

Black Holes, Bandwidths and Beethoven

Achim Kempf

Institute for Fundamental Theory, Department of Physics
University of Florida, Gainesville, FL 32611, USA

Email: kempf@phys.ufl.edu

Abstract

It is usually believed that a function $\phi(t)$ whose Fourier spectrum is bounded can vary at most as fast as its highest frequency component ω_{max} . This is in fact not the case, as Aharonov, Berry and others drastically demonstrated with explicit counter examples, so-called superoscillations. It has been claimed that even the recording of an entire Beethoven symphony can occur as part of a signal with 1Hz bandwidth. Bandlimited functions also occur as ultraviolet regularized fields. Their superoscillations have been suggested, for example, to resolve the transplanckian frequencies problem of black hole radiation.

Here, we give an exact proof for generic superoscillations. Namely, we show that for every fixed bandwidth there exist functions which pass through any finite number of arbitrarily prespecified points. Further, we show that, in spite of the presence of superoscillations, the behavior of bandlimited functions can be characterized reliably, namely through an uncertainty relation: The standard deviation ΔT of samples $\phi(t_n)$ taken at the Nyquist rate obeys: $\Delta T \geq 1/4\omega_{max}$. This uncertainty relation generalizes to variable bandwidths. We identify the bandwidth as the in general spatially variable finite local density of the degrees of freedom of ultraviolet regularized fields.

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1 Introduction

Functions which contain only frequencies up to a certain maximum frequency occur in various contexts from theoretical physics to the applied sciences. For example, in quantum field theory the method of “ultraviolet” regularization by energy-momentum cut-off yields fields which are frequency limited. Frequency limited functions also occur for example as so-called “bandlimited signals” in communication engineering.

Intuitively, one may expect that a frequency limited function, $\phi(t)$, can vary at most as fast as its highest frequency component, ω_{max} . In fact, this is not the case.

Aharonov et al [1] and Berry [2], gave explicit examples - which they named superoscillations - which drastically demonstrate that frequency limited functions are able to oscillate for arbitrarily long finite intervals arbitrarily faster than the highest frequency component which they contain. It has even been conjectured, in [2], that for example 5000 seconds of a 20 KHz bandwidth recording of a symphony of Beethoven can be part of a 1Hz bandlimited signal.

Before we begin our investigation of superoscillations a few remarks on our use of terminology may be in place.

Frequency limited functions and superoscillations not only occur in ultraviolet cut-off quantum field theory and in information theory but also in a whole variety of other physical contexts. Superoscillation are known to occur for example with evanescent waves and quantum billiards, see [3], and for example with effects of apparent superluminal propagation in unstable media such as media with inverted level populations, see e.g. [4]. We will here mostly be concerned with the general properties of superoscillations and we could therefore use as our terminology the language of any one of these contexts where superoscillations occur.

Our choice of terminology here will be to use both the language of quantum field theory and the language of information theory.

We make this choice because, on the one hand, our main interest is in the implications of superoscillations in ultraviolet regular quantum field theories. On the other hand, we will often find it advantageous to use the concrete and intuitive terminology of information theory. We will for example often use terms such as “signal” and “bandwidth” where we mean “field” and “ultraviolet cut-off”. For our purposes, the main advantage of the language of information theory will be that this language contains several useful terms which describe properties of bandlimited signals - and which by correspondence also describe properties of ultraviolet cut-off fields - for which there does not seem to exist an established corresponding terminology in the language of quantum field theory. These will be terms such as “data transmission rate”, “noise”, or “signal reconstruction from samples”. We will introduce these terms as needed, and we will discuss their corresponding meaning in quantum field theory.

Concerning the physics of superoscillations in ultraviolet regularized quantum field the-

ories, an interesting possibility has been discussed in [5, 6]. These authors suggest that the existence of superoscillations may resolve the transplanckian energies paradox of black hole radiation:

Let us recall how this paradox arises for black hole radiation as derived in the standard free field theory formalism. One considers a Hawking photon which is observed at asymptotic distance from the black hole and which has some typical energy E . The calculation of the redshift shows that the same photon, when it was still close to the horizon, say at a Planckian distance, should have had a proper energy of the order of $Ee^{\alpha M^2}$ where α is of order one and where for a macroscopic black hole, in Planckian units, $M \approx 10^{40}$. The paradox is that the assumptions which went into the derivation of the existence of Hawking radiation (no interactions or backreactions) likely do not hold true at those far transplanckian energies. See for example [7]-[12].

This, therefore, raises the question whether the phenomenon of Hawking radiation is dependent on assumptions about the physics at transplanckian energies. In particular, a fundamental ultraviolet cut-off may well exist at a Planck- or string scale, and the question arises whether Hawking radiation is compatible with the existence of a natural ultraviolet cut-off. A number of studies have therefore investigated the problem of Hawking radiation in the presence of various kinds of ultraviolet cut-off. For reviews, see e.g. [9] or [12]. An intuitive example is Unruh's consideration of the dumb hole [13, 14], which is an acoustic analog of the black hole, where a "horizon" forms where a stationarily flowing fluid's velocity exceeds the velocity of the sound waves.

The consensus in the literature appears to be that Hawking radiation is indeed a robust phenomenon - if only for example for thermodynamical reasons. However, there does not seem to exist a consensus about how precisely the transplanckian frequencies paradox is to be resolved. The recent work by Rosu [5] and Reznik [6] which we mentioned above aims at resolving the transplanckian energies paradox by employing superoscillations. Their main argument is that even fields with a strict ultraviolet cut-off at some maximum frequency can still display arbitrarily high frequency oscillations, indeed any transplanckian frequencies, in some finite region, e.g. close to the horizon, namely if the field superoscillates.

In this context, as in all contexts where an argument is based on the phenomenon of superoscillations, the argument can only be as good as our understanding of the properties of superoscillations.

We will therefore address here three points concerning the general properties of superoscillations:

Firstly, we will apply methods recently developed in [15] to obtain exact results about the extent to which frequency limited functions can superoscillate. Namely, we will show that among the functions with frequency cut-off ω_{max} there always exist functions which pass through any finite number of arbitrarily prespecified points. We will also show that superoscillations cannot be prespecified on any continuous interval. We

can translate this result into the language of information theory:

The implication is that a 20KHz recording of a Beethoven symphony cannot occur as part of a 1Hz bandlimited signal - but that 1 Hz bandlimited signals can indeed always be found which coincide with the Beethoven symphony at arbitrarily many discrete points in time.

Our result on the extent to which frequency limited functions can superoscillate shows, in particular, that frequency limited functions can indeed not be reliably characterized as varying slower than their highest Fourier component. This raises the problem of finding a reliable characterization of the effect of frequency limitation on the “behavior” of functions.

