \dots}}}$$

He waited for a solution to be offered in three issues, over six months, but failed to receive any. At the end, Ramanujan supplied the solution to the problem himself. On page 105 of his first notebook, he formulated an equation that could be used to solve the infinitely nested radicals problem.

$$x + n + a = \sqrt{ax + (n + a)^2 + x\sqrt{a(x + n) + (n + a)^2 + (x + n)\sqrt{\dots}}}$$

Using this equation, the answer to the question posed in the *Journal* was simply 3.

V. LIFE IN ENGLAND AND RETURN TO INDIA

Ramanujan sent a copy of his works to some of the greatest mathematicians of this time, but that didn't help him find his way further. After having read 'Orders of infinity' by

G.H. Hardy in 1913, he also wrote to him. Hardy, along with Littlewood, went through his works and to Ramanujan's delight, Hardy replied to him. In May 1913, 'The Board of Studies in Mathematics bestowed Ramanujan with a scholarship of Rs.75 per month for his two-year study in University of Madras. The following year Ramanujan went to Trinity College, Cambridge, with the help of Hardy. This gave way to extra ordinary collaboration. Ramanujan left India on 17 March 1914 and arrived in London on 14 April 1914. Along with hardy, Ramanujan was able to prove some important results. Ramanujan had some health issues in the early winter season in March 1915, which stopped him from publishing anything for five months. Hence, on 16 March 1916, he graduated from Cambridge and acquired a 'Bachelor of Science degree by Research.

Plagued by health problems throughout his life, living in a country far away from home, and obsessively involved with his mathematics, Ramanujan's health worsened in England, perhaps exacerbated by stress and by the scarcity of vegetarian food during the First World War. He was diagnosed with tuberculosis and a severe vitamin deficiency and was confined to a sanatorium.

Ramanujan returned to Kumbakonam, Madras Presidency in 1919 and died soon thereafter at the age of 32. His widow, S. Janaki Ammal, moved to Mumbai, but returned to Chennai (formerly Madras) in 1950, where she lived until her death in 1994.

A 1994 analysis of Ramanujan's medical records and symptoms by Dr. D.A.B. Young concluded that it was much more likely he had hepatic amoebiasis, a parasitic infection of the liver widespread in Madras, where Ramanujan had spent time. He had two episodes of dysentery before he left India. When not properly treated, dysentery can lie dormant for years and lead to hepatic amoebiasis,^[5] a difficult disease to diagnose, but once diagnosed readily cured.^[5]

VI. CONTRIBUTIONS TO MATHEMATICS

In mathematics, there is a distinction between having an insight and having a proof. Ramanujan's talent suggested a plethora of formulae that could then be investigated in depth later. It is said by G. H. Hardy that Ramanujan's discoveries are unusually rich and that there is often more to them than initially meets the eye. As a by-product, new directions of research were opened up. Examples of the most interesting of these formulae include the intriguing infinite series for π , one of which is given below

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}$$

This result is based on the negative fundamental discriminant $d = -4 \times 58 = -232$ with class number $h(d) = 2$ (note that $5 \times 7 \times 13 \times 58 = 26390$ and that $9801 = 99 \times 99$; $396 = 4 \times 99$) and is related to the fact that

$$e^{\pi\sqrt{58}} = 396^4 - 104.000000177\dots$$

Compare to Heegner numbers, which have class number 1 and yield similar formulae. Ramanujan's series for π

converges extraordinarily rapidly (exponentially) and forms the basis of some of the fastest algorithms currently used to calculate π . Truncating the sum to the first term also gives the approximation $9801\sqrt{2}/4412$ for π , which is correct to six decimal places. See also the more general Ramanujan-Sato series.

One of his remarkable capabilities was the rapid solution for problems. He was sharing a room with P. C. Mahalanobis who had a problem, "Imagine that you are on a street with houses marked 1 through n. There is a house in between (x) such that the sum of the house numbers to left of it equals the sum of the house numbers to its right. If n is between 50 and 500, what are n and x?" This is a bivariate problem with multiple solutions. Ramanujan thought about it and gave the answer with a twist: He gave a continued fraction. The unusual part was that it was the solution to the whole class of problems. Mahalanobis was astounded and asked how he did it. "It is simple. The minute I heard the problem, I knew that the answer was a continued fraction. Which continued fraction, I asked myself. Then the answer came to my mind", Ramanujan replied.

His intuition also led him to derive some previously unknown identities, such as

$$\left[1 + 2 \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{\cosh(n\pi)}\right]^{-2} + \left[1 + 2 \sum_{n=1}^{\infty} \frac{\cosh(n\theta)}{\cosh(n\pi)}\right]^{-2} = \frac{2\Gamma^4\left(\frac{3}{4}\right)}{\pi}$$

for all θ , where $\Gamma(z)$ is the gamma function. Expanding into series of powers and equating coefficients of θ^0, θ^4 , and θ^8 gives some deep identities for the hyperbolic secant.

