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HOMOGENEOUS GRAPHS AND STABILITY

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Abstract

Let Γ be a graph with finite vertex set V. Γ is *homogeneous* if whenever $U_1, U_2 \subseteq V$ are such that the vertex subgraphs $\langle U_1 \rangle$, $\langle U_2 \rangle$ are isomorphic, then every isomorphism from $\langle U_1 \rangle$ to $\langle U_2 \rangle$ extends to an automorphism of Γ; homogeneous graphs were studied by Sheehan (1974) and were classified by the author. Γ is *locally homogeneous* if whenever $U \subseteq V$, then every automorphism of $\langle U \rangle$ extends to an automorphism of Γ. We prove that every locally homogeneous graph is homogeneous.

We study finite, undirected, loopless graphs $\Gamma = (V, E)$, with vertex set $V = V\Gamma$, edge set $E \subseteq V \times V$, and automorphism group $\mathrm{Aut}\Gamma = G$. If $U \subseteq V$, then the vertex subgraph $\langle U \rangle$ has vertex set U and edge set $(U \times U) \cap E$. We have a natural metric ∂ on V and denote by d the diameter of Γ. Set

$$\Gamma_i := \{(u, v) \in V \times V : \partial(u, v) = i\}, \qquad 0 \leqq i,$$

and for $u \in V$

$$\Gamma_i(u) := \{v \in V : (u, v) \in \Gamma_i\}.$$

We write $\Gamma(u) := \Gamma_1(u)$.

K_r denotes the complete graph on r vertices, $K_{k,k}$ denotes the complete bipartite graph of valency k, $K_{t:r}$ denotes the complete t-partite graph with blocks of size r, C_n denotes the circuit of length n, 0_3 denotes Petersen's graph. (These graphs are described in Wilson (1972), to which the reader is referred for the general graph theoretical background.)

If Γ is a graph, then $t \cdot \Gamma$ denotes the disjoint union of t copies of Γ, Γ^c denotes the complement of Γ, and $L(\Gamma)$ denotes the line graph of Γ.

If $U \subseteq V$, then G_U and $G_{\{U\}}$ denote respectively the pointwise and setwise stabilisers of U; if $U = \{u_1, u_2, \cdots, u_t\}$, then we simply write $G_{u_1 u_2 \cdots u_t}$ and $G_{\{u_1, u_2, \cdots, u_t\}}$. S_t denotes the symmetric group on t symbols. Basic facts about permutation groups can be found in Wielandt (1964).

A graph is *homogeneous* if whenever $\langle U_1\rangle$, $\langle U_2\rangle$ are isomorphic subgraphs of Γ each isomorphism from $\langle U_1\rangle$ to $\langle U_2\rangle$ extends to an automorphism of Γ. Finite homogeneous graphs were studied by Sheehan (1974) and were classified by the present author.

A graph is *locally homogeneous* if whenever $U \subseteq V$, each isomorphism from $\langle U\rangle$ to $\langle U\rangle$ extends to an automorphism of Γ. Clearly Γ is locally homogeneous if and only if Γ^c is locally homogeneous. Each homogeneous graph is locally homogeneous by definition. We prove the converse by classifying locally homogeneous graphs.

THEOREM. *A finite locally homogeneous graph is homogeneous.*

Let Γ be a graph and let $v_1, v_2, \cdots, v_t \in V$; then we define $\Gamma_{v_1} := \langle V - \{v_1\}\rangle$, and for each i, $1 \leqq i < t$, $\Gamma_{v_1v_2\cdots v_{i+1}} := (\Gamma_{v_1v_2\cdots v_i})_{v_{i+1}}$.

A graph Γ is *stable* if for some enumeration $(v_1, v_2, \cdots, v_n)$ of the vertex set V, $G_{v_1v_2\cdots v_i} = \mathrm{Aut}(\Gamma_{v_1v_2\cdots v_i})$ for each i, $1 \leqq i \leqq n$. Γ is *totally stable* if for each enumeration $(v_1, v_2, \cdots, v_n)$ of the vertex set V, $G_{v_1v_2\cdots v_i} = \mathrm{Aut}(\Gamma_{v_1v_2\cdots v_i})$ for each i, $1 \leqq i \leqq n$.