Therefore, secondly, we will show that a reliable characterization of the effect of frequency limitation on the behavior of functions is in terms of an uncertainty relation: If a strictly frequency limited function, $\phi(t)$, superoscillating or not, is sampled at the so-called Nyquist rate, then the standard deviation ΔT of its samples $\phi(t_n)$ is bounded from below by $\Delta T > 1/4\omega_{max}$. We will therefore conclude that a frequency limit is not a limit to how quickly a function can vary, but that instead a frequency limit is a limit to how much a function’s Nyquist rate samples can be peaked.

Thirdly, we will explain how this characterization of frequency limited functions generalizes to time-varying frequency limits $\omega_{max}(t)$. We will apply these results to a recently developed generalized Shannon sampling theorem [15].

Translated into the language of quantum field theory, our results will show, for example, that a frequency cut-off can possess many of the advantages of lattice regularizations without needing to break translation invariance. For example, the Shannon sampling theorem and its generalization implies that the number of degrees of freedom per unit volume (we assume a euclidean formulation) is literally finite for ultraviolet cut-off fields: the fields are fully determined if they are specified on any one of a family of equivalent lattices whose spacing is determined by the ultraviolet-cut-off. The family of lattices is covering the entire continuous space so that translational invariance is not broken. We will cover the general case where the density of degrees of freedom is spatially varying.

2 Examples of Superoscillations

Let us consider functions, $\phi(t)$, which are frequency limited with a maximum frequency ω_{max} , i.e. which contain only plane waves up to this frequency. We can write such functions in the form

$$\phi(t) = \int_{-\infty}^{+\infty} du r(u) e^{it\omega(u)} \quad (1)$$

where $\omega(u)$ is a real-valued function which obeys

$$|\omega(u)| \leq \omega_{max} \quad \text{for all } u \in \mathbb{R} , \quad (2)$$

and where $r(u)$ is a complex-valued function.

Berry [2] gives the explicit example (among other examples)

$$\omega(u) := \frac{\omega_{max}}{1 + u^2} \quad (3)$$

and

$$r(u) := \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{(u-ic)^2}{2\epsilon}} , \quad (4)$$

where ϵ and c are positive constants. The claim is that for suitable choices of ϵ and c the resulting function $\phi(t)$ displays superoscillations, i.e. that in some interval it oscillates faster than ω_{max} .

There is a simple argument for why this should be true. Berry reports this argument to be due to Aharonov:

Namely, for sufficiently small ϵ the function $r(u)$ should effectively become a Gaussian approximation to a Dirac δ -function which is peaked around the imaginary value $u = ic$. Therefore, the factor $r(u)$ in Eq.1 should effectively project out the value of the integrand at $u = ic$. Due to Eq.3, this value of u corresponds to the frequency:

$$\omega(ic) = \frac{\omega_{max}}{1 - c^2} . \quad (5)$$

Clearly, for suitable choices of the parameter c , this frequency can be made arbitrarily larger than the bandwidth ω_{max} .

Thus, the situation is that on the one hand, $\phi(t)$ certainly contains only frequencies up to ω_{max} because $\omega(u) \leq \omega_{max}$ for all *real* values of u , and the intergration in Eq.1 is over real u only. On the other hand, for *imaginary* values of u the value of $\omega(u)$ can become much larger than ω_{max} . Indeed, the behavior of $r(u)$ indicates that the integral should effectively be peaked around the imaginary value $u = ic$. This suggests that, for choices of c close enough to 1, in some interval the function $\phi(t)$ could display superoscillations with frequencies around $\omega_{so} \approx 1/(1 - c^2) > \omega_{max}$.

This intuitive argument for superoscillations has been confirmed, in [2], both by asymptotic analysis and by numerical calculations. Berry also explains in [2] that the price for a function to have this type of a superoscillating period is that the function also possesses a period with exponentially large amplitudes - nevertheless, the whole function is square integrable. We remark that another method for constructing examples of superoscillations has been found in [16].

3 To which extent are frequency limited functions able to superoscillate?

3.1 Definitions

Let us in the following refer to frequency limited functions $\phi(t)$ as “signals” and to the variable t as “time”.

More precisely, we define the class of *signals* ϕ with *bandwidth* ω_{max} as the Hilbert space of square integrable functions on the interval $[-\omega_{max}, \omega_{max}]$ in frequency space

$$H_{\omega_{max}} = L^2(-\omega_{max}, \omega_{max}) \quad (6)$$

with the usual scalar product:

$$(\phi_1, \phi_2) = \int_{-\omega_{max}}^{\omega_{max}} d\omega \tilde{\phi}_1(\omega)^* \tilde{\phi}_2(\omega) \quad (7)$$

We then define the set $B_{\omega_{max}}$ of *strictly bandlimited signals* with bandwidth ω_{max} as the set of all functions $\tilde{\phi}(\omega)$ on frequency space for which there exists a $c(\phi) < \omega_{max}$ such that

$$\tilde{\phi}(\omega) = 0 \quad \text{if } |\omega| > c(\phi) \quad (8)$$

and whose derivatives $d^n \tilde{\phi}(\omega)/d\omega^n$ are square integrable for all $n \in \mathbb{N}$.

Clearly, the strictly bandlimited signals are dense in the Hilbert space of bandlimited signals $H_{\omega_{max}}$:

$$H_{\omega_{max}} = \overline{B_{\omega_{max}}} \quad (9)$$

3.2 Proposition

We claim that each Hilbert space of bandlimited signals $H_{\omega_{max}}$ contains signals such that the Fourier transform of $\tilde{\phi}(\omega)$, i.e. the signal $\phi(t)$, passes through any finite number of arbitrarily prespecified points.

Explicitly, we can fix a value for the bandwidth, ω_{max} . Then, we choose N arbitrary times $\{t_i\}_{i=1}^N$ and N arbitrary amplitudes $\{a_i\}_{i=1}^N$. The claim is that there always exist signals of bandwidth ω_{max} which obey:

$$\phi(t_i) = a_i \quad \text{for all } i = 1, 2, \dots, N \quad (10)$$

In field theory language, we are claiming that for any choice of an ultraviolet cut-off frequency there are fields which obey the cut-off and which at an arbitrary finite number of points in space take arbitrary prespecified values. In particular, since these points and the amplitude of the field at these points can be chosen arbitrarily we claim that even fields which obey a cut-off can vary arbitrarily wildly over any finite interval.

3.3 Proof

Let us first outline the proof. We will begin by considering the simple symmetric operator $T : \phi(t) \rightarrow t\phi(t)$ on $B_{\omega_{max}}$. Its self-adjoint extensions, $T(\alpha)$, then yield a set of Hilbert bases $\{\mathbf{t}_n(\alpha)\}$ of $H_{\omega_{max}}$ as their eigenbases. The amplitudes of bandlimited signals $\phi(t)$ can be written as scalar products with these eigenvectors: $\phi(t) = (\mathbf{t}, \phi)$. The proof of the proposition will consist in showing that any finite set $\{\mathbf{t}_i\}_{i=1}^N$ of basis vectors among all eigenvectors of the self-adjoint extensions is linearly independent.