In 1918, Hardy and Ramanujan studied the partition function $P(n)$ extensively and gave a non-convergent asymptotic series that permits exact computation of the number of partitions of an integer. Hans Rademacher, in 1937, was able to refine their formula to find an exact convergent series solution to this problem. Ramanujan and Hardy's work in this area gave rise to a powerful new method for finding asymptotic formulae, called the circle method.

VII. MOCK THETA FUNCTION

In his last letter to Hardy, Ramanujan defined 17 Jacobi theta function-like functions $F(q)$ with $|q| < 1$ which he called "mock theta functions" (Watson 1936ab, Ramanujan 1988, pp. 127-131; Ramanujan 2000, pp. 354-355). These functions are q -series with exponential singularities such that the arguments terminate for some power t^N . In particular, if $f(q)$ is not a Jacobi theta function, then it is a mock theta function if, for each root of unity ρ , there is an approximation of the form

$$f(q) = \sum_{\mu=1}^M t^{k\mu} \exp\left(\sum_{\nu=1}^N c_{\mu\nu} t^\nu\right) + O(1) \tag{1}$$

as $t \rightarrow 0^+$ with $q = \rho e^{-t}$ (Gordon and McIntosh 2000).

If, in addition, for every root of unity ρ there are modular forms $h_j^{(\rho)}(q)$ and real numbers α_j and $1 \leq j \leq J(\rho)$ such that

$$f(q) = \sum_{j=1}^{J(\rho)} q^{\alpha_j} h_j^{(\rho)}(q) \tag{2}$$

is bounded as q radially approaches ρ , then $f(q)$ is said to be a strong mock theta function (Gordon and McIntosh 2000).

Ramanujan found an additional three mock theta functions in his "lost notebook" which were subsequently rediscovered by Watson (1936ab). The first formula on page 15 of Ramanujan's lost notebook relates the functions which Watson calls $\rho(-q)$ and $\omega(-q)$ (equivalent to the third equation on page 63 of Watson's 1936 paper), and the last formula on page 31 of the lost notebook relates what Watson calls $\nu(-q)$ and $\omega(q^2)$ (equivalent to the fourth equation on page 63 of Watson's paper). The orders of these and Ramanujan's original 17 functions were all 3, 5, or 7.

For many years these functions were a mystery, but they are now known to be the holomorphic parts of harmonic weak Maass forms.

VIII. PARTITION FUNCTION CONGRUENCES

The discovery of some partition function congruences by Ramanujan, and subsequent research motivated by these congruences as well as some of his questions and conjectures, have brought forth a beautiful flower in 'Ramanujan's Garden'.

The fraction of odd values of the partition function $P(n)$ is roughly 50%, independent of n , whereas odd values of $Q(n)$ occur with ever decreasing frequency as n becomes large. Kolberg (1959) proved that there are infinitely many even and odd values of $P(n)$.

Leibniz noted that $P(n)$ is prime for $n = 2, 3, 4, 5, 6$, but not 7. In fact, values of n for which $P(n)$ is prime are 2, 3, 4, 5, 6, 13, 36, 77, 132, 157, 168, 186, ... (OEIS A046063), corresponding to 2, 3, 5, 7, 11, 101, 17977, 10619863, ... (OEIS A049575). Numbers which cannot be written as a product of $P(n)$ are 13, 17, 19, 23, 26, 29, 31, 34, 37, 38, 39, ... (OEIS A046064), corresponding to numbers of nonisomorphic Abelian groups which are not possible for any group order.

Ramanujan conjectured a number of amazing and unexpected congruences involving $P(n)$. In particular, he proved

$$P(5m + 4) \equiv 0 \pmod{5} \tag{1}$$

using Ramanujan's identity (Darling 1919; Hardy and Wright 1979; Drost 1997; Hardy 1999, pp. 87-88; Hirschhorn 1999). Ramanujan (1919) also showed that

$$P(25m + 24) \equiv 0 \pmod{5^2}, \tag{2}$$

and Krečmar (1933) proved that

$$P(125m + 99) \equiv 0 \pmod{5^3}. \tag{3}$$

Watson (1938) then proved the general congruence

$$P(n) \equiv 0 \pmod{5^a} \text{ if } 24n \equiv 1 \pmod{5^a} \tag{4}$$

(Gordon and Hughes 1981; Hardy 1999, p. 89). For $a = 1, 2, \dots$, the corresponding minimal values of n are 4, 24, 99, 599,

2474, 14974, 61849, ... (OEISA052463). However, the even more general congruences

$$P(125m + 74, 99, 124) \equiv 0 \pmod{5^3} \tag{5}$$

$$P(3125m + 1849, 2474, 3099) \equiv 0 \pmod{5^5} \tag{6}$$

seem also to hold.

