Wavelet Transform Method of Waveform Estimation for Hilbert Transform of Fractional Stochastic Signals with Noise*

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Abstract In this paper, those splendid characters of the Hilbert transform let the processes that taking wavelet transform after taking Hilbert transform for the statistic self-similarity processes FBM [$B_H(t)$] become another processes, that firstly taking Hilbert transform for the wavelet function $\phi(t)$ and forming a new wavelet function $\psi(t)$, secondly taking the wavelet transform for $B_H(t)$. Then, we use the optimum threshold to estimate the $\hat{B}_H(t)$ embedded in additive white noise. Typical computer simulation results to demonstrate the viability and the effectiveness of the Hilbert transform in the signal's estimation of the statistic self-similarity process.

1 Introduction

Hilbert transform has some splendid characters, for example, the statistic self-similarity fixed character. By using it, a real signal can be described into a complex signal. This not only make the theory easier but also make we can be studying the real signal's instantaneous phase and instantaneous frequency. Now, the Hilbert transform will become one of the important tools in communication researching theory. Here, we take the estimation of original signal's Hilbert transform it is the complex signal's imaginary part, temporarily not of the complex signal. This paper is an earlier stage achievement in the series of discussing.

In this paper, those splendid characters of the Hilbert transform let the processes that taking wavelet transform after taking Hilbert transform for the statistic self-similarity processes FBM [$B_H(t)$] become another processes, that firstly taking Hilbert

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transform for the wavelet function $\phi(t)$ and forming a new wavelet function $\psi(t)$, secondly taking the wavelet transform for $B_H(t)$. These are principal results in this paper. Then, we use the optimum threshold to estimate the $\hat{B}_H(t)$ embedded in additive white noise. Typical computer simulation results to demonstrate the viability and the effectiveness of the Hilbert transform in the signal's estimation of the statistic self-similarity process.

2 Second-Order Statistics of Hilbert Transform of FBM Wavelet Coefficients

Fractional Brownian motion (FBM) is a Gaussian zero-mean non-stationary stochastic process $B_H(t)$, indexed by a single scalar parameter $0 < H < 1$. The non-stationary character of FBM is evidenced by its covariance structure:

$$E[B_H(t)B_H(s)] = \frac{\sigma^2}{2}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}] \tag{1}$$

It follows from the above equation that the variance of FBM is of the type

$$\operatorname{var}[B_H(t)] = \sigma^2 |t|^{2H} \tag{2}$$

As a non-stationary process, FBM does not admit a spectrum in the usual sense. However, it is possible to attach to it an average spectrum

$$S_{B_H}(\omega) = \frac{\sigma^2}{|\omega|^{2H+1}} \tag{3}$$

Let $f(t)$ is a real signal, so its Hilbert transform $\hat{f}(t)$ is defined as following:

$$\hat{f}(t) = f(t) * \frac{1}{\pi t} = \int_{-\infty}^{\infty} \frac{f(\tau)}{\pi(t-\tau)} d\tau \tag{4}$$

Hilbert transform has some splendid characters, and can give beautiful results in the signal estimation.

Theorem 1 If $X(t)$ is a statistic self-similarity process, $\hat{X}(t)$ is the Hilbert transform of $X(t)$, then $\hat{X}(t)$ is also a statistic self-similarity process.

Proof: Because $X(t)$ is a statistic self-similarity process, so

$$E(X(t)) = a^{-H} E(X(at))$$
$$E(X(t)X(s)) = a^{-2H} E(X(at)X(as)) \quad a > 0$$

for $\hat{X}(t)$, we have

$$E\left(\hat{X}(t)\right) = E\int_R X(s)\frac{1}{\pi(t-s)}ds = a^{-H}E\left[\hat{X}(at)\right]$$

$$E\left[\hat{X}(t_1)\hat{X}(t_2)\right] = E\left[\int_R X(s_1)\frac{1}{\pi(t_1-s_1)}ds_1\int_R X(s_2)\frac{1}{\pi(t_2-s_2)}ds_2\right]$$

$$= a^{-2H}E\left[\hat{X}(at_1)\hat{X}(at_2)\right]$$

From above, we know that $\hat{X}(t)$ satisfied the definition of statistic self-similarity process, so the theory is proofed. #

FBM is a typical example in statistic self-similarity process, its Hilbert transform $\hat{B}_H(t)$ also has statistic self-similarity.

