



# DYNAMICS IN TIME-DELAY RECURRENTLY COUPLED OSCILLATORS

CAN JIANG and SHANGJIANG GUO\*

*College of Mathematics and Econometrics,  
Hunan University, Changsha, Hunan 410082, P. R. China*  
\*shangjguo@hnu.cn

YIGANG HE

*College of Electric and Information Engineering,  
Hunan University, Changsha, Hunan 410082, P. R. China*

Received February 8, 2010; Revised June 16, 2010

A model of time-delay recurrently coupled spatially segregated neural oscillators is proposed. Each of the oscillators describes the dynamics of average activities of excitatory and inhibitory populations of neurons. Bifurcation analysis shows the richness of the dynamical behaviors in a biophysically plausible parameter region. We find oscillatory multi-stability, hysteresis, and stability switches of the rest state provoked by the time delay as well as the strength of the connections between the oscillators. Then we derive the equation describing the flow on the center manifold that enables us to determine the bifurcation direction and stability of bifurcated periodic solutions and equilibria. We also give some numerical simulations to support our main results.

*Keywords:* Bifurcation; neural network; time delay; periodic solutions; stability.

## 1. Introduction

Time delays can be the source of instabilities and bifurcations in dynamical systems and are frequently observed in biological systems such as neural networks. Introducing delays in neuronal models can be due to the synaptic connections between neurons. In recent years, a great number of mathematicians have carried out deep investigations on neural networks. These investigations include the global asymptotic and exponential stability and the existence and stability of periodic solutions, see, e.g. [Bungay & Campbell, 2007; Campbell *et al.*, 2005; Guo, 2005; Guo & Huang, 2003, 2005, 2007; Guo & Yuan, 2009; Huang & Wu, 2003; Mao & Hu, 2008, 2009; Song *et al.*, 2005;

Wu *et al.*, 1999; Wu, 1998; Wei *et al.*, 2002; Wei & Velarde, 2004; Xu & Wang, 2009; Yu & Cao, 2006; Yuan & Campbell, 2004; Yuan, 2007] and references therein.

There are also many papers devoted to the investigation of the dynamical behavior of two coupled neural oscillators [Aronson *et al.*, 1990; Ermentrout & Kopell, 1991]. The main functional unit of oscillatory neural networks is a neural oscillator. Some neural oscillator models consider oscillations as an endogenous property of a pacemaker neuron, such as the Van der Pol model [Kawato *et al.*, 1979], the Hindmarsh–Rose model [Hindmarsh & Rose, 1982], and the Hodgkin–Huxley model [Hansel *et al.*, 1993]. Another approach

---

\* Author for correspondence

suggests that oscillations arise as a result of the interactions between neural populations, for example, between excitatory and inhibitory populations, such as the Wilson–Cowan model [Wilson & Cowan, 1972], *integrate and fire* model [Kryukov et al., 1990], and McGregor model [MacGregor, 1987].

In the present paper, the model of a single neural oscillator is presented by a system of two autonomous differential equations describing the dynamics of average activities of the excitatory and inhibitory populations (measured as a portion of firing neurons in each population). Denoting these activities by  $E$  and  $I$ , respectively, the model reads

$$\begin{cases} E'(t) = -E(t) + af(-c_1 I(t - \tau)), \\ I'(t) = -I(t) + af(c_2 E(t - \tau)), \end{cases} \quad (1)$$

where  $a$ ,  $c_1$ ,  $c_2$ ,  $\tau$  are positive constants, and  $f$  is the activation function, which is usually adopted as  $f(x) = \tanh x$ . Without loss of generality, throughout this paper, we always assume that  $f: \mathbb{R} \rightarrow \mathbb{R}$  is  $C^1$ -smooth function with  $f(0) = 0$  and  $f'(0) = 1$ . For a number of two-neuron models and their linear stability analysis, we refer to the works of Marcus and Westervelt [1989], Babcock and Westervelt [1987], Gospalsamy and Leung [1996]. Consider two identical neural oscillators given by (1) and which are coupled using terms that may be interpreted as an additional external input. This system of coupled oscillators has the form

$$\begin{cases} E'_1(t) = -E_1(t) + af(-c_1 I_1(t - \tau) + P_{12}(t - \tau)), \\ I'_1(t) = -I_1(t) + af(c_2 E_1(t - \tau) + Q_{12}(t - \tau)), \\ E'_2(t) = -E_2(t) + af(-c_1 I_2(t - \tau) + P_{21}(t - \tau)), \\ I'_2(t) = -I_2(t) + af(c_2 E_2(t - \tau) + Q_{21}(t - \tau)), \end{cases} \quad (2)$$

where  $(E_1, I_1)$  and  $(E_2, I_2)$  describe activities of the first and the second oscillators, respectively; the terms  $P_{12}, P_{21}, Q_{12}, Q_{21}$  describe connections between the oscillators. In general, both  $P_{12}$  and  $Q_{12}$  depend on  $(E_2, I_2)$ , both  $P_{21}$  and  $Q_{21}$  depend on  $(E_1, I_1)$ . Jiang and Guo [2010] considered system (2) with  $P_{12} = P_{21} = 0$  and  $Q_{12} = \alpha E_2$  and  $Q_{21} = \alpha E_1$ . Peng and Guo [2010] considered the case where  $P_{12} = P_{21} = 0$  and  $Q_{12} = -\alpha I_2$  and  $Q_{21} = -\alpha I_1$ . Zhang et al. [2010] considered system (2) with  $P_{12} = \alpha E_2$  and  $P_{21} = \alpha E_1$  and  $Q_{12} = Q_{21} = 0$ . Xiao and Guo [2010] considered system (2) with inhibitory connections  $P_{12} = Q_{12} = -\alpha I_2$  and  $P_{21} = Q_{21} = -\alpha I_1$ .

In this paper, we consider the connections from the excitatory population  $E_1$  to the excitatory population  $E_2$  and from the inhibitory population  $I_2$  to the inhibitory population  $I_1$ , i.e.  $Q_{12} = -\alpha I_2$ ,  $P_{21} = \alpha E_1$ ,  $P_{12} = Q_{21} = 0$ . Namely, we consider the following model:

$$\begin{cases} E'_1(t) = -E_1(t) + af(-c_1 I_1(t - \tau)), \\ I'_1(t) = -I_1(t) + af(c_2 E_1(t - \tau) - \alpha I_2(t - \tau)), \\ E'_2(t) = -E_2(t) + af(-c_1 I_2(t - \tau) + \alpha E_1(t - \tau)), \\ I'_2(t) = -I_2(t) + af(c_2 E_2(t - \tau)), \end{cases} \quad (3)$$

where  $\alpha$  represents the strength of the connections between subpopulations related to different oscillators.

On one hand, it is natural to consider the effect of time delay  $\tau$  on the dynamics of (3) by regarding  $\tau$  as the bifurcation parameter while the connection topology is fixed. On the other hand, realistic modeling of networks inevitably requires careful design and variation of the connection topology, and the fact that a wide range of different behaviors can be established by varying the coupling strength and structure has important implications for neural networks, since synaptic coupling can change through learning. The observation provides us the motivation of this study to regard connection topology as the bifurcation parameter while the time delay  $\tau$  is fixed. Therefore, our focus in this paper is on the stability and bifurcation phenomena of system (3) in different regions of the space of normalized parameters  $(\gamma, \eta, \tau)$ , where  $\gamma = a\sqrt{c_1 c_2}$  and  $\eta = a\alpha$ . The study of such problems is important in various areas, for example, in the applications of content addressable memories where a stable solution can be used as coded information of a memory of the system to be stored and retrieved.

Our main technical tools are local Hopf bifurcation theory of delay differential equations, the normal forms on center manifold of functional differential equations, and stability theory of periodic solutions of ordinary differential equations.

The paper is organized as follows. In Sec. 2, we will discuss the distribution of roots of characteristic equation of the linearized system of (3). Then, in Sec. 3, the bifurcation direction and the stability of bifurcated periodic solutions are determined by using normal form method and center manifold reduction. Section 4 is devoted to fold bifurcation. In Sec. 5, some numerical simulations are

accomplished to illustrate our main results. Finally, we draw some conclusions in Sec. 6.

### 2. Linear Stability

Let  $\mathcal{C} = C([-τ, 0], \mathbb{R}^4)$  denote the Banach space of all continuous mappings from  $[-τ, 0]$  into  $\mathbb{R}^4$  equipped with the supremum norm  $\|\phi\| = \sup_{-τ \leq \theta \leq 0} |\phi(\theta)|$  for  $\phi \in \mathcal{C}$ . As usual, if  $\sigma \in \mathbb{R}, A \geq 0$  and  $u : [\sigma - \tau, \sigma + A] \rightarrow \mathbb{R}^4$  is a continuous mapping, then  $u_t \in \mathcal{C}$  for  $t \in [\sigma, \sigma + A]$  is defined by  $u_t(\theta) = u(t + \theta)$  for  $-\tau \leq \theta \leq 0$ .

Linearizing system (3) at the trivial solution leads to the following linear system

$$\dot{u}(t) = -u(t) + Mu(t - \tau), \tag{4}$$

where

$$M = \begin{pmatrix} 0 & -ac_1 & 0 & 0 \\ ac_2 & 0 & 0 & -a\alpha \\ a\alpha & 0 & 0 & -ac_1 \\ 0 & 0 & ac_2 & 0 \end{pmatrix}. \tag{5}$$

The characteristic matrix for (4) is

$$\Delta(\gamma, \eta, \tau, \lambda) = (\lambda + 1)\text{Id}_4 - Me^{-\lambda\tau} \tag{6}$$

and hence the characteristic equation is  $\det \Delta(\gamma, \eta, \tau, \lambda) = 0$ , that is,

$$[(\lambda + 1)^2 + \gamma^2 e^{-2\lambda\tau}]^2 - [\gamma\eta e^{-2\lambda\tau}]^2 = 0. \tag{7}$$

Then  $\det \Delta(\gamma, \eta, \tau, \lambda)$  can be decomposed as  $\det \Delta(\gamma, \eta, \tau, \lambda) = \Delta_+ \cdot \Delta_-$ , where  $\Delta_{\pm} = \lambda^2 + 2\lambda + 1 + \gamma(\gamma \pm \eta)e^{-2\lambda\tau}$ . It is well known that the trivial solution of system (3) is locally asymptotically stable if all roots  $\lambda$  of the characteristic equation (7) satisfy  $\text{Re}(\lambda) < 0$ . It follows that it is natural to choose  $\gamma$  and  $\eta$  as the bifurcation parameters. As a result, it is necessary to determine when the infinitesimal generator  $\mathcal{A}$  of the  $C^0$ -semigroup generated by the linear system (4) has eigenvalues lying on the imaginary axis. For this purpose, we first consider

$$P(\lambda) = \lambda^2 + 2\lambda + 1 + ze^{-2\lambda\tau}, \tag{8}$$

where  $z \in \mathbb{R}$ . We have the following results.

