Journal of Graph Algorithms and Applications
http://jgaa.info/ vol. 20, no. 2, pp. 347–362 (2016)
DOI: 10.7155/jgaa.00396

The (3,1)-ordering for 4-connected planar triangulations

Therese Biedl[1] *Martin Derka*[1]

[1]David R. Cheriton School of Computer Science, University of Waterloo, Ontario, Canada

Abstract

Canonical orderings of planar graphs have frequently been used in graph drawing and other graph algorithms. In this paper we introduce the notion of an (r, s)-canonical order, which unifies many of the existing variants of canonical orderings. We then show that $(3, 1)$-canonical ordering for 4-connected triangulations always exist; to our knowledge this variant of canonical ordering was not previously known. We use it to give much simpler proofs of two previously known graph drawing results for 4-connected planar triangulations, namely, rectangular duals and rectangle-of-influence drawings.

Submitted:	Reviewed:	Revised:	Accepted:	Final:
November 2015	April 2016	May 2016	May 2016	June 2016
		Published: June 2016		
	Article type: Regular paper		Communicated by: G. Liotta	

Research supported by NSERC. The second author was supported by Vanier CGS.

E-mail addresses: biedl@uwaterloo.ca (Therese Biedl) mderka@uwaterloo.ca (Martin Derka)

1 Background

A canonical ordering of a planar graph is a way of building the graph by iteratively attaching vertices to some "basic graph" (such as an edge) while preserving some connectivity invariant after each iteration. This concept was introduced in the late 1980's by de Fraysseix, Pach and Pollack [4]. They used the canonical ordering to show that planar graphs can be drawn on a grid of size $(2n-4) \times (n-2)$. Subsequently, canonical orderings became one of the main tools in graph drawings, e.g. for drawing graphs in grids of small dimensions (see, e.g., [4, 2]), rectangular duals [9], and also graph algorithms such as encoding planar graphs [7] or finding k-disjoint trees in planar graphs [12, 11].

Our contribution There is now a number of variations of canonical orderings, depending on the connectivity of the graph and whether it is triangulated or not (we will review these below). In this paper, we show the existence yet another canonical ordering, this one for planar 4-connected triangulations. It is substantially different from the canonical ordering for such graphs that was presented by Kant and He [9]. We call this the $(3,1)$-*canonical ordering*. More generally, we introduce the concept of an (r,s)-*canonical ordering*, which (roughly speaking) means that the partial graph must be r-connected and the rest-graph must be s-connected; the existing canonical orders all fit into this framework.

We use the $(3,1)$-canonical ordering to provide alternate proofs of two previously known results about 4-connected planar triangulations: they have rectangular duals (Section 4.1) and rectangle-of-influence drawings (Section 4.2). These proofs are significantly shorter than previous proofs, provided the existence of a $(3,1)$-canonical ordering is treated as a black box.

2 Review of existing canonical orderings

We assume that the reader is familiar with planar graphs (refer, e.g., to [5]). We use the term *triangulation* for a maximal planar simple graph, i.e., a graph in which all faces are triangles and which has $3n-6$ edges of which none is a multiple edge or a loop. Such a graph has a unique planar embedding; we further assume that one face has been fixed as the outer face and then refer to it as a *plane graph*. We begin our review of canonical ordering with the one for triangulations introduced by de Fraysseix et al. [4]. We paraphrase their definition to the following one (which is easily shown to be equivalent):

Definition 1 (Canonical ordering for plane triangulations [4]) *Let G be a plane triangulation with outer face u_1, u_2, u_3. A vertex ordering $v_1, \ldots, v_n$ is called a* canonical ordering *if*

- $v_1 = u_1$, $v_2 = u_2$, $v_n = u_3$,

- *For every* $1 < k < n$ *the subgraph* G_k *of* G *induced by vertices* $v_1, v_2, \dots, v_k$ *is* 2*-connected.*[1]

