LEGENDRE RAMANUJAN SUMS TRANSFORM

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ABSTRACT

In this paper, Legendre Ramanujan Sums transform(LRST) is proposed and derived by applying DFT to the complete generalized Legendre sequence (CGLS) matrices. The original matrix based Ramanujan Sums transform (RST) by truncating the Ramanujan Sums series is non-orthogonal and lack of fast algorithm, the proposed LRST has orthogonal property and O($N\log_2 N$) complexity fast algorithm. The LRST transform matrix is a sparse matrix and can be calculated with only additions and multiplications with more improvement in efficiency. It is suitable for image compression and transform coding. Meanwhile the LRST is useful to analyze to periodic signal especially for already known periodic sequences.

Index Terms**—**Generalized Legendre sequence, Ramanujan sum, Image transform coding

1. INTRODUCTION

The Ramanujan Sums (RS) are useful to many research areas such as digital signal processing, image processing, time-frequency analysis, and so on. The important property of the RS is that the values of RS are all integers despite from the original definition that the RS are sums of primitive roots of unity which are usually complex irrational numbers.

The Ramanujan Sums transform (RST) use the component of RS as basis to decompose the input signal. The transform is used to represent arithmetical functions by an infinite series expansion and are considered as the basic building blocks of number-theoretic functions. In addition to being integer valued, the coefficients possess orthogonal property, and this makes them even more attractive as a signal processing tool. In [1], it shows that the Fourier coefficients of RS are real-valued and can be calculated by weighted average of the signal values using integer-valued coefficients. In [2], RS is used to low frequency noise filtering. In [3], the Ramanujan Fourier Transform (RFT) based transform functions are introduced, constituted by RS basis. In [4][5], it shows that RS can be used to extract periodic components in discrete-time signals and the author develops this theory in detail. In [6], the odd-symmetric length-4N periodic signals are studied, and it is shown how the odd RS are used as weighting coefficients to compute their pure imaginary DFT integer-valued coefficients. In [7], it combines the wavelet transform with the RS transform in order to create a new representation of signals. In [8], RST is used for the assessment of T-Wave alternant (TWA).

Since the RS are all integers, we can reduce quantization error on implementation. However, lack of closed form inverse RST representation makes it hard to realization. In [9], the authors truncate the Ramanujan Sums series into matrix form to construct RST. Using matrix-based transform representation, the inverse RST can be easily calculated by taking matrix inverse. Meanwhile, it is proved that the RST matrix is invertible for non-zero determinant. It is also applied to image transform coding and image recovering without errors. However, non-orthogonal and lack of fast algorithm makes it not suitable for signal processing applications. Meanwhile, when treating the input as periodic signal, the original RST transform results do not match the expectation. Therefore, in this paper we propose the Legendre Ramanujan sums transform (LRST) as a new definition to replace the original RST. We will use complete generalized Legendre sequence (CGLS) [10] as basis and take DFT to the CGLS matrix to get the LRST kernel. It is showed that the basis of LRST contain Ramanujan Sum sequences when the column index of the kernel matrix is divisor of N. Meanwhile, we will give an example using radix-2 fast algorithm to implement the LRST with complexity O($N\log_2 N$).

The organization of this letter is as follows. In Section II we review the original matrix-based RST and state its drawbacks. In section III we review the properties of the CGLS and the relationship between the CGLS matrix and the RS. In section IV we propose LRST by taking DFT to the CGLS matrix and use signal flow graph to show its fast algorithm implementation. In section V, we apply the LRST to image transform coding. Finally, we give conclusion in section VI.