**Lemma 1**

- (i) If  $|z| < 1$ , then all zeros of  $P(\lambda)$  have negative real parts;
- (ii) If  $|z| > 1$ , then at and only at  $\tau = \tau_s(z), s \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ ,  $P(\lambda)$  has a pair of purely imaginary zeros  $\lambda = \pm i\omega$ , where  $\omega = \sqrt{|z| - 1}$  and

$$\tau_s(z) = \begin{cases} \frac{\left[ (2s + 1)\pi - \arcsin \frac{2\omega}{z} \right]}{2\omega}, & \text{for } z \in (-\infty, -2], \\ \frac{\left[ 2(s + 1)\pi + \arcsin \frac{2\omega}{z} \right]}{2\omega}, & \text{for } z \in (-2, -1), \\ \frac{\left[ (2s + 1)\pi - \arcsin \frac{2\omega}{z} \right]}{2\omega}, & \text{for } z \in (1, 2), \\ \frac{\left[ 2s\pi + \arcsin \frac{2\omega}{z} \right]}{2\omega}, & \text{for } z \in [2, \infty); \end{cases} \tag{9}$$

- (iii) For  $|z| > 1$  and each fixed  $s \in \mathbb{N}$ , there exist  $\delta > 0$  and an analytical function  $\lambda : (\tau_s - \delta, \tau_s + \delta) \rightarrow \mathbb{C}$  such that  $\lambda(\tau_s) = i\omega$  and  $\lambda(\tau)$  is a zero of  $P(\lambda)$  for all  $\tau \in (\tau_s - \delta, \tau_s + \delta)$ . Moreover,  $\text{Re}\lambda'(\tau_s) > 0$ ;
- (iv)  $P(\lambda) = 0$  has only solutions with negative real parts if  $\tau \geq 0$  and  $|z| < 1$  or  $0 \leq \tau < \tau_0(z)$

---

with  $z > 1$ , exactly one solution with a positive real part if  $0 \leq \tau \leq \tau_0(z)$  with  $z < -1$ , exactly  $2s$  solutions with positive real parts if  $\tau_{s-1}(z) < \tau \leq \tau_s(z)$  with  $z > 1$  and, exactly  $2s + 1$  solutions with positive real parts if  $\tau_{s-1}(z) < \tau \leq \tau_s(z)$  with  $z < -1, s \in \mathbb{N}$ .

*Proof*

(i) Let  $\lambda = u + iv$  be a zero of  $P(\lambda)$ . Then, we get

$$(u + 1)^2 - v^2 = -ze^{-2\tau u} \cos(2\tau v)$$

and

$$2v(u + 1) = ze^{-2\tau u} \sin(2\tau v),$$

from which it follows that

$$|z|e^{-2\tau u} = v^2 + (u + 1)^2.$$

Consequently,  $u < 0$  for otherwise, the left-hand side of the above equality is strictly less than 1 while the right-hand side is larger than or equal to 1.

(ii) Obviously,  $\lambda = \pm iv$  ( $v > 0$ ) is a pair of imaginary zeros of  $P(\lambda)$  if and only if

$$\begin{aligned} 1 - v^2 + z \cos(2\tau v) &= 0 \quad \text{and} \\ 2v - z \sin(2\tau v) &= 0. \end{aligned} \tag{10}$$

Eliminating  $\tau$  from (10) yields  $v = \omega$ . This, together with (10), implies that  $\tau = \tau_s$  for some  $s \in \mathbb{N}_0$ .

(iii) The existence of  $\delta$  and the mapping  $\lambda$  follows from the implicit function theorem. We now differentiate the equality  $P(\lambda(\tau)) = 0$  with respect to  $\tau$  to get

$$\lambda'(\tau_s) = \frac{\omega^2 - i\omega(1 + \tau_s + \tau_s\omega^2)}{(1 + \tau_s)^2 + (\tau_s\omega)^2}.$$

Thus,  $\operatorname{Re}\lambda'(\tau_s) > 0$ .

(iv) We first notice the fact that there exist at most a finite number of zeros of  $P(\lambda)$  in right half-plane for each  $\tau \geq 0$ . Indeed, for any zero  $\lambda$  of  $P(\lambda)$ ,

$$|\lambda + 1| = \sqrt{|z|} \exp\{-\tau \operatorname{Re}\lambda\}.$$

This implies that there is a real number  $\alpha$  such that all zeros of  $P(\lambda)$  satisfy  $\operatorname{Re}\lambda < \alpha$ . Clearly,  $P(\lambda)$  is an entire function. Hence, there can only be a finite number of zeros of  $P(\lambda)$  in any compact set. Namely, there exist only a finite number of zeros in any vertical strip in the complex plane.

In what follows, for  $|z| > 1$ , we consider the distribution of zeros of  $P(\lambda)$ . We can regard  $\lambda$  as the continuous function of  $\tau$  according to the implicit function theorem. If  $\tau = 0$ , then  $P(\lambda)$  has exactly two zero points:

$$\lambda_{\pm}(0) = -1 \pm \sqrt{-z}. \tag{11}$$

If  $z > 1$  then both of  $\lambda_{\pm}(0)$  have negative real parts. However, if  $z < -1$  then  $\lambda_+(0)$  and  $\lambda_-(0)$  are positive and negative numbers, respectively. Recall the fact that all zeros of  $P(\lambda)$  are simple and continuously depend on  $\tau$ , then there exists a critical value  $\tau_0^*$  such that all zeros of  $P(\lambda)$  have negative real parts if  $\tau \in [0, \tau_0^*)$ , and that as  $\tau$  increases and passes through  $\tau_0^*$ , only a pair of zero points of  $P(\lambda)$ , denoted by  $\lambda^*(\tau)$  and  $\overline{\lambda^*(\tau)}$ , vary from a pair of conjugated complex numbers with negative real parts to a pair of conjugated purely imaginary number and then to a pair of conjugated complex numbers with positive real parts. In fact, the proof of conclusion (ii) yields that  $\tau_0^* = \tau_0$ . Therefore,  $P(\lambda)$  has only solutions with negative real parts if  $z > 1$  and  $0 \leq \tau < \tau_0$ , and exactly one solution with a positive real part if  $z < -1$  and  $0 \leq \tau < \tau_0$ .

We can repeat the same analysis to conclude that there exists next critical value  $\tau_1^*$  such that  $\lambda^*(\tau)$  and  $\overline{\lambda^*(\tau)}$  have positive real parts and all other zeros of  $P(\lambda)$  have negative real parts if  $\tau \in (\tau_0, \tau_1^*)$ , and that as  $\tau$  increases and passes through  $\tau_1^*$ , another pair of zero points of  $P(\lambda)$  vary from a pair of conjugated complex numbers with negative real parts to a pair of conjugated purely imaginary number and then to a pair of conjugated complex numbers with positive real parts. Similarly, it follows from the proof of conclusion (ii) that  $\tau_1^* = \tau_1$ . Therefore,  $P(\lambda)$  has only a pair of solutions with positive real parts if  $z > 1$  and  $\tau_0 < \tau \leq \tau_1$ , and exactly three solutions with positive real parts if  $z < -1$  and  $0 \leq \tau < \tau_0$ .

By induction, we can draw the conclusion that  $P(\lambda)$  has exactly  $2k$  solutions with positive real parts if  $z > 1$  and  $\tau_{k-1} < \tau \leq \tau_k$ , and exactly  $2k + 1$  solutions with positive real parts if  $z < -1$  and  $\tau_{k-1} < \tau \leq \tau_k$ ,  $k \in \mathbb{N}$ . ■

We divide the whole  $(\gamma, \eta)$ -plane into three parts (see Fig. 1):

$$\mathcal{D}_1 = \left\{ (\gamma, \eta) \mid |\eta| > \frac{1 + \gamma^2}{\gamma} \right\}$$

$$\mathcal{D}_2 = \left\{ (\gamma, \eta) \mid \frac{\gamma^2 - 1}{\gamma} < \eta < \frac{1 - \gamma^2}{\gamma}, \gamma \in [0, 1] \right\},$$

$$\mathcal{D}_3 = \left\{ (\gamma, \eta) \mid \frac{1 - \gamma^2}{\gamma} < |\eta| < \frac{1 + \gamma^2}{\gamma} \right\}.$$

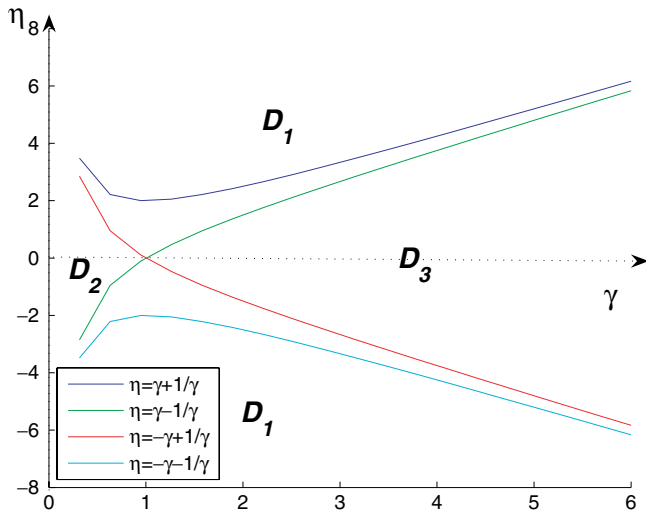


Fig. 1. Stability region of the trivial solution of system (3). In  $\mathcal{D}_1$  the trivial solution of (3) is unstable for all  $\tau \geq 0$ ; in  $\mathcal{D}_2$  the trivial solution of (3) is stable for all  $\tau \geq 0$ ; in  $\mathcal{D}_3$  the trivial solution of (3) is stable for  $\tau \in (0, \tau_0)$ .