The “time operator” T

We define the operator T on the domain $D_T := B_{\omega_{max}}$ as the operator which acts on strictly bandlimited signals $\phi(t)$ by multiplication with the time variable:

$$T : \phi(t) \rightarrow T\phi(t) = t\phi(t) \quad (11)$$

The operator T maps strictly bandlimited functions into strictly bandlimited functions:

$$T : B_{\omega_{max}} \rightarrow B_{\omega_{max}} \quad (12)$$

This is because T acts in the Fourier representation as

$$T : \tilde{\phi}(\omega) \rightarrow T\tilde{\phi}(\omega) = -i\frac{d}{d\omega}\tilde{\phi}(\omega) \quad (13)$$

and, clearly, if $\tilde{\phi}(\omega)$ obeys the bandwidth condition, Eq.8, so does its derivative $\partial_\omega\tilde{\phi}(\omega)$.

The elements $\phi \in D_T$ are strictly bandlimited and they therefore obey, in particular:

$$\tilde{\phi}(-\omega) = 0 = \tilde{\phi}(\omega) \quad (14)$$

Thus, for all $\phi \in B_{\omega_{max}}$:

$$\int_{-\omega_{max}}^{\omega_{max}} d\omega \tilde{\phi}_1(\omega)^* (-i\partial_\omega)\tilde{\phi}_2(\omega) = \int_{-\omega_{max}}^{\omega_{max}} d\omega \left((-i\partial_\omega)\tilde{\phi}_1 \right)^* \tilde{\phi}_2(\omega) \quad (15)$$

Consequently,

$$(\phi, T\phi) = (T\phi, \phi) = (\phi, T\phi)^* \quad \forall \phi \in B_{\omega_{max}} \quad (16)$$

and therefore,

$$(\phi, T\phi) \in \mathbb{R} \quad \forall \phi \in B_{\omega_{max}} \quad (17)$$

which means that T is a symmetric operator.

Nevertheless, T is not self-adjoint. Indeed, T possesses no (normalizable nor nonnormalizable) eigenvectors. This is because the only candidates for eigenvectors, namely

the plane waves $e^{2\pi i t \omega}$ do not obey Eqs.8,14. Thus, the plane waves are not strictly bandlimited and therefore they are not in the domain $D_T = B_{\omega_{max}}$ of T . On the other hand, while the plane waves are not strictly bandlimited, they are nevertheless bandlimited, i.e. they are elements of the Hilbert space $H_{\omega_{max}}$. Indeed, the domain of T can be suitably enlarged to yield a whole family of self-adjoint extensions of T , each with a discrete subset of the plane waves as an eigenbasis. We will derive these self-adjoint extensions below. For a standard reference on the functional analysis of self-adjoint extensions, see e.g. [17].

The self-adjoint extensions $T(\alpha)$ of T , and their eigenbases

There exists a $U(1)$ - family of self-adjoint extensions $T(\alpha)$ of T :

The self-adjoint operator $T(\alpha)$ is obtained by enlarging the domain of T by signals, ϕ , which obey the boundary condition:

$$\tilde{\phi}(-\omega) = e^{i\alpha} \tilde{\phi}(\omega) \quad (18)$$

To be precise: We first close the operator T . Then, the domain D_{T^*} of T^* consists of all those signals $\phi \in H$ for which also $-i\partial_\omega \tilde{\phi}(\omega) \in H_{\omega_{max}}$. The signals $\phi \in D_{T^*}$ are not required to obey any boundary conditions. Thus, all plane waves are eigenvectors of T^* . Note that while some plane waves are orthogonal, most are not. This is consistent because T^* is not a symmetric operator: due to the lack of boundary conditions in its domain, T^* also has complex expectation values. Any self-adjoint extension $T(\alpha)$ of T is a restriction of T^* by imposing a boundary condition of the form of Eq.18:

$$D_{T(\alpha)} = \{\phi \in D_{T^*} \mid \tilde{\phi}(-\omega) = e^{i\alpha} \tilde{\phi}(\omega)\}. \quad (19)$$

For each choice of a phase $e^{i\alpha}$ we obtain an operator $T(\alpha)$ which is self-adjoint and diagonalizable. Its orthonormal eigenvectors, $\{\mathbf{t}_n^{(\alpha)}\}_{n=-\infty}^{+\infty}$, obeying

$$T(\alpha)\mathbf{t}_n(\alpha) = t_n(\alpha)\mathbf{t}_n(\alpha), \quad (20)$$

form a Hilbert basis for $H_{\omega_{max}}$. In frequency space, they are the plane waves

$$\tilde{\mathbf{t}}_n^{(\alpha)}(\omega) = \frac{e^{2\pi i t_n(\alpha)\omega}}{\sqrt{2\omega_{max}}} \quad (21)$$

which correspond to the $T(\alpha)$ -eigenvalues:

$$t_n(\alpha) = \frac{n}{2\omega_{max}} - \frac{\alpha}{4\pi\omega_{max}}, \quad n \in \mathbb{Z} \quad (22)$$

As mentioned before, each eigenvector of a self-adjoint extension is also an eigenvector of T^* , the adjoint of T :

$$T^*\mathbf{t}_n(\alpha) = t_n(\alpha) \mathbf{t}_n(\alpha) \quad \forall n, \alpha \quad (23)$$

The eigenvalues of T^* , i.e. the eigenvalues of all the extensions $T(\alpha)$, together, cover the real line exactly once, i.e. for each $t \in \mathbb{R}$ there exists exactly one $e^{i\alpha}$ and one n such that $t = t_n(\alpha)$. We will therefore occasionally write simply \mathbf{t} for $\mathbf{t}_n(\alpha)$. In this notation, Eq.23 reads:

$$T^*\mathbf{t} = t \mathbf{t} \quad (24)$$

Using the scalar product, Eq.7, the signal $\phi(t)$, i.e. the Fourier transform of the function $\tilde{\phi}(\omega)$, can then be written simply as:

$$\phi(t) = (\mathbf{t}, \phi) \quad (25)$$

Thus, the signal as a time-dependent function $\phi(t)$ is the expansion of the abstract signal ϕ in an overcomplete set of vectors, namely in all the eigenbases of the family of operators $T(\alpha)$.