As we will see later, it will be convenient to define $V_k := \{v_k\}$ for every $1 \leq k \leq n$ and so $V_1 \cup \dots \cup V_n$ becomes a partition of the vertex set. For any such partition and an index k, we use the notation G_k for the subgraph induced by $V_1 \cup \dots \cup V_k$ and we let the *complement* $\overline{G_k}$ of G_k be the subgraph induced by the vertices $V \setminus (V_1 \cup \dots \cup V_{k-1})$. Note that vertex set V_k belongs to both G_k and $\overline{G_k}$. [2]

One can observe that in a canonical ordering for a triangulation, the complement $\overline{G_k}$ is a connected graph for all $k < n$. This holds because any vertex $v_k \neq u_1, u_2, u_3$ is not on the outer face and so there must exist some minimal $k' > k$ where v_k is not on the outer face of $G_{k'}$. Due to the triangular faces, v_k receives an edge to $v_{k'}$, and iterating the argument, hence has a path within $\overline{G_k}$ that leads to v_n.

We note here, without giving details, that this canonical ordering has been generalized to 3-connected plane graphs that are not necessarily triangulated [8], and also to non-planar 3-connected graphs (see [13] and the references therein).

In 1997, Kant and He [9] showed that one can define a different canonical ordering for 4-connected plane triangulations, and used it to construct visibility representations of 4-connected plane graphs. Its definition, slightly paraphrased, is as follows:

Definition 2 Canonical ordering for 4-connected plane triangulations [9] *Let* G *be a 4-connected plane triangulation with outer face* u_1, u_2, u_3. *A vertex order* $v_1, \dots, v_n$ *is called a* canonical ordering for 4-connected triangulations *if*

- $v_1 = u_1$, $v_2 = u_2$, $v_n = u_3$,
- *For every* $1 < k < n$, *graphs* G_k *and* $\overline{G_k}$ *are* 2*-connected.*

This canonical ordering was extended to a canonical ordering for all plane 4-connected graphs (not necessarily triangulated) by Nakano, Rahman and Nishizeki [12]. Versions of a canonical order for 4-connected non-planar graphs are also known [3].

Going one higher in connectivity, Nagai and Nakano [11] introduced a canonical ordering for 5-connected plane triangulations. Here, vertices are added in sets that are sometimes more than a singleton. We need a definition. Let G be a graph where all interior faces are triangles. A *fan* of G is a subset of vertices $z_1, \dots, z_f$ that induces a path with $deg(z_i) = 3$ for all $i = 1, \dots, f$. We will only apply this concept if all vertices in the fan belong to the outer face of G. Since interior faces are triangles, it follows that for all z_i the third neighbor (i.e., the one not on the outer face) is the same vertex. See also Figure 1(b).

[1] We consider the complete graph K_n to be k-connected and in particular, an edge $\{v_1, v_2\}$ to be 2-connected.

[2] Some references instead define $\overline{G_k}$ to be the subgraph induced by $V - (V_1 \cup \dots \cup V_k)$. This complicates stating some of the conditions.

Definition 3 (Canonical ordering for 5-connected plane triangulations [11]) *Let G be a 5-connected plane triangulation with outer face u_1, u_2, u_3. A partition of the vertices $V = V_1 \cup \cdots \cup V_L$ is called a* canonical ordering for 5-connected triangulations *if*

- $V_1 = \{u_1, u_2\}$,
- V_2 *consists of all neighbors of* u_1 *and* u_2,
- $V_L = \{u_3\}$,
- V_{L-1} *consists of all neighbors of* u_3,
- *For* $2 < k < L-1$, *vertex set* V_k *is either a single vertex or a fan,*
- *For every* $2 < k < L$, *graph* G_k *is* 3*-connected and graph* $\overline{G_k}$ *is* 2*-connected.*

This canonical ordering was used to find 5 independent spanning trees in 5-connected triangulations [11]. To our knowledge, it has not been generalized to planar 5-connected (not necessarily triangulated) graphs, and not to non-planar 5-connected graphs either. Since no planar graph is 6-connected, no canonical orderings for higher connectivity can exist for planar graphs.