If  $|\gamma^2 \pm \gamma\eta| > 1$ , then for  $s \in \mathbb{N}_0$ , define  $\tau_s^\pm$  as

$$\tau_n^\pm = \tau_s(\gamma^2 \pm \gamma\eta),$$

where  $\tau_s(\cdot)$  is defined as (9). Then, applying Lemma 1 to each factor of  $\Delta(\gamma, \eta, \tau, \lambda)$  in (6), we have the following.

**Theorem 1**

- (i) If  $(\gamma, \eta) \in \mathcal{D}_1$ , then the trivial solution of system (3) is unstable for all  $\tau \geq 0$ .
- (ii) If  $(\gamma, \eta) \in \mathcal{D}_2$ , then the trivial solution of system (3) is asymptotically stable for all  $\tau \geq 0$ .
- (iii) If  $(\gamma, \eta) \in \mathcal{D}_3$ , then the trivial solution of system (3) is asymptotically stable for all  $\tau \in [0, \tau_0^*)$  but is unstable for  $\tau \geq \tau_0^*$ , where  $\tau_0^*$  is equal to  $\tau_0^+$  if  $\gamma(\gamma + \eta) > 1 > \gamma(\gamma - \eta)$ , to  $\tau_0^-$  if  $\gamma(\gamma - \eta) > 1 > \gamma(\gamma + \eta)$ , and to  $\min\{\tau_0^+, \tau_0^-\}$  if both of  $\gamma(\gamma \pm \eta)$  is greater than 1.
- (iv) If  $(\gamma, \eta) \in \mathcal{D}_1 \cup \mathcal{D}_3$ , then at each of critical values  $\tau = \tau_n^\pm, n \in \mathbb{N}_0$ , system (3) undergoes a Hopf bifurcation near the trivial solution.
- (v) System (3) undergoes a fold bifurcation along the curves  $\gamma(\gamma \pm \eta) = 1$ .

*Proof.* In view of Lemma 1, all solutions of (7) have negative real parts if and only if  $|\gamma(\gamma \pm \eta)| < 1$ , i.e.  $(\gamma, \eta) \in \mathcal{D}_2$ . Moreover, (7) has at least one solution with positive real parts for all  $\tau \geq 0$  if and only if either  $|\gamma(\gamma + \eta)| < -1$  or  $|\gamma(\gamma - \eta)| < -1$ , i.e.  $(\gamma, \eta) \in \mathcal{D}_1$ . Thus, conclusions (i) and (ii) follow.

If  $(\gamma, \eta) \in \mathcal{D}_3$  then  $\gamma(\gamma \pm \eta) > -1$  and either of  $\gamma(\gamma + \eta)$  and  $\gamma(\gamma - \eta)$  is greater than 1. In view of Lemma 1(iv), if  $\gamma(\gamma \pm \eta) > 1$  then  $\lambda^2 + 2\lambda + 1 + \gamma(\gamma \pm \eta)e^{-2\lambda\tau}$  has only solutions with negative real parts for  $\tau \in [0, \tau_0^*)$ , and at least one solution with a positive real part for  $\tau > \tau_0^*$ .

If  $(\gamma, \eta) \in \mathcal{D}_1 \cup \mathcal{D}_3$ , then either of  $|\gamma(\gamma + \eta)|$  and  $|\gamma(\gamma - \eta)|$  is greater than 1. Hence, conclusion (iv) is a direct consequence of Lemma 1.

Finally, if  $\gamma(\gamma \pm \eta) = 1$ , then (7) has a zero point  $\lambda = 0$ , which implies that fold bifurcation may occur. ■

The remaining part of this section is devoted to the effect of  $(\gamma, \eta)$  on the distribution of zeros of  $\det \Delta(\gamma, \eta, \tau, \cdot)$  when the time delay  $\tau$  is fixed. Define a curve  $\Sigma$  with the following parametric equations.

$$\begin{cases} u(t) = (t^2 - 1) \cos 2t\tau + 2t \sin 2t\tau, \\ v(t) = (t^2 - 1) \sin 2t\tau - 2t \cos 2t\tau, \end{cases} \quad t \in \mathbb{R}. \quad (12)$$

It is not difficult to see that the curve  $\Sigma$  is symmetric about the  $u$ -axis. Let  $\theta(t) = v(t)/u(t)$  then  $\theta'(t) = 2(t^2 + 1)[\tau(1 + t^2) + 1]/u^2(t) > 0$  for all  $t \in \mathbb{R}$  such that  $u(t) \neq 0$ . This implies that, as  $t$  increases, the corresponding point  $(u(t), v(t))$  on the curve  $\Sigma$  moves anticlockwise around the origin. Moreover, it follows from  $u^2(t) + v^2(t) = (1 + t^2)^2$  that  $\Sigma^+ = \{(u(t), v(t)) : t \in \mathbb{R}^+\}$  is simple, i.e. it cannot intersect itself. Let  $\{t_n\}_{n=0}^\infty$  be the monotonic increasing sequence of the non-negative zeros of  $v(t)$ , and  $b_n = u(t_n)$  for all  $n \in \mathbb{N}_0$ . Obviously, we have  $t_0 = 0$ . Therefore, the curve  $\Sigma$  intersects with the  $u$ -axis at  $(b_n, 0), n \in \mathbb{N}_0$ . It follows from the anticlockwise property of the curve  $\Sigma$  that  $(-1)^n b_n < 0$  for all  $n \in \mathbb{N}_0$ . Besides, we have  $|b_n| = 1 + t_n^2$ , which implies that  $b_n = (-1)^{n-1}(1 + t_n^2)$  for all  $n \in \mathbb{N}$  and  $\{|b_n|\}_{n \in \mathbb{N}_0}$  is an increasing sequence. In particular,  $b_0 = -1$ . The following lemma plays an important role in analyzing the distribution of the roots of (7).

**Lemma 2.** Consider  $P(\lambda)$  defined in (8) with  $z \in \mathbb{R}$ , then

- (i)  $P(\lambda)$  has a pair of purely imaginary zeros  $\pm it_n$  if and only if  $z = b_n$  for some  $n \in \mathbb{N}$ .
- (ii) For each fixed  $s \in \mathbb{N}_0$ , there exist  $\delta > 0$  and an analytical function  $\lambda : (b_n - \delta, b_n + \delta) \rightarrow \mathbb{C}$  such that  $\lambda(b_n) = it_n$  and  $\lambda(z)$  is a zero of  $P(\lambda)$  for all  $z \in (b_n - \delta, b_n + \delta)$ . Moreover,  $b_n \operatorname{Re} \lambda'(b_n) > 0$ .

- (iii)  $P(\lambda)$  has only zeros with strictly negative real parts if and only if  $z \in (-1, b_1)$ ; exactly  $2j$  zeros with positive real parts if  $z \in (b_{2j-1}, b_{2j+1})$ , and exactly  $2j+1$  zeros with positive real parts if  $z \in (b_{2j+2}, b_{2j})$ ,  $j \in \mathbb{N}_0$ .

*Proof*

- (i) The sufficiency is obvious. It suffices to verify the necessity. If  $P(\lambda)$  has a pair of purely imaginary zeros  $\pm it$  with  $t > 0$  then  $(1 + it)^2 e^{2it\tau} + z = 0$ . In view of  $z \in \mathbb{R}$ , we have  $v(t) = 0$  and hence  $t = t_n$  for some  $n \in \mathbb{N}$ . This implies that  $z = u(t_n) = b_n$ .
- (ii) Since  $i\theta_n$  is a simple zero of  $P(\lambda)$  with  $z = b_n$ , the existence of the positive constant  $\delta$  and the analytical function  $\lambda$  follows from the Implicit Function Theorem. We now differentiate the equality (8) with respect to  $z$  to get

$$\lambda'(z) = \frac{\lambda + 1}{2z(1 + \tau(\lambda + 1))}$$

Noting that  $P(it_n) = 0$  when  $z = b_n$ , we have  $\text{Re}\lambda'(b_n) = (1 + \tau + \tau t_n^2)/2b_n[(\tau + 1)^2 + (\tau t_n)^2] \neq 0$ . Obviously,  $b_n \text{Re}\lambda'(b_n) > 0$ .

- (iii) Since  $P(\lambda)$  depends on the parameter  $z$ , we rewrite  $P(\lambda)$  as  $P_z(\lambda)$ . Note that  $P_0(\lambda)$  has exactly one double zero  $-1$ , which obviously has a negative real part. Since zeros of  $P_z(\lambda)$  depend continuously on  $z$ , there exists a real interval  $I_0$  containing  $z = 0$  such that for  $z \in I_0$ , all zeros of  $P_z(\lambda)$  have negative real parts. Moreover, as  $z \in I_0$  varies and passes through the two extreme points, either only one real zero point of  $P_z(\lambda)$  vary from a negative number to zero and then to a positive number, or two zero points of  $P_z(\lambda)$  vary from complex numbers with negative real parts to purely imaginary numbers and then to complex numbers with positive real parts. By (i), these two extreme points are  $-1$  and  $b_1$ . Therefore,  $P_z(\lambda)$  has only zeros with negative real parts if  $z$  is inside the interval  $(-1, b_1)$ .

We can repeat the same analysis to conclude that there exists the next critical value  $b^*$  such that  $\lambda^*(z)$  has a positive real part and all other zeros of  $P_z(\lambda)$  have negative real parts if  $b \in (b^*, -1)$ , and that as  $z$  decreases and passes through  $b^*$ , another two zero points of  $P_z(\lambda)$  vary from complex numbers with negative real parts to purely imaginary numbers and then to complex numbers with positive real

parts. Similarly, it follows from the proof of conclusion (i) that  $b^* = b_2$ . Therefore,  $P_z(\lambda)$  has only one solution with positive real parts if  $b_2 < z < -1$ .