As an immediate consequence we recover the Shannon sampling theorem:

The Shannon sampling theorem, and its translation into field theory terminology

The Shannon sampling theorem states that if the amplitudes of a strictly bandlimited signal $\phi(t)$ are known at discrete points in time with spacing

$$t_{n+1} - t_n = 1/2\omega_{max} \quad (26)$$

which is the so-called Nyquist rate, then the signal $\phi(t)$ can already be calculated for all t :

Namely, let us fix one α . Then, to know the values $\phi(t_n(\alpha))$ of the function $\phi(t)$ at the discrete set of eigenvalues $t_n(\alpha)$ (whose spacing, from Eq.22, is $1/2\omega_{max}$), is to know the coefficients of the vector ϕ in the Hilbert basis $\{\mathbf{t}_n(\alpha)\}$. Thus, ϕ is fully determined as a vector in the Hilbert space $H_{\omega_{max}}$. Therefore, its coefficients can be calculated in any arbitrary Hilbert basis. Thus, in particular, the values of $\phi(t) = (\mathbf{t}, \phi)$ can be calculated for all t :

$$\phi(t) = \sum_{n=-\infty}^{\infty} (\mathbf{t}, \mathbf{t}_n(\alpha)) \phi(t_n(\alpha)) \quad (27)$$

Clearly, Eq.27 is obtained simply by inserting the resolution of the identity $1 = \sum_{n=-\infty}^{\infty} \mathbf{t}_n(\alpha) \otimes \mathbf{t}_n^*(\alpha)$ on the RHS of Eq.25. We note that while for each fixed α the set of vectors $\{\mathbf{t}_n(\alpha)\}$ forms an orthonormal Hilbert basis in H , the basis vectors belonging to different self-adjoint extensions are not orthogonal:

$$(\mathbf{t}_n(\alpha), \mathbf{t}_m(\alpha')) \neq 0 \quad \text{for } \alpha \neq \alpha'. \quad (28)$$

In the sampling formula Eq.27 we need this scalar product, i.e. $(\mathbf{t}, \mathbf{t}_n(\alpha))$, and it is easily calculated for all values of the arguments:

$$(\mathbf{t}, \mathbf{t}') = \int_{-\omega_{max}}^{\omega_{max}} d\omega \frac{e^{2\pi i(t-t')\omega}}{2\omega_{max}}$$

(29)

$$= \frac{\sin(2\pi(t-t')\omega_{max})}{2\pi(t-t')\omega_{max}}$$

Note that the sampling kernel $(\mathbf{t}, \mathbf{t}')$ is real and continuous which means that we describe real, continuous (in fact, entire) signals $\phi(t)^* = \phi(t)$, which would not be the case for other choices of the phases of the eigenvectors \mathbf{t} .

The Shannon sampling theorem has an interesting translation into the language of field theory: Consider first fields, say scalar fields, without an ultraviolet cut-off. These fields possess at each point in space one degree of freedom: the amplitude. Thus, the field possesses an infinite number of degrees of freedom per unit volume.

The Shannon sampling theorem shows that an ultraviolet cut-off field is already determined everywhere if it is known only on any one of a family of discrete lattices. In other words, fields which are ultraviolet cut-off, in the original sense of a frequency cut-off, are continuous fields, which can however be represented without loss of information on certain discrete lattices. This also means that for ultraviolet cut-off fields the number of degrees of freedom of per unit volume is literally finite: it is given by the number of sampling points needed per unit volume in order to be able to reconstruct the field everywhere. The field theoretic meaning of the information theory term ‘‘Nyquist rate’’ is the spatial density of the degrees of freedom of fields. We will later discuss a generalization of the Shannon sampling theorem for classes of signals whose Nyquist rate is time-varying. This theorem will translate into the statement that these signals with time-varying bandwidth correspond to fields whose spatial density of degrees of freedom is spatially varying. These are continuous fields which are representable without loss of information on families of lattices whose minimum spacing is spatially varying.

Superoscillations

We can now prove that for every bandwidth ω_{max} there always exist bandlimited signals $\phi \in H_{\omega_{max}}$, which pass through any finite number of prespecified points.

To this end we choose N arbitrary distinct times t_1, \dots, t_N and N amplitudes a_1, \dots, a_N . We must show that for each such choice and for each bandwidth ω_{max} there exist bandlimited signals $\phi \in H_{\omega_{max}}$ which pass at the times t_i through the values a_i :

$$\phi(t_i) = (\mathbf{t}_i, \phi) = a_i \quad \forall i = 1, \dots, N \quad (30)$$

We recall that the eigenbases of the self-adjoint extensions $T(\alpha)$ of T each yield a resolution of the identity:

$$1 = \sum_{n=-\infty}^{+\infty} \mathbf{t}_n(\alpha) \otimes \mathbf{t}_n^*(\alpha) \quad (31)$$

Inserting one of these resolutions of the identity into Eq.30 we obtain an explicit inhomogeneous system of linear equations:

$$\sum_{n=-\infty}^{+\infty} (\mathbf{t}_i, \mathbf{t}_n(\alpha)) (\mathbf{t}_n(\alpha), \phi) = a_i \quad \forall i = 1, \dots, N. \quad (32)$$

Solutions to Eq.32 exist, i.e. there are bandlimited signals which go through all the specified points, exactly if the matrix $(\mathbf{t}_i, \mathbf{t}_n(\alpha))$ is of full rank

$$\text{rank} \left((\mathbf{t}_i, \mathbf{t}_n(\alpha))_{i=1, n=-\infty}^{i=N, n=+\infty} \right) = N, \quad (33)$$

which is the case exactly if the set of vectors $\{\mathbf{t}_i\}$ is linearly independent.

In order to prove that indeed every finite set of distinct eigenvectors \mathbf{t}_i of T^* is linearly independent, let us now assume the opposite. Namely, let us assume that there does exist a set of N eigenvectors \mathbf{t}_i of T^* , and complex coefficients λ_i which are not all zero, such that:

$$\sum_{i=1}^N \lambda_i \mathbf{t}_i = 0 \quad (34)$$

Since the sum is a finite sum, we can repeatedly apply T^* to Eq.34, to obtain:

$$\sum_{i=1}^N \lambda_i t_i^n \mathbf{t}_i = 0 \quad \forall n \in \mathbb{N} \quad (35)$$

The first N equations yield:

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ t_1 & t_2 & t_3 & \dots & t_N \\ & & \vdots & & \\ t_1^{N-1} & t_2^{N-1} & t_3^{N-1} & \dots & t_N^{N-1} \end{pmatrix} \begin{pmatrix} \lambda_1 \mathbf{t}_1 \\ \lambda_2 \mathbf{t}_2 \\ \lambda_3 \mathbf{t}_3 \\ \vdots \\ \lambda_N \mathbf{t}_N \end{pmatrix} = 0 \quad (36)$$

This $N \times N$ matrix is a Vandermonde matrix and its determinant is known to take the form:

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ t_1 & t_2 & t_3 & \dots & t_N \\ & & \vdots & & \\ t_1^{N-1} & t_2^{N-1} & t_3^{N-1} & \dots & t_N^{N-1} \end{vmatrix} = \prod_{1 \leq j < k \leq N} (t_k - t_j) \quad (37)$$

In particular, the determinant does not vanish, since the \mathbf{t}_i are by assumption distinct, i.e. $t_k \neq t_j$ for all $k \neq j$. Thus, the Vandermonde matrix has an inverse. Multiplying

this inverse from the left onto Eq.36 we obtain that $\lambda_i \mathbf{t}_i = 0$, $\forall i = 1, \dots, N$, i.e. we can conclude that $\lambda_i = 0 \forall i = 1, \dots, N$.