Note that the three canonical orderings listed here are very similar, with the essence being the connectivity that is required of the subgraphs and their complements. In light of this, we aim to unify the three definitions with the following:

Definition 4 ((r, s)-canonical ordering) *Let G be a plane triangulation with outer face $\{u_1, u_2, u_3\}$. We say that a vertex partition $V_1 \cup \ldots \cup V_L$ is an (r, s)-*canonical ordering *if*

- u_1 *belongs to* V_1 *and* u_3 *belongs to* V_L, *and*
- *for every* $1 < k < L$, *graph* G_k *is* r*-connected and* $\overline{G_k}$ *is* s*-connected.*

Note that this definition is deliberately vague on the exact form that the vertex sets V_k must have. In particular, nothing prevents us (yet) from setting $L = 1$ and $V_1 = V$, which satisfies all conditions. The existing canonical orderings restrict V_k to be a singleton or, for 5-connected triangulations, fans. Thus the above definition should be viewed as a class of definitions, to be refined further by stating restrictions on the vertex sets V_k.

Rephrasing the existing canonical orders in the above terms, the canonical order for triangulations becomes a $(2, 1)$-canonical ordering with only singletons, the one for 4-connected triangulations becomes a $(2, 2)$-canonical ordering with only singletons, and the one for 5-connected triangulations becomes a $(3, 2)$-canonical ordering with only singletons or fans. The reader will notice that the sum of the two numbers equals the connectivity of the graph. Pushing this further, one may ask whether any $(r+s)$-connected triangulation has an (r, s)-canonical ordering such that each V_k has some simple form. Note that we may assume that $r \geq s$, since a reversal of an (r, s)-canonical ordering gives an (s, r)-canonical ordering. We study here $(3, 1)$-canonical ordering for 4-connected triangulations, under the restriction that each V_k is a singleton or a fan. To our knowledge no such ordering was known before.

3 $(3, 1)$-canonical orderings

We have already given the broad idea of a $(3, 1)$-canonical ordering earlier. We re-state it here and give the specific restrictions imposed on the vertex sets. See also Figure 1.

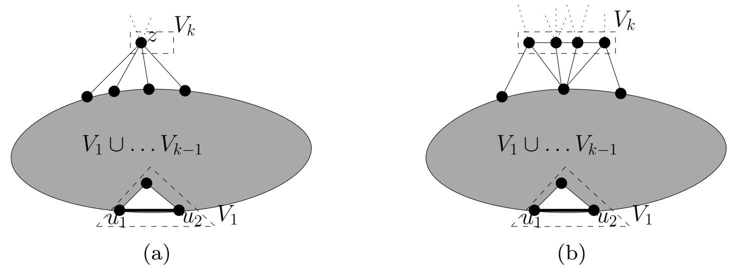

Figure 1: A singleton V_k and a fan V_k in a $(3, 1)$-canonical ordering.

Definition 5 *Let G be a 4-connected plane triangulation with outer face $\{u_1, u_2, u_3\}$. A* $(3, 1)$-canonical order with singletons and fans *is a partition $V = V_1 \cup \cdots \cup V_L$ such that*

- *$V_1 = \{u_1, u_2, z\}$, where z is the third vertex of the interior face adjacent to (u_1, u_2).*
- *$V_L = \{u_3\}$.*
- *For any $1 < k < L$, set V_k is either a singleton or a fan.*
- *For any $1 < k < L$, graph G_k is 3-connected and $\overline{G_k}$ is connected.*

In what follows, we will omit the "with singletons and fans", as we will not study any other version of $(3, 1)$-canonical orderings. Our main goal is to show that every 4-connected triangulation has such a $(3, 1)$-canonical ordering. The proof of this proceeds by induction, and we state the crucial lemma for the induction step separately first. We need a few definitions.