By induction, we can draw the conclusion that  $P_z(\lambda)$  has exactly  $2k + 1$  solutions with positive real parts if  $b_{2k+2} < z \leq b_{2k}$ ,  $k \in \mathbb{N}$ . One can continue in this manner to finish the remaining proof also with the help of conclusions (i) and (ii). ■

Applying Lemma 2 to each factor of  $\det \Delta(\gamma, \eta, \tau, \cdot)$ , we can get

**Corollary 2.1**

- (i) All zeros of  $\det \Delta(\gamma, \eta, \tau, \cdot)$  have negative real parts if and only if  $-1 < \gamma(\gamma \pm \eta) < b_1$ .
- (ii) If and only if  $\gamma(\gamma + \eta) = b_j$  or  $\gamma(\gamma - \eta) = b_j$  for some  $j \in \mathbb{N}$ ,  $\det \Delta(\gamma, \eta, \tau, \cdot)$  has a pair of simple conjugate purely imaginary zeros, which are  $\pm it_j$ .
- (iii) If and only if  $\gamma(\gamma + \eta) = -1$  or  $\gamma(\gamma - \eta) = -1$ ,  $\det \Delta(\gamma, \eta, \tau, \cdot)$  has a simple zero  $\lambda = 0$ .

From Lemma 2 and Corollary 2.1, we can obtain the following

**Theorem 2**

- (i) If  $-1 < \gamma(\gamma \pm \eta) < b_1$ , then the trivial solution of system (3) is asymptotically stable.
- (ii) System (3) undergoes a Hopf bifurcation along  $\gamma(\gamma + \eta) = b_j$  and  $\gamma(\gamma - \eta) = b_j$ ,  $j \in \mathbb{N}$ .
- (iii) System (3) undergoes a fold bifurcation along  $\gamma(\gamma + \eta) = -1$  and  $\gamma(\gamma - \eta) = -1$ .

*Remark 2.1.* From Theorem 2, we can see that the stability of the trivial solution of system (3) switches as either of  $\gamma(\gamma \pm \eta)$  passes through  $b_j$ ,  $j \in \mathbb{N}_0$ . In the subsequent sections, we shall discuss two kinds of bifurcations: Hopf bifurcation and fold bifurcation.

**3. Hopf Bifurcation**

We start with the analysis of Hopf bifurcation induced by  $\gamma$  and  $\eta$  when the time delay  $\tau$  is a given positive constant. For a given  $j \in \mathbb{N}$ , let  $\mu = \gamma(\gamma \pm \eta) - b_j$  and rewrite  $\Delta(\gamma, \eta, \tau, \lambda)$  in (6) as  $\Delta(\mu, \lambda)$ . Theorem 2 shows that system (3) undergoes a Hopf bifurcation at  $\mu = 0$ , that is,  $\det \Delta(0, \pm i\omega_0) = 0$ , where  $\omega_0 = t_j$ . This means that there exists a family of periodic solutions bifurcated from the trivial solution for sufficiently small  $\mu$ . In order to investigate the stability and direction of the Hopf bifurcation, we need to compute the normal form of

system (3) on the associated two-dimensional center manifold. For this purpose, we further assume that  $f \in C^3(\mathbb{R}, \mathbb{R})$ .

Letting that  $u(t) = (E_1(t), I_1(t), E_2(t), I_2(t))^T$  and  $u_t(\theta) = u(t + \theta)$  for  $\theta \in [-\tau, 0]$ , we rewrite (3) as

$$\dot{u}(t) = L_\mu u_t + F(\mu, u_t), \tag{13}$$

where  $L_\mu : \mathcal{C} \rightarrow \mathbb{R}^4$  is a linear operator,  $F(\mu, \cdot) \in C^3(\mathcal{C}, \mathbb{R}^4)$  is a nonlinear operator. There exists a  $4 \times 4$  matrix-valued function  $\Xi : [-\tau, 0] \rightarrow \mathbb{R}^{16}$  with bounded variation in  $\theta \in [-\tau, 0]$  for each  $\mu$ , such that

$$L_\mu \varphi = \int_{-\tau}^0 d\Xi(\theta, \mu) \varphi(\theta) \quad \text{for } \varphi \in \mathcal{C}, \tag{14}$$

where  $\Xi(\theta, \mu) = \delta_0 I - \delta_\tau M$  with  $\delta_\tau(\theta) = \delta(\theta + \tau)$  is the Dirac distribution at the point  $\theta = -\tau$ .

Define the infinitesimal generator  $\mathcal{A}_\mu : \mathcal{C}^1 \rightarrow \mathcal{C}$  by

$$(\mathcal{A}_\mu \phi)(\theta) = \phi'(\theta)$$

for  $\theta \in [-\tau, 0]$  and  $\phi \in \text{dom}(\mathcal{A}_\mu) := \{\phi \in \mathcal{C}^1, \phi(0) = L_\mu \phi\}$ . The spectrum  $\sigma(\mathcal{A}_\mu)$  of  $\mathcal{A}_\mu$  is

$$\sigma(\mathcal{A}_\mu) = \{\lambda \in \mathbb{C} : \Delta(\mu, \lambda)v = 0, \text{ for } v \in \mathbb{C}^4 \setminus \{0\}\}.$$

Let

$$q(\theta) = \begin{cases} \left(1, -d, -\frac{ac_1}{\gamma}d, -\frac{\gamma}{ac_1}\right)^T e^{i\omega_0\theta}, & \text{if } \mu = \gamma(\gamma + \eta) - b_j, \\ \left(1, -d, \frac{ac_1}{\gamma}d, \frac{\gamma}{ac_1}\right)^T e^{i\omega_0\theta}, & \text{if } \mu = \gamma(\gamma - \eta) - b_j \end{cases} \tag{15}$$

for  $\theta \in [-\tau, 0]$ , where  $d = ((1 + i\omega_0)/ac_1)e^{i\omega_0\tau}$ . Then,  $q(\theta)$  is an eigenvector of  $\mathcal{A}(0)$  associated with the eigenvalue  $i\omega_0$ .

To construct coordinates to describe the center manifold  $\mathcal{C}_\mu$  of (13) near  $\mu = 0$ , we define the following adjoint bilinear form

$$\langle \psi, \varphi \rangle = \bar{\psi}(0)\varphi(0) - \int_{\theta=-\tau}^0 \int_{\xi=0}^\theta \bar{\psi}(\xi - \theta) d\Xi(\theta, 0) \varphi(\xi) d\xi \tag{16}$$

for  $\psi \in C([0, \tau], \mathbb{R}^{4*})$  and  $\varphi \in C([-\tau, 0], \mathbb{R}^4)$ . The adjoint operator  $\mathcal{A}^*(\mu_0)$  is

$$(\mathcal{A}_\mu^* \psi)(\xi) = \begin{cases} -\frac{d\psi(\xi)}{d\xi}, & \xi \in [0, \tau], \\ \int_{-\tau}^0 \psi(-\xi) d\Xi(\theta, \mu), & \xi = 0. \end{cases}$$

Using the formal adjoint theory, we see that  $-i\omega_0$  is an eigenvalue of  $\mathcal{A}^*(\mu_0)$  and

$$(\mathcal{A}_0^* p)(\xi) = -i\omega_0 p(\xi),$$

where  $p(\xi), \xi \in [0, \tau]$  is a nonzero row-vector function. It is easy to check

$$\langle \psi, \mathcal{A}_0 \varphi \rangle = \langle \mathcal{A}_0^* \psi, \varphi \rangle.$$

Normalize  $q$  and  $p$  by  $\langle p, q \rangle = 1$  and  $\langle p, \bar{q} \rangle = 0$ , then we obtain

$$p(\xi) = \begin{cases} \bar{D} \left( -\bar{d}, 1, \frac{\gamma}{ac_1}, \frac{ac_1}{\gamma} \bar{d} \right) e^{i\omega_0 \xi}, & \text{if } \mu = \gamma(\gamma + \eta) - b_j, \\ \bar{D} \left( -\bar{d}, 1, -\frac{\gamma}{ac_1}, -\frac{ac_1}{\gamma} \bar{d} \right) e^{i\omega_0 \xi}, & \text{if } \mu = \gamma(\gamma - \eta) - b_j \end{cases} \tag{17}$$

for  $\xi \in [0, \tau]$ , where  $D = -[4d(1 + \tau(1 + i\omega_0))]^{-1}$ .

For each  $x \in \text{Dom}(\mathcal{A}_\mu)$  with sufficiently small  $|\mu|$ , we associate it with the pair  $(z, w)$ , where  $z = \langle p, x \rangle$  and  $w = x - zq - \bar{z}\bar{q} = x - 2\text{Re}\{zq\}$ . Then, on the center manifold  $\mathcal{C}_\mu$  with sufficiently small  $|\mu|$ , the reduced equation of (3) takes the form

$$\dot{z}(t) = i\omega_0 z + G(z, \bar{z}, \mu). \tag{18}$$

Equation (18) can be transformed by an invertible parameter-dependent change of complex coordinates into an equation with only one cubic term:

$$\dot{z} = \lambda^*(\mu)z + \frac{1}{2}\vartheta(\mu)z^2\bar{z} + O(|z|^4), \tag{19}$$

where  $\lambda^*(\mu) = i\omega_0 + \mu\lambda'(b_j) + O(|\mu|^2)$  and

$$\vartheta(0) = \frac{i}{\omega_0} \left[ G_{20}G_{11} - 2|G_{11}|^2 - \frac{1}{3}|G_{02}|^2 \right] + G_{21},$$

where  $G_{jk} = (\partial^{j+k}/\partial z^j \partial \bar{z}^k)G(z, \bar{z}, 0)$ ,  $j, k \in \mathbb{N}_0$ . Let  $z = re^{i\theta}$ , then (19) can be rewritten as

$$\dot{r} = r\mu\mathcal{R}e\{\lambda'(b_j)\} + \frac{1}{2}r^3\mathcal{R}e\{\vartheta(0)\} + \text{h.o.t.},$$

$$\dot{\theta} = \omega_0 + \mu \mathcal{I}m\{\lambda'(b_j)\} + \frac{1}{2}r^2 \mathcal{I}m\{\vartheta(0)\} + \text{h.o.t.} \tag{20}$$