COROLLARY [Yap (1974) Theorem 4]. *The only totally stable graphs are the complete and the null graphs.*

We assume throughout that Γ is some locally homogeneous graph.

LEMMA 1. *If Γ is disconnected, then $\Gamma \cong t \cdot K_r$ for some $t, r \geqq 1$.*

PROOF. Let $V = U_1 \cup U_2 \cup \cdots \cup U_t$, $t \geqq 2$, be the decomposition of V into connected components. Choose $u_i \in U_i$, $1 \leqq i \leqq t$; then $\langle u_1, u_2, \cdots, u_t\rangle \cong t \cdot K_1$, so $G_{[u_1,u_2,\cdots,u_t]}$ induces the full symmetric group S_t on $\langle u_1, u_2, \cdots, u_t\rangle$. Thus all the connected components of Γ are isomorphic. If $\langle U_1\rangle$ is not a complete graph choose u_0, u_1 such that $u_0 \in \Gamma_2(u_1)$; then $G_{[u_0,u_1,\cdots,u_t]}$ induces the full symmetric group S_{t+1} on $\langle u_0, u_1, \cdots, u_t\rangle \cong (t+1)\cdot K_1$. Hence $t = 1$.

LEMMA 2. *If Γ is connected, then (G, V) is transitive and either $\Gamma \cong K_{k+1}$, or $d = 2$.*

PROOF. If $u \in V$, $v \in \Gamma(u)$, then $G_{[u,v]}$ induces the symmetric group S_2 on $\langle u, v\rangle$. Thus for each $u \in V$, $v \in \Gamma(u)$, G contains an element g_v for which $u^{g_v} = v$; since Γ is connected, (G, V) must be transitive. In particular $|\Gamma(u)| = k$ and $|\Gamma_2(u)| = k_2$ are independent of $u \in V$. If $d = 1$, then $\Gamma \cong K_{k+1}$. Assume $d \geqq 2$ and choose $u \in V$, $v \in \Gamma_2(u)$. If $d = 4$, choose $w \in \Gamma_4(u)$; then $G_{[u,v,w]}$ must induce the full symmetric group S_3 on $\langle u, v, w\rangle$, which is evidently impossible. Suppose $d = 3$. If we can choose $w \in \Gamma_3(u) - \Gamma(v)$, then we obtain a contradiction as for $d = 4$. Thus $\Gamma_3(u) \subseteq \Gamma(v)$ for each $v \in \Gamma_2(u)$, $\Gamma_2(u) \subseteq \Gamma(w)$

for each $w \in \Gamma_3(u)$; then $\Gamma_2(w) \supseteq \Gamma(u)$, so since (G, V) is transitive we have $\Gamma_2(u) = \Gamma(w)$, $\Gamma(u) = \Gamma_2(w)$, and $\Gamma_3(u) = \{w\}$. But then for $x \in \Gamma(u)$, $\Gamma_3(x) = \{y\} \subseteq \Gamma_2(u)$, $\langle u, x, y\rangle = K_2 \cup K_1$. and no element of $G_{\{u,x,y\}}$ can fix y and interchange $u \in \Gamma_2(y)$ and $x \in \Gamma_3(y)$. Thus $d = 2$.

LEMMA 3. *If Γ has diameter $d = 2$, then for each $u \in V$, G_u acts transitively on $\Gamma(u)$ and on $\Gamma_2(u)$ (in other words: Γ is a rank three graph).*

PROOF. Let $v \in \Gamma(u)$. For each $w \in \Gamma(v) \cap \Gamma(u)$, $G_{\{u,v,w\}}$ induces the full symmetric group S_3 on $\langle u, v, w\rangle$ so G_u contains an element interchanging v and w. Thus G_u acts transitively on each connected component of $\langle\Gamma(u)\rangle$. Further if v_1, v_2 lie in distinct connected components of $\langle\Gamma(u)\rangle$, then $G_{\{u,v_1,v_2\}}$ contains an element fixing u and interchanging v_1 and v_2. Hence G_u acts transitively on the connected components of $\langle\Gamma(u)\rangle$. The result for $\Gamma_2(u)$ follows by considering the complement of Γ.

Thus we may assume that Γ is a connected (rank three) graph of diameter $d = 2$.

