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ACYCLIC SUM-LIST-COLOURING OF GRIDS AND OTHER CLASSES OF GRAPHS

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Abstract. In this paper we consider list colouring of a graph G in which the sizes of lists assigned to different vertices can be different. We colour G from the lists in such a way that each colour class induces an acyclic graph. The aim is to find the smallest possible sum of all the list sizes, such that, according to the rules, G is colourable for any particular assignment of the lists of these sizes. This invariant is called the *D_1-sum-choice-number* of G. In the paper we investigate the D_1-sum-choice-number of graphs with small degrees. Especially, we give the exact value of the D_1-sum-choice-number for each grid $P_n \Box P_m$, when at least one of the numbers n, m is less than five, and for each generalized Petersen graph. Moreover, we present some results that estimate the D_1-sum-choice-number of an arbitrary graph in terms of the decycling number, other graph invariants and special subgraphs.

Keywords: sum-list colouring, acyclic colouring, grids, generalized Petersen graphs.

Mathematics Subject Classification: 05C30, 05C15.

1. MOTIVATION AND PRELIMINARIES

In the realities of the world around us we often meet objects that are in some specific conflict relationships. It could be a computer network or water supply as well as telecommunication, distribution and social networks. Sometimes, network objects benefit from access to resources. Moreover, maintaining the availability of the resource is burdened by a unit cost. In this case, the aim of the study is to determine the smallest possible total cost of the availability of resources throughout all objects so that in any unit of time, by any allocation of resources in accordance with the size of the access, the network works without any conflict. The description of this problem by the graph theory notions first appeared in 2002 [7] in connection of studies on sum-list-colouring of graphs. This concept has generalized two previously well-known concepts of list and sum colourings of graphs [6, 8]. An overview of the recent state of research in this area is given in the Ph.D. Thesis of Lastrina [9].

In the standard investigation of this type (sum-list-colouring) a set of objects (vertices) is not in a conflict when it induces the edgeless graph. We consider the situation in which a set of objects (vertices) is not in a conflict when it induces an acyclic graph (acyclic sum-list-colouring).

To give precise definitions of main graph theory objects used in the paper, and to state the main results we have to recall or introduce some notions and notations.

Throughout this paper we follow the notations and terminology of [3]. Almost the entire paper we consider finite and undirected graphs G with vertex set $V(G)$ and edge set $E(G)$ that are loopless and have no multiple edges. Exceptionally, $M(n,f)$ (see Construction 5.16) is defined as a finite multigraph. Referring to $M(n,f)$ we always allow the existence of loops or multiply edges.

For any graph G and $S \subseteq V(G)$ we write $G[S]$ to denote the subgraph of G induced by S. Using the symbol $G - S$ we mean the graph $G[V(G) \setminus S]$. The set $S \subseteq V(G)$ is *stable* in a graph G if $G[S]$ is an edgeless graph. The *degree of a vertex* v in a graph G, $deg_G(v)$, is the number of edges incident with v in G. By $\Delta(G)$ we denote $\max\{deg_G(v) : \ v \in V(G)\}$. A $(u-v)$-path $(u, v \in V(G))$ is a path in G, represented by the sequence of its vertices without repetitions, whose ends are u and v. The *length* of a path is the number of vertices in the sequence decreased by one. By $c(G)$ we denote the number of connected components of G.

Let G_1, G_2 be graphs. The union $G_1 \cup G_2$ of two disjoint graphs G_1, G_2 is defined as a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. We adopt the convention $kG = \underbrace{G \cup \cdots \cup G}_{k}$.

The symbol $\mathbb{N}$ stands for the set of positive integers. For $n \in \mathbb{N}$ we denote by K_n and P_n a complete graph and a path with n vertices, respectively.

By D_1 we mean the class of all acyclic graphs. We use this notation following [2]. A *list assignment* $\mathbf{L}$ of a graph G is a collection $\{L(v)\}_{v \in V(G)}$ of nonempty subsets of $\mathbb{N}$. The graph G is $(\mathbf{L}, D_1)$-*colourable* if there exists a mapping (colouring) $c : V(G) \to \mathbb{N}$, such that $c(v) \in L(v)$ for each $v \in V(G)$ and for each $i \in \mathbb{N}$ the graph induced in G by vertices coloured i belongs to D_1. Such a mapping c is called $(\mathbf{L}, D_1)$-*colouring* of G. Next let $f : V(G) \to \mathbb{N}$ be a function which assigns list sizes to the vertices of G (in many cases f will be called a size function for G). The graph G is (f, D_1)-*choosable* if for every list assignment $\mathbf{L}$ whose sizes are specified by f ($|L(v)| = f(v)$ for all $v \in V(G)$) the graph G is $(\mathbf{L}, D_1)$-*colourable*.

The D_1-*sum-choice-number* $\chi_{sc}^{D_1}(G)$ of a graph G is the minimum of the sum of sizes in f taken over all f such that G is (f, D_1)-choosable. Thus

$$\chi_{sc}^{D_1}(G) = \min_f \left\{ \sum_{v \in V(G)} f(v) : \ G \text{ is } (f, D_1)\text{-choosable} \right\}.$$

The main results of this paper give the exact values of the D_1-sum-choice-numbers for all grids $P_n \Box P_m$, when one of the numbers n, m is less than five (Corollary 4.2(ii) and Theorems 5.10, 5.21) and estimate $\chi_{sc}^{D_1}(P_n \Box P_m)$ for remaining grids (Corollary 6.1). Also, we present some upper and lower bounds on the D_1-sum-choice-number of an

arbitrary graph in terms of the decycling number, other graph invariants and special subgraphs (Theorems 3.1, 3.3 and Corollary 5.9). As a consequence we conclude the exact values of the D_1-sum-choice-numbers of all graphs G for which $\Delta(G) \leq 3$, including all generalized Petersen graphs (Corollaries 4.1, 4.2).

2. ESTIMATION OF $\chi_{sc}^{D_1}(G)$

In this short section, we present some properties and estimating results connected with $\chi_{sc}^{D_1}(G)$ that were obtained in [4].

Remark 2.1 ([4])**.** If G_1, G_2 are graphs and G_1 is a subgraph of G_2, then

(i) $\chi_{sc}^{D_1}(G_1) + |V(G_2)| - |V(G_1)| \leq \chi_{sc}^{D_1}(G_2)$, and
(ii) if $f : V(G_2) \to \mathbb{N}$ and G_2 is (f, D_1)-choosable, then G_1 is $(f|_{V(G_1)}, D_1)$-choosable.

Theorem 2.2 ([4])**.** *For every graph G it holds*

$$\chi_{sc}^{\mathcal{D}_1}(G) \leq |E(G)| + c(G).$$

Theorem 2.3 ([4])**.** *Let G be a graph and $B_1, \ldots, B_t$ be disjoint subsets of $V(G)$. If for each cycle C of G there exists $i \in \{1, \ldots, t\}$ such that C has at least two vertices in B_i, then*

$$\chi_{sc}^{D_1}(G) \leq \sum_{i=1}^{t} \binom{|B_i|+1}{2} + |V(G)| - \sum_{i=1}^{t} |B_i|.$$

Next bounds on $\chi_{sc}^{D_1}(G)$ depend on some specific degrees of vertices of G.

Definition 2.4. A β-degree of a vertex v in a graph G, denoted by $deg_G^{\beta}(v)$, is the maximum number of cycles of a graph G, each of which contains the vertex v and such that v is the unique common vertex for any two of them.

In [4] the following theorem was shown.

Theorem 2.5 ([4])**.** *Let $v_1, \ldots, v_{|V(G)|}$ be an ordering of vertices of a graph G and let $G_i = G[\{v_1, \ldots, v_i\}]$. If $f : V(G) \to \mathbb{N}$ is a mapping such that $f(v_i) = deg_{G_i}^{\beta}(v_i) + 1$, then G is (f, D_1)-choosable and consequently*

$$\chi_{sc}^{D_1}(G) \leq \sum_{i=1}^{|V(G)|} deg_{G_i}^{\beta}(v_i) + |V(G)|.$$

Note also the lower bound on the D_1-sum-choice number of a graph.

Theorem 2.6 ([4])**.** *If G is a graph and $v \in V(G)$, then*

$$deg_G^{\beta}(v) + |V(G)| \leq \chi_{sc}^{D_1}(G).$$

3. EXPRESSION OF $\chi_{sc}^{D_1}(G)$ BY THE DECYCLING NUMBER OF G

A set S of vertices of a graph G for which $G-S$ contains no cycles is a *decycling set of* G. The minimum cardinality of a decycling set of G is called the *decycling number of* G (or the *feedback* of G) and it is denoted by $\nabla(G)$.