**Theorem 3.** *System (19) has a branch of bifurcated periodic solutions for sufficiently small  $\mu > 0$  (resp.,  $\mu < 0$ ) when  $(-1)^j \text{Re}\{\vartheta(0)\} > 0$  (resp.,  $(-1)^j \text{Re}\{\vartheta(0)\} < 0$ ). Moreover, the bifurcated periodic solutions are orbitally stable (resp., unstable) if  $\text{Re}\{\vartheta(0)\} < 0$  (resp.,  $\text{Re}\{\vartheta(0)\} > 0$ ). Moreover, the periods of the bifurcated periodic solutions are  $\geq 2\pi/\omega_0$  (resp.,  $< 2\pi/\omega_0$ ) if  $T_\vartheta \leq 0$  (resp.,  $> 0$ ), where*

$$T_\vartheta = \mathcal{I}m\{\vartheta(0)\} - \frac{\omega_0}{1 + \tau + \tau\omega_0^2} \mathcal{R}e\{\vartheta(0)\}.$$

*Proof.* Consider the following truncated equation of (20).

$$\dot{r} = r\mu \mathcal{R}e\{\lambda'(b_j)\} + \frac{1}{2}r^3 \mathcal{R}e\{\vartheta(0)\}, \tag{21}$$

$$\dot{\theta} = \omega_0 + \mu \mathcal{I}m\{\lambda'(b_j)\} + \frac{1}{2}r^2 \mathcal{I}m\{\vartheta(0)\}. \tag{22}$$

We first consider the amplitude equation (21) as it is decoupled from  $\theta$ . Equation (21) always has the trivial equilibrium  $r_0 = 0$ . Other equilibria  $r$  of (21) satisfy  $2\mu \text{Re}\{\lambda'(b_j)\} + r^2 \text{Re}\{\vartheta(0)\} = 0$ , which has exactly one positive solution

$$r_1 = \sqrt{\frac{-2\mu \text{Re}\{\lambda'(b_j)\}}{\text{Re}\{\vartheta(0)\}}} \tag{23}$$

if and only if  $\mu \text{Re}\{\lambda'(b_j)\} \text{Re}\{\vartheta(0)\} < 0$ . Obviously,  $r_1 \rightarrow 0$  as  $\mu \rightarrow 0$ . This implies that system (19) has a branch of periodic solutions bifurcated from the origin and existing for  $\mu > 0$  (resp.,  $\mu < 0$ ) when  $\text{Re}\{\lambda'(b_j)\} \text{Re}\{\vartheta(0)\} < 0$  (resp.,  $\text{Re}\{\lambda'(b_j)\} \text{Re}\{\vartheta(0)\} > 0$ ).

Note that the eigenvalue of the linearized operator of the right-hand side of (21) at  $r = r_1$  is  $r_1^2 \text{Re}\{\vartheta(0)\}$ . As the stability of the bifurcated periodic solutions is the same as that of  $r_1$ , it follows that the bifurcated periodic solutions are stable if  $\text{Re}\{\vartheta(0)\} < 0$  and unstable otherwise.

Finally, we consider the phase equation (22) for the bifurcated periodic solution of (19) corresponding to the equilibrium  $r_1$ . It follows from (23) that

$$\mu = -\frac{r_1^2 \text{Re}\{\vartheta(0)\}}{2 \text{Re}\{\lambda'(b_j)\}}.$$

Substituting the above expression of  $\mu$  into Eq. (22) yields

$$\dot{\theta} = \omega_0 + \frac{1}{2}r_1^2 T_\vartheta.$$

Then the remaining conclusion of the theorem follows immediately. ■

If we further assume that:

**(H)**  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a sufficiently smooth odd function satisfying  $xf''(x) < 0$  for all  $x \neq 0$ .

Then, we have  $f''(0) = 0$  and hence  $G_{20} = G_{11} = G_{02}$  and

$$\vartheta(0) = G_{21}(0) = f'''(0)N(1 + \tau + \tau\omega_0^2 + i\omega_0)$$

where  $N = 1/4(c_2 + c_1)((1/c_1) + (1/c_2)|d|^2)[(1 + \tau)^2 + (\tau\omega)^2]^{-1} > 0$ . It follows that  $\text{Re}\{\vartheta(0)\} = f'''(0)N(1 + \tau + \tau\omega_0^2)$  and  $\mathcal{I}m\{\vartheta(0)\} = f'''(0)N\omega_0$ . Therefore, we have the following results.

**Corollary 3.1.** *At each pair  $(\gamma, \eta)$  satisfying  $\gamma(\gamma \pm \eta) = b_j$  for some  $j \in \mathbb{N}$ , system (3) undergoes a Hopf bifurcation. Namely, there exists a branch of periodic solutions, which approaches the trivial solution as  $\gamma(\gamma \pm \eta) \rightarrow b_j$ . Their period  $\omega(\gamma, \eta)$  satisfies  $\omega(\gamma, \eta) \rightarrow 2\pi/t_j$  as  $\gamma(\gamma \pm \eta) \rightarrow b_j$ . Furthermore, under assumption (H), the direction of the Hopf bifurcation and stability of the bifurcated periodic solutions are determined by  $\text{sign}\{f'''(0)\}$ . More precisely,*

- (i) *If  $(-1)^j f'''(0) > 0$  (resp.,  $< 0$ ), then each branch of the bifurcated periodic solutions exists for  $(\gamma, \eta)$  satisfying  $\gamma(\gamma \pm \eta) > b_j$  (resp.,  $\gamma(\gamma \pm \eta) < b_j$ ).*
- (ii) *If  $f'''(0) < 0$  (resp.,  $> 0$ ), then the bifurcated periodic solutions have the same stability as the trivial solution had before the bifurcation (resp., is unstable).*

*Remark 3.1.* We can see that (7) has at least one root with positive real parts when  $\gamma(\gamma \pm \eta) = b_j$  for some  $j > 1$ . Then the bifurcating periodic orbits must be unstable in the whole phase space even if they are stable on the center manifold. At the same time, if  $-1 < \gamma(\gamma - \eta) < \gamma(\gamma + \eta) = b_1$  or  $-1 < \gamma(\gamma + \eta) < \gamma(\gamma - \eta) = b_1$ , besides a pair of purely imaginary roots, all other roots of (7) have negative real parts. Thus, bifurcating periodic solutions are stable if and only if  $f'''(0) < 0$ .

In what follows, we consider Hopf bifurcation induced by delay when  $(\gamma, \eta)$  is a fixed point in



$\mathcal{D}_1 \cup \mathcal{D}_3$ . Theorem 1 shows that, near each fixed  $\tau = \tau_s, s \in \mathbb{N}$ , the trivial solution of system (3) undergoes Hopf bifurcation giving rise to a unique branch of periodic solutions. By using a similar argument as above, we obtain the following result.

**Theorem 4.** *Suppose that  $(\gamma, \eta) \in \mathcal{D}_1 \cup \mathcal{D}_3$  and condition (H) holds. Then near each fixed  $\tau = \tau_s, s \in \mathbb{N}$ , the trivial solution of system (3) undergoes Hopf bifurcation giving rise to a unique branch of periodic solutions, which exists for  $\tau > \tau_s$  (resp.,  $\tau < \tau_s$ ) and has the same stability as the trivial solution had before the bifurcation (resp., is unstable) if  $f'''(0) < 0$  (resp.,  $f'''(0) > 0$ ).*

### 4. Fold Bifurcation

In this section, we shall discuss the fold bifurcation, which may occur at  $(\gamma, \eta)$  satisfying either  $\gamma(\gamma + \eta) = -1$  or  $\gamma(\gamma - \eta) = -1$ . Let  $\mu = \gamma(\gamma \pm \eta) + 1$  and  $\mathcal{A}_\mu$  be the infinitesimal generator of the  $C^0$ -semigroup generated by the linear system (4). Obviously,  $\mathcal{A}_0$  has a simple eigenvalue zero. Let

$$q(\theta) = \begin{cases} \left(1, -d_0, -\frac{ac_1}{\gamma}d_0, -\frac{\gamma}{ac_1}\right)^T, & \text{if } \mu = \gamma(\gamma + \eta) + 1, \\ \left(1, -d_0, \frac{ac_1}{\gamma}d_0, \frac{\gamma}{ac_1}\right)^T, & \text{if } \mu = \gamma(\gamma - \eta) + 1 \end{cases} \quad (24)$$

for  $\theta \in [-\tau, 0]$ , where  $d_0 = 1/ac_1$ . Then,  $q(\theta)$  is an eigenvector of  $\mathcal{A}(0)$  associated to the eigenvalue 0. Let

$$p(\xi) = \begin{cases} \bar{D}_0 \left(-\bar{d}_0, 1, -\frac{\gamma}{ac_1}, -\frac{ac_1}{\gamma}\bar{d}_0\right), & \text{if } \mu = \gamma(\gamma + \eta) + 1, \\ \bar{D}_0 \left(-\bar{d}_0, 1, -\frac{\gamma}{ac_1}, -\frac{ac_1}{\gamma}\bar{d}_0\right), & \text{if } \mu = \gamma(\gamma - \eta) + 1 \end{cases} \quad (25)$$

for  $\xi \in [0, \tau]$ , where  $D_0 = -[4d_0(1 + \tau)]^{-1}$ . Obviously,  $\langle p, q \rangle = 1$ .