By the way, if $W(t)$ is zero-mean Gaussian white noise with variance σ_w^2, $\hat{W}(t)$ is the Hilbert transform of $W(t)$, then $\hat{W}(t)$ is also the zero-mean Gaussian white noise, and has the same spectrum σ_w^2 with $W(t)$.

Let $\hat{B}_H(t)$ is the Hilbert transform of FBM $B_H(t)$, the self-correlation function for $\hat{B}_H(t)$ is

$$E\left[\hat{B}_H(t)\hat{B}_H(s)\right] = \frac{\sigma^2}{2}\iint_{R^2}\left[|\tau_1|^{2H}+|\tau_2|^{2H}-|\tau_1-\tau_2|^{2H}\right]\frac{1}{\pi(t-\tau_1)}\frac{1}{\pi(s-\tau_2)}d\tau_1 d\tau_2$$

It is clear from the previous section that $\hat{B}_H(t)$ is also a non-stationary stochastic process.

The wavelet mean-square representation of $\hat{B}_H(t)$ is

$$\hat{B}_H(t) = 2^{-J/2}\sum_{n=-\infty}^{\infty}a_J[n]\varphi\left(2^{-J}t-n\right)+\sum_{j=-\infty}^{J}2^{-j/2}\sum_{n=-\infty}^{\infty}d_j[n]\phi(2^{-j}t-n)$$

where $\phi(t)$ is the basic wavelet, $\varphi(t)$ is scaling function associated with $\phi(t)$,

$$d_j[n] = 2^{-j/2}\int_{-\infty}^{\infty}\hat{B}_H(t)\phi\left(2^{-j}t-n\right)dt \quad a_j[n] = 2^{-j/2}\int_{-\infty}^{\infty}\hat{B}_H(t)\varphi(2^{-j}t-n)dt$$

$$j,n\in Z$$

An important character of the wavelet coefficients $d_j[n]$ is that, for each scale j, the serial $\{d_j[n],\quad n\in Z\}$ is similar to a stationary serial. Later, we will use it.

Theorem 2 Let $\hat{\phi}(t)$ is the Hilbert transform of basic wavelet $\phi(t)$, then $\hat{\phi}(t)$ is also the basic wavelet.

Proof: From the definition of Hilbert transform, there is

$$\hat{\phi}(t) = \phi(t) * \frac{1}{\pi t}$$

so the Fourier transform of $\hat{\phi}(t)$ is

$$F(\hat{\phi}(t)) = \Phi(\omega) \cdot H(\omega)$$

and

$$\int_R \frac{\left|F(\hat{\phi}(t))\right|^2}{|\omega|} d\omega = \int_R \frac{|\Phi(\omega)|^2}{|\omega|} d\omega < \infty$$

So $\hat{\phi}(t)$ is the basic wavelet. #

For sign's convenience, let

$$\begin{aligned} \psi_{j,m}(\tau) = 2^{-j/2} \psi(2^{-j}\tau - m) &= \int_R 2^{-j/2} \phi(2^{-j} t - m) \cdot \frac{1}{\pi(t-\tau)} dt \\ &= (-2^{-j/2} \phi(2^{-j}\tau - m)) * \frac{1}{\pi\tau} \end{aligned} \quad (5)$$

From the theorem 2, we know $2^{-j/2}\psi(2^{-j}\tau - m)$ is the basic wavelet. When $j = 0$, we get

$$\psi(\tau - m) = -\phi(\tau - m) * \frac{1}{\pi\tau} \quad (6)$$

So $d_j[n]$ can be expressed by another way as following:

$$d_j[n] = 2^{-j/2} \int_{-\infty}^{\infty} B_H(t)\psi\left(2^{-j} t - n\right) dt \qquad j, n \in Z$$

Now we consider $R_j[n]$, which is the correlation function of $d_j[n]$.

$$R_j[n] = E\left[d_j[m+n] d_j[m]\right] = \frac{\sigma^2}{2}\left(-\int_R A_\psi(1, \tau - n)|\tau|^{2H} d\tau\right)(2^j)^{2H+1}$$

where

$$A_\psi(1,\tau) = \int_R \psi(t)\psi(t-\tau)dt$$

Especially, when $n = 0$, the variance of wavelet coefficients $d_j[n]$ is

$$R_j[0] = \operatorname{var}(d_j[n]) = \frac{\sigma^2}{2} V_\psi(H)(2^j)^{2H+1} = \sigma_c{}^2 2^{\gamma j}$$

where

$$V_\psi(H) = -\int_R A_\psi(1,\tau)|\tau|^2 d\tau$$

From above equations, we can see that through comparing the wavelet transform equations of $\hat{B}_H(t)$ with the wavelet transform equations of $B_H(t)$, all but the basic wavelet are the same.