Theorem 3.1. *If S is a decycling set of a graph G of the cardinality $\nabla(G)$, then*

$$\chi_{sc}^{D_1}(G) \leq \sum_{v\in S}\left\lfloor \frac{deg_G(v)}{2}\right\rfloor + |V(G)| \leq \frac{1}{2}\nabla(G)\Delta(G) + |V(G)|.$$

Proof. Let $v_1,\ldots,v_{|V(G)|}$ be an ordering of vertices of G, that starts with the vertices of the set $V(G)\setminus S$ and next labels other vertices. Let $G_i = G[\{v_1,\ldots,v_i\}]$. Observe that $deg^{\beta}_{G_i}(v_i) = 0$ for $i \in \{1,\ldots,|V(G)\setminus S|\}$. Moreover, for $i \in \{|V(G)\setminus S|+1, \ldots,|V(G)|\}$

$$deg^{\beta}_{G_i}(v_i) \leq \left\lfloor \frac{deg_{G_i}(v_i)}{2}\right\rfloor \leq \left\lfloor \frac{deg_G(v)}{2}\right\rfloor \leq \frac{1}{2}\Delta(G).$$

It implies the assertion by Theorem 2.5. □

Lemma 3.2. *Let G be a graph and $f : V(G)\to\mathbb{N}$. If G is (f,D_1)-choosable, then $\{v\in V(G) : f(v)\geq 2\}$ is a decycling set of G.*

Proof. Let $S=\{v\in V(G) : f(v)\geq 2\}$. For a contradiction, suppose that S is not a decycling set, which means $G-S$ is not an acyclic graph. So, there is at least one cycle in $G-S$ whose vertices all have list sizes equal to one. Suppose that V_1 is a vertex set of one among such cycles and a is a fixed element in $\mathbb{N}$. Consider a list assignment $\mathbf{L}=\{L(v)\}_{v\in V(G)}$ such that $L(v)=\{a\}$ for $v\in V_1$ and $L(v)=\{1,\ldots,f(v)\}$ for other vertices. Observe that G is not $(\mathbf{L},D_1)$-colourable. Hence G is not (f,D_1)-choosable, a contradiction. □

Using just shown Lemma 3.2 we obtain the next fact.

Theorem 3.3. *For every graph G it holds*

$$\nabla(G) + |V(G)| \leq \chi_{sc}^{D_1}(G).$$

Proof. Let $f : V(G)\to\mathbb{N}$ be any mapping such that G is (f,D_1)-choosable and $\sum_{v\in V(G)} f(v) = \chi_{sc}^{D_1}(G)$. Next let $S=\{v\in V(G) : f(v)\geq 2\}$. By Lemma 3.2, we obtain $|S|\geq\nabla(G)$. Since

$$\sum_{v\in V(G)} f(v) = \sum_{v\in S} f(v) + \sum_{v\in V(G)\setminus S} f(v)$$

and $f(v)\geq 1$ for each $v\in V(G)$, we have

$$\sum_{v\in V(G)} f(v) \geq \nabla(G) + |V(G)|.$$

□

From the definition of the D_1-sum-choice-number of a graph we have $\chi_{sc}^{D_1}(G_1 \cup G_2) = \chi_{sc}^{D_1}(G_1) + \chi_{sc}^{D_1}(G_2)$. Using the general lower bound on $\nabla(G)$ ([1]) of any connected graph G with at least three vertices we can note one more observation.

Corollary 3.4. *If every connected component of an n-vertex graph G has at last three vertices, then*

$$\frac{|E(G)| + |V(G)|(\Delta(G) - 2) + c(G)}{\Delta(G) - 1} \leq \chi_{sc}^{D_1}(G).$$

Proof. From Theorem 3.3, $\chi_{sc}^{D_1}(G) \geq \nabla(G) + |V(G)|$. By [1] we know that

$$\nabla(H) \geq \frac{|E(H)| - |V(H)| + 1}{\Delta(H) - 1}$$

for any connected graph H of order at least three. Hence, assuming that $G_1, \ldots, G_s$ are connected components of G, with $s = c(G)$, and taking into account that $\Delta(G) \geq \Delta(G_i)$ for $i \in \{1, \ldots, s\}$, we have

$$\begin{aligned}\chi_{sc}^{D_1}(G) &\geq \nabla(G) + n \\ &\geq \frac{|E(G_1)| - |V(G_1)| + 1}{\Delta(G_1) - 1} + \ldots + \frac{|E(G_s)| - |V(G_s)| + 1}{\Delta(G_s) - 1} + |V(G)| \\ &\geq \frac{|E(G)| + |V(G)|(\Delta(G) - 2) + c(G)}{\Delta(G) - 1}.\end{aligned}$$

□

4. GRAPHS WITH SMALL DEGREES

A graph G is called *subcubic* if $\Delta(G) \leq 3$.

Corollary 4.1. *If G is a subcubic graph, then $\chi_{sc}^{D_1}(G) = \nabla(G) + |V(G)|$.*

Proof. By Theorem 3.1, we know that

$$\chi_{sc}^{D_1}(G) \leq \sum_{v \in S} \left\lfloor \frac{deg_G(v)}{2} \right\rfloor + |V(G)|,$$

when S is a decycling set of the cardinality $\nabla(G)$. The required upper bound on $\chi_{sc}^{D_1}(G)$ follows from this inequality and the fact that for each $v \in V(G)$ we have $deg_G(v) \leq 3$. The corresponding lower bound is implied by Theorem 3.3. □

A *generalized Petersen graph* $P_{n,k}$ is a 3-regular graph on $2n$ vertices with

$$V(P_{n,k}) = \{a_i, b_i : 0 \leq i \leq n - 1\}$$

and

$$E(P_{n,k}) = \{a_i b_i, a_i a_{i+1(\bmod n)} : 0 \leq i \leq n - 1\} \cup \{b_i b_{i+k(\bmod n)} : 0 \leq i \leq n - 1\}.$$

Observe that $P_{5,2}$ is the well-known Petersen graph, which in what follows will be denoted by P.

Let G_1 and G_2 be graphs such that $V(G_1) = \{x_1, \ldots, x_n\}$ and $V(G_2) = \{y_1, \ldots, y_m\}$. The *Cartesian product of* G_1 *and* G_2 is a graph $G_1 \Box G_2$, whose vertex set is $V(G_1) \times V(G_2) = \{v_{i,j} : 1 \leq i \leq n,\ 1 \leq j \leq m\}$ and vertices $v_{i,j}$ and $v_{l,k}$ are adjacent in $G_1 \Box G_2$ if either $x_i = x_l$ and $y_j y_k \in E(G_2)$ or $y_j = y_k$ and $x_i x_l \in E(G_1)$.

By Q_3 we mean the 3-*cube*, which is the graph $(P_2 \Box P_2) \Box P_2$. It is very easy to see that $\nabla(Q_3) = 3$.

The next result is an immediate consequence of the equality $\nabla(P_n \Box P_2) = \lfloor \frac{n}{2} \rfloor$ that was shown in [1], the equality

$$\nabla(P_{n,k}) = \begin{cases} \lceil \frac{n+1}{2} \rceil, & \text{if } n \neq 2k \\ \lceil \frac{k+1}{2} \rceil, & \text{if } n = 2k \end{cases},$$

that was shown in [5] and Corollary 4.1.

Corollary 4.2. *If* $n \in \mathbb{N}$, *then*

(i) $\chi_{sc}^{D_1}(Q_3) = 11$, *and*
(ii) $\chi_{sc}^{D_1}(P_n \Box P_2) = 2n + \lfloor \frac{n}{2} \rfloor$, *and*
(iii) $\chi_{sc}^{D_1}(P_{n,k}) = \begin{cases} \lceil \frac{n+1}{2} \rceil + 2n, & \text{if } n \neq 2k \\ \lceil \frac{k+1}{2} \rceil + 2n, & \text{if } n = 2k \end{cases}$. *Especially* $\chi_{sc}^{D_1}(P) = 13$.

Note that the Petersen graph achieves the lower bound on $\chi_{sc}^{D_1}(P)$, given in Corollary 3.4. It is so because

$$\chi_{sc}^{D_1}(P) = \frac{|E(P)| + |V(P)|(\Delta(P) - 2) + 1}{\Delta(P) - 1} = \frac{15 + 10(3-2) + 1}{2} = 13.$$

Corollary 4.2 concerns subcubic graphs. Theorems 3.1, 3.3 imply a simple but general statement on graphs G with $\Delta(G) \leq 4$.

Corollary 4.3. *If* G *is a graph such that* $\Delta(G) \leq 4$, *then*

$$\nabla(G) + |V(G)| \leq \chi_{sc}^{D_1}(G) \leq 2\nabla(G) + |V(G)|.$$

5. GRIDS

Note that Corollary 4.3 estimates $\chi_{sc}^{D_1}(P_n \Box P_m)$ with the usage of the decycling number $\nabla(P_n \Box P_m)$. Unfortunately, this parameter is unknown for almost all pairs n, m. On the other hand, applying Theorem 3.3 and knowing the inequality

$$\nabla(P_n \Box P_m) \geq \left\lceil \frac{mn - m - n + 2}{3} \right\rceil,$$

that for $n, m \geq 2$ was proven in [10], we have the following conclusion.

Corollary 5.1. *If $n, m \in \mathbb{N}$ and $n, m \geq 2$, then*

$$mn + \left\lceil \frac{mn - m - n + 2}{3} \right\rceil \leq \chi_{sc}^{D_1}(P_n \Box P_m).$$

Since each path is an acyclic graph, we have $\chi_{sc}^{D_1}(P_n \Box P_1) = n$. Next $\chi_{sc}^{D_1}(P_n \Box P_2)$ is known by Corollary 4.2(ii). Thus the assumption $n, m \geq 3$ is natural, and it does not limit of the generality of considerations. Now we apply Theorem 2.5 to obtain the upper bound on $\chi_{sc}^{D_1}(P_n \Box P_m)$.