For each  $u \in \text{Dom}(\mathcal{A}_\mu)$  with sufficiently small  $|\mu|$ , we associate it with the pair  $(x, w)$ , where  $x = \langle p, u \rangle$  and  $w = u - xq$ . Then, on the center manifold  $\mathcal{C}_\mu$  with sufficiently small  $|\mu|$ , the reduced

equation of (3) takes the form

$$\dot{x} = G(x, \mu), \quad x \in \mathbb{R}, \quad (26)$$

where the smooth real function  $G(x, \mu)$  satisfies  $G(x, 0) = p(0)F(0, qx + W(0, x))$  with  $W(x)$  satisfying

$$\dot{W} = \mathcal{A}_0 W + H(x) \quad (27)$$

and

$$H(x)(\theta) = \begin{cases} -G(x, 0)q(\theta), & \text{if } \theta \in [-\tau, 0), \\ F(0, xq + W(x)) - G(x, 0)q(0), & \text{if } \theta = 0. \end{cases}$$

Let

$$G(x, \mu) = \sum_{j \geq 2} \frac{1}{j!} G_j(\mu) x^j$$

and

$$W(x) = \sum_{j \geq 2} \frac{1}{j!} W_j x^j.$$

By a direct computation, we have

$$\begin{aligned} G_1(\mu) &= \mu \lambda'(-1) + (|\mu|^2), \\ G_2(0) &= p(0)\mathcal{F}_2(q, q) \\ &= -\frac{1}{4} f''(0)(1 + \tau)^{-1} \left( \frac{1 + \gamma}{ac_1} + \frac{1 - \gamma}{\gamma} \right), \\ G_3(0) &= p(0)\{3\mathcal{F}_2(q, W_2) + \mathcal{F}_3(q, q, q)\}, \end{aligned}$$

where  $\lambda(\cdot)$  is an implicit function determined by equation  $\det \Delta(\mu, \lambda) = 0$  near  $(\mu, \lambda) = (0, 0)$ , and  $\mathcal{F}_j$  is the  $j$ th Fréchet-derivative of  $F(0, \cdot)$ . It follows from (27) that

$$\dot{W}_2(\theta) = qG_2(0), \quad \theta \in [-\tau, 0) \quad (28)$$

and

$$L_0 W_2(0) = qG_2(0) - \mathcal{F}_2(q, q). \quad (29)$$

Solving (28) for  $W_2(\theta)$ , we get

$$W_2(\theta) = G_2(0)\theta q(\theta) + \varepsilon, \quad (30)$$

where  $\varepsilon$  is a four-dimensional vector and it can be determined by (29). In fact, substituting (30) into (29) gives

$$\begin{aligned} \Delta(\mu_0, 0)\varepsilon &= \xi \\ &:= \mathcal{F}_2(q, q) + L_0(\theta q(\theta)G_2(0)) - qG_2(0). \end{aligned} \quad (31)$$

Notice that  $\Delta(0,0)q(0) = 0$  and  $p(0)\Delta(0,0) = 0$ , as  $\Delta(0,0)$  is singular, we extend to a five-dimensional nonsingular bordered system [Keller, 1977] as follows

$$\begin{bmatrix} \Delta(0,0) & q(0) \\ p(0) & 0 \end{bmatrix} \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} = \begin{bmatrix} \xi \\ 0 \end{bmatrix}.$$

By doing so, we can obtain

$$\varepsilon = \begin{cases} e_0 \left( 1, d_0, -\frac{ac_1}{\gamma}d_0, \frac{\gamma}{ac_1} \right)^T, & \text{if } \mu = \gamma(\gamma + \eta) + 1, \\ e_0 \left( 1, d_0, \frac{ac_1}{\gamma}d_0, -\frac{\gamma}{ac_1} \right)^T, & \text{if } \mu = \gamma(\gamma - \eta) + 1, \end{cases} \quad (32)$$

where  $e_0 = -f''(0)(1 + \gamma)/(4ac_1)$ . If  $f''(0) \neq 0$ , then the truncated form of (26) is

$$\dot{x}(t) = \mu\lambda'(-1)x + \frac{1}{2}G_2(0)x^2, \quad (33)$$

where  $\lambda'(-1) = -1/(2 + 2\tau) < 0$  (see Lemma 2). Thus, we see that (33) has two equilibria:  $x_1 = 0$  and  $x_2 = -4(1 + \gamma^2 \pm \gamma\eta)/[f''(0)1/4f''(0)(1 + \tau)^{-1}(1 + \gamma)/ac_1 + (1 - \gamma)/\gamma]$ . Furthermore, if  $\gamma^2 \pm \gamma\eta + 1 > 0$  then  $x_1$  is stable and  $x_2$  unstable; if  $\gamma^2 \pm \gamma\eta + 1 < 0$  then  $x_1$  is unstable and  $x_2$  stable. The two equilibria coalesce at  $\gamma^2 \pm \gamma\eta + 1 = 0$ . Therefore, we obtain a transcritical bifurcation of equilibria of system (3).

**Theorem 5.** *If  $f''(0) \neq 0$ , then at each pair  $(\gamma_0, \eta_0)$  satisfying  $\gamma_0(\gamma_0 + \eta_0) + 1 = 0$  (or  $\gamma_0(\gamma_0 - \eta_0) + 1 = 0$ ), system (3) undergoes a transcritical bifurcation near  $(\gamma, \eta, x) = (\gamma_0, \eta_0, 0)$ . Namely, in addition to the trivial solution, system (3) has a nonzero equilibrium, which is continuously dependent on  $(\gamma, \eta)$  for all small enough  $|\gamma^2 \pm \gamma\eta + 1|$ . Moreover, this nonzero equilibrium is stable if  $\gamma(\gamma + \eta) + 1 < 0$  and  $-1 < \gamma_0(\gamma_0 - \eta_0) < b_1$  (resp.,  $\gamma(\gamma - \eta) + 1 < 0$  and  $-1 < \gamma_0(\gamma_0 + \eta_0) < b_1$ ) and is unstable otherwise.*

In what follows, we consider the case where (H) holds. Now,  $f''(0) = 0$  and so  $G_2(0) = 0$  and  $G_3(0) = f'''(0)\zeta/[4(1 + \tau)]$ , where  $\zeta = (1 + \gamma^2)[(ac_1)^{-2} + \gamma^{-2}] > 0$ . Thus, the truncated form of (26) is

$$2(1 + \tau)\dot{x}(t) = -(\gamma^2 \pm \gamma\eta + 1)x + \frac{1}{12}f'''(0)\zeta x^3. \quad (34)$$

It is not difficult to see that (34) has only one equilibrium  $x_1 = 0$  if  $f'''(0)(1 + \gamma^2 \pm \gamma\eta) < 0$ , and three equilibria  $x_1 = 0$  and  $x_{2,3} = \pm\sqrt{12(1 + \gamma^2 \pm \gamma\eta)/[f'''(0)\zeta]}$  otherwise. Namely, there exists a pitchfork bifurcation at  $(\gamma, \eta)$  satisfying  $\gamma^2 \pm \gamma\eta + 1 = 0$ . More precisely, if  $f'''(0) < 0$ , then, for  $\gamma^2 \pm \gamma\eta + 1 > 0$ , (34) has a stable equilibrium  $x_1$ ; for  $\gamma^2 \pm \gamma\eta + 1 < 0$ ,  $x_1$  is still an equilibrium, but two new equilibria  $x_2$  and  $x_3$  appear. In this process,  $x_1$  becomes unstable for  $\gamma^2 \pm \gamma\eta + 1 < 0$  while the other two equilibria are stable. Therefore,

**Theorem 6.** *Under assumption (H), system (3) undergoes a pitchfork bifurcation near pair  $(\gamma, \eta)$  satisfying  $\gamma(\gamma \pm \eta) + 1 = 0$ . More precisely, we have the following statements.*

- (i) *If  $f'''(0) < 0$ , two nontrivial equilibria exist for  $(\gamma, \eta)$  with  $\gamma^2 \pm \gamma\eta + 1 < 0$  (which both have the same stability as the trivial solution had before the bifurcation) and only the trivial equilibrium continues to exist for  $\gamma^2 \pm \gamma\eta + 1 > 0$ . Moreover, the two nontrivial equilibria coalesce into zero as  $\gamma^2 \pm \gamma\eta + 1$  goes to 0.*
- (ii) *If  $f'''(0) > 0$ , two nontrivial equilibria exist for  $(\gamma, \eta)$  with  $\gamma^2 \pm \gamma\eta + 1 > 0$  (which are unstable) and only the trivial equilibrium continues to exist for  $\gamma^2 \pm \gamma\eta + 1 < 0$ . Moreover, the two nontrivial equilibria coalesce into zero as  $\gamma^2 \pm \gamma\eta + 1$  goes to 0.*

### 5. Numerical Simulation

Specify the transfer function and take  $f(x) = \tanh(x)$  throughout this section. Namely, we consider the following system

$$\begin{cases} x'_1(t) = -x_1(t) + \tanh(-c_1x_2(t - \tau)), \\ x'_2(t) = -x_2(t) \\ \quad + \tanh(c_2x_1(t - \tau) - \alpha x_4(t - \tau)), \\ x'_3(t) = -x_3(t) \\ \quad + \tanh(-c_1x_4(t - \tau) + \alpha x_1(t - \tau)), \\ x'_4(t) = -x_4(t) + \tanh(c_2x_3(t - \tau)). \end{cases} \quad (35)$$

It is easy to check that  $\tanh'(0) = 1$ ,  $\tanh''(0) = 0$ , and  $\tanh'''(0) = -2 < 0$ , which satisfy all conditions

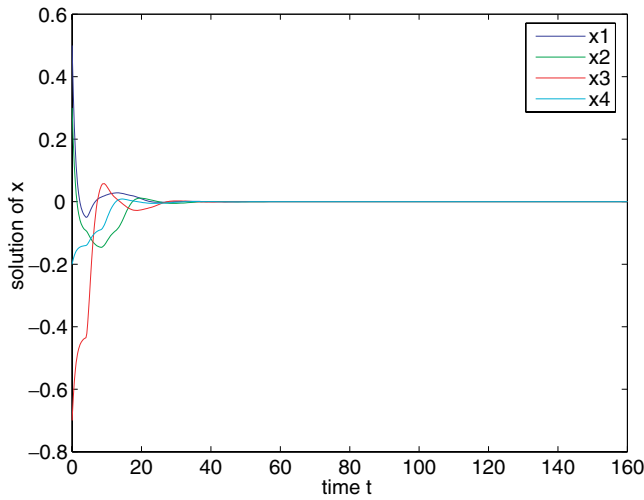


Fig. 2. Numerical simulation of a solution of system (35) with  $c_1 = 0.2, c_2 = 0.2, \alpha = -2$  and  $\tau = 5$ .

stated in the previous sections. Then, we shall provide some numerical simulation to verify the analytic results presented in the previous sections.

In view of Theorem 1, we first focus on  $(\gamma, \eta) \in \mathcal{D}_2 \cup \mathcal{D}_3$  to consider the periodic oscillations induced by the time delay  $\tau$ .

Then we choose  $c_1 = 0.2, c_2 = 0.2$ , and  $\alpha = -2$ . By an easy calculation, we have  $\gamma = 0.2 \in [0, 1]$  and  $\eta = -2$ , and hence  $\gamma(\gamma + |\eta|) = 0.44 < 1$ , which implies that  $(\gamma, \eta) \in \mathcal{D}_2$ . It follows from Theorem 1 that the trivial solution is delay-independently asymptotically stable. This can be shown in Fig. 2.

