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Particular Solutions of the Confluent Hypergeometric Differential Equation by Using the Nabla Fractional Calculus Operator

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Abstract: In this work; we present a method for solving the second-order linear ordinary differential equation of hypergeometric type. The solutions of this equation are given by the confluent hypergeometric functions (CHFs). Unlike previous studies, we obtain some different new solutions of the equation without using the CHFs. Therefore, we obtain new discrete fractional solutions of the homogeneous and non-homogeneous confluent hypergeometric differential equation (CHE) by using a discrete fractional Nabla calculus operator. Thus, we obtain four different new discrete complex fractional solutions for these equations.

Keywords: discrete fractional calculus; confluent hypergeometric equation; Nabla operator

1. Introduction

Due to their potential applications fractional and discrete fractional differential equations have attracted much attention in recent years [1–5]. Recently, many papers on discrete fractional calculus (DFC) have been published. For example, Atici and Eloe introduced in [6] the discrete Laplace transform method for a family of finite fractional difference equations. In [7] they defined the initial value problems in DFC. Atici and Eloe [8] studied the properties of DFC with the Nabla operator. They developed exponential laws and the product rule for the forward fractional calculus. Atici and Sengul [9] developed the Leibniz rule in DFC. Bastos and Torres [10] presented the more general discrete fractional operator and this operator was defined by the Delta and Nabla fractional sums. Holm [11] introduced fractional sums and difference operators. Jarad and Tas [12] defined the generalized discrete Sumudu transform and its essential properties. Mohan [13] discussed the differentiability properties of solutions of Nabla fractional difference equations of non-integral order. Mohan [14] established sufficient conditions for the global existence and uniqueness of the nonlinear fractional Nabla difference system.

We recall that a given confluent hypergeometric function means a solution for the confluent hypergeometric equation, which represents a degenerate form of a hypergeometric differential equation such that two of the three regular singularities merge into an irregular one. In this manuscript we studied the confluent hypergeometric differential equation (CHE) [15], namely:

$$w'' r + (\eta - r)\, w' - \delta w = 0, \tag{1}$$

where δ and η are real constants and r is an independent variable. We recall that this equation was found by Kummer [15] and the confluent hypergeometric equation originates in physical problems. For example, it rises in connection with cylindrical waves and their extensions. Singularities of the differential equations are regular, except for sound in a flow [16]. Bearing this in mind we conclude that this type of equation involves complex calculations. Solutions of Equation (1) are defined by the confluent hypergeometric functions [16]. Akimoto and Suziki [17] acquired new generalized entropies by using the confluent hypergeometric function of the first type.

Fractional calculus plays an important part in entropy and many other works. Entropy is utilized in the analysis of anomalous diffusion process and fractional diffusion equations. Many numerical methods were presented to investigate the problem: the homotopy perturbation method and homotopy analysis method, the collocation methods, the finite element method. Entropy was presented in thermodynamics by Clausius and Boltzmann. These advances activated the formulation of novel entropy indices and fractional operators allowing their implementation in complex dynamical systems [18,19]. Machado investigated the entropy analysis of fractional derivatives and their approximation [20].

Magin *et al.* [21] characterized anomalous diffusion in porous biological tissues using fractional order derivatives and entropy. Ingo *et al.* [22] applied entropy for the case of anomalous diffusion governed by the time and space fractional order diffusion equation. Thus, they acquired a new perspective for fractional order models.

The aim of this paper to get new discrete fractional solutions of the homogeneous and nonhomogeneous CHEs by means of the Nabla discrete fractional operator. This paper is organized as follows: in Section 2, the basic definitions of the discrete fractional calculus are presented. Our results are then given in Section 3. Some conclusions and future perspectives are given in the last Section.

2. Preliminaries

In this section, we present some essential information about discrete fractional calculus theory. We use some notations $\mathbb{N}_k = \{k, k+1, k+2, \cdots\}$ for $k \in \mathbb{Z}$. Let $\mu(n)$ and $\varpi(n)$ be a real-valued functions defined on $\mathbb{N}_0^+$. These and other related results can be found in [6–14].