2. ORIGINAL MATRIX-BASED RST

Ramanujan Sums (RS) is a function of two positive integer variables q and n defined by the formula

$$c_q(n) = \sum_{p=1;(p,q)=1}^{q} \exp\left(2\pi i \frac{p}{q} n\right) \tag{1}$$

where the greatest common divisor $(p, q) = 1$ means that p only takes on values co-prime to q. Note that $c_q(n)$ is a periodic sequence with period q. The orthogonal property of RS shows that for different q and q', $c_q(n)$ and $c_{q'}(n)$ are orthogonal by:

$$\sum_{n=1}^{qq'} c_q(n)\, c_{q'}(n) = 0 \quad \text{for } (q\ q') = 1 \tag{2}$$

Note that the summation range of index n is from 1 to qq' which is the least common multiple (lcm) of q and q'. From (2) we can realize that unlike the common orthogonal property which is defined by fixed sequence length inner product space, the summation range of the inner product between different RS is from 1 to infinity. Truncating the infinite RS sequence ruins the orthogonality of the RS sequence set.

In [9], the authors proposed Ramanujan Sums transform matrix A by truncating the Ramanujan Sums series into M by M matrix as:

$$A(q,j) = \frac{1}{\phi(q)M} c_q(\mathrm{mod}(j-1,q)+1) \tag{3}$$

Where $q, j = 1,2,\ldots,M$ and mod(.) represents modular operation and ϕ is Euler's totient function which means the number of integers smaller and co-prime to q. For example, let $M = 10$, we can get the Ramanujan Sums matrix A_{10} as in (4). The authors prove that the RST matrix is invertible by showing that the determinant of A is not zero. They also apply the RST to 2D image coding. Truncating the existing RS sequences seems intuitively to create RST matrix. However, there are two problems: First, the RS matrix A is not orthogonal which can be easily proved by directly computing the inverse matrix of A. Second, physically speaking, we usually treat input vector as periodic sequence.

$$A_{10} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 2 & -1 & -1 & 2 & -1 & -1 & 2 & -1 \\ 0 & -2 & 0 & 2 & 0 & -2 & 0 & 2 & 0 & -2 \\ -1 & -1 & -1 & -1 & 4 & -1 & -1 & -1 & -1 & 4 \\ 1 & -1 & -2 & -1 & 1 & 2 & 1 & -1 & -2 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 6 & -1 & -1 & -1 \\ 0 & 0 & 0 & -4 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & -3 & 0 & 0 & 6 & 0 \\ 1 & -1 & 1 & -1 & -4 & -1 & 1 & -1 & 1 & 4 \end{bmatrix} \tag{4}$$

For example, when we apply DFT to a specific length N sequence $x(n)$, actually we regard $x(n)$ as a periodic sequence with period N. Therefore, when we use matrix form to represent the RST, we should consider the fact that the input sequences are actually the periodic signals. That means, the projection from the input vector to the RS sequence period q co-prime to N should be 0. We prove it by computing the length Nq inner product between $x(n)$ and $c_q(n)$ in Eq. (5)

However, using the original RST definition, the projection results do not match the expectation values as described before. Therefore, we will define a LRST by complete generalized Legendre sequence (CGLS) as basis to construct the LRST matrix and we will show that the LRST is orthogonal.

$$\begin{aligned} &\sum_{n=1}^{Nq} x(n)c_q(n) \\ &= x(1)\left(c_q(1)+c_q(2)+c_q(3)+\ldots+c_q(q)\right) \\ &+x(2)\left(c_q(1)+c_q(2)+c_q(3)+\ldots+c_q(q)\right) \\ &+x(3)\left(c_q(1)+c_q(2)+c_q(3)+\ldots+c_q(q)\right) \\ &+\ldots+x(N)\left(c_q(1)+c_q(2)+c_q(3)+\ldots+c_q(q)\right) \\ &= 0 \end{aligned} \tag{5}$$

3. PAGE TITLE SECTION

In [10], the authors use complete generalized Legendre sequences (CGLS) as building blocks to generate closed form orthogonal complete DFT eigenvectors and its finite field version is introduced in [11] to solve number theoretic transform (NTT) eigenvector problem. In this section, we briefly review the definition of the CGLS matrix. First the generalized Legendre Sequence (GLS) is defined where the sequence length N equals power of prime number, i.e., $N = p^l$.