Taking  $c_1 = 2, c_2 = 0.2$ , and  $\alpha = -2$ , we have  $\gamma = 0.6325, \eta = -2$ , and hence  $(\gamma, \eta) \in \mathcal{D}_3$ . By Theorem 1, we see that the trivial solution of (3) is asymptotically stable for all  $\tau \in (0, \tau_0)$  and

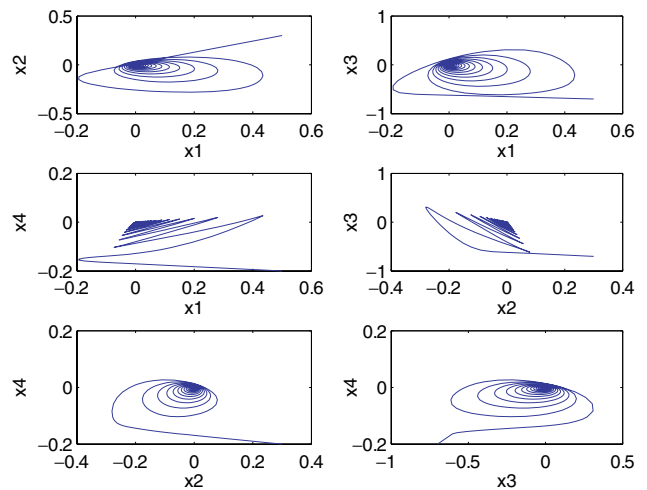
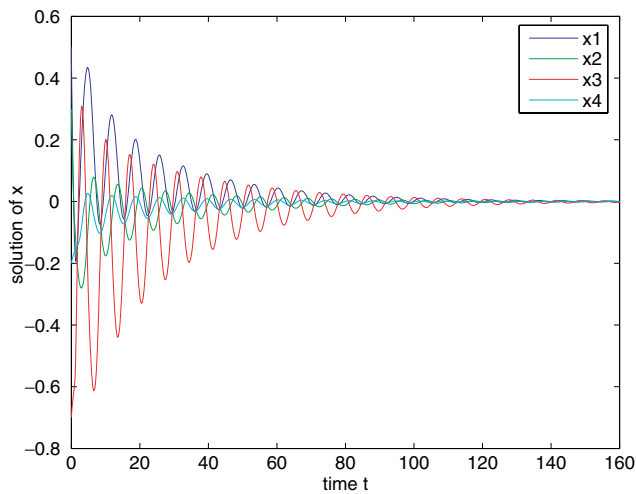


Fig. 3. Numerical simulations of a solution of system (35) with  $c_1 = 2, c_2 = 0.2, \alpha = -2$  and  $\tau = 1.0$ .

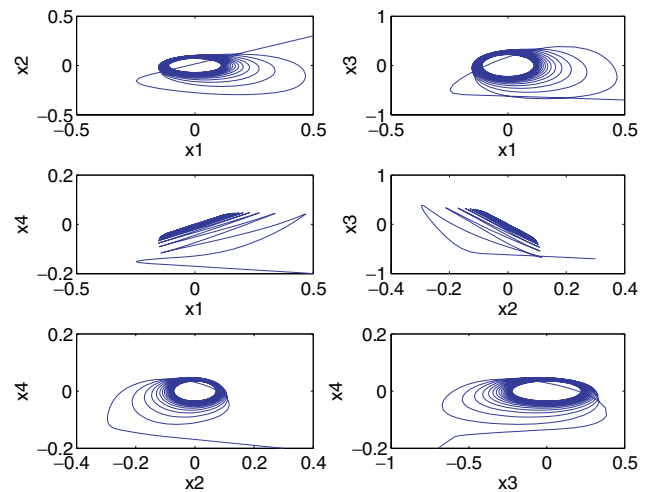
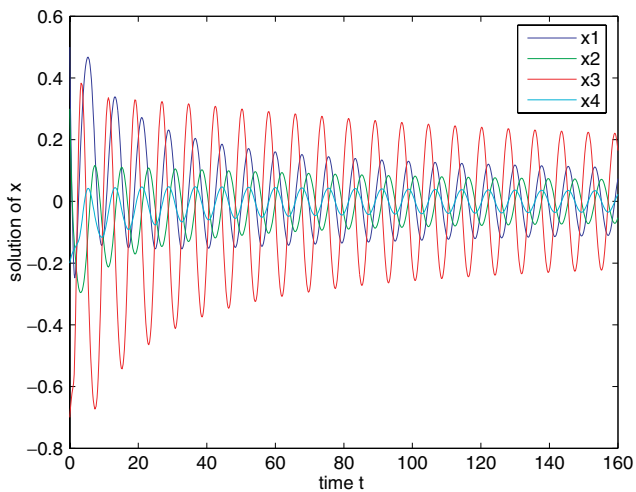


Fig. 4. Numerical simulations of a solution of system (35) with  $c_1 = 2, c_2 = 0.2, \alpha = -2$  and  $\tau = 1.1$ .

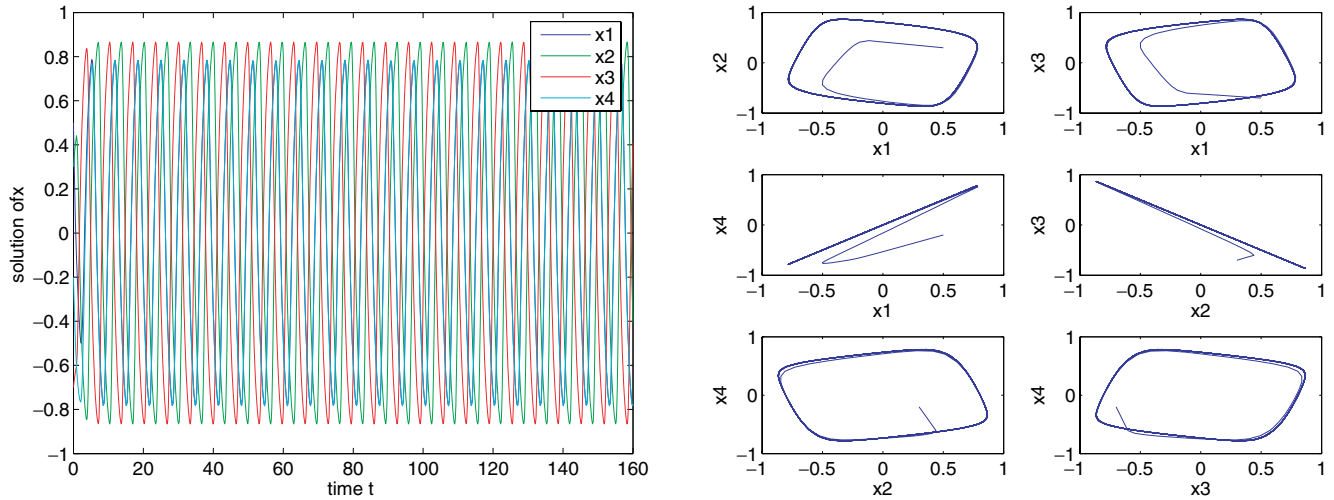


Fig. 5. Numerical simulations of system (35) showing that system (35) has a periodic solution for  $c_1 = 2$ ,  $c_2 = 2$ ,  $\alpha = -2$ , and  $\tau = 0.9$ .

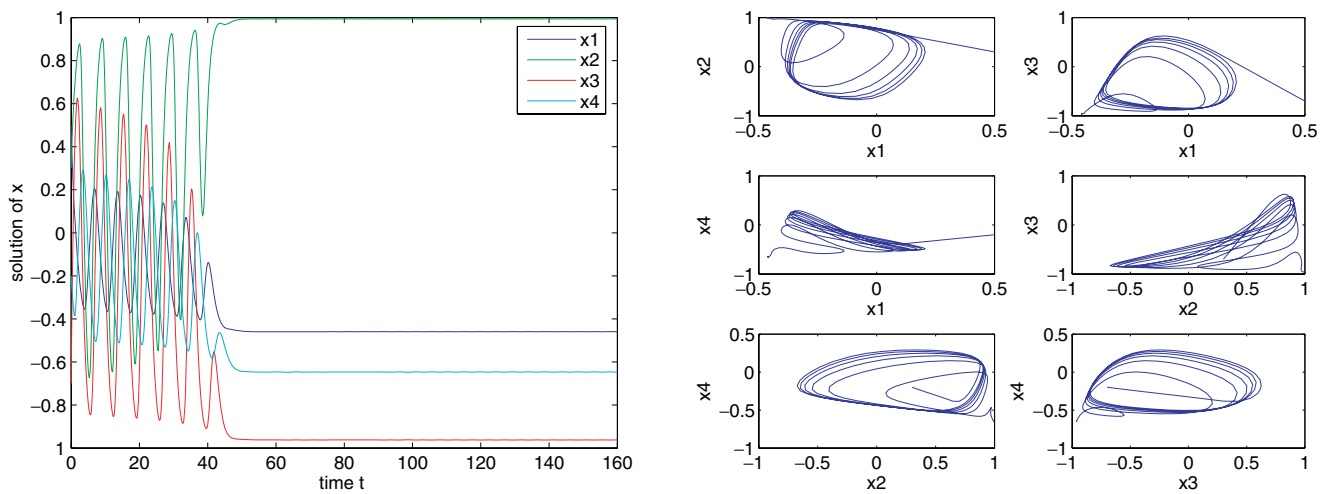


Fig. 6. Numerical simulations of system (35) with  $c_1 = 0.5$ ,  $c_2 = 0.8$ ,  $\alpha = 5$ , and  $\tau = 0.9$ .

unstable for  $\tau > \tau_0$ , where it follows from (9) that  $\tau_0 \approx 1.0865$ . Figure 3 shows that the trivial solution of (35) with  $\tau = 1.0 < \tau_0$  is asymptotically stable. When  $\tau = 1.1 > \tau_0$ , it follows from Theorems 1 and 4 that the trivial solution of (35) loses its stability and undergoes a supercritical Hopf bifurcation giving rise to a branch of stable periodic solutions, which can be shown in Fig. 4.