A plane graph is called a *triangulated disk* if every interior face is a triangle and the outer face is a simple cycle. A triangulated disk is called *internally 4-connected* if its outer face has no chord, and every triangle is a face. Observe that a triangle is an internally 4-connected triangulated disk, and so is any 4-connected triangulation. Also observe that a subgraph of an internally 4-connected triangulated disk is again an internally 4-connected triangulated disk if and only if its outer face is a simple cycle that has no chord.

Lemma 1 *Let G be an internally 4-connected triangulated disk with $n \geq 4$. Let (u_1, u_2) be an edge on the outer face. Then there exists a vertex set V' such that*

- *V' contains only outer face vertices, and none of u_1, u_2.*
- *$G - V'$ is an internally 4-connected triangulated disk.*
- *V' is a singleton or a fan.*

Proof: [3] Enumerate the outer face vertices of G as $u_1 = c_1, c_2, \ldots, c_\ell = u_2$ in clockwise order. Define a *2-leg* to be a path $c_i - x - c_j$ where $i < j - 1$ and x is not on the outer face. Vertex x is called a *2-leg-center*. We always have at least one 2-leg (namely, the one consisting of $u_1 = c_1, u_2 = c_\ell$ and their common neighbor at the interior face incident to (u_1, u_2); this vertex is interior since G has no chord and at least 4 vertices).

We say that a 2-leg-center x *dominates* a 2-leg-center y if vertex y is strictly inside the cycle $x - c_i - c_{i+1} - \cdots - c_j - x$ formed by some 2-leg $\{c_i, x, c_j\}$ with center-vertex x. See also Figure 2(a). The dominance-relationship is acyclic since any 2-leg with center-vertex y must enclose strictly fewer faces than the 2-leg $\{c_i, x, c_j\}$. Therefore we must have some *minimal* 2-leg-centers, which are the ones that do not dominate any other 2-leg-center.

By definition for any 2-leg $\{c_i, x, c_j\}$, we have $j \geq i + 2$ and so there exists at least one vertex between c_i and c_j on the outer face. We say that a 2-leg $\{c_i, x, c_j\}$ is *basic* if the vertices $c_{i+1}, \ldots, c_{j-1}$ all have degree 3, and *complex* otherwise. Note that if $\{c_i, x, c_j\}$ is basic, then $c_{i+1}, \ldots, c_{j-1}$ form a fan and their common neighbor is x.

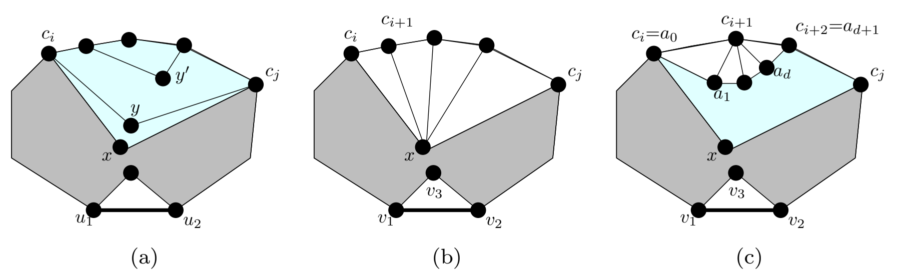

Figure 2: (a) 2-leg center x dominates both y and y'. (b) If all 2-legs containing x are basic, then we can remove a fan. (c) If $\{c_i, x, c_j\}$ is complex, then removing c_{i+1} leaves an internally 4-connected triangulated disk.

Let x be a minimal 2-leg center. We have two cases:

- All 2-legs containing x are basic.

[3]The proof is strongly inspired of the one for a $(3, 2)$-canonical order in 5-connected graphs [11]. Since we demand less on our $(3, 1)$-canonical order, we can simplify the exposition somewhat.