1) If $p > 2$,

$$\begin{aligned} &\eta_{a,l}(n) = \exp(i2\pi\, a(ind_g n)/\phi(p^l)) \qquad \text{for } (n,p)=1 \\ &\eta_{a,l}(n) = 0 \qquad\qquad \text{for } (n,p)\neq 1 \\ &a = 0,1,\ldots..\phi(p^l)-1, \quad n = 0,1,\ldots,p^l-1 \end{aligned} \tag{6}$$

The symbol ϕ denotes the Euler's totient function defined in (3). The ind_g operator denotes the logarithm over the modulus operation, i.e., $ind_g(n) = m$ if $n = g^m \bmod(p^l)$. Note that the multiplication property of the logarithm operator that $ind_g ab = ind_g a + ind_g b$ also holds. The index g is a primitive root modulo p^l, that is, $g^q \neq 1 \bmod(p^l)$ for $0 < q < (\phi(p^l))$.

2) If $p = 2$, the GLS is defined as follows:

$$\begin{aligned} &l = 1,\ \eta_{0,l}(n) = [0\ \ 1]. \\ &l = 2,\ \eta_{0,l}(n) = [0\ \ 1\ \ 0\ \ 1],\ \eta_{1,l}(n) = [0\ \ 1\ \ 0\ \ -1]. \end{aligned} \tag{7}$$

$l > 2$, we can find an integer b so that

$$\eta_{a,c,l}(n) = \begin{cases} (-1)^{(n-1)c/2}\exp(i2\pi ab/2^{l-2}) & n \in odd \\ 0 & n \in even \end{cases} \tag{8}$$

for $c = 0,1$ $a = 0, 1, \ldots, 2^{l-2}-1$, where b is an integer that satisfies the following equation:

$$n \equiv (-1)^{(n-1)c/2} 5^b \quad \mathrm{mod}(2^l) \quad c = 0,1 \text{ for odd } n. \tag{9}$$

To get the complete set, we insert $(p-1)$ zeros between each neighbor element to $\eta_{a,(l-1)}(n)$, then we insert (p^2–1) zeros between each neighbor element to $\eta_{a,(l-2)}(n)$, and so on. The CGLS is defined by adding parameter ***s*** to the GLS as follows:

$$\chi_{a,l,s}(n) = \eta_{a,(l-s)}(k) \quad \text{for } n = kp^s \text{ and} \\ k = 1,2,3,\cdots, p^{l-s}-1 \quad \text{and } (k,p)=1 \tag{10}$$

$$\chi_{a,l,s}(n) = 0 \qquad \text{otherwise}$$

where $a = 0, 1, \ldots, \varphi(p^{l-s}) - 1$, $s = 0, 1, \ldots, l - 1$. When $s = l$, a must be zero and

$$\chi_{0,l,0}(0) = 1, \quad \chi_{0,l,l}(n) = 0 \quad \text{when } n \neq 0. \tag{11}$$

In the case where $p = 2$ and $s < l–2$, the equality $\chi_{a,l,s}(n) = \eta_{a,(l-s)}(k)$ in (10) should be changed as $\chi_{a,c,l,s}(n) = \eta_{a,c,(l-s)}(k)$. Moreover, in this case, $a = 0, 1, 2, \ldots, 2^{l-s-2}-1$ and $c = 0, 1$.

The CGLS for arbitrary length $N = p_1^{l_1} p_2^{l_2} \ldots p_k^{l_k}$ is defined by multiplying the CGLS defined for each $p_i^{l_i}$ as:

$$\chi_{a_1,a_2..a_k,l_1,l_2..l_k,s_1,s_2..s_k}(n) = \chi_{a_1,l_1,s_1}(n)\chi_{a_2,l_2,s_2}(n)..\chi_{a_k,l_k,s_k}(n) \tag{12}$$

Note that, in (12), $\chi_{a_i,l_i,s_i}(n)$ is repeated ($p_1^{l_1} p_2^{l_2} \ldots p_k^{l_k} / p_i^{l_i}$) times. For example when $N = 10$, we can show the CGLS matrix content by

$$\mathrm{L}_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & i & -1 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -i & -1 & i \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & i & -1 & -i \\ 0 & 1 & -i & -1 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix} \tag{13}$$