Therefore, any finite set of distinct eigenvectors \mathbf{t} of T^* is indeed linearly independent and consequently Eq.33 is obeyed.

Thus, for any arbitrarily chosen bandwidth ω_{max} , there are indeed signals $\phi \in H_{\omega_{max}}$ which pass through any finite number of arbitrarily prespecified points.

3.4 Beethoven at 1Hz ?

Let us now address the question whether a recording of a Beethoven symphony could indeed appear as part of a 1Hz bandlimited signal. Correspondingly, the question is whether fields on a space with this particular ultraviolet cut-off are free to take pre-specified values on a finite interval.

More explicitly, let us ask for example whether it is possible to take say 5000 seconds of a 20 KHz recording of a Beethoven symphony and to append a suitable function before and suitable function after the symphony, so that the whole signal ranging from time $t = -\infty$ to $t = +\infty$ is a 1Hz bandlimited signal.

If the question is posed in this form, the answer is no. To see this, we recall that bandlimited functions are always entire functions. Entire functions are Taylor expandable everywhere, and with infinite radius of convergence. Thus if an entire function $\phi(t)$ is known on even a tiny interval $[t_i, t_f]$ of the time axis, then we can calculate at a point $t_0 \in [t_i, t_f]$ in that interval all derivatives $d^n/dt^n \phi(t_0)$. This yields a Taylor series expansion of $\phi(t)$ around the time t_0 with infinite radius of convergence. Thus, if a bandlimited function is known on any finite interval then it is already determined everywhere.

One consequence is that a bandlimited signal cannot vanish on any finite interval, since this would mean that it vanishes everywhere. Thus, for example, if the original signal of the Beethoven recording is truly 20KHz bandlimited, then it is an entire function and therefore it does not vanish on any finite interval between $t = -\infty$ and $t = +\infty$. On the other hand, we are only interested in an interval of length about 5000s. Now the question is whether these 5000 seconds of the 20KHz bandlimited recording can occur as a superoscillating period of a signal which is bandlimited, say by 1Hz. The answer is negative because this 1Hz bandlimited signal, if existing, would also be entire - but clearly two entire functions which coincide on a finite interval coincide everywhere.

It is therefore not possible to arbitrarily prespecify the exact values of a 1Hz bandlimited signal on any finite interval. We are left with the question whether there are topologies with respect to which approximations may converge. On the other hand, it is clear that if we wish to prespecify precise values of the signal then the most that may be possible is to arbitrarily prespecify the values of a 1Hz bandlimited signal at arbitrary discrete times. This would mean, for example, that one can find 1Hz bandlimited signals which coincide with the 20KHz Beethoven recording at arbitrarily many

discrete points in time. That it is indeed possible to prespecify the signals' values at an arbitrary finite number of discrete points in time is what we proved in the previous section.

3.5 Superoscillations for data compression?

As is well-known, the bandwidth of a communication channel limits its maximal data transmission rate. We have just seen, however, that signals with fixed bandwidth can superoscillate and exhibit for example arbitrarily fine ripples and arbitrarily sharp spikes. This suggests that it should be possible to encode and transmit an arbitrarily large amount of information in an arbitrarily short time interval of a 1Hz bandlimited signal - because there is always a 1Hz bandlimited signal which passes through any number of arbitrarily prespecified points.

Thus, this raises the question whether superoscillations are able to circumvent the bandwidth limitations of communication channels - and whether, as Berry suggested, superoscillations may for example be used for data compression.

Here, we need to recall that the bandwidth alone does not fix the maximal data transmission rate. It is known that, in the absence of noise, every channel - with any arbitrary bandwidth - can carry an infinite amount of information in any arbitrarily short amount of time.

In practise, every channel has noise and this prevents us from measuring the signal to ideal precision. Essentially, the effect of the noise is that only a finite number of amplitude levels can be resolved. Now if the information is encoded in V different amplitude levels (i.e. binary would be two levels, $V = 2$), then the maximum baud rate b in bits/second is

$$b = 2\omega_{max} \ln_2 V. \quad (38)$$

This follows immediately from the Shannon sampling theorem: Each amplitude measurement yields one out of V possible outcomes, i.e. each measurement yields $\ln_2 V$ bits of information. This yields Eq.38 because by the Shannon theorem we need to measure only $2\omega_{max}$ samples per second to capture all of the signal. We only remark here that for the example of white noise the maximal data transmission rate can be expressed directly in the signal to noise ratio S/N :

$$b_{noise} = \omega_{max} \ln_2 \left(1 + \frac{S}{N} \right) \quad (39)$$

For the precise definitions and the proof, see e.g. the classic text by Shannon, [18].

For us interesting here is that Eqs.38,39 show that indeed even in the presence of noise the data transmission rate can be made arbitrarily large, for any fixed bandwidth - though at a cost. The price to be paid is that in order to increase the baud rate to bandwidth ratio the maximal signal amplitude must be improved exponentially as

compared to the resolution of the amplitude, or more precisely as compared to the noise level.

Let us consider the implications for superoscillations. Superoscillations, in spite of their peculiar behavior, do obey the bandlimit, ω_{max} . Therefore, superoscillations cannot violate the limits on the baud rate in Eqs.38, 39. Indeed, conversely, from the validity of the limits on the baud rate we can deduce properties of superoscillations: If large amounts of information are to be sent over a low bandwidth channel, e.g. by employing superoscillations, this necessitates an exponentially large dynamical range of the superoscillating signal. Indeed, Berry conjectured that superoscillations necessarily occur with exponentially large dynamical ranges.

This is essentially the same as saying that it is difficult to stabilize superoscillations under perturbations:

We showed that it is not possible to prespecify superoscillations on any *continuous time interval*. For example, there is no 1Hz bandlimited function which coincides with a symphony's recording on a continuous interval of say 5000 seconds. On the other hand, we showed that it is possible to prespecify superoscillations at any number of *discrete points in time*. For example, there do exist 1Hz bandlimited functions which coincide with the 20KHz bandlimited Beethoven recording at 10^{1000} points in time during the 5000 seconds of the recording.