Lemma 5.2. *If $n, m \in \mathbb{N}$, then*

$$\chi_{sc}^{D_1}(P_n \Box P_m) \leq \frac{3}{2}mn - \frac{m+n}{2} + 1.$$

Moreover, if at least one of the numbers m, n is odd, then

$$\chi_{sc}^{D_1}(P_n \Box P_m) \leq \frac{3}{2}mn - \frac{m+n}{2} + \frac{1}{2}.$$

Proof. The discussion before the lemma confirms the assertion in the case when $n \leq 2$ or $m \leq 2$. Suppose that $n, m \geq 3$. Without loss of generality we can assume that either n is odd or both n and m are even. Let $V(P_n \Box P_m) = \{v_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$. Now we put $S' = \{v_{2l,j} : 1 \leq l \leq \lfloor \frac{n-1}{2} \rfloor, 2 \leq j \leq m\}$ and $S'' = \{v_{n,2l} : 1 \leq l \leq \lfloor \frac{m}{2} \rfloor\}$. Let $S = S'$ if n is odd and $S = S' \cup S''$ otherwise (see Figures 1, 2). Observe that S is a decycling set of $P_n \Box P_m$.

Let $x_1, \ldots, x_{nm}$ be the new ordering of vertices of $P_n \Box P_m$ starting with the vertices in $V(P_n \Box P_m) \setminus S$, next labelling the vertices in the set S': first the vertices in the set $\{v_{2l,2} : 1 \leq l \leq \lfloor \frac{n-1}{2} \rfloor\}$, next the vertices in $\{v_{2l,3} : 1 \leq l \leq \lfloor \frac{n-1}{2} \rfloor\}$ and so on, until $\{v_{2l,m} : 1 \leq l \leq \lfloor \frac{n-1}{2} \rfloor\}$. If n is odd, then, at this moment, all the vertices have labels, otherwise we label, in the next step, the vertices in the set S''. Let f be a size function for $P_n \Box P_m$ such that $f(x_i) = deg_{G_i}^{\beta}(x_i) + 1$, where $G_i = P_n \Box P_m[\{x_1, \ldots, x_i\}]$. By Theorem 2.5 we know that $P_n \Box P_m$ is (f, D_1)-choosable. Observe that $f(v_i) = 2$ for vertices in the decycling set S and $f(v_i) = 1$ for other vertices. Hence for n odd we obtain

$$\sum_{i=1}^{nm} f(x_i) = mn + \frac{n-1}{2}(m-1) = \frac{3}{2}mn - \frac{m+n}{2} + \frac{1}{2}.$$

For n and m even we obtain

$$\sum_{i=1}^{nm} f(x_i) = mn + \frac{n-2}{2}(m-1) + \frac{m}{2} = \frac{3}{2}mn - \frac{m+n}{2} + 1. \qquad \square$$

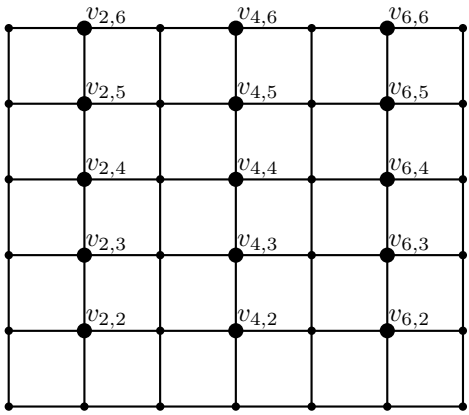

Fig. 1. The decycling set $S = S'$ of $P_7 \Box P_6$ used in the proof of Lemma 5.2

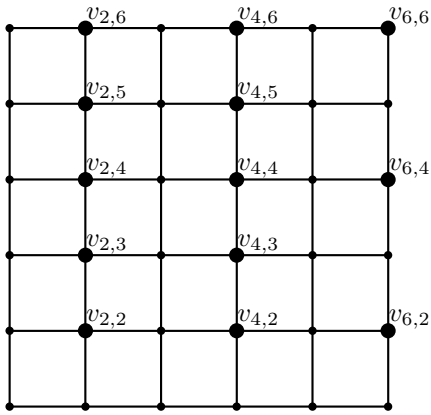

Fig. 2. The decycling set $S = S' \cup S''$ of $P_6 \Box P_6$ used in the proof of Lemma 5.2

In the case $n, m \geq 3$, Corollary 5.1 and Lemma 5.2 give a relatively narrow interval of possible values of $\chi_{sc}^{D_1}(P_n \Box P_m)$, but still disappointing for large n, m. This fact provokes the formulation of the following problem.

Problem 5.3. *What are the exact values of* $\chi_{sc}^{D_1}(P_n \Box P_m)$ *for all* $n, m \in \mathbb{N}$*?*

In the next part of this section we solve this problem when one of the numbers n, m is less than five.

As usually, by an *identification* of two nonadjacent vertices v_1 with v_2 in a graph G (into a vertex w) we mean the result of the following operations on G: the removal of vertices v_1, v_2, the addition of a new vertex w and the addition of the edges vw for all $v \in N_G(v_1) \cup N_G(v_2)$.

Definition 5.4. Let $C(1)$ denote a class of all cycles and let for each $s \geq 2$ the symbol $C(s)$ stand for a class of all graphs that are obtained from two disjoint graphs G_1, G_2, such that $G_1 \in C(s-1)$ and $G_2 \in C(1)$, by the identification of an arbitrary vertex of G_1 with an arbitrary vertex of G_2 in a graph $G_1 \cup G_2$ (see Figure 3).

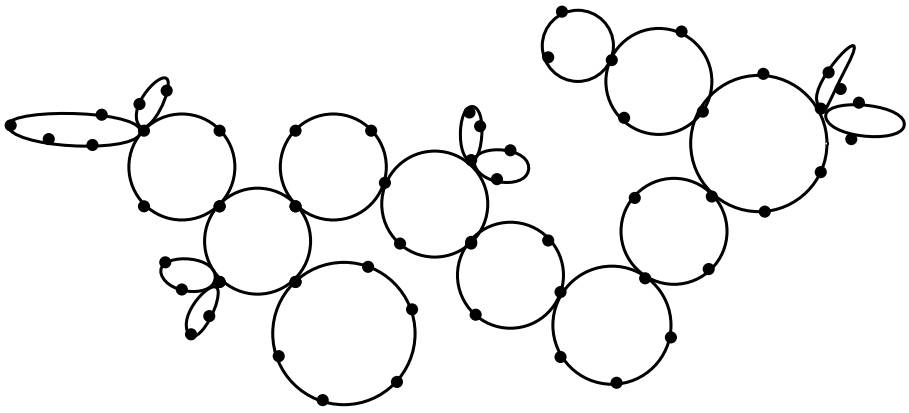

Fig. 3. An example of a graph in $C(19)$

Observe that given a graph G in $C(s)$ and the procedure its recursive construction we know s disjoint cycles, say $C^1, \ldots, C^s$ such that G is a result of $(s-1)$-times repeated application of Definition 5.4 in the following way. First we take C^1 and, for $j \in \{2, \ldots, s\}$, in the j^{th} step we identify one of the vertices of a graph $G_1 \in C(j-1)$ constructed from $C^1, \ldots, C^{j-1}$ with a vertex of C^j in a graph $G_1 \cup C^j$. To emphasize this knowledge, sometimes we write $G = G(C^1, \ldots, C^s)$. On the other hand if $G = G(C^1, \ldots, C^s) \in C(s)$ and $s \geq 2$, then there is at least one reordering $(i_1, \ldots, i_s)$ of the ordering $(1, \ldots, s)$ such that G can be represented as $G(C^{i_1}, \ldots, C^{i_s})$.

If $G = G(C^1, \ldots, C^s)$, then, for simplicity, we say that $v \in V(G)$ is a vertex of the cycle C^i when actually v is a vertex of C^i or v is a result of the identification of some vertex of C^i with another vertex.

A *cut-vertex* of a graph is a vertex whose removal increases the number of connected components.

Lemma 5.5. *If $s \in \mathbb{N}$ and $G = G(C^1, \ldots, C^s)$ is a graph in $C(s)$, then the only cycles of G are $C^1, \ldots, C^s$ and*

$$\sum_{v \in V(G)} deg^{\beta}_G(v) = |V(G)| + s - 1.$$

Proof. Clearly, each vertex of G that is a result of one among identifications implied by the recursive usage of Definition 5.4 is a cut-vertex. Hence, the only cycles of G are $C^1, \ldots, C^s$.

Now we focus on the equality of the statement. Trivially it holds for $s = 1$ and suppose that it also holds for parameters less than s, $s \geq 2$. Using Definition 5.4 for $G \in C(s)$ we know graphs $G_1 \in C(s-1)$ and $G_2 \in C(1)$ such that G is obtained by the identification of one vertex, say v_1, of G_1 with a vertex of G_2, say v_2, into the vertex w.

