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ORIGINAL PAPER

Existence of solutions for generalized vector quasi-equilibrium problems

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Abstract This paper deals with three classes of generalized vector quasi-equilibrium problems with or without compact assumptions. Using the well-known Fan-KKM theorems, their existence theorems for them are established. Some examples are given to illustrate our results.

Keywords Generalized vector quasi-equilibrium problem · Weak type C_x-diagonal quasi-convex and strong type C_x-diagonal quasi-convex · Fan-KKM theorem

Mathematics Subject Classification (2000) 49K30 · 90C29

1 Introduction

Equilibrium problems were introduced by Blum and Oettli [1] and by Noor and Oettli [2] in 1994 as generalizations of variational inequalities and optimization problems. The equilibrium problem theory provides a novel and united treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization. This theory has had a great impact and influence in the development of several branches of pure and applied sciences. As a result of this interaction, we have a variety of techniques to study existence results for equilibrium problems. Among many techniques, the auxiliary principle technique has been used to suggest and analyze a number of iterative methods for solving many different types of equilibrium problems and variational inequalities, see [3–9] and the

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references therein. Another technique is well-known KKM theorems or fixed pointed theorems which have been used to study the existence results of various classes of equilibrium problems for the past years. For the details, we can refer to [10–20] and the references therein.

Throughout this paper, unless otherwise specified, let X, Y and Z be three real topological spaces, let X and Y be Hausdorff, $E \subseteq X$ and $D \subseteq Z$ be two non-empty subsets. We also assume that $C : X \to 2^Y$ is a set-valued mapping such that $C(x)$ is a proper, closed and convex cone of Y with $\text{int}C(x) \neq \emptyset$, for each $x \in X$, where $\text{int}C(x)$ denotes the interior of $C(x)$. Let $S : E \to 2^E, T : E \to 2^D$ and $F : E \times D \times E \to 2^Y$ be three set-valued maps.

Consider the following three classes of generalized vector quasi-equilibrium problems:

(GVQEP1) Find $\bar{x} \in E$ and $\bar{z} \in T(\bar{x})$ such that

$$\bar{x} \in S(\bar{x}) \quad \text{and} \quad F(\bar{x}, \bar{z}, y) \not\subseteq -\text{int}C(\bar{x}), \quad \forall y \in S(\bar{x}).$$

(GVQEP2) Find $\bar{x} \in E$ and $\bar{z} \in T(\bar{x})$ such that

$$\bar{x} \in S(\bar{x}) \quad \text{and} \quad F(\bar{x}, \bar{z}, y) \cap -\text{int}\, C(\bar{x}) = \emptyset, \quad \forall y \in S(\bar{x}).$$

(GVQEP3) Find $\bar{x} \in E$ and $\bar{z} \in T(\bar{x})$ such that

$$\bar{x} \in S(\bar{x}) \quad \text{and} \quad F(\bar{x}, \bar{z}, y) \subseteq -C(\bar{x}), \quad \forall y \in S(\bar{x}).$$

In [19], Li et al. used the Ky Fan minimax inequality theorem [21] and nonlinear scalarization function [10] to obtain existence results for (GVQEP2) and (GVQEP3). In [13], applying the Fan-KKM theorem Lin and Chen investigated the special cases of (GVQEP2) and (GVQEP3) when $S(x) = E$. In [16] Ansari and Flores-Bazan studied a special case of (GVQEP1) when $T(x)$ is singleton. By using the fixed point theorem in [14] and maximal element theorem in [22], they obtained some existence results under conditions with or without involving ϕ-condensing maps.

In this paper, we first introduce four classes of generalized diagonal quasi-convex set-valued maps. By using the well-known Fan-KKM theorems with or without the compact assumption of E, we obtain existence results for (GVEQEP1), (GVQEP2) and (GVQEP3), respectively. Then we give two examples to explain that our existence results with the compact assumption of E are different from corresponding results in [15,16,19]. We also illustrate that our existence result without the compact assumption of E is a generalization of the corresponding one in [13].

The rest of the paper is organized as follows. In Sect. 2, we first introduce four classes of C_x-diagonal quasi-convexity with respect to a set-valued map, and discuss their properties. Then, we recall some basic definitions and the well-known Fan-KKM theorems. In Sect. 3, we show existence results of solutions for (GVQEP1), (GVQEP2) and (GVQEP3), respectively.