The properties of the CGLS are shown in [10], including orthogonality, conjugate DFT pairs, even or odd symmetric, and so on. Especially, let $a = 0$, the Legendre sequence degenerate to even function which only contains values 0 and 1 and we can show that

$$\chi_{0,l,s}(n) = \eta_{0,(l-s)}(k) \quad \text{for } n = kp^s \text{ and} \\ k = 1,2,3,\cdots, p^{l-s}-1 \quad \text{and } (k,p)=1 \tag{14}$$

These Legendre sequences in (14) form DFT pairs with Ramanujan Sums sequences and we can prove it by applying DFT to these CGLS such that

$$c_{N'}(k) = \sum_{n=1;\gcd(n,N')=1}^{N'-1} \exp\left(\frac{2\pi i n k}{N'}\right) \tag{15}$$

For $N' = N / p_1^{r_1} p_2^{r_2} \ldots p_k^{r_k}$

where $r_1 = 0,1,\ldots l_1$, $r_2 = 0,1,\ldots l_2$, … $r_k = 0,1,\ldots l_k$.

For example, when $N = 10 = 2*5$, we can see that the factors of N can be 1, 2, 5, 10 and exactly there are four sequences of the CGLS matrix whose DFT results map to RS sequences such that

$$\mathrm{L}_{10}(:,1) = [1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]^{\mathrm{T}} \tag{16}$$

$$\mathrm{F}\{\ \mathrm{L}_{10}(:,1)\ \} = c_1(k) = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]^{\mathrm{T}} \tag{17}$$

$$\mathrm{L}_{10}(:,2) = [0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0]^{\mathrm{T}} \tag{18}$$

$$\mathrm{F}\{\ \mathrm{L}_{10}(:,2)\ \} = c_2(k) = [1\ \text{-}1\ 1\ \text{-}1\ 1\ \text{-}1\ 1\ \text{-}1\ 1\ \text{-}1]^{\mathrm{T}} \tag{19}$$

$$\mathrm{L}_{10}(:,5) = [0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 1]^{\mathrm{T}} \tag{20}$$

$$\mathrm{F}\{\ \mathrm{L}_{10}(:,5)\ \} = c_5(k) = [4\ \text{-}1\ \text{-}1\ \text{-}1\ \text{-}1\ 4\ \text{-}1\ \text{-}1\ \text{-}1\ \text{-}1]^{\mathrm{T}} \tag{21}$$

$$\mathrm{L}_{10}(:,10) = [0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0]^{\mathrm{T}} \tag{22}$$

$$\mathrm{F}\{\ \mathrm{L}_{10}(:,10)\ \} = c_{10}(k) = [4\ 1\ \text{-}1\ 1\ \text{-}1\ \text{-}4\ \text{-}1\ 1\ \text{-}1\ 1]^{\mathrm{T}} \tag{23}$$

Therefore, we can realize that the CGLS are closed related to the RS sequences and it inspires us to define the LRST by the CGLS.

4. DEFINITION of LRST

In this section, we propose the Legendre Ramanujan sums transform (LRST) matrix by applying DFT to the CGLS matrices and show that it has good properties such as orthogonality. Let the DFT matrix F and the CGLS matrix L, the proposed LRST matrix $\mathbf{A_L}$ is defined as follows:

$$\mathbf{A_L} = \mathrm{FL} \tag{24}$$

For example, when $N = 10$, taking DFT to the 10 points CGLS matrix we can get that

$$\mathrm{A}_{\mathrm{L},10} = \mathrm{FL}_{10} = \mathrm{K}_{10}\mathrm{D} \tag{25}$$

The matrix D is a diagonal matrix and K_{10} is similar to the CGLS matrix but it contains Ramanujun sums sequence in the specific columns shown by

$$\mathrm{D} = \mathrm{Diag}\{1,\ 1,\ w,\ \sqrt{10},\ \text{-}w^*,\ 1,\ 1,\ \text{-}w,\ \sqrt{10},\ w^*\} \tag{26}$$

where $w = 1.9021 + 1.1756\mathrm{i}$

$$\mathrm{K}_{10} = \begin{bmatrix} 1 & 4 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -i & 1 & i & 1 & -1 & i & 1 & -i \\ 1 & -1 & i & 1 & -i & -1 & 1 & i & -1 & -i \\ 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 4 & 0 & 0 & 0 & -1 & -4 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -i & 1 & i & -1 & 1 & -i & -1 & i \\ 1 & -1 & i & 1 & -i & 1 & -1 & -i & 1 & i \\ 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \tag{27}$$