Thus

$$deg^{\beta}_G(v) = \begin{cases} deg^{\beta}_{G_1}(v), & v \in V(G_1) \setminus \{v_1\}, \\ deg^{\beta}_{G_1}(v_1) + 1, & v = w, \\ 1, & v \in V(G_2) \setminus \{v_2\}. \end{cases}$$

Hence

$$\begin{aligned} \sum_{v \in V(G)} deg^{\beta}_G(v) &= deg^{\beta}_{G_1}(v_1) + 1 + \sum_{v \in V(G_1) \setminus \{v_1\}} deg^{\beta}_{G_1}(v) + \sum_{v \in V(G_2) \setminus \{v_2\}} 1 \\ &= \sum_{v \in V(G_1)} deg^{\beta}_{G_1}(v) + |V(G_2)|. \end{aligned}$$

By induction hypothesis applied to G_1 and since $|V(G)| = |V(G_1)| + |V(G_2)| - 1$, we have

$$\sum_{v \in V(G)} deg^{\beta}_G(v) = |V(G_1)| + s - 2 + |V(G_2)| = |V(G)| + s - 1. \qquad \square$$

Lemma 5.6. *If $s \in \mathbb{N}$ and G is a graph in $C(s)$, then $|V(G)| + s \leq \chi^{D_1}_{sc}(G)$.*

Proof. We proceed by induction on s. Trivially, the assertion holds for $s = 1$. Suppose that the statement is true for all parameters less than s with $s \geq 2$, and there is a graph $G \in C(s)$ such that $\chi^{D_1}_{sc}(G) \leq |V(G)| + s - 1$.

Let f be a size function for G that realizes $\chi^{D_1}_{sc}(G)$, so G is (f, D_1)-choosable and

$$\chi^{D_1}_{sc}(G) = \sum_{v \in V(G)} f(v) \leq |V(G)| + s - 1.$$

Definition 5.4 implies that there exist two disjoint graphs $G_1 \in C(s-1)$ and $G_2 \in C(1)$ with vertices $v_1 \in V(G_1)$, $v_2 \in V(G_2)$, respectively, such that G can be obtained from G_1, G_2 by the identification of v_1 with v_2 in $G_1 \cup G_2$. Thus $|V(G)| = |V(G_1)| + |V(G_2)| - 1$ and next

$$\chi^{D_1}_{sc}(G) = \sum_{v \in V(G)} f(v) \leq |V(G_1)| + |V(G_2)| - 1 + s - 1.$$

Since $\chi_{sc}^{D_1}(G_1) \geq |V(G_1)| + s - 1$, by the induction hypothesis and because G_1 is $(f|_{V(G_1)}, D_1)$-choosable by Remark 2.1(ii), it follows

$$\sum_{v \in V(G) \setminus V(G_1)} f(v) \leq |V(G_2)| - 1.$$

Next, because $|V(G) \setminus V(G_1)| = |V(G_2)| - 1$ and since $f(v) \geq 1$ for each $v \in V(G)$ we have

$$\sum_{v \in V(G) \setminus V(G_1)} f(v) \geq |V(G_2)| - 1.$$

It gives that $f(v) = 1$ for each $v \in V(G) \setminus V(G_1)$ and $\sum_{v \in V(G_1)} f(v) = |V(G_1)| + s - 1$.

Let w be a common vertex of G_1 and G_2 in G (obtained by the identification of v_1 with v_2). Let f' be the size function for G_1 defined by $f'(v) = f(v)$ for each $v \in V(G_1) \setminus \{v_1\}$ and $f'(v_1) = f(w) - 1$. Because

$$\sum_{v \in V(G_1)} f'(v) \leq |V(G_1)| + s - 2$$

G_1 is not (f', D_1)-choosable, by the induction hypothesis. Consequently, there exists a list assignment $\mathbf{L}' = \{L'(v)\}_{v \in V(G_1)}$ that satisfies $|L'(v)| = f'(v)$ for each $v \in V(G_1)$ such that G_1 is not $(\mathbf{L}', D_1)$-colourable. Let $a \notin L'(v_1)$. Obviously G is not $(\mathbf{L}, D_1)$-colourable for $\mathbf{L} = \{L(v)\}_{v \in V(G)}$ such that $L(v) = L'(v)$ for $v \in V(G_1) \setminus \{v_1\}$, $L(v) = \{a\}$ for $v \in V(G) \setminus ((V(G_1) \setminus \{v_1\}) \cup \{w\}))$ and $L(w) = L'(v_1) \cup \{a\}$. Note that $|L(v)| = f(v)$ for each $v \in V(G)$, which implies that, contrary to our assumptions, G is not (f, D_1)-choosable. Hence $\chi_{sc}^{D_1}(G) \geq |V(G)| + s$. □

Lemma 5.7. *If $s \in \mathbb{N}$ and G is a graph in $C(s)$, then $\chi_{sc}^{D_1}(G) \leq |V(G)| + s$.*

Proof. Since $G \in C(s)$ we know the disjoint cycles $C^1, \ldots, C^s$ such that $G = G(C^1, \ldots, C^s)$ (see discussion after Definition 5.4). We label vertices of G, say $v_1, \ldots, v_{|V(G)|}$, starting with the vertices of C^1 (in an arbitrary order). Next, for $j \in \{2, \ldots, s\}$, in the j^{th} step we consequently label vertices of C^j (in an arbitrary order) with the exception of one vertex which was labelled previously.

Let $|V(C^j)| = n_j$. Observe that if $G_i = G[\{v_1, \ldots, v_i\}]$, then

$$deg_{G_i}^{\beta}(v_i) = \begin{cases} 1, & \text{if } i \in \{n_1 + \ldots + n_j - j + 1 : 1 \leq j \leq s\}, \\ 0, & \text{otherwise.} \end{cases}$$

Since exactly s vertices have the β-degrees equal to one and remaining vertices have the β-degrees equal to zero, by Theorem 2.5, it holds $\chi_{sc}^{D_1}(G) \leq |V(G)| + s$. □

As a consequence of Lemmas 5.6 and 5.7 we have the next result.

Theorem 5.8. *If $s \in \mathbb{N}$ and G is a graph in $C(s)$, then $\chi_{sc}^{D_1}(G) = |V(G)| + s$.*

Remark 2.1(i) and Theorem 5.8 yield the following fact.

Corollary 5.9. *Let $s \in \mathbb{N}$. If G is a graph that contains a subgraph in $C(s)$, then*

$$|V(G)| + s \leq \chi_{sc}^{D_1}(G).$$

Consider a graph $G_1 \in C(s)$, which is obtained by $s-1$ successive identifications of vertices into one vertex. The existence of such a specific subgraph G_1 in a graph G is equivalent to the existence of a vertex $v \in V(G)$ such that $deg_G^{\beta}(v) \geq s$. It means that Corollary 5.9 generalizes Theorem 2.6. Another consequence of Corollary 5.9 is given in the next theorem.

Theorem 5.10. *If $n \in \mathbb{N}$, then $\chi_{sc}^{D_1}(P_n \Box P_3) = 4n - 1$.*

Proof. By Lemma 5.2 we know that

$$\chi_{sc}^{D_1}(P_n \Box P_3) \leq \frac{3}{2}3n - \frac{n+3}{2} + \frac{1}{2} = 4n - 1.$$

Thus we focus on the opposite inequality. The discussion after Corollary 5.1 says that we can assume $n \geq 3$. Let $V(P_n \Box P_3) = \{v_{i,j} : 1 \leq i \leq n, 1 \leq j \leq 3\}$. Consider a subgraph G of $P_n \Box P_3$ obtained by:

- the removal of two vertices $v_{1,3}$ and either $v_{n,1}$ (when n is odd) or $v_{n,3}$ (when n is even), and next
- if $n \geq 4$, then, the removal

$$\begin{cases} \text{for all } j \in \{1, \ldots, \frac{n-3}{2}\} \text{ all the edges } v_{2j,1}v_{2j+1,1} \text{ and } v_{2j+1,3}v_{2j+2,3}, & \text{if } n \text{ is odd,} \\ \text{for all } j \in \{1, \ldots, \frac{n-2}{2}\} \text{ all the edges } v_{2j,1}v_{2j+1,1} \text{ and} \\ \text{for all } j \in \{1, \ldots, \frac{n-4}{2}\} \text{ all the edges } v_{2j+1,3}v_{2j+2,3}, & \text{if } n \text{ is even.} \end{cases}$$

It is easy to see that $G \in C(n-1)$ (see Figure 4). Thus by Corollary 5.9 we have $\chi_{sc}^{D_1}(P_n \Box P_3) \geq 3n + n - 1 = 4n - 1$. □

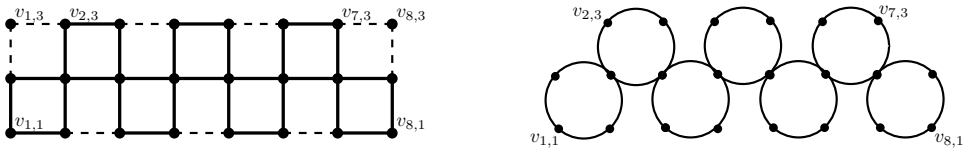

Fig. 4. The subgraph of $V(P_8 \Box P_3)$ that is in $C(7)$

Assuming that $s \in \mathbb{N}$ and $G \in C(s)$, by Theorem 5.8, we know that the graph G is not (f, D_1)-choosable for each size function $f : V(G) \to \mathbb{N}$, such that

$$\sum_{v \in V(G)} f(v) \leq |V(G)| + s - 1.$$

In the next theorem we construct a specific list assignment $\mathbf{L} = \{L(v)\}_{v \in V(G)}$ satisfying that G is not $(\mathbf{L}, D_1)$-colourable and $\sum_{v \in V(G)} |L(v)| = |V(G)| + s - 1$. The knowledge on this $\mathbf{L}$ shall help us in the solution of other problems.