Definition 2.1. [6] The rising factorial power is given by:

$$\chi^{\overline{n}} = \chi(\chi+1)(\chi+2)\ldots(\chi+n-1), \; n \in \mathbb{N}, \; \chi^{\overline{0}} = 1. \tag{2}$$

Let α a real number. Then $\chi^{\overline{\alpha}}$ is defined as:

$$\chi^{\overline{\alpha}} = \frac{\Gamma(\chi+\alpha)}{\Gamma(\chi)}, \tag{3}$$

where $\chi \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$ and $0^{\overline{\alpha}} = 0$. Let us note that:

$$\nabla\left(\chi^{\overline{\alpha}}\right) = \alpha\chi^{\overline{\alpha-1}}, \tag{4}$$

where $\nabla u(\tau) = u(\tau) - u(\tau-1)$. For $\sigma = 2, 3, \ldots$, define ∇^{σ} in deductively by $\nabla^{\sigma} = \nabla\nabla^{\sigma-1}$.

Definition 2.2. [6] The $\alpha - \chi h$ order fractional sum of ϖ is given by:

$$\nabla_k^{-\alpha}\varpi(\chi) = \sum_{s=k}^{\chi} \frac{(\chi-\delta(\chi))^{\overline{\alpha-1}}}{\Gamma(\alpha)}\varpi(s), \tag{5}$$

where $\chi \in \mathbb{N}_k$, $\delta(\chi) = \chi - 1$ is backward jump operator of the time scale calculus.

Theorem 2.1. [13] Let $\mu(n)$ and $\varpi(n)$ $: N_0^+ \to R$, $\gamma, \varphi > 0$ and h, v are scalars. The following equality holds:

$$1. \quad \nabla^{-\gamma}\nabla^{-\varphi}\mu(n) = \nabla^{-(\gamma+\varphi)}\mu(n) = \nabla^{-\varphi}\nabla^{-\gamma}\mu(n), \tag{6}$$

$$2. \quad \nabla^{\gamma}[h\mu(n) + v\varpi(n)] = h\nabla^{\gamma}\mu(n) + v\nabla^{\gamma}\varpi(n), \tag{7}$$

$$3. \quad \nabla\nabla^{-\gamma}\mu(n) = \nabla^{-(\gamma-1)}\mu(n), \tag{8}$$

$$4. \quad \nabla^{-\gamma}\nabla\mu(n) = \nabla^{(1-\gamma)}\mu(n) - \begin{pmatrix} n+\gamma-2 \\ n-1 \end{pmatrix}\mu(0). \tag{9}$$

Lemma 2.1. [14] (Leibniz Rule). For any $\alpha > 0$, $\alpha - \tau h$ order fractional difference of the product $\mu\varpi$ is given by:

$$\nabla_0^{\alpha}(\mu\varpi)(\chi) = \sum_{n=0}^{\chi}\begin{pmatrix} \alpha \\ n \end{pmatrix}\left[\nabla_0^{\alpha-n}\mu(\chi-n)\right]\left[\nabla^{n}\varpi(\chi)\right]. \tag{10}$$

Lemma 2.2. [23] If the function $\mu(n)$ is single-valued and analytic then we have:

$$\left(\mu_{\gamma}(n)\right)_{\rho} = \mu_{\gamma+\rho}(n) = \left(\mu_{\rho}(n)\right)_{\gamma} \quad \left(\mu_{\gamma}(n) \neq 0;\, \mu_{\rho}(n) \neq 0;\, \gamma, \rho \in \mathbb{R};\, n \in \mathbb{N}\right). \tag{11}$$

3. Main Results

In this section, we give two theorems for the discrete fractional solutions of the nonhomogeneous and homogeneous CHEs by using the Nabla DFC operator.