LEMMA 4. *$\langle\Gamma(u)\rangle$ is locally homogeneous.*

PROOF. Let $U \subseteq \Gamma(u)$. Then each automorphism φ of $\langle U\rangle$ corresponds to a unique automorphism $\hat{\varphi}$ of $\langle U \cup \{u\}\rangle$ fixing u, and $\hat{\varphi}$ extends to an automorphism of Γ (fixing u) which leaves $\Gamma(u)$ invariant. Thus φ extends to an automorphism of $\langle\Gamma(u)\rangle$.

Thus if Γ is locally homogeneous we may choose $u \in V = V\Gamma$ and pass to the locally homogeneous graph $\Gamma^1 := \langle\Gamma(u)\rangle$, then choose $u_1 \in V\Gamma^1$ and pass to the locally homogeneous graph $\Gamma^2 := \langle\Gamma^1(u_1)\rangle$, etc., until we finally obtain some graph Γ^i isomorphic to $t \cdot K_r$ for some t, r. We must thus determine the minimal class of graphs which contains all the graphs $t \cdot K_r$ and which is closed with respect to 'extension'. Let $u \in V$.

LEMMA 5. *If $\langle\Gamma(u)\rangle \cong k \cdot K_1$, $k \geqq 2$, then $\Gamma \cong C_5$ or $\Gamma \cong K_{k,k}$.*

PROOF. We assume $k \geqq 3$, $d = 2$. For each $w \in \Gamma_2(u)$, $|\Gamma(w) \cap \Gamma(u)| = c_2$ is constant. If $c_2 = 1$, then Γ is a Moore graph admitting a rank three group, and so is either 0_3 or the Hoffman-Singleton graph; however 0_3 contains a vertex subgraph isomorphic to $3 \cdot K_2$ on which $\mathrm{Aut}\, 0_3 \cong S_5$ does not induce the full wreath product $S_2 \wr S_3$, and the Hoffman-Singleton graph has vertex stabiliser $G_u \cong S_7$ whereas if $v \in \Gamma(u)$, then G_{uv} induces $S_{k-1} \times S_{k-1}$ on $\Gamma(u) \cup \Gamma(v)$. If $k > c_2 \geqq 2$, then $G_{\{\{u\} \cup \Gamma(u)\}}$ induces the full symmetric group S_k on $\Gamma(u)$ so each c_2-subset of $\Gamma(u)$ corresponds to a vertex of $\Gamma_2(u)$ and

$$\binom{k}{c_2} \leqq k_2 = \frac{k(k-1)}{c_2}.$$

Moreover if $x \in \Gamma(w) \cap \Gamma_2(u)$, then $\Gamma(x) \cap \Gamma(u) \cap \Gamma(w) = \varnothing$, so $2c_2 \leqq k$. Hence $c_2 = 2$. $G_{[\{u,w\} \cup (\Gamma(u) - \Gamma(w))]}$ induces the full symmetric group S_{k-2} on $\Gamma(u) - \Gamma(w)$ so each 2-subset of $\Gamma(u) - \Gamma(w)$ corresponds to a vertex of $\Gamma(w) \cap \Gamma_2(u)$. Thus $\binom{k-2}{2} = k - 2$, so $k = 5$, $|V| = 16$, $\langle \Gamma_2(u) \rangle \cong 0_3$, and 0_3 is not locally homogeneous. Thus $c_2 = k$, $\Gamma \cong K_{k,k}$.

LEMMA 6. *If* $\langle \Gamma(u) \rangle \cong K_k$, *then* $\Gamma \cong t \cdot K_{k+1}$, *for some* t.

LEMMA 7. *If* $\langle \Gamma(u) \rangle \cong t \cdot K_r$, $r \geqq 2$, $t \geqq 2$, *then* $\Gamma \cong L(K_{3,3})$.