Theorem 5.11. *Let $s \in \mathbb{N}$, and $G = G(C^1, \ldots, C^s)$ be a graph in $C(s)$. Next let $g : \{C^1, \ldots, C^s\} \to \mathbb{N}$ be a mapping such that $g(C^i) \neq g(C^j)$ when C^i, C^j have common vertex in G. If $\mathbf{L} = \{L(v)\}_{v \in V(G)}$ is a list assignment for G such that*

$$L(v) = \{g(C^i) : v \text{ is a vertex of the cycle } C^i\},$$

then G is not $(\mathbf{L}, D_1)$-colourable. Moreover, $\sum_{v \in V(G)} |L(v)| = |V(G)| + s - 1$.

Proof. We start with proving the last assertion of the theorem. By Lemma 5.5, the only cycles of G are $C^1, \ldots, C^s$. The assumptions on g imply that for each vertex $v \in V(G)$, it holds $|L(v)| = deg^{\beta}_G(v)$. Using Lemma 5.5 once again we have

$$\sum_{v \in V(G)} |L(v)| = \sum_{v \in V(G)} deg^{\beta}_G(v) = |V(G)| + s - 1.$$

Now we focus on proving the $(\mathbf{L}, D_1)$-non-colourability of G. To do it we proceed by induction on s. Let $s = 1$. It means $G = C^1$ and $L(v) = \{g(C^1)\}$ for each vertex $v \in V(G)$. Consequently, G is not $(\mathbf{L}, D_1)$-colourable and the assertion holds. Thus $s \geq 2$. Let $\{C^{i_1}, \ldots, C^{i_r}\}$ be a subset of $\{C^1, \ldots, C^s\}$ consisting of all the cycles whose all, except one, vertices have β-degrees in G equal to one. Next for each $j \in \{1, \ldots, r\}$ let v_{i_j} be this exceptional vertex of C^{i_j}. Observe that $s \geq 2$ guarantees $r > 0$. Realize that v_{i_p} can be the same as v_{i_q} for $p \neq q$. Now, let V_1 be the set of all vertices of the cycles $C^{i_1}, \ldots, C^{i_r}$, except vertices $v_{i_1}, \ldots, v_{i_r}$ (because of possible repetitions, $|\{v_{i_1}, \ldots, v_{i_r}\}| \leq r$). Consider a graph $H = G - V_1$. If $r = s$, then $H = K_1$ and G is a graph obtained from s disjoint cycles $C^1, \ldots, C^s$ by $s - 1$ successive identifications of vertices $v_{i_1}, \ldots, v_{i_s}$ into one vertex. If $r < s$, then $H \in C(s - r)$ and moreover, H can be recursively constructed from the cycles in the set $\{C^1, \ldots, C^s\} \setminus \{C^{i_1}, \ldots, C^{i_r}\}$. Observe that H and $g' = g|_{\{C^1, \ldots, C^s\} \setminus \{C^{i_1}, \ldots, C^{i_r}\}}$ and $\mathbf{L}' = \{L'(v)\}_{v \in V(H)}$ such that

$$L'(v) = \{g'(C^i) : v \text{ is a vertex of a cycle } C^i\}$$

satisfy the assumptions of the theorem, but $H \in C(r - s)$. Hence H is not $(\mathbf{L}', D_1)$-colourable by the induction hypothesis.

Now suppose, for a contradiction, that G is $(\mathbf{L}, D_1)$-colourable and $c : V(G) \to \mathbb{N}$ is an $(\mathbf{L}, D_1)$-colouring of G. For each $v \in V_1$ we have $L(v) = \{g(C^{i_j})\}$, where v is a vertex of C^{i_j}. Thus $c(v_{i_j})$ must be different from $g(C^{i_j})$. Suppose $r = s$. Hence $v_{i_1} = \cdots = v_{i_r} = x$ and $L(x) = \{g(C^1), \ldots, g(C^r)\}$, which means $c(x) \in \{g(C^1), \ldots, g(C^r)\}$ contrary to $c(v_{i_j}) \neq g(C^{i_j})$ for each $j \in \{1, \ldots, r\}$. Thus $r < s$. In this case $c|_{V(H)}$ has to be an $(\mathbf{L}', D_1)$-colouring of H. From the earlier consideration, H is not $(\mathbf{L}', D_1)$-colourable, which implies that the required $c|_{V(H)}$ does not exist. It completes the proof. □

Now we have to define some objects that will help us in proving the main result of the paper.

Definition 5.12. Let $n \geq 3$ and $V(P_n \Box P_4) = \{v_{i,j} : 1 \leq i \leq n, 1 \leq j \leq 4\}$. By $\mathcal{A}(n)$ we mean a family of cycles in $P_n \Box P_4$, each of which has length four, such that

$$\mathcal{A}(n) = \{C^{1,1}, \ldots, C^{1,\lfloor \frac{n}{2} \rfloor}, C^{2,1}, \ldots, C^{2,\lfloor \frac{n-1}{2} \rfloor}, C^{3,1}, \ldots, C^{3,\lfloor \frac{n}{2} \rfloor}\},$$

where for all permissible k we put $C^{1,k} = (v_{2k-1,2}, v_{2k,2}, v_{2k,1}, v_{2k-1,1})$, $C^{3,k} = (v_{2k-1,4}, v_{2k,4}, v_{2k,3}, v_{2k-1,3})$, and $C^{2,k} = (v_{2k,3}, v_{2k+1,3}, v_{2k+1,2}, v_{2k,2})$.

Definition 5.13. Let $n \geq 3$ and $V(P_n \Box P_4) = \{v_{i,j} : 1 \leq i \leq n, 1 \leq j \leq 4\}$. By $H(n)$ we denote a graph whose vertex set is $\mathcal{A}(n)$ (see Definition 5.12), in which two vertices (cycles) $C', C'' \in \mathcal{A}(n)$ are adjacent when they have common vertex in $P_n \Box P_4$.

It is worth noting here that $H(n)$ has $|\mathcal{A}(n)|$ vertices, which is equal to $\frac{3n-3}{2}$ when n is odd and $\frac{3n-2}{2}$ when n is even. The forms of both $\mathcal{A}(n)$ and $H(n)$ are illustrated in Figure 5.

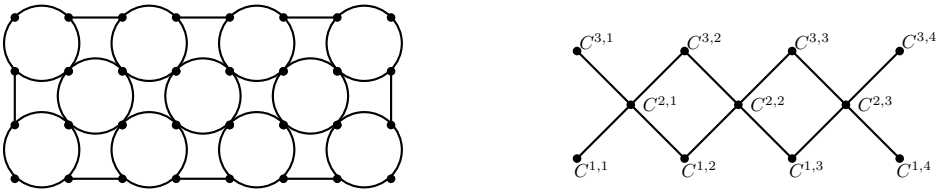

Fig. 5. The illustration of the family $\mathcal{A}(8)$ in $P_8 \Box P_4$ and the graph $H(8)$

Lemma 5.14. *Let $n \geq 3$ and let T be a subgraph of the graph $H(n)$ (see Definition 5.13) that is a tree. Next let T^* be a graph induced in $H(n)$ by $V(T)$. There is $g : V(T^*) \to \mathbb{N}$ such that $g(C') \neq g(C'')$ when $C'C'' \in E(T)$ and $g(C') = g(C'')$ when $C'C'' \in E(T^*) \setminus E(T)$.*

Proof. For simplicity, $H = H(n)$ and $E_1 = E(T)$ and $E_2 = E(T^*) \setminus E(T)$. Now we define the relation ρ on $V(T^*)$ (recall that $V(T^*) = V(T)$) in the following way: $(x, y) \in \rho$ if there exists an $(x - y)$-path in T^* whose all the edges are in E_2.

Observe that ρ is the equivalence relation on $V(T^*)$. We know that each equivalence relation provides a partition of a set on which it is described into equivalence classes. Thus we have the corresponding partition of $V(T^*)$, say $V_1, \ldots, V_k$.

Claim 5.15. *For each $i \in \{1, \ldots, k\}$, the set V_i is stable in T.*

Proof. Suppose, without restriction of generality and for a contradiction, that V_1 includes two vertices C', C'' that are adjacent in T. It follows that there is a $(C' - C'')$-path P^* in T^* whose all the edges are in E_2. Since T^* is a bipartite graph (as an induced subgraph of a bipartite graph H) the length of P^* is odd. First assume that the length of P^* is at least three and without loss of generality its form is $(C', w_1, \ldots, w_p, C'')$. It means that at least two vertices w_1, w_2 are different and different from both C', C''. It follows that for $i \in \{1, 2\}$ the vertex w_i has at least two neighbours in T^* joined with w_i by edges from E_2. Moreover, w_i, as a vertex of T, has at least one neighbour in T. Hence the degree of w_i in T^*, and consequently in H, is at least three. Note that w_1, w_2 are consecutive vertices of the path in T^*, which implies that w_1, w_2 are neighbours in T^* and consequently they are neighbours in H. Thus w_1, w_2 are adjacent vertices of H, both with degree at least three, a contradiction with the construction of H (see Figure 5). It follows that the length of P^* is one. In

this case C', C'' are joined by two edges, one from E_1 and the second one from E_2, a contradiction with the construction of T^*, which is not a multigraph. □

By Claim 5.15 we found a partition of $V(T)$ into sets $V_1, \ldots, V_k$ (the equivalence classes of ρ) so that for each $i \in \{1, \ldots, k\}$ the set V_i is stable in T.