Thus, a 1Hz bandlimited function which coincides with a symphony's recording at 10^{1000} points on a 5000s interval, can only be 1Hz bandlimited because of very fine-tuned cancellations in the calculation of its Fourier spectrum - cancellations which depend on small details of the function. We therefore conclude that tiny perturbations of such a 1Hz bandlimited superoscillating function are able to induce very high frequency components. Thus, superoscillations are in this sense unstable, and they are therefore likely to be difficult to make practical use of in imperfect communication channels.

On the other hand, as the reverse side of the coin, important phenomena in signal processing are instabilities in the reconstruction procedures of signals which are over-sampled, i.e. which are sampled at a rate higher than the Nyquist rate. The instabilities in the reconstruction arise because small imprecisions in the measurement of the then overdetermined samples of an ordinary (i.e. in general non-superoscillating) signal can lead to the reconstruction of a deviant signal which, in our terminology, possesses superoscillations. This connection was pointed out already by Berry [2], quoting I. Daubechi. For a general reference on oversampling see e.g. [19].

In terms of models of fields at the Planck scale, the instabilities of superoscillations suggest that in ultraviolet cut-off quantum field theories the interaction of particles whose fields superoscillate could easily destroy their superoscillations. In concrete cases this effect is likely to depend on the type interactions of the field theory that one considers. Studies in this direction could be worth pursuing since these instabilities could have implications for example for the viability of the Rosu-Reznik approach to su-

peroscillations in black hole radiation when treated within a framework of interacting fields.

4 A strict bandlimit is a lower bound to how much the samples of a signal can be peaked - an uncertainty relation for ultraviolet cut-off fields

4.1 The minimum standard deviation

The existence of superoscillations shows that bandlimited functions cannot be characterized reliably as varying at most as fast as their highest Fourier component. Indeed, we have just proved that for any fixed bandwidth there always exist functions which possess arbitrarily fine ripples and arbitrarily sharp spikes. Let us therefore look for a better, i.e. for a reliable characterization of the effect of bandlimitation on the behavior of functions.

Our proposition is that, while a bandlimit does not imply a bound on how much bandlimited signals can *locally* be peaked, a bandlimit does imply a bound to how much strictly bandlimited signals can be peaked *globally*. Equivalently, our proposition is that while an ultraviolet cut-off does not imply a bound to how much the fields can be peaked locally in space, the cut-off does imply a lower bound to how much the fields can be peaked globally.

Our motivation derives from the Heisenberg uncertainty principle: If we read T as the momentum operator of a particle in a (one-dimensional) box then, because the position uncertainty is bounded from above by the size of the box, we expect the momentum uncertainty (here $\Delta T(\phi)$) to be bounded from below.

To be precise, consider a normalized, strictly bandlimited signal $\phi \in B_{\omega_{max}}$. Then,

$$\bar{T}(\phi) := (\phi, T\phi) \tag{40}$$

is the T -expectation value, or the time-mean or the “center of mass” of the signal ϕ on the time axis. A measure of how much the signal is overall peaked around this time is the formal standard deviation:

$$\Delta T(\phi) := \sqrt{(\phi, (T - \bar{T}(\phi))^2 \phi)} \tag{41}$$

We note that both, $\bar{T}(\phi)$ and $\Delta T(\phi)$ are not sensitive to local features of $\phi(t)$, such as fine ripples and sharp spikes. Instead, being the first and second moment of T , the time $\bar{T}(\phi)$ is simply the signal’s global average position on the time axis and $\Delta T(\phi)$ is the global spread of the signal around that position.

Our claim is that strictly bandlimited signals, $\phi \in B_{\omega_{max}}$, are always globally spread by at least a certain minimum amount:

$$\Delta T(\phi) > \frac{1}{4\omega_{max}} \quad \text{for all } \phi \in B_{\omega_{max}} \quad (42)$$

In field theory language, our claim is that there exists a formal finite minimum uncertainty in position for ultraviolet cut-off fields.

4.2 The minimum standard deviation as a property of the Nyquist rate samples

Let us now rewrite $\bar{T}(\phi)$ and $\Delta T(\phi)$ as explicit expressions in the signals $\phi(t)$ as functions of time. To this end, we can use any one of the resolutions of the identity $1 = \sum_{n=-\infty}^{+\infty} \mathbf{t}_n(\alpha) \otimes \mathbf{t}_n^*(\alpha)$ which are induced by the self-adjoint extensions $T(\alpha)$ of T . Inserting one of the resolutions of the identity into Eq.40 we obtain, restricting attention to signals $\phi \in B_{\omega_{max}}$ which are real, $\phi(t)^* = \phi(t)$:

$$\bar{T}(\phi) = \sum_{n=-\infty}^{\infty} \phi(t_n(\alpha))^2 t_n(\alpha), \quad (\text{independently of } \alpha) \quad (43)$$

Thus, $\bar{T}(\phi)$ is the “mean” of the discrete set of samples of the signal, when sampled on one of the time-lattices of Eq.22, i.e., $\bar{T}(\phi)$ is the time around which the discrete samples of the signal ϕ are centered. Indeed, for each set of samples taken at the Nyquist rate (i.e. for each time lattice corresponding to some fixed α), the time $\bar{T}(\phi)$ around which the samples are centered is the same. This is because in order to calculate $\bar{T}(\phi)$ from Eq.41 we can equivalently use any one of the resolutions of the identity $1 = \sum_{n=-\infty}^{+\infty} \mathbf{t}_n(\alpha) \otimes \mathbf{t}_n^*(\alpha)$.

Similarly, we obtain an explicit expression for how much the samples are spread around the value $\bar{T}(\phi)$ by inserting a resolution of the identity into the expression for the standard deviation, Eq.41:

$$\Delta T(\phi) = \sqrt{\sum_{n=-\infty}^{\infty} \phi(t_n(\alpha))^2 (t_n(\alpha) - \bar{T}(\phi))^2} \quad (44)$$

Again, also the standard deviation does not depend on which sampling lattice $\{t_n(\alpha)\}$ has been chosen. We remark that, clearly, not only the mean and standard deviation, but indeed also all higher moments of a bandlimited signal’s Nyquist rate samples are independent of the choice of the lattice of sampling times. We can therefore refer to the mean, the standard deviation and to the higher moments of a signal ϕ without needing to specify the choice of a sampling lattice.

On the other hand, let us emphasize that the values of $\bar{T}(\phi)$ and $\Delta T(\phi)$ are not the

usual mean and standard deviation of a continuous curve as conventionally calculated in terms of integrals rather than sums. Instead, while the strictly bandlimited signals are of course continuous, $\bar{T}(\phi)$ and $\Delta T(\phi)$ are the mean and the standard deviation of their discrete Nyquist rate samples.