PROOF. Let $\Gamma(u) = U_1 \cup U_t \cup \cdots \cup U_t$ be the decomposition of $\langle \Gamma(u) \rangle$ into connected components. If $v \in \Gamma_2(u)$, then $|\Gamma(v) \cap U_i| \leqq 1$, so $c_2 \leqq t$. $G_{[\{u\} \cup \Gamma(u)]}$ induces the full wreath product $S_r \wr S_t$ on $\Gamma(u)$. Thus each subgraph $c_2 \cdot K_1$ of $\langle \Gamma(u) \rangle$ corresponds to some vertex of $\Gamma_2(u)$. Hence

$$\binom{t}{c_2} r^{c_2} \leqq k_2 = \frac{tr(tr-r)}{c_2},$$

so $c_2 \leqq 2$. On the other hand $\Gamma(u) \cap \Gamma_2(v)$ contains a subset U with $\langle U \rangle = t \cdot K_{r-1}$, so $G_{[\{u,v\} \cup U]}$ induces the full wreath product $S_{r-1} \wr S_t$ on U and fixes both u and v. Hence $t = c_2 = 2$, Γ is a line graph, $\Gamma = L(\Delta)$, and Δ is bipartite of diameter two and valency $r + 1$. Hence $\Gamma \cong L(K_{r+1,r+1})$. If $s \geqq 4$, then $L(K_{s,s})$ contains a vertex subgraph $2 \cdot C_4$ on which the full automorphism group $D_8 \wr S_2$ is not induced. Hence $r = 2 = t$, $\Gamma \cong L(K_{3,3})$.

LEMMA 8. *If* $\langle \Gamma(u) \rangle \cong K_{t;r}$, $r \geqq 2$, $t \geqq 2$, *then* $\Gamma \cong K_{t+1;r}$.

PROOF. Choose $v \in \Gamma(u)$ and set $\Gamma(v) \cap \Gamma_2(u) =: W$. In $\langle \Gamma(v) \rangle \cong K_{t;r}$ we see that $\langle \{u\} \cup W \rangle \cong r \cdot K_1$ and that for each $v_1 \in \Gamma(u) \cap \Gamma(v)$, $W \subseteq \Gamma(v_1)$. Since $\langle \Gamma(u) \rangle$ is connected we obtain $W = \Gamma_2(u)$, $\Gamma \cong K_{t+1;r}$.

LEMMA 9. $\langle \Gamma(u) \rangle \not\cong C_5$, $L(K_{3,3})$.

PROOF. Suppose $\langle \Gamma(v) \rangle \cong C_5$ for each $v \in V$. Considering each $v \in \Gamma(u)$ in turn forces $\langle \Gamma_2(u) \rangle \cong C_5$, whence Γ is the isosahedron, contrary to $d = 2$. Suppose $\langle \Gamma(u) \rangle \cong L(K_{3,3})$. Choose $w \in \Gamma_2(u)$. If $v \in \Gamma(u) \cap \Gamma(w)$, then $|\Gamma(u) \cap \Gamma(v) \cap \Gamma(w)| = 2$, and applying this to $v_1 \in \Gamma(u) \cap \Gamma(w) \cap \Gamma(v)$ implies $c_2 \geqq 4$. Since $9 \cdot 4/c_2 = k_2$ we have either (a) $c_2 = 4$, or (b) $c_2 = 6$. If $c_2 = 4$, then $|V| = 19$, so Γ is a graph of valency 9 on 19 vertices which is impossible. If $c_2 = 6$, then $\langle \Gamma_2(u) \rangle$ is a trivalent graph on six vertices, so $\langle \Gamma_2(u) \rangle \cong K_{3,3}$. But Γ^c then contradicts Lemma 7.

Our induction is thus complete: a finite locally homogeneous graph is one of the following: (i) $t \cdot K_r$, $t \geqq 1$, $r \geqq 1$, (ii) $K_{t;r}$, $t \geqq 1$, $r \geqq 1$, (iii) C_5, (iv) $L(K_{3,3})$. But each of these graph is also homogeneous. Thus we have the

THEOREM. *The following conditions on a finite graph* Γ *are equivalent*:

(a) Γ *is homogeneous*,

(b) Γ *is locally homogeneous*,

(c) Γ *is one of the graphs* $t \cdot K_r$, $t \geqq 1$, $r \geqq 1$; $K_{t;r}$, $t \geqq 1$, $r \geqq 1$; C_5; $L(K_{3,3})$.

REMARK. Our results do not in fact need the full force of the finiteness assumption; the same results, with the same proofs, hold for locally finite graphs.

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