For $C \in V(T)$, let $g(C) = i$, where i is the unique index such that $C \in V_i$. Claim 5.15 implies that $g(C') \neq g(C'')$ for any two adjacent vertices C', C'' of T and moreover, by the definition of ρ, it holds $g(C') = g(C'')$ for any two vertices C', C'' satisfying $C'C'' \in E(T^*) \setminus E(T)$. □

Construction 5.16. Let $n \geq 3$ and $V(P_n \Box P_4) = \{v_{i,j} : 1 \leq i \leq n, 1 \leq j \leq 4\}$. Next let $\mathcal{A}(n)$ be the family of cycles in $P_n \Box P_4$ given by Definition 5.12.

Let V' be the set of vertices of a graph G induced by all the edges of all the cycles in $\mathcal{A}(n)$ and let $f : V' \to \mathbb{N}$.

By $M(n, f)$ we mean a multigraph with vertex set $\mathcal{A}(n)$ whose edges are implied by values of $f(v)$ taken over all $v \in V'$. Precisely, if v is a vertex of exactly one among the cycles in $\mathcal{A}(n)$, say C, then it implies the creation of $f(v) - 1$ loops in $M(n, f)$ containing the vertex C. If v is a common vertex of two among the cycles in $\mathcal{A}(n)$, say C', C'', then it implies the creation of $f(v) - 1$ edges joining C', C'' in $M(n, f)$. The set of edges $E(M(n, f))$ is the union of pairwise disjoint sets of edges implied by all $v \in V'$ (see Figure 6).

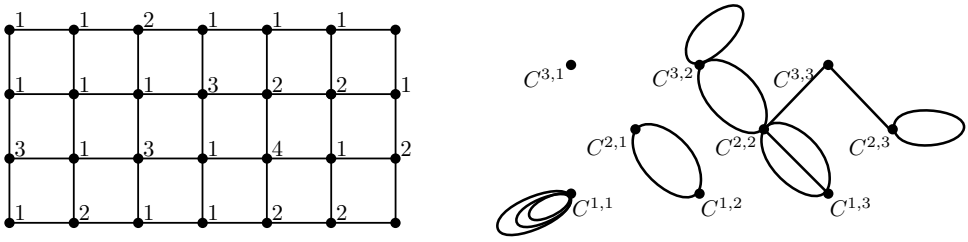

Fig. 6. The graph $P_7 \Box P_4$ with the values of $f(v)$ given for all $v \in V'$ and represented right at the vertices, and the multigraph $M(7, f)$

It is worth noting some properties of the graph $H(n)$ and the multigraph $M(n, f)$.

Remark 5.17. Let $n \geq 3$ and $V(P_n \Box P_4) = \{v_{i,j} : 1 \leq i \leq n, 1 \leq j \leq 4\}$. Next let $f : V' \to \mathbb{N}$, where V' is the set defined in Construction 5.16.

(i) $V' = V(P_n \Box P_4) \setminus \{v_{n,1}, v_{n,4}\}$ when n is odd and $V' = V(G)$ when n is even.
(ii) Each vertex $v \in V'$ is included either in exactly one or in exactly two of the cycles in $\mathcal{A}(n)$ (there is no common vertex of more than two cycles).
(iii) The loops consisting of the vertex $C \in \mathcal{A}(n)$ in $M(n, f)$ can be implied by one, two or even three vertices in V'. All the edges joining two different vertices (cycles) $C', C'' \in \mathcal{A}(n)$ in $M(n, f)$ are implied by only one vertex in V'.

(iv) The number of edges in $M(n,f)$ (including loops) is equal to $\sum_{v\in V'}(f(v)-1)$.
(v) If a component of the multigraph $M(n,f)$ is a graph, then it is a subgraph of $H(n)$.

The statement v) of Remark 5.17 is only one, which should be explained. Indeed $V(M(n,f)) = V(H(n))$. If G^* is a subgraph of $M(n,f)$ and $e = C'C''$ is an edge of G^*, then there exists a common vertex v of C', C'' in $P_n\Box P_4$, which was the reason of the creation of e in $M(n,f)$, so v implies the existence of an edge in $H(n)$. It confirms that G^* is a subgraph of $H(n)$.

Lemma 5.18. *Let G be a graph and $\mathbf{L}' = \{L'(v)\}_{v\in V(G)}$ be a list assignment for G. Next, let $a \in \mathbb{N}$ and v_1, v_2 be two nonadjacent vertices of G for which $L'(v_1) = L'(v_2) = \{a\}$. Let H be a graph obtained from G by the identification of v_1 with v_2 into w and $\mathbf{L} = \{L(v)\}_{v\in V(H)}$ be the list assignment for H defined by $L(v) = L'(v)$ for $v \in V(H)\setminus\{w\}$ and $L(w) = \{a\}$. If H is $(\mathbf{L}, D_1)$-colourable, then G is $(\mathbf{L}', D_1)$-colourable or equivalently if G is not $(\mathbf{L}', D_1)$-colourable, then H is not $(\mathbf{L}, D_1)$-colourable.*

Proof. We give the proof of the first among two equivalent statements. Let $c : V(H) \to \mathbb{N}$ be an $(\mathbf{L}, D_1)$-colouring of H. Define $c' : V(G) \to \mathbb{N}$ in the following way: $c'(v) = c(v)$ for $v \in V(G)\setminus\{v_1, v_2\}$ and $c'(v_1) = c'(v_2) = a$. To finish the proof we shall show that c' is an $(\mathbf{L}', D_1)$-colouring of G. Suppose, for a contradiction, that it does not occur. Obviously, for each $v \in V(G)$ we have $c'(v) \in L'(v)$. It means there is a cycle in G, say $(x_1, x_2, \ldots, x_p)$ and $b \in \mathbb{N}$, such that $c'(x_i) = b$ for each $i \in \{1, \ldots, p\}$. If $|\{v_1, v_2\} \cap \{x_1, \ldots, x_p\}| = 0$, then $(x_1, x_2, \ldots, x_p)$ is a monochromatic cycle of H in c, a contradiction. If $|\{v_1, v_2\} \cap \{x_1, \ldots, x_p\}| = 1$, then, without loss of generality $x_p = v_1$ and $(x_1, x_2, \ldots, x_{p-1}, w)$ is a monochromatic cycle of H in c, a contradiction. If $|\{v_1, v_2\} \cap \{x_1, \ldots, x_p\}| = 2$, then without loss of generality, assume that $x_1 = v_1$ and $x_s = v_2$ $(s \le p)$, and consequently $(x_1 = w, x_2, \ldots, x_s = w)$ is a monochromatic cycle of H in c. It also gives a contradiction in this case. □

The presentation of the proof of the next crucial lemma seems to be very difficult. Therefore, for the convenience of readers, Figures 7, 8, 9 illustrate the consecutive steps of the proof.

Lemma 5.19. *Let $n \ge 3$. If $f : V(P_n\Box P_4) \to \mathbb{N}$ is a mapping such that $M(n,f)$ has a connected component that is a tree, then $P_n\Box P_4$ is not (f, D_1)-choosable.*

Proof. Let T be a fixed connected component of $M(n,f)$ that is a tree (see Figure 7). If $|V(T)| = 1$, then, by the definition of $M(n,f)$, there is a cycle (v_1, v_2, v_3, v_4) in $\mathcal{A}(n)$ such that $f(v_1) = \cdots = f(v_4) = 1$. Clearly, $P_n\Box P_4$ is not $(\mathbf{L}, D_1)$-colourable, when $\mathbf{L} = \{L(v)\}_{v\in V(P_n\Box P_4)}$ and $L(v_1) = \cdots = L(v_4) = \{a\}$ for some $a \in \mathbb{N}$. Thus $P_n\Box P_4$ is not (f, D_1)-choosable in this case. Next let $V(T) = \{C^1, \ldots, C^p\}$, where $p \ge 2$ and the labels in $V(T)$ are given in such a way that for $i \in \{2, \ldots, p\}$, the graph induced by $C^1, \ldots, C^i$ is a tree and the cycle C^i is a leaf of this induced subtree of T (it is very easy to see that for each tree at least one such an ordering always exists, see Figure 8*a*)). Now, for simplicity, let $G = P_n\Box P_4$ and let G' be the graph induced by all the edges of all the cycles in $V(T)$. By V' we denote $V(G')$.