Let $i \geq 1$ be minimal and $j \leq \ell$ be maximal such that x is adjacent to c_i and c_j. See also Figure 2(b). Since x is a 2-leg-center, we have $i < j-1$. By case assumption the 2-leg $\{c_i, x, c_j\}$ is basic, so $V' = \{c_{i+1}, \ldots, c_{j-1}\}$ is a fan. We verify that $G' := G - V'$ is an internally 4-connected triangulated disk:

- The outer face of G' consists of the one of G, minus the vertices in V', plus x. By definition of a 2-center x was not on the outer face, so G' is a triangulated disk.
- Since G had no chord, the only possible chord of G' would be incident to vertex x. But by choice of i and j the only neighbors of x on the outer face of G' are c_i and c_j. So G' has no chord.

• Some 2-leg $\{c_i, x, c_j\}$ is complex.

We assume that i has been chosen maximally, i.e., so that $\{c_{i+1}, x, c_j\}$ is not a complex 2-leg. We claim that in this case $V' = \{c_{i+1}\}$ is a suitable vertex set.

We first show that c_{i+1} cannot be adjacent to x. If this was a case, $\{c_{i+1}, x, c_j\}$ would be a 2-leg. However, it cannot be complex by the maximality assumption, and it cannot be basic either as $\{c_i, x, c_{i+1}\}$ would be a separating triangle (recall that $\{c_{i+1}, x, c_j\}$ is a *minimal* complex 2-leg). Thus, c_{i+1} cannot be adjacent to x, and in particular, $c_{i+1} \neq c_j$.

Let $c_i = a_0, a_1, \ldots, a_d, a_{d+1} = c_{i+2}$ be the neighbors of c_{i+1} in ccw order. See also Figure 2(c). None of $a_1, \ldots, a_d$ can be on the outer face of G, else G would have a chord. The outer face of $G' := G - V'$ consists of $c_1, \ldots, c_i, a_1, \ldots, a_d, c_{i+2}, \ldots, c_\ell$, and so this is a simple cycle and G' is a triangulated disk. Further, we can show that it has no chord:

- If a chord of G' connected two vertices in $c_1, \ldots, c_i, c_{i+2}, \ldots, c_\ell$, then it would also be a chord in G, which is excluded.
- If a chord connected two non-consecutive vertices in $c_i{=}a_0, \ldots, a_{d+1}{=} c_{i+2}$, then in G there would be an edge between two non-consecutive neighbors of c_{i+1}, implying a triangle that is not a face.
- If a chord connected some a_s, $1 \leq s \leq d$, with some c_h, $i+2 < h \leq j$, then $\{c_{i+1}, a_s, c_h\}$ would be a 2-leg in G. By minimality of x hence $a_s = x$, but this contradicts that c_{i+1} is not adjacent to x.
- If a chord connected some a_s, $1 \leq s \leq d$, with some c_h, $1 \leq h < i$ or $j < h \leq \ell$, then by $a_s \neq x$ it would have to cross (c_i, x) or (x, c_j), contradicting planarity.

So G' is an internally 4-connected triangulated disk.

Observe that in both cases $V' \subseteq \{c_{i+1}, \ldots, c_{j-1}\}$ for some $1 \leq i < j \leq \ell$, and so V' does not contain u_1 or u_2 as desired. □

Theorem 1 *Let G be a 4-connected plane triangulation with at least 4 vertices. Then G has a $(3,1)$-canonical order.*

Proof: We choose the vertex set in reverse order. Let $\{u_1, u_2, u_3\}$ be the outer face and choose $V_L := \{u_3\}$; this satisfies all conditions since u_3 has at least 3 neighbors. We do not at this point know the correct value of L, but simply assign indices backwards and shift indices at the end so that the vertex sets are numbered $V_1, \ldots, V_L$.