The discuss in this segment tell us, through the Hilbert transform, a new wavelet function $\psi(t)$ is formed by the original wavelet function $\phi(t)$, the wavelet transform of $\hat{B}_H(t)$ become another wavelet transform of $B_H(t)$ with the new wavelet function $\psi(t)$. At the same time, all the equations in the wavelet transform of $\hat{B}_H(t)$ and $B_H(t)$ have very beautiful similarity. So we can select existing methods for the waveform estimation. This is the major result.

Now, we give the wavelet mean-square representation of $\hat{B}_H(t)$ which have N_0 sample points:

$$\hat{B}_H(k) = 2^{-J/2} \sum_{n=0}^{(N_0/2^J)-1} a_J[n]\varphi(2^{-J}k-n) + \sum_{j=1}^{J} 2^{-j/2} \sum_{n=0}^{(N_0/2^j)-1} d_j[n]\phi(2^{-j}k-n) \quad k = 0,\ldots\ldots, N_0 - 1$$

3 Optimum Threshold Method of Wavelet Estimation for Hilbert Transform of FBM

Consider the received signal

$$y(k) = B_H(k) + W(k) \tag{7}$$

where $B_H(k)$ is FBM, $W(k)$ is additive Gaussian white noise with zero-mean and variance σ_w^2. The Hilbert transform of equation (7) is

$$\hat{y}(k) = \hat{B}_H(k) + \hat{W}(k) \tag{8}$$

We know that $\hat{B}_H(k)$ is the statistic self-similarity process and $\hat{W}(t)$ is also the zero-mean Gaussian white noise with variance σ_w^2.

A simple existing method to estimate $\hat{B}_H(k)$ from the equation (3.2) is that, setting a threshold to the wavelet coefficients of the signal $\hat{y}(k)$, using the inverse wavelet transform to combine the signal $\hat{B}_H(k)$ only with that marked wavelet coefficients beyond the threshold. However, setting a appropriate threshold has a restricted condition: that increasing the threshold to decrease the influence of noise will make the distortion of signal's estimates increasing, because large threshold restricts the smaller wavelet coefficients which do not be used to the inverse wavelet transform: on the contrary, decreasing the threshold to decrease the noise's distortion, but make the influence from the noise increasing, because the large noise wavelet coefficients that beyond the threshold will be used to the inverse wavelet transform. So, we need an optimum threshold for the estimation of the signal.

Let $a_0[k] = \hat{y}(k)$ then

$$\hat{y}[n] = d_j[n] + \hat{W}_j[n] \quad j = 1,\ldots\ldots,J \tag{9}$$

where $d_j[n]$ and $\hat{W}_j[n]$ is the wavelet coefficients serial of $\hat{B}_H(k)$ and $\hat{W}(t)$, and $\{W_j[n], \quad n \in N\}, j = 1,\ldots\ldots,J$ is white noise serial with variance σ_w^2. When the scale J is abundant large, we have

$$\tilde{\hat{B}}_H(k) = \sum_{j=1}^{J} 2^{-j/2} \sum_{n=0}^{\left(N_0/2^j\right)-1} d_j[n]\phi\left(2^{-j}k - n\right) \qquad k = 1,\ldots..N_0 - 1$$

In the following, we use the optimum threshold to estimate $d_j[n]$. Let $\tilde{d}_j[n]$ named the estimation of $d_j[n]$, and L the optimum threshold, then

$$\tilde{d}_j[n] = \begin{cases} 0 & j \le L \\ \hat{y}_j[n] & j > L \end{cases} \tag{10}$$

To make the mean-variance estimation of the wavelet coefficients serial of the scale j smallest, namely

$$\sigma_j^2 \triangleq E\{(d_j[n] - \tilde{d}_j[n])^2\} = \min(R_j(0), \sigma_w^2\}$$

$$= \begin{cases} \sigma_w^2 & \sigma_w^2 < R_j(0) \\ R_j(0) & \sigma_w^2 \ge R_j(0) \end{cases} \tag{11}$$