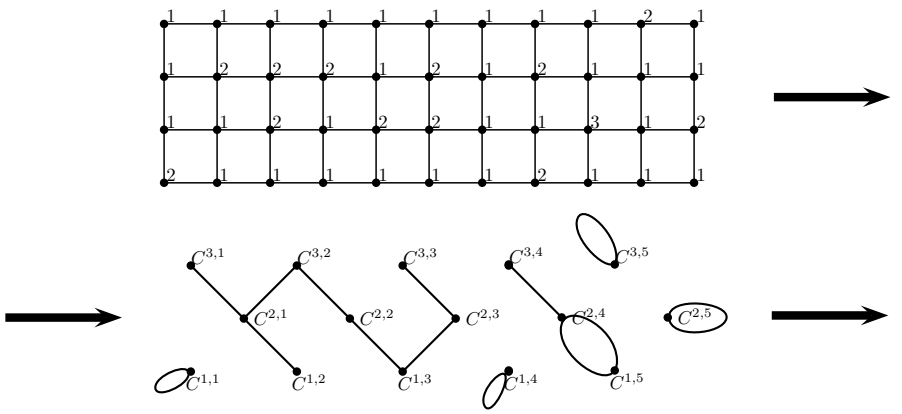

Fig. 7. The graph $P_{11}\square P_4$ and the mapping $f : V(P_{11}\square P_4) \to \mathbb{N}$ for which the multigraph $M(11, f)$ contains a connected component that is a tree

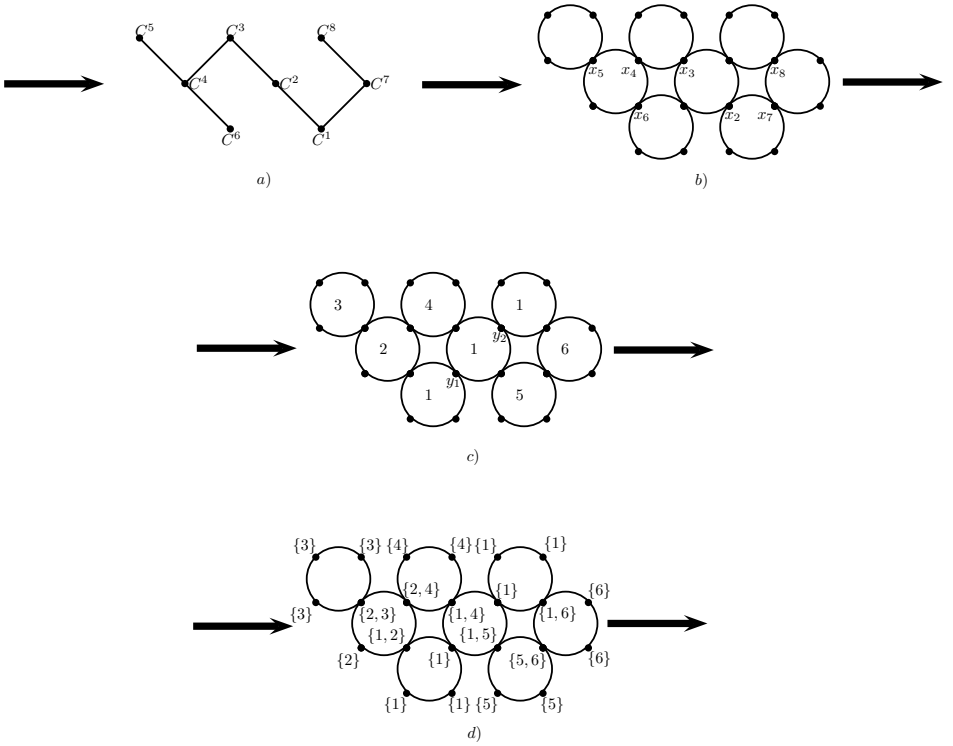

Fig. 8. (a) The labels of vertices in $V(T)$. (b) The set W_1. (c) The values of the mapping g. (d) The list assignment $\mathbf{L}$ for G' used in the proof of Lemma 5.19 and given according to the mapping g that was constructed in Lemma 5.14

First we shall show that G' is not $(f|_{V'}, D_1)$-choosable. It will imply, by Remark 2.1(ii), the statement of the theorem.

Because T is a tree, which is a connected component of $M(n, f)$, for each vertex $v \in V'$ we have $f(v) \leq 2$. We divide the set V' into W_1, W_2, W_3 such that:

$$W_1 = \{v \in V' : \ v \text{ is a vertex of exactly two among cycles in } V(T) \text{ and } f(v) = 2\},$$
$$W_2 = \{v \in V' : \ v \text{ is a vertex of exactly two among cycles in } V(T) \text{ and } f(v) = 1\},$$
$$W_3 = V_1 \setminus (W_1 \cup W_2).$$

The fact that there is no loops in T implies that

$$W_3 = \{v \in V' : \ v \text{ is a vertex of exactly one among cycles in } V(T) \text{ and } f(v) = 1\}.$$

Since in each tree the number of edges is one less than the number of vertices and because each edge of T corresponds to some vertex in W_1 we have $|W_1| = p - 1$. Next, for $i \in \{2, \ldots, p\}$, by x_i we denote the vertex in W_1 that corresponds to the edge, which joins in T the vertex C^i with one of the vertices $C^1, \ldots, C^{i-1}$ (recall that C^i is a leaf of the subtree of T induced by $C^1, \ldots, C^i$, which means that x_i is precisely given). Thus $W_1 = \{x_2, \ldots, x_p\}$ (see Figure 8b)). Now we use Remark 5.17 (v) to see that T is a subgraph of $H(n)$. According to Lemma 5.14 we find a mapping $g : V(T^*) \to \mathbb{N}$ such that $g(C') \neq g(C'')$ when $C'C'' \in E(T)$ and $g(C') = g(C'')$ when $C'C'' \in E(T^*) \setminus E(T)$, where, as before (in Lemma 5.14), T^* is a graph induced by $V(T)$ in $H(n)$ (see Figure 8c)).

We define the list assignment $\mathbf{L} = \{L(v)\}_{v \in V'}$ in the following way:

$$L(v) = \{g(C^i) : \text{v is a vertex of } C^i\} \quad \text{(see Figure 8d)).}$$

Our next aim is to observe that $|L(v)| = f(v)$ for each $v \in V'$. Indeed, if $v \in W_1 \cup W_2$, then v is a vertex of exactly two cycles in $V(T)$, say C', C''. So, $L(v) = \{g(C'), g(C'')\}$. Thus, in this case, $|L(v)| = 2$ when $g(C') \neq g(C'')$ and $|L(v)| = 1$ when $g(C') = g(C'')$. Actually, because v is a vertex of both C', C'', there is $f(v) - 1$ edges that joins C', C'' in $M(n, f)$ and next in T. Consequently, $f(v) = 2$ guarantees that $C'C''$ is an edge in T ($g(C') \neq g(C'')$) and $f(v) = 1$ guarantees that $C'C''$ is an edge in $E(T^*) \setminus E(T)$ ($g(C') = g(C'')$). Thus $|L(v)| = f(v)$ for each $v \in W_1 \cup W_2$. Finally, if $v \in W_3$, then v is a vertex of exactly one among cycles in $V(T)$, which implies $|L(v)| = 1 = f(v)$ in this case.

To finish the proof we shall show that G' is not $(\mathbf{L}, D_1)$-colourable, which will imply $(f|_{V'}, D_1)$-non-choosability of G', as we required.

Duplicate each vertex $v \in W_1 \cup W_2$ to v' and v''. Next, each vertex v of a fixed cycle C^i that is from the set $W_1 \cup W_2$ substitute by the vertex v', when v is the vertex of another C^j with $i < j$ and by v'', otherwise. The set of disjoint cycles obtained by application of these rules is $\{C^{1*}, \ldots, C^{p*}\}$, when C^{i*} corresponds to C^i (see Figure 9a)). We construct a graph $G^* = G^*(C^{1*}, \ldots, C^{p*}) \in C(p)$ in $p - 1$ steps according to Definition 5.4. First we take a graph C^{1*}. For $j \in \{2, \ldots, p\}$, in the j^{th} step we identify the pair of nonadjacent vertices x'_j, x''_j into x_j. Note that the ordering $(1, \ldots, p)$, which is in the connection with the form of T guarantees that

$G^* = G^*(C^{1*}, \ldots, C^{p*})$ is well defined. Indeed C^j is a leaf of the subtree of T induced by $C^1, \ldots, C^{j-1}$ (see Figure 9b)).

Put $g(C^{i*}) = g(C^i)$ for $i \in \{1, \ldots, p\}$. Consider a list assignment $\mathbf{L}^* = \{L^*(v)\}_{v \in V^*}$, where V^* is the set obtained from V' by the duplication of each vertex $v \in W_2$ to corresponding v', v'' (these vertices are not identified) or equivalently $V^* = V(G^*)$.

$$L^*(v) = \{g(C^{i*}) : \text{v is a vertex of } C^{i*}\} \quad \text{(see Figure 9c)).}$$

It should be observed that $g(C^{i*}) \neq g(C^{j*})$, when C^{i*}, C^{j*} have common vertex in G^*. Hence, Theorem 5.11 shows that G^* is not $(\mathbf{L}^*, D_1)$-colourable.

Also note that, for each $v \in W_2$ the pair of vertices v', v'' obtained by the duplication of v satisfies $L^*(v') = L^*(v'')$ and $|L^*(v')| = |L^*(v'')| = 1$.