Observe that $G - u_3$ is an internally 4-connected triangulated disk, because the neighbors of u_3 form a simple cycle without chord which would form a separating triangle at u_3. Assume now some $V_{k+1}, \ldots, V_L$ have been chosen already such that the remaining graph $G_k := G-(V_{k+1} \cup \cdots \cup V_L)$ is an internally 4-connected triangulated disk with (u_1, u_2) on the outer face. If G_k has at least 4 vertices, then apply Lemma 1 to find the next V_k. Graph $G_k - V_k$ is again internally 4-connected, so we can continue choosing vertex sets until only 3 vertices, including u_1 and u_2, are left. Since the graph is still internally 4-connected, these vertices must be a triangle, and hence a face of G. So setting V_1 to be the three vertices of this triangle gives the desired ordering.

To observe that the required connectivity holds, note that any internally 4-connected graph is 3-connected since it is a triangulated disk without a chord. To see that $\overline{G_k}$ is connected, it suffices to show that every vertex except u_3 has a neighbor in a later vertex set; the set of these edges then forms a spanning tree in $\overline{G_k}$. The argument for this is nearly the same as for $(2,1)$-orderings. Clearly each of u_1, u_2 are adjacent to u_3. For any vertex $z \neq u_1, u_2, u_3$, vertex z is not on the outer face of G, and hence there must exist some k' such that z is on the outer face of $G_{k'-1}$, but not on the outer face of $G_{k'}$. Since faces are triangles, this implies that z is adjacent to some vertex in $V_{k'}$. By the above hence $\overline{G_k}$ is connected for any $1 < k < L$. □

3.1 Linear time algorithm

The proofs of the above results are constructive and lead to a polynomial time algorithm for finding a $(3,1)$-canonical ordering. In this section, we show that by keeping suitable lists and counters, a $(3,1)$-canonical ordering can be found in linear time. The algorithm to do so does not exactly follow the above proofs; instead we use counters to find vertices that can be used for the next vertex set, and the proofs then are used to show that such vertices always exists.

We are in the same setup as in Lemma 1, i.e., we work on a subgraph of G that is internally 4-connected, and we would like to find repeatedly a vertex set V' that we can remove (i.e., make the next vertex set) while maintaining an internally 4-connected graph. In what follows, we use z for the unique vertex on the inner face adjacent to the fixed edge (u_1, u_2). We keep track of the following information:

- Every vertex v stores its degree $deg(v)$. We need to update this whenever we remove a vertex x from the graph. For any such vertex x we need

$O(deg(x))$ time to update the degree of all its remaining neighbors, but since every vertex is removed at most once, this is $O(n)$ time overall.

- Every vertex stores whether it is on the outer face or not. We need to update this whenever we remove a vertex x from the graph, since all neighbors of x that were not previously on the outer face now are there. This takes $O(n)$ time overall as before.

- Every vertex v keeps a list $outer(v)$, which stores the neighbors of v that are on the outer face (in no particular order). This needs to be updated whenever a vertex x newly becomes a vertex on the outer face. For any such vertex x we need $O(deg(x))$ time to add x to $outer(v)$ of all its neighbors v, but since every vertex comes to the outer face at most once, this is $O(n)$ time overall.

  The list also needs to be updated whenever a vertex x is removed, because any such removed vertex is on the outer face. We assume that removal from a list can be done in constant time, and that x keeps track of all places where it was stored in the lists of its neighbors. Then this update takes $O(deg(x))$ time, which is $O(n)$ time overall since every vertex is removed only once.

- With this, every vertex $x \neq z$ can check in $O(1)$ time whether it is a 2-leg center: This is true if and only if x is not on the outer face, $|outer(x)| \geq 2$, and if $|outer(x)| = 2$ then the two vertices in it are not consecutive on the outer face.