Since $R_j(0) = \sigma_c^2 2^{\gamma j}$

so when $j \le \frac{1}{\gamma}\log_2 \frac{\sigma_w^2}{\sigma_c^2}$ and $\sigma_w^2 \ge R_j^2(0)$, let $L = [\frac{1}{\gamma}\log_2 \frac{\sigma_w^2}{\sigma_c^2}]$, where [ ] means reserving the integer part, and we have

$$\tilde{d}_j[n] = \begin{cases} 0 & j \le L \\ \hat{y}_j[n] & j > L \end{cases} \tag{12}$$

4 Simulation

In the signal estimation of statistic self-similarity process, to demonstrate the viability and the effectiveness of the Hilbert transform, we use computer simulation to rehabilitate the $1/f$-type fractional signal. Figure 1 and figure 5 indicate Gaussian zero-mean $1/f$-type fractional signal, which comes from random Weiestrass function, where the number of sample points is 1000, $\gamma = 1.7$, and bases on Harr wavelet. Figure 2 indicates the received signal in Gaussian white noise, in which the fractional signal has a figure that x=0dB. Figure 3 indicates the Hilbert transform of the received signal. Figure 4 indicates the estimation of by the optimum threshold. Figure 6 indicates the signal by inverse Hilbert transform of the signal in figure 4, where M=2, and the error of signal's estimation deta=0.5591. So the viability and the effectiveness of the Hilbert transform have been demonstrated.

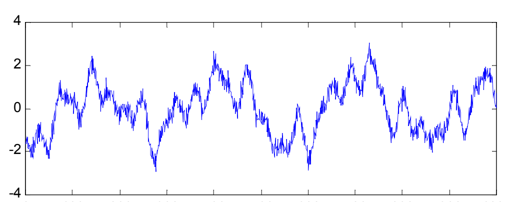

Figure1. 1/f Signal

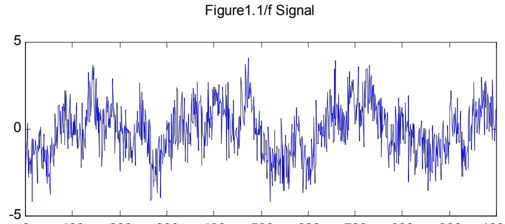

Figure2. 1/f Signal in Noise

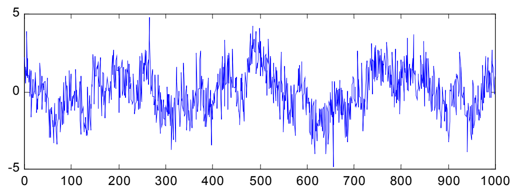

Figure3. Signal through Hilbert Transform

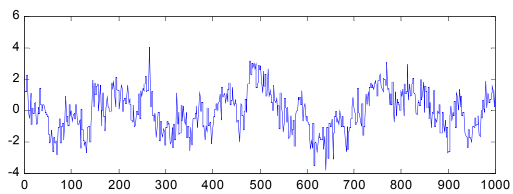

Figure4. Estimated Signal by the Optimum Threshold Technique

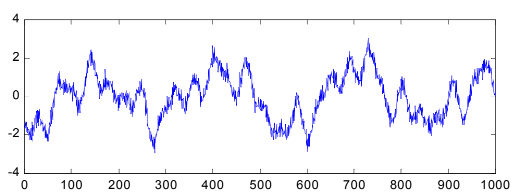

Figure5. 1/f Signal

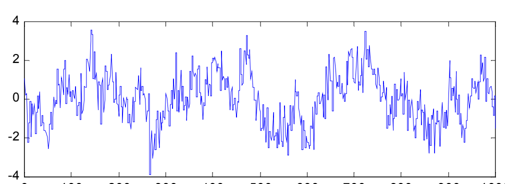

Figure6.Estimated Signal through Inverse Hilbert Transform

5 Conclusion

In this paper, those splendid characters of the Hilbert transform let the processes that taking wavelet transform after taking Hilbert transform for the statistic self-similarity processes FBM [$B_H(t)$] become another processes, that firstly taking Hilbert transform for the wavelet function $\phi(t)$ and forming a new wavelet function $\psi(t)$, secondly taking the wavelet transform for $B_H(t)$. Then, we use the optimum threshold to estimate the $\hat{B}_H(t)$ embedded in additive white noise. Typical computer simulation results to demonstrate the viability and the effectiveness of the Hilbert transform in the signal's estimation of the statistic self-similarity process. So this paper is the fundamental work, later we will take part in the estimation of complex signals.

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