Our proposition of above, i.e. Eq.42, if expressed explicitly in terms of the strictly bandlimited signal's Nyquist rate samples, is therefore that the standard deviation $\Delta T(\phi)$ of these samples is bounded from below by $1/4\omega_{max}$.

4.3 Calculation of the maximally peaked signals/fields

In order to prove the lower bound on the standard deviation expressed in Eq.42, let us now explicitly solve the variational problem of finding signals ϕ which minimize $\Delta T(\phi)$. To this end, we minimize $(\phi, T^2\phi)$ while enforcing the constraints $(\phi, T\phi) = t$ and $(\phi, \phi) = 1$.

We work in frequency space, where T acts on the strictly bandlimited signals as the symmetric operator $T = -i d/d\omega$.

Introducing Lagrange multipliers k_1, k_2 , the functional to be minimized reads:

$$S[\phi] := \int_{-\omega_{max}}^{\omega_{max}} d\omega \left\{ -(\partial_\omega \tilde{\phi}^*)(\partial_\omega \tilde{\phi}) + k_1(\tilde{\phi}^* \tilde{\phi} - c_1) + k_2(-i\tilde{\phi}^* \partial_\omega \tilde{\phi} - c_2) \right\}, \quad (45)$$

Setting $\delta S[\phi]/\delta\phi = 0$ yields the Euler-Lagrange equation:

$$\partial_\omega^2 \tilde{\phi} + k_1 \tilde{\phi} - i\partial_\omega \tilde{\phi} = 0 \quad (46)$$

Imposing the boundary condition, Eq.14, which is obeyed by all strictly bandlimited signals, we obtain exactly one (up to phase) normalized solution $\Phi_{\bar{T}}$ for each value of the mean \bar{T} :

$$\tilde{\Phi}_{\bar{T}}(\omega) = \frac{1}{\sqrt{2\pi\omega_{max}}} \cos\left(\frac{\pi\omega}{2\omega_{max}}\right) e^{2\pi i\bar{T}\omega} \quad (47)$$

The standard deviations, $\Delta T(\Phi_{\bar{T}})$, of these solutions are straightforward to calculate in Fourier space, to obtain:

$$\Delta T(\phi_t) = \frac{1}{4\omega_{max}} \quad \text{for all } t \quad (48)$$

Since the signals $\tilde{\Phi}_{\bar{T}}(\omega)$ which minimize ΔT are not themselves strictly bandlimited - they do not obey Eq.8 - we can conclude that all strictly bandlimited signals, or ultraviolet cut-off fields, obey the strict bound given in Eq.42.

5 Generalization to time-varying bandwidths - or spatially varying ultraviolet cut-offs

5.1 Superoscillations and the concept of time-varying bandwidth

Intuitively, it is clear that the bandwidths of signals can vary with time. One might therefore expect to be able to define the time-varying bandwidth of signals for example in terms of the highest frequency components which they contain in intervals centered around different times. This approach encounters difficulties, however, due to the existence of superoscillations:

We recall that a signal $\phi(t)$ obeys a *constant* bandlimit ω_{max} if its Fourier transform

$$\tilde{\phi}(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} dt \phi(t) \exp(2\pi i\omega t) \quad (49)$$

has support only in the interval $[-\omega_{max}, \omega_{max}]$. The integration in Eq.49 ranges over the entire time axis. This means that the bandlimit is a *global* property of the signal.

If it were true that bandlimited signals could nowhere vary faster than their highest frequency component then this would mean that the bandwidth is also a *local* property of the signal. Namely, one might then expect that if we consider the same signal on some finite interval, $[t_i, t_f]$, and if we calculate its Fourier expansion on that interval then we will find that its Fourier coefficients are nonzero only for frequencies smaller or equal than ω_{max} . If so, we could indeed define time-varying bandwidths as time-varying upper limits on the local frequency content, as indicated above. Indeed, in practise, windowed Fourier transforms and in particular the more sophisticated Wigner transforms or wavelet decompositions, are generally very useful [20, 21, 22].

However, the existence of superoscillations shows that any *local* definition of a time-varying bandwidth must contain counterintuitive features: This is because whatever the overall bandwidth ω_{max} , there are always signals with this bandwidth which superoscillate in any given interval $[t_i, t_f]$. In practise, of course, strongly superoscillating signals will rarely occur because they are very fine-tuned. But their existence shows that there do exist low bandwidth signals which locally possess arbitrarily high frequency components - where “local frequency components” are defined e.g. by windowed Fourier transforms - in any finite length interval.

In field theory terminology this means that even if a field is varying wildly in some spatial region, this does not imply that the field necessarily possesses a large cut-off frequency or, equivalently, that it possesses a high density of degrees of freedom. Instead, even at small cut-off frequencies there are fields which locally display fast oscillations. Even these superoscillating fields are fully determined everywhere (by the sampling theorem) if known only on any one of the family of lattices whose lattice spacing is as large as is consistent with the ultraviolet cut-off.

5.2 The time-varying bandwidth as a limit to how much the samples of signals can be peaked around different times

We saw that a finite bandwidth does not impose a limit to how much signals can be *locally* peaked around say a time t . However, we also saw that a finite bandwidth does impose a limit ΔT_{min} to how much the signals can be *globally* peaked, around any time t . Indeed, this characterization of the effect of bandlimitation naturally generalizes to time-varying bandwidths: Namely, the limit to how much signals can be peaked may in general depend on the time t around which they are peaked:

We found that a constant bandwidth can be understood as a minimum standard deviation of the signals' Nyquist rate samples: If a strictly bandlimited signal $\phi \in B_{\omega_{max}}$ is centered around a time $t = \bar{T}(\phi)$, then its standard deviation around the time t is always bounded from below by the uncertainty relation $\Delta T(\phi) > 1/4\omega_{max}$.

We were then only discussing the case of constant bandwidth. Accordingly, we found that the standard deviation of signals $\phi \in B_{\omega_{max}}$ which are centered around a time t_1 obey the same lower bound $1/4\omega_{max}$ as do signals $\phi' \in B_{\omega_{max}}$ which are centered around some other time t_2 .

This suggests to try to define the notion of time-varying bandwidth in such a way that a class of strictly bandlimited signals with a time-varying bandwidth is simply a class of signals for which the minimum standard deviation ΔT_{min} depends on the time t around which the signals are centered. This would mean that the uncertainty relation Eq.42 becomes time dependent:

$$\Delta T(\phi) > \Delta T_{min}(\bar{T}(\phi)) \tag{50}$$

Correspondingly, we would expect the Nyquist rate to be time-varying.

To this end, let us recall the functional analytic structure of the Hilbert space of bandlimited signals which we discussed in Sec.3.3: The operator T is a simple symmetric operator with deficiency indices $(1, 1)$, whose self-adjoint extensions have purely discrete and *equidistant* spectra.