- Every vertex v keeps a counter $outerDeg3(v)$, which stores how many vertice in $outer(v)$ have degree 3. This needs to be updated whenever a vertex x newly becomes a vertex on the outer face and now has degree 3, or whenever a vertex x on the outer face has its degree reduced to 3. Either event happens to any vertex x at most once, and then takes $O(deg(x))$ time to handle, so takes $O(n)$ time overall.

- With this, any vertex $x \neq z$ can test in $O(1)$ time whether it is a 2-leg-center for which all 2-legs containing x are basic. For any such 2-leg, we must have $outerDeg3(x) = |outer(x)| - 2$, because the first and last neighbor of x on the outer face have degree ≥ 4 if $x \neq z$, and all others have degree 3. Vice versa, one easily verifies that $outerDeg3(x) = |outer(x)| - 2$ implies that all 2-legs at x are basic.

  So during any update to $outer(x)$ or $outerDeg3(x)$, we immediately check whether x is now or continues to be a 2-leg-center for which all 2-legs are basic. We keep a list L_b of all such 2-leg-centers; the updates to L_b are then overhead to the updates that triggered the change to L_b.

- Every vertex v keeps a counter $2legDeg(v)$, which stores how many neighbors x of v are 2-leg-centers. This needs to be updated whenever a vertex x newly becomes a 2-leg-center. For any such vertex x we need $O(deg(x))$

time to add x to *2legDeg*(v) of all its neighbors v, but since every vertex becomes a 2-leg center at most once (it remains a 2-leg center until it is on the outer face), this is $O(n)$ time overall. This counter also needs to be updated whenever a vertex x that used to be a 2-leg-center newly becomes part of the outer face (and hence no longer is a 2-leg-center); this takes $O(n)$ time overall since every vertex comes to the outer face at most once.

- We keep a list L_c that contains all those vertices v that are on the outer face and for which *2legDeg*$(v) = 0$. This needs to be updated whenever a vertex newly becomes a vertex on the outer face, or is removed, or whenever *2legDeg*(v) changes; this update is an $O(1)$ overhead to these changes.

With these counters and lists in place, finding a $(3,1)$-canonical ordering now consists of two simple steps:

1. If L_b is non-empty, then let x be the first vertex in it. We know that x is a 2-leg-center and all 2-legs containing x are basic (which implies that x is minimal). Use all degree-3 vertices in *outer*(x) as next vertex set.

2. If L_c is non-empty, then let c_{i+1} be the first vertex in it. We claim that $\{c_{i+1}\}$ can be used as vertex set V' in Lemma 1. Indeed c_{i+1} is a single vertex on the outer face. Since none of the neighbors of c_{i+1} is a 2-leg-center, one proves as in Lemma 1 that removing c_{i+1} leaves an internally 4-connected graph.

While $n \geq 4$, one of the above two situations must be true due to Lemma 1, because there exists a minimal 2-leg-center x, and at it we either find complex 2-leg and vertex c_{i+1} (which would be in L_c) or all 2-legs are basic (then x would be in L_b).

Thus we can find the next vertex set in $O(1)$ amortized time, where "amortized" hides the terms of $O(n)$ time overall that are needed to execute updates to the various lists and counters whenever vertex becomes a 2-leg-center, comes to the outer face, has its degree reduced to 3, or is removed. The counters and lists can clearly be initialized in $O(n)$ time for a triangulated graph, and we end with $n \leq 3$ and the first vertex group, which can be handled in $O(1)$ time. We conclude:

Lemma 2 *We can find a* $(3,1)$*-canonical ordering in linear time.*

4 Applications

In this section, we demonstrate two uses for the $(3,1)$-canonical ordering in graph drawing. Both results proved here were known before, but in our opinion using the $(3,1)$-canonical ordering as a black box simplifies the proof of these results.