Theorem 3.1. Let $w \in \{w : 0 \neq |w_{\alpha}| < \infty; \alpha \in \mathbb{R}\}$ and $\psi \in \{\psi : 0 \neq |\psi_{\alpha}| < \infty; \alpha \in \mathbb{R}\}$. Then the non-homogeneous CHE:

$$w_2 r + w_1(\eta - r) - \delta w = \psi \ (r \neq 0), \tag{12}$$

has particular solutions of the form:

$$w = \left(\left[\psi_{-E^{-1}\delta}e^{-r}r^{\eta-\delta-1}\right]_{-1}e^{r}r^{\delta-\eta}\right)_{E^{-1}\delta-1} \equiv w^{\iota}, \tag{13}$$

$$w = r^{1-\eta}\left\{\left[\left(\psi\, r^{\eta-1}\right)_{E^{-1}(\eta-\delta-1)}e^{-r}r^{-\delta}\right]_{-1}e^{r}r^{\delta-1}\right\}_{-1+E^{-1}(\delta-\eta+1)} \equiv w^{\iota\iota}, \tag{14}$$

where $w_n = d^n w/dr^n$ $(n = 0, 1, 2)$, $w_0 = w = w(r)$, $r \in \mathbb{R}$.

Proof. (i) Operating ∇^{α} to the both sides of Equation (12) gives:

$$\nabla^{\alpha}(w_2 r) + \nabla^{\alpha}[w_1(\eta - r)] - \nabla^{\alpha}(w)\,\delta = \nabla^{\alpha}\psi. \tag{15}$$

Using Equations (4) to (13), we get:

$$\nabla^{\alpha}(w_2 r) = w_{2+\alpha}r + \alpha\, E w_{1+\alpha}, \tag{16}$$

where E is a shift operator [24]. We have:

$$\nabla^{\alpha}[w_1(\eta - r)] = w_{1+\alpha}(\eta - r) - \alpha\, E w_{\alpha}. \tag{17}$$

We may write Equation (15) in the following form:

$$w_{2+\alpha}r + w_{1+\alpha}(\alpha\, E + \eta - r) - w_{\alpha}(\alpha\, E + \delta) = \psi_{\alpha}, \tag{18}$$

by using relations (16) and (17). Choose α such that:

$$\alpha = -E^{-1}\delta. \tag{19}$$

We have:

$$w_{2-E^{-1}\delta}r + w_{1-E^{-1}\delta}\left(-\delta + \eta - r\right) = \psi_{-E^{-1}\delta}, \tag{20}$$

from Equation (18).

Therefore, setting:

$$w_{1-E^{-1}\delta} = y = y\left(r\right) \quad \left(w = y_{E^{-1}\delta-1}\right), \tag{21}$$

we obtain:

$$y_1 + y\left(\frac{\eta - \delta}{r} - 1\right) = \psi_{-E^{-1}\delta}r^{-1}, \tag{22}$$

from Equation (20). This is an ordinary differential equation of the first order which has a particular solution:

$$y = \left[\psi_{-E^{-1}\delta}e^{-r}r^{\eta-\delta-1}\right]_{-1} e^{r}r^{\delta-\eta}. \tag{23}$$

Therefore, we obtain the solution (13) from Equations (21) and (23).

(ii) Set:

$$w = r^{\kappa}\phi, \;\; \phi = \phi\left(r\right). \tag{24}$$

Therefore:

$$w_1 = \kappa\, r^{\kappa-1}\phi + r^{\kappa}\,\phi_1, \tag{25}$$

$$w_2 = \kappa\left(\kappa - 1\right)\, r^{\kappa-2}\phi + 2\kappa r^{\kappa-1}\phi_1 + r^{\kappa}\phi_2. \tag{26}$$

Substitute Equations (24)–(26) into (12) we have:

$$\phi_2 r + \phi_1\left(2\kappa + \eta - r\right) + \phi\left[\frac{\kappa\left(\kappa + \eta - 1\right)}{r} - \kappa - \delta\right] = \psi r^{-\kappa}. \tag{27}$$

Here, we choose κ such that:

$$\kappa\left(\kappa + \eta - 1\right) = 0, \tag{28}$$

that is $\kappa_1 = 0$, $\kappa_2 = 1 - \eta$. □

In the case $\kappa = 0$, we have the same results as proof (i).