Taking  $c_1 = 2, c_2 = 2$ , and  $\alpha = -2$ , we obtain the critical value  $\tau_0 \approx 0.1349$ . When  $\tau = 0.9 > \tau_0$ , it follows from Remark 3.1 and Theorems 1 and 4 that there exists Hopf bifurcation in the neighborhood of trivial solution, and the trivial solution is unstable for  $\gamma(\gamma - \eta) = 8 > c_1$  and  $\gamma(\gamma + \eta) = 0$ , which is shown in Fig. 5.

Taking  $c_1 = 0.5, c_2 = 0.8$ , and  $\alpha = 5$ , we have  $\tau_0 \approx 0.3485$  and  $(\gamma, \eta) \in \mathcal{D}_1$ . It follows from Theorem 1 that the trivial of solutions of system (35) are unstable for all  $\tau > 0$  and hence the bifurcated periodic solution of system (35) is unstable too. This implies that almost all the solutions of system (35), no matter how close to the bifurcated periodic solutions, converge to other equilibria, which can be shown in Fig. 6.

## 6. Conclusions

Delayed coupling neural networks are complex and large-scale nonlinear dynamical systems. Although there are numerous articles studying the stability

and bifurcations for the delayed neural networks, few papers are concerned with neural networks with not only the intrinsic delay but also the coupling delay. So, in this paper, we consider a neural network coupled by two oscillators, each describing the average activities of excitatory and inhibitory populations of neurons, with delayed coupling. We aim to explore the capabilities of a rather simple neural network to exhibit different dynamical behavior and to further understand the collective behavior of coupled complex networks. By analyzing the corresponding characteristic equation, we see that the trivial solution may lose its stability and a Hopf (or fold) bifurcation occurs from the trivial solution as parameters (including the time delay) cross the critical values. On one hand, we consider the effect of time delay  $\tau$  and then divide the half-plane  $\{(\gamma, \eta) | \gamma > 0\}$  into three regions (see Fig. 1). The trivial solution is unstable in  $\mathcal{D}_1$ , delay-independently stable in region  $\mathcal{D}_2$ , and conditionally stable in regions  $\mathcal{D}_3$ . In particular, we obtained some results about the stability and bifurcation direction of bifurcated periodic solutions of system (3) when  $(\gamma, \eta) \in \mathcal{D}_1 \cup \mathcal{D}_3$ . Then, we also discussed the bifurcation induced by connection weights. On the other hand, we considered the effect of connection weights on the dynamics of system (3) by regarding  $\gamma$  and  $\eta$  as bifurcation parameters. In particular, the bifurcation direction and the stability of the bifurcated periodic solutions and equilibria are determined by the sign of  $f'''(0)$  under the condition (H).

There are a few questions that are worthy of further investigation. In this paper, we only consider a particular form of coupling which, for a given oscillator, just add the (weighted) output from one or more other oscillators to the external input of this oscillator. This choice led to the model (2), which generally takes into account four connection types: excitatory to excitatory, excitatory to inhibitory, inhibitory to excitatory, inhibitory to inhibitory. In this paper, we only investigate one excitatory to the other excitatory connection and the second inhibitory to the first inhibitory connection. However, to consider each connection type separately — the excitatory to inhibitory and inhibitory to excitatory and so on — further discussion is needed. Another interesting problem to be investigated is the possibility of double limit cycle bifurcation, which is beyond the scope of this paper.

## Acknowledgments

This work was partially supported by NNSF of China (Grant Nos. 10971057 & 50925727), by the Program for NCET (Grant No. NCET-07-0264), the SRF for ROCS (Grant No. [2008]890), and the Key Project (Grant No. [2009]41) of Chinese Ministry of Education, by the Hunan Provincial Natural Science Foundation (Grant No. 10JJ1001), and by the Scientific Research Fund of Hunan Provincial Education Department.

## References

- Aronson, D., Ermentrout, G. B. & Kopell, N. [1990] “Amplitude response of coupled oscillators,” *Physica D* **41**, 403–449.
- Babcock, K. L. & Westervelt, R. M. [1987] “Dynamics of simple electronic neural networks with added inertia,” *Physica D* **23**, 464–469.
- Bungay, S. & Campbell, S. A. [2007] “Patterns of oscillation in a ring of identical cells with delayed coupling,” *Int. J. Bifurcation and Chaos* **17**, 3109–3125.
- Campbell, S. A., Yuan, Y. & Bungay, S. D. [2005] “Equivariant Hopf bifurcation in a ring of identical cells with delayed coupling,” *Nonlinearity* **18**, 2827–2846.
- Ermentrout, G. B. & Kopell, N. [1991] “Multiple pulse interactions and averaging in systems of coupled neural oscillators,” *J. Math. Biol.* **29**, 195–217.
- Gopalsamy, K. & Leung, I. [1996] “Delay induced periodicity in a neural netlet of excitation and inhibition,” *Physica D* **89**, 395–426.
- Guo, S. & Huang, L. [2003] “Hopf bifurcating periodic orbits in a ring of neurons with delays,” *Physica D* **183**, 19–44.
- Guo, S. [2005] “Spatio-temporal patterns of nonlinear oscillations in an excitatory ring network with delay,” *Nonlinearity* **18**, 2391–2407.
- Guo, S. & Huang, L. [2005] “Global continuation of nonlinear waves in a ring of neurons,” *Proc. Roy. Soc. Edinburgh A* **135**, 999–1015.
- Guo, S. & Huang, L. [2007] “Stability of nonlinear waves in a ring of neurons with delays,” *J. Diff. Eqs.* **236**, 343–374.
- Guo, S. & Yuan, Y. [2009] “Pattern formation in a ring network with delay,” *Math. Mod. Meth. Appl. Sci.* **19**, 1797–1852.
- Hansel, D., Mato, G. & Meunier, C. [1993] “Phase dynamics for weakly coupled Hodgkin–Huxley neurons,” *Europhys. Lett.* **23**, 367–372.
- Hindmarsh, J. L. & Rose, R. M. [1982] “A model of the nerve impulse using two first-order differential equations,” *Nature* **296**, 162–164.

- Huang, L. & Wu, J. [2003] "Nonlinear waves in networks of neurons with delayed feedback: Pattern formation and continuation," *SIAM J. Math. Anal.* **34**, 836–860.
- Jiang, Y. & Guo, S. [2010] "Linear stability and Hopf bifurcation in a delayed two-coupled oscillator with excitatory-to-inhibitory connection," *Nonlin. Anal.: Real World Appl.* **11**, 2001–2015.
- Kawato, M., Sokabe, M. & Suzuki, R. [1979] "Synergism and antagonism of neurons caused by an electrical synapse," *Biol. Cybern.* **34**, 81–89.
- Keller, H. [1977] "Numerical solution of bifurcation and nonlinear eigenvalue problems," in *Applications of Bifurcation Theory*, ed. Rabinowitz, P. (Academic Press, NY).
- Kryukov, V. I., Borisyuk, G. N., Borisyuk, R. M., Kirillov, A. B. & Kovalenko, E. L. [1990] "Metastable and unstable states in the brain," in *Stochastic Cellular Systems: Ergodicity, Memory, Morphogenesis*, eds. Dobrushin, R. L., Kryukov, V.I. & Toom, A. L. (Manchester University Press, Manchester), pp. 225–357.
- MacGregor, R. J. [1987] *Neural and Brain Modelling* (Academic Press, NY).
- Mao, X. C. & Hu, H. Y. [2008] "Stability and Hopf bifurcation of a delayed network of four neurons with a short-cut connection," *Int. J. Bifurcation and Chaos* **18**, 3053–3072.
- Mao, X. C. & Hu, H. Y. [2009] "Hopf bifurcation analysis of a four-neuron network with multiple time delays," *Nonlin. Dyn.* **55**, 95–112.
- Marcus, C. M. & Westervelt, R. M. [1989] "Stability of analog neural networks with delay," *Phys. Rev. A* **39**, 347–359.
- Peng, J. & Guo, S. [2010] "Synchronous dynamics of two coupled oscillators with inhibitory-to-inhibitory connection," *Commun. Nonlin. Sci. Numer. Simulat.* **15**, 4131–4148.
- Song, Y., Han, M. & Wei, J. [2005] "Stability and Hopf bifurcation analysis on a simplified BAM neural network with delays," *Physica D* **200**, 185–204.
- Wilson, H. R. & Cowan, J. D. [1972] "Excitatory and inhibitory interactions in localized populations of model neurons," *Biophys. J.* **12**, 1–24.
- Wei, J., Velarde, M. G. & Makarov, V. A. [2002] "Oscillatory phenomena and stability of periodic solutions in a simple neural network with delay," *Nonlin. Phenom. Compl. Syst.* **5**, 407–417.
- Wei, J. & Velarde, M. G. [2004] "Bifurcation analysis and existence of periodic solutions in a simple neural network with delays," *Chaos* **14**, 940–953.
- Wu, J. [1998] "Symmetric functional-differential equations and neural networks with memory," *Trans. Amer. Math. Soc.* **350**, 4799–4838.
- Wu, J., Faria, T. & Huang, Y. S. [1999] "Synchronization and stable phase-locking in a network of neurons with memory," *Math. Comput. Model.* **30**, 117–138.
- Xiao, K. & Guo, S. [2010] "Synchronization for two coupled oscillators with inhibitory connection," *Math. Meth. Appl. Sci.* **33**, 892–903.
- Xu, X. & Wang, Z. H. [2009] "Effects of small world connection on the dynamics of a delayed ring network," *Nonlin. Dyn.* **56**, 127–144.
- Yu, W. & Cao, J. [2006] "Stability and Hopf bifurcation analysis on a four-neuron BAM neural network with time delays," *Phys. Lett. A* **351**, 64–78.
- Yuan, Y. & Campbell, S. A. [2004] "Stability and synchronization of a ring of identical cells with delayed coupling," *J. Dyn. Diff. Eqs.* **16**, 709–744.
- Yuan, Y. [2007] "Dynamics in a delayed-neural network," *Chaos Solit. Fract.* **33**, 443–454.
- Zhang, P., Guo, S. & He, Y. [2010] "Dynamics of a delayed two-coupled oscillator with excitatory-to-excitatory connection," *Appl. Math. Comput.* **216**, 631–646.