4.1 Rectangular duals

A *rectangular dual drawing* (or *RD-drawing* for short) of a planar graph G consists of a set of interior-disjoint rectangles assigned to the vertices of G in such a way that the union of the rectangles forms a rectangle without holes, and the rectangles assigned to vertices v and w touch in a non-zero-length line segment if and only if (v, w) is an edge. The following theorem has been proved repeatedly:

Theorem 2 ([15, 14, 9]) *Let G be a 4-connected plane triangulation, and let e be an edge on the outer face of G. Then $G - e$ has a rectangular dual.*

Previous proofs on this result usually used the $(2, 2)$-canonical ordering (or some equivalent characterization, such as regular edge labellings). We give here a different proof using the $(3, 1)$-canonical ordering.

Proof: Let the outer face be $\{u_1, u_2, u_3\}$, chosen such that $e = (u_1, u_2)$. Find a $(3, 1)$-canonical ordering $V_1 \cup \cdots \cup V_L$ of G. We now build the rectangular-dual drawing of $G - e$ by drawing $G_k - e$ for $k = 1, \ldots, L$. By construction, $e = (u_1, u_2)$ is an edge on the outer face of G_k, and we can hence enumerate the outer face of G_k as $c^k_1, \ldots, c^k_{\ell_k}$ with $c^k_1 = u_1$ and $c^k_{\ell_k} = u_2$. We maintain the invariant that in the RD-drawing of G_k, the rectangles of $c^k_1, \ldots, c^k_{\ell_k}$ all attach at the top side of the bounding box, in this order.

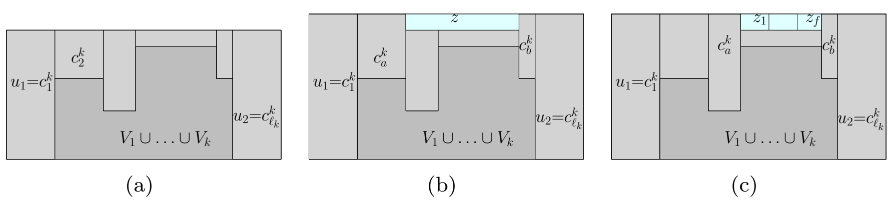

Figure 3: (a) The invariant. (Middle and right) Adding a singleton and a fan.
‘

Such a drawing is easily created for $G_1 - e$, since G_1 is a triangle and so $G_1 - e$ is a path $u_1 - z - u_2$, where z is the third vertex of the interior face at (u_1, u_2). Now assume G_k is drawn and consider adding either a singleton or a fan V_{k+1}. Let a and b be the smallest and largest index such that c^k_a and c^k_b are adjacent to a vertex in V_{k+1}.

Extend all rectangles of $c^k_1, \ldots, c^k_a$ and $c^k_b, \ldots, c^k_{\ell_k}$ upward by one unit. This leaves a "gap" where the rectangles of $c^k_{a+1}, \ldots, c^k_{b-1}$ ended. There is at least one such rectangle since $b \geq a + 2$ by properties of the $(3, 1)$-canonical ordering (else G_{k+1} would not be 3-connected). If V_{k+1} is a singleton z, then we insert the rectangle for z into this gap. If V_{k+1} is a fan $\{z_1, \ldots, z_f\}$, then $b = a + 2$ and so the gap consists exactly of the top of c^k_{a+1}. Split this range into f pieces and assign rectangles for $z_1, \ldots, z_f$ in this place. One easily verifies that this represents all added edges as contacts and satisfies the invariant. So we have the desired RD-drawing. □

We note one interesting property of our rectangular dual drawings: any maximal horizontal segment s is 1-sided in the sense that all vertical segments that end at s either all attach from top or all attach from bottom. 1-sidedness is of interest for rectangular duals with prescribed face areas (see [6]), though unfortunately our vertical segments are not necessarily 1-sided.

We also note that the algorithm as described gives integral y-coordinate (no bigger than n), but not necessarily integral x-coordinates. To achieve the latter, we proceed similarly as