Let $\kappa = 1 - \eta$. From Equations (24) and (26) we have:

$$w = r^{1-\eta}\phi, \tag{29}$$

and:

$$\phi_2 r + \phi_1\left(2 - \eta - r\right) + \phi\left(\eta - \delta - 1\right) = \psi r^{\eta-1}, \tag{30}$$

respectively.

Applying the operate ∇^{α} to both members of Equation (32) gives:

$$\phi_{2+\alpha}\, r + \phi_{1+\alpha}\left(\alpha\, E - \eta - r + 2\right) + \phi_{\alpha}\left(-\alpha E + \eta - \delta - 1\right) = \left(\psi\, r^{\eta-1}\right)_{\alpha}. \tag{31}$$

Choose α such that:

$$\alpha = E^{-1}\left(\eta - \delta - 1\right), \tag{32}$$

we have:

$$\phi_{2+E^{-1}\Xi}r + \phi_{1+E^{-1}\Xi}\left(-\delta + 1 - r\right) = \left(\psi\, r^{\eta-1}\right)_{E^{-1}\Xi}, \tag{33}$$

from Equation (31) where $\Xi = \left(\eta - \delta - 1\right)$.

Therefore, setting:

$$\phi_{1+E^{-1}\Xi} = \varpi = \varpi\,(r) \quad \left(\phi = \varpi_{-1+E^{-1}\Xi}\right), \tag{34}$$

it gives:

$$\varpi_1 + \varpi\left(\frac{1-\delta}{r} - 1\right) = \left(\psi r^{\eta-1}\right)_{E^{-1}\Xi} r^{-1}, \tag{35}$$

From Equation (33). A particular solution of the ordinary differential Equation (35) is given by:

$$\varpi = \left[\left(\psi\, r^{\eta-1}\right)_{E^{-1}\Xi} e^{-r} r^{-\delta}\right]_{-1} e^r r^{\delta-1}. \tag{36}$$

Thus, we obtain the solution (13) from Equations (31) and (36).

Theorem 3.2. Let $w \in \{w : 0 \neq |w_\alpha| < \infty, \alpha \in \mathbb{R}\}$. Then, the homogeneous CHE:

$$w_2 r + w_1\,(\eta - r) - w\delta = 0 \;\; (r \neq 0)\,, \tag{37}$$

has solutions of the form:

$$w = h\left(e^r r^{\delta-\eta}\right)_{E^{-1}\delta-1} \equiv w^{(I)}, \tag{38}$$

$$w = h r^{1-\eta}\left(e^r r^{\delta-1}\right)_{-1+E^{-1}(\delta-\eta+1)} \equiv w^{(II)}, \tag{39}$$

where h is an arbitrary constant.

Proof. When $\psi = 0$ in Theorem 1, we conclude that:

$$y_1 + y\left(\frac{\eta-\delta}{r} - 1\right) = 0, \tag{40}$$

$$\varpi_1 + \varpi\left(\frac{1-\delta}{r} - 1\right) = 0, \tag{41}$$

instead of Equations (22) and (35), respectively. □

As a result, we obtain Equation (38) for (40) and Equation (39) for (41).

4. Conclusions

We have obtained some new discrete fractional solutions of the homogeneous and nonhomogeneous CHEs in this work. The Nabla fractional calculus operator was used to apply the integration of this equation as it was pointed out in the classical methods. Therefore, we obtain many different discrete fractional solutions for these equations. We believe that this type of solutions for such an equation will be useful in future investigations. These new discrete fractional solutions can be used to define new entropies.

As it is known solving a general case of non-homogeneous DFC requires a huge computational effort, therefore we believe that the reported solutions will be very useful in the future applications to some real world problems. We will obtain discrete fractional solutions of the same type equations by using the combined Delta-Nabla sum operator in DFC in our future works.

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