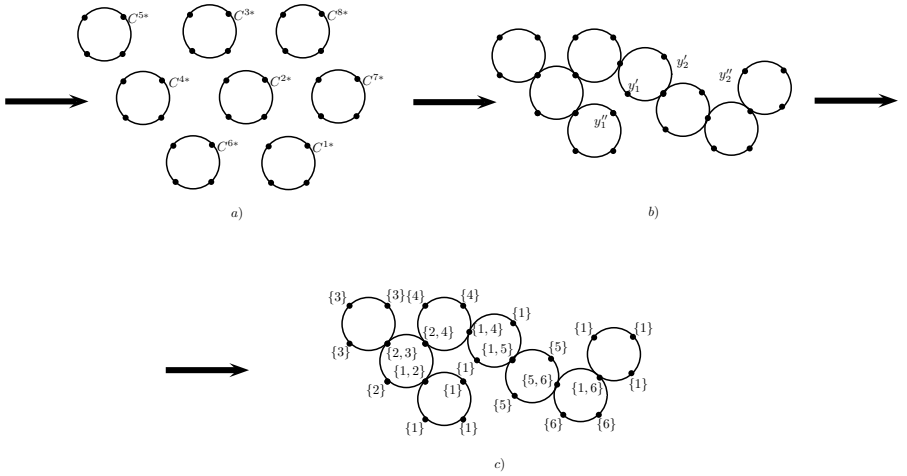

Fig. 9. (a) The cycles $C^{1*}, \ldots, C^{p*}$. (b) The graph $G^*(C^{1*}, \ldots, C^{p*})$ and its duplicated vertices from the set W_2. (c) The list assignment $\mathbf{L}^*$ for $G^*(C^{1*}, \ldots, C^{p*})$

Now we start with the graph $G^* = G_0$ and the list assignment $\mathbf{L}^* = \{L^*(v)\}_{v \in V(G^*)} = \mathbf{L}_{G_0}$. If $W_2 = \emptyset$, then obviously $G^* = G'$ and $\mathbf{L}^* = \mathbf{L}$. If $W_2 \neq \emptyset$, then additionally assume that $W_2 = \{y_1, \ldots, y_q\}$ (see Figure 8c)). For each $j \in \{1, \ldots, q\}$, in the j^{th} step we apply Lemma 5.18 to the graph G_{j-1} (it plays the role of G in the lemma), the list assignment $\mathbf{L}_{G_{j-1}}$ and the graph G_j (it plays the role of H in the lemma), obtained by the identification of two nonadjacent vertices y'_j, y''_j into y_j and the list assignment $\mathbf{L}_{\mathbf{G_j}}$ given in accordance with the assumptions of Lemma 5.18. It is possible because for each $j \in \{1, \ldots, q\}$ it holds $L^*(y'_j) = L^*(y''_j) = \{a\}$ for some $a \in \mathbb{N}$ and moreover, the vertices $y'_1, y''_1, \ldots y'_q, y''_q$ are pairwise different. Thus after q-time application of Lemma 5.18 we have a graph G_q which is not $(\mathbf{L}_{G_q}, D_1)$-colourable. The current observation is that G_q is a subgraph of G' (actually $G_q = G'$, but this stronger fact is redundant) and $\mathbf{L}_{G_q} = \mathbf{L}$. The equality $\mathbf{L}_{G_q} = \mathbf{L}$ follows immediately by the constructions of $\mathbf{L}^* = \mathbf{L}_{G_0}$ and all $\mathbf{L}_{G_j}$

for $j \in \{1, \ldots, q\}$. The constructive argument also simply gives $V(G_q) = V'$ (first we duplicated some of the vertices in V', next each of the duplicated pairs was identified to the source vertex). Hence, to observe that G_q is a subgraph of G' it is enough to see that each edge of G_q is present in G'. Note that G_q is a graph obtained from disjoint cycles $\{C^{1*}, \ldots, C^{p*}\}$ by identification of some pairs of their vertices. It means that G_q can have only edges of the cycles $\{C^1, \ldots, C^p\}$. All of them are in G'. Hence G_q is a subgraph of G' and moreover, $V(G_q) = V(G') = V'$. It follows, by Remark 2.1(ii) that G' is not $(\mathbf{L}, D_1)$-colourable, which means G' is not $(f|_{V'}, D_1)$-choosable. We use Remark 2.1(ii) once again to observe (f, D_1)-non-choosability of G, which completes the proof. □

Lemma 5.20. *If $n \in \mathbb{N}$, then*

$$\frac{11n-3}{2} \leq \chi_{sc}^{D_1}(P_n \Box P_4).$$

Proof. Let $G = P_n \Box P_4$. Suppose, for a contradiction, that there is a mapping $f : V(G) \to \mathbb{N}$ such that G is (f, D_1)-choosable and $\sum_{v \in V(G)} f(v) < \frac{11n-3}{2}$. Precisely, $\sum_{v \in V(G)} f(v) \leq \frac{11n-5}{2}$ for odd n and $\sum_{v \in V(G)} f(v) \leq \frac{11n-4}{2}$ for even n. Consider a multigraph $M(n, f)$ defined in Construction 5.16. Recall that

$$|V(M(n,f))| = \begin{cases} \frac{3n-3}{2}, & \text{if } n \text{ is odd,} \\ \frac{3n-2}{2}, & \text{if } n \text{ is even.} \end{cases}$$

By Remark 5.17(iv), the number of edges of $M(n, f)$ is equal to $\sum_{v \in V'}(f(v) - 1)$, where V' is the set precisely described in Remark 5.17(i). Since $f(v) - 1 \geq 0$ for each $v \in V(G)$ we have for odd n

$$|E(M(n,f))| \leq \sum_{v \in V(G)} (f(v) - 1) \leq \frac{11n-5}{2} - 4n = \frac{3n-5}{2},$$

and for even n

$$|E(M(n,f))| = \sum_{v \in V(G)} (f(v) - 1) \leq \frac{11n-4}{2} - 4n = \frac{3n-4}{2}.$$

It yields $|E(M(n,f))| \leq |V(M(n,f))| - 1$. Hence, there is at least one connected component of $M(n, f)$ that is a tree. It follows from Lemma 5.19 that G is not (f, D_1)-choosable, a contradiction. □

Theorem 5.21. *If $n \in \mathbb{N}$, then*

$$\chi_{sc}^{D_1}(P_n \Box P_4) = \left\lceil \frac{11n-3}{2} \right\rceil.$$

Proof. Lemmas 5.2 and 5.20 lead to the statement equality. □

6. CONCLUDING REMARKS

Several supporting results presented in the paper can be used as tools for other research in this field. Some of them could help us to establish the exact values of $\chi_{sc}^{D_1}(P_n \Box P_m)$, when both $n, m \geq 5$. We are able to calculate some special numbers of this type but, in general, Problem 5.3 is still open. The method that is used in this work when $\min\{n, m\} \leq 4$ fails in the possible analogue of Lemma 5.14.

On the other hand, Theorem 5.21 implies the improvement of Corollary 5.1. Actually, graphs $P_n \Box P_m$ with $n \geq 1,\ m \geq 4$ contain subgraphs $G_{n,m}$ of the forms

$$G_{n,m} = \begin{cases} \lfloor \frac{m}{4} \rfloor (P_n \Box P_4), & \text{if } n \equiv 0 \pmod 4, \\ \lfloor \frac{m}{4} \rfloor (P_n \Box P_4) \cup (P_n \Box P_1), & \text{if } n \equiv 1 \pmod 4, \\ \lfloor \frac{m}{4} \rfloor (P_n \Box P_4) \cup (P_n \Box P_2), & \text{if } n \equiv 2 \pmod 4, \\ \lfloor \frac{m}{4} \rfloor (P_n \Box P_4) \cup (P_n \Box P_3) & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

As we mentioned previously, $\chi_{sc}^{D_1}(G_1 \cup G_2) = \chi_{sc}^{D_1}(G_1) + \chi_{sc}^{D_1}(G_2)$, which gives

$$\chi_{sc}^{D_1}(G_{n,m}) = \begin{cases} \lfloor \frac{m}{4} \rfloor \lceil \frac{11n-3}{2} \rceil, & \text{if } n \equiv 0 \pmod 4, \\ \lfloor \frac{m}{4} \rfloor \lceil \frac{11n-3}{2} \rceil + n, & \text{if } n \equiv 1 \pmod 4, \\ \lfloor \frac{m}{4} \rfloor \lceil \frac{11n-3}{2} \rceil + (2n + \lfloor \frac{n}{2} \rfloor), & \text{if } n \equiv 2 \pmod 4, \\ \lfloor \frac{m}{4} \rfloor \lceil \frac{11n-3}{2} \rceil + (4n-1), & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Careful calculation of these numbers yields

$$\chi_{sc}^{D_1}(G_{n,m}) = nm + \left\lceil \frac{(n-1)(m-1)}{2} \right\rceil - \left\lfloor \frac{m}{5} \right\rfloor \left\lceil \frac{(n-1)}{2} \right\rceil, \text{ for } n \geq 1, m \geq 4.$$

Observe that $V(G_{n,m}) = V(P_n \Box P_m)$. Thus Remark 2.1(i) implies

$$\chi_{sc}^{D_1}(G_{n,m}) \leq \chi_{sc}^{D_1}(P_n \Box P_m) \text{ for } n \geq 1, m \geq 4.$$

Note that $\chi_{sc}^{D_1}(G_{n,m})$ is an asymptotically better lower bound on $\chi_{sc}^{D_1}(P_n \Box P_m)$ than that one which was obtained in Corollary 5.1.

We turn out Lemma 5.2 to write its two alternative statements by one inequality.

$$\chi_{sc}^{D_1}(P_n \Box P_m) \leq nm + \left\lceil \frac{(n-1)(m-1)}{2} \right\rceil \text{ for } m, n \in \mathbb{N}.$$

Now we combine all the above observations and known values of $\chi_{sc}^{D_1}(P_n \Box P_m)$ that were discussed in the previous section in the following result.

Corollary 6.1. *If $n, m \in \mathbb{N}$, then*

$$nm + \left\lceil \frac{(n-1)(m-1)}{2} \right\rceil - \left\lfloor \frac{m}{5} \right\rfloor \left\lceil \frac{(n-1)}{2} \right\rceil \leq \chi_{sc}^{D_1}(P_n \Box P_m) \leq nm + \left\lceil \frac{(n-1)(m-1)}{2} \right\rceil.$$

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