Indeed, the theory of simple symmetric operators with deficiency indices $(1, 1)$, whose self-adjoint extensions have discrete *but not necessarily equidistant* spectra, has been shown to yield a generalized Shannon sampling theorem in [15], and it is indeed exactly the theory of time-varying bandwidths in the sense which we just indicated. For example, the nonequidistant spectra yield time-varying Nyquist rates. The time-varying Nyquist rate can be calculated from the time-varying minimum standard deviation $\Delta T_{min}(t)$ and vice versa. This is worked out in detail in [23].

In terms of field theory, the time-varying bandwidth means, as we mentioned already, a spatially varying ultraviolet cut-off. We remark that this is a nontrivial generalization of the concept of frequency (or energy-momentum) cut-off in field theories. An ordinary energy-momentum cut-off affects fields globally, i.e. the cut-off scale is the

same everywhere in space. But we may ask: how could an energy momentum cut-off be implemented such that the cut-off-scale is spatially varying? For example, the cut-off scale may be dynamically generated, e.g. through an interplay of gravity and ordinary forces. In such a scenario, the actual cut-off scale may be dynamic and spatially varying, e.g. determined by the value of some field. In our approach to defining spatially varying cut-offs the cut-off is understood as a formal minimum uncertainty or standard deviation in position. For constant bandwidths this is an equivalent definition to the usual definition as a frequency cut-off. We then found that the notion of formal minimum position uncertainty generalizes ‘naturally’ to the situation where the value of the formal minimum position uncertainty depends on the position, i.e. on the formal position expectation value of the field. As we will discuss in the last section, there is in fact very little arbitrariness in the definition of these short-distance structures.

6 Outlook

Functions with a bounded Fourier spectrum appear in numerous contexts from theoretical physics to the experimental sciences and engineering applications. A priori, the phenomena of superoscillations can play a role in each of these contexts.

Our aim here has been to investigate the general properties of superoscillations. In particular, we found precise results about the extent to which frequency limited functions can superoscillate. Further, we gave a reliable characterization of the effect of frequency limitation on the behavior of functions, in terms of uncertainty relations.

We formulated much of our discussion in the concrete and intuitive language of information theory but, of course, our results can easily be translated into all those physical contexts where frequency limited functions occur. Here, we chose to always translate our results into the context of ultraviolet cut-off fields, where the ultraviolet cut-off is understood in the original sense of a high frequency cut-off. We mentioned that superoscillations in field theory have been suggested, by Rosu and Reznik, to resolve the transplanckian frequencies paradox of black hole radiation. In this context, our results showed that while generic superoscillations of arbitrarily high frequencies do exist, they could be too unstable under perturbations by interactions. This problem should be worth further pursuing.

We also obtained the general result that strictly bandlimited signals obey a lower bound ΔT_{min} on the standard deviation of their Nyquist rate samples and we generalized to time-varying bandwidths. In terms of ultraviolet cut-off quantum field theory these results mean that the fields in ultraviolet cut-off field theories obey a formal minimum spatial uncertainty ΔX_{min} , where the minimum value of this formal position uncertainty can be spatially varying. In particular, we found that in ultraviolet cut-off quantum field theories the Nyquist rate for signals corresponds exactly to the in general spatially varying density of local degrees of freedom.

So-far in our discussion we assumed this short-distance cut-off to arise from the crude ultraviolet cut-off obtained by cutting off high momenta. However, interestingly, the same short-distance structure can also arise for example in theories with effective Heisenberg uncertainty relations which contain correction terms of the form:

$$\Delta X \Delta P \geq \frac{\hbar}{2} \left(1 + k(\Delta P)^2 + \dots \right), \quad (51)$$

As is easy to verify, for a suitable small positive constant k , Eq.51 indeed yields a lower bound

$$\Delta X_{min} = \hbar \sqrt{k} \quad (52)$$

which could be at a Planck- or at a string scale. This type of uncertainty relation implies that the momentum stays unbounded. This means that the short-distance structure which we have here considered - a formal finite minimum uncertainty in position - is not tied to putting an upper bound to momentum. Indeed, correction terms to the uncertainty relations of the type of Eq.51 have appeared in various studies in the context of quantum gravity and string theory. For reviews, see e.g. [24, 25]. For recent discussion of potential physical origins of this type of uncertainty relations see e.g. [26, 27].

Quantum mechanical and quantum field theoretical models which display such uncertainty relations have been investigated in detail. For example, the ultraviolet regularity of loop graphs in such field theories has been shown. See [28, 29, 30].

In work by Brout et al, [11], it has been shown that this type of short-distance cut-off without energy-momentum cut-off, when built into quantum theory could resolve the transplanckian energies paradox of black hole radiation - without invoking superoscillations. Since as we now see, both, the approaches of Rosu and Reznik, [5, 6], and of Brout et al, [11], assume in fact the same short-distance structure it should be very interesting to investigate their relationship. A recent reference in this context is [31].

Finally, we remark that it is not necessarily surprising that various different studies in quantum gravity and in string theory have led to the same short-distance structure, namely the short distance structure that arises for example from the uncertainty relation Eq.51. In a certain sense it is not even surprising that the same type of minimum uncertainty structure also appears in communication engineering:

This is because, as has been shown in [32], in any theory, any real degree of freedom which is described by an operator which is *linear* can only display very few types of short-distance structures. The basic possibilities are continua, lattices and two basic types of unsharp short distance structures, which have been named “fuzzy-A” and “fuzzy-B”. All others are mixtures of these. Technically, the unsharp real degrees of freedom are those described by simple symmetric operators with nonzero (and, for the two types fuzzy-A and fuzzy-B either equal or unequal) deficiency indices. The “time” degree of freedom of electronic signals is real, the corresponding time operator T therefore had to fall into this classification, and among the few possibilities it happened to

be of the type fuzzy-A.

But equally, we can consider for example in the matrix model of string theory the coordinates of D0-branes. These are encoded in (the diagonal of) self-adjoint matrices X_i . The quantization and the limit for the matrix size $N \rightarrow \infty$ are difficult, but it is clear that the X_i will eventually be operators which are at least symmetric i.e. that their formal expectation values are real. The short distance structure which these X_i display will therefore fall into this classification which we mentioned. Since there are only these few basic possibilities continuous, discrete or “fuzzy”, they may well be found to be of one of the fuzzy, types.

In the present paper we have been concerned with short-distance structures which are characterized by a formal finite minimum uncertainty. The classification given in [32] shows that all such degrees of freedom are of the type fuzzy-A.

We can therefore also view our present results on superoscillations as clarifying aspects of one of these very general classes of short-distance structures of real degrees of freedom. Our results on superoscillations translate in any theory which contains unsharp degrees of freedom of this type.

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