Using the LRST definition, we can show that the LRST matrix is orthogonal since the CGLS and DFT matrices are all orthogonal matrix so that the LRST matrix is also orthogonal. Meanwhile, we can show that the LRST has fast algorithm by the fact that the CGLS and DFT matrices both have $\boldsymbol{O}(N\log_2 N)$ fast algorithms, the combination of these two matrices will also has $\boldsymbol{O}(N\log_2 N)$ fast algorithm. Especially from the DFT conjugate pairs property of the CGLS shown in [10] we can realize that the LRST matrix can be implemented by CGLS matrix plus sparse linear combination. For example in (25), we can plot the signal flow graph as in Fig. 1 and we can calculate the LRST need only additions and multiplications. Compared with the original RST which has $\boldsymbol{O}(N^2)$ complexity, the proposed LRST has far more improvement in efficiency.

Finally, although the kernel elements of $\mathbf{A_L}$ are not all integers, the proposed LRST is still suitable to analyze the periodic signals especially for already known period input sequences.

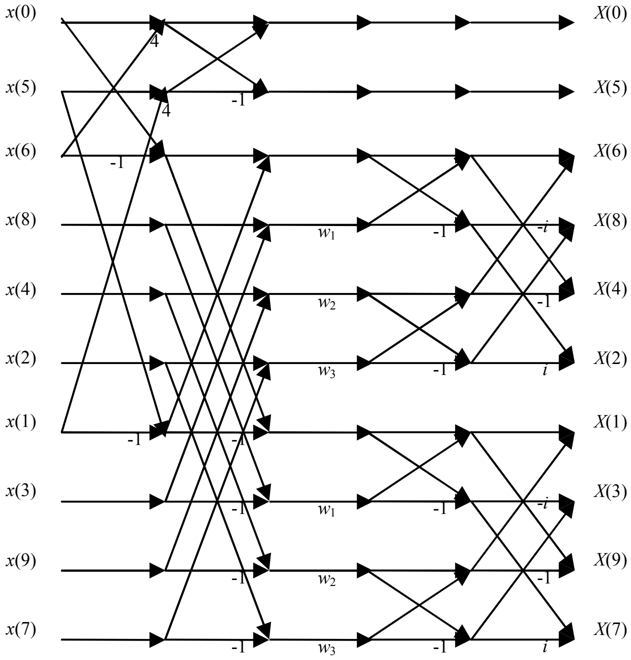

Fig.1 10 points orthogonal LRST signal flow graph

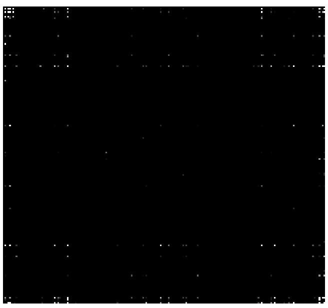

Fig. 2 The LRST spectrum of “Lena” Image

5. IMAGE TRANSFORM CODING APPLICATION

We propose the application of the LRST in image coding. We use the “Lena” image of size 256 by 256 pixels for 2-D LRST in Fig. 2 and we show the LRST spectrum of “Lena” and realize that the LRST coefficients matrix is a sparse matrix and it is suitable for image compression. Meanwhile, it also has potential application on pattern recognition.

6. CONCLUSION

We propose Legendre Ramanujan sums transform (LRST) derived by applying DFT to complete generalized Legendre sequence (CGLS) matrix. The proposed RST has orthogonality and fast algorithm properties with $\boldsymbol{O}(N\log_2 N)$ complexity. Compared with the original matrix based RST the LRST is suitable for periodic sequence analysis.

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