Ramanujan showed that

$$P(7m + 5) \equiv 0 \pmod{7} \tag{7}$$

(Darling 1919), which can be derived using the Euler identity and Jacobi triple product (Hardy 1999, pp. 87-88), and also that

$$P(49m + 47) \equiv 0 \pmod{7^2} \tag{8}$$

(Hardy 1999, p. 90). He conjectured that in general

$$P(n) \equiv 0 \pmod{7^b} \text{ if } 24n \equiv 1 \pmod{7^b} \text{ [incorrect]} \tag{9}$$

(Gordon and Hughes 1981, Hardy 1999), although Gupta (1936) showed that this is false when $b = 3$. Watson (1938) subsequently formulated and proved the modified relation

$$P(n) \equiv 0 \pmod{7^b} \text{ if } 24n \equiv 1 \pmod{7^{2b-2}} \tag{10}$$

for $b \geq 2$. For $b = 1, 2, \dots$, the corresponding minimal values of n are 0, 47, 2301, 112747, ... (OEIS A052464). However, the even more general congruences

$$P(49m + 19, 33, 40, 47) \equiv 0 \pmod{7^2} \tag{11}$$

appear to hold.

Ramanujan showed that

$$P(11m + 6) \equiv 0 \pmod{11} \tag{12}$$

holds (Gordon and Hughes 1981; Hardy 1999, pp. 87-88), and conjectured the general relation

$$P(n) \equiv 0 \pmod{11^c} \text{ if } 24n \equiv 1 \pmod{11^c}. \tag{13}$$

This was finally proved by Atkin (1967). For $c = 1, 2, \dots$, the corresponding minimal values of n are 6, 116, 721, 14031, ... (OEIS A052465).

Atkin and O'Brien (1967) proved

$$P(169n - 7) \equiv \kappa_d P(n) \pmod{13^d} \text{ if } 24n \equiv 1 \pmod{13^d},$$

PARTITION OF WHOLE NUMBERS: Partition of whole numbers is another similar problem that captured Ramanujan attention. Subsequently Ramanujan developed a formula for the partition of any number, which can be made to yield the required result by a series of successive approximation. Example $3=3+0=1+2=1+1+1$;

NUMBERS: Ramanujan studied the highly composite numbers also which are recognized as the opposite of prime numbers. He studies their structure, distribution and special forms.

FERMAT THEOREM: He also did considerable work on the unresolved Fermat theorem, which states that a prime number of the form $4m+1$ is the sum of two squares.

RAMANUJAN NUMBER: 1729 is a famous Ramanujan number. It is the smaller number which can be expressed as the sum of two cubes in two different ways- $1729 = 1^3 + 12^3 = 9^3 + 10^3$

CUBIC EQUATIONS AND QUADRATIC EQUATION: Ramanujan was shown how to solve cubic equations in 1902 and he went on to find his own method to solve the quadratic. The following year, not knowing that the quintic could not be solved by radicals, he tried (and of course failed) to solve the quintic.

EULER'S CONSTANT: By 1904 Ramanujam had began to undertake deep research. He investigated the series $(1/n)$ and calculated *Euler's constant* to 15 decimal places.

HYPO GEOMETRIC SERIES: He worked hypo geometric series, and investigated relations between integrals and series. He was to discover later that he had been studying elliptic functions. Ramanujan's own works on partial sums and products of hyper-geometric series have led to major development in the topic.

Applications of S.Ramanujan Mathematical Method to Computer Science, Physics etc.

According to an eminent mathematician, all the numbers were actually the intimate friends of S. Ramanujan. Ramanujan was so close to the numbers that he made the number 1729 as the 'Ramanujan number', as the other mathematicians call it so in his honour. The main reason behind this is that S. Ramanujan gave its fine characteristics in an anecdote involving G. H. Hardy, who had visited him in a sanatorium by hiring a taxi having this number. In order to calculate the value of pi (π) up to 17 million places using a computer, the present day mathematicians actually use S. Ramanujan's fastest step-by-step method. The mathematical contributions of S. Ramanujan have also been widely used in solving various problems in higher scientific fields of specialisation. The diverse specialised higher scientific fields include the likes of particle physics, statistical mechanics, computer science, space science, cryptology, polymer chemistry and medical science. The strange thing is that some of these fields were not even in existence during his lifetime. Apart from the above fields, S. Ramanujan's mathematical methods are being used in designing better blast furnaces for smelting metals and splicing telephone cables for communications, as well.

IX. CONCLUSION

Thus Ramanujan left behind several notebooks of arcane but unexplained mathematical calculation, which have been studied as a mystery to be unraveled in the decades since his death. His findings made him the namesake of the Ramanujan graph, Ramanujan modular functions, Ramanujan prime numbers, Ramanujan's sum, Ramanujan's tau function, and Ramanujan theta function, and his work has influenced areas of mathematics including automorphic functions, ciophantine analysis, continued fractions, definite integrals, elliptic functions, highly composite numbers, modular forms, molecular physics, multiplicative number theory, probability theory, q-series theorems, statistical mechanics, the theory and partition of numbers, and theta and mock theta functions.

Since 1997 the academic Springer Verlag press has published The Ramanujan Journal, dedicated to developments related to his work. Thus Ramanujan can be remembered as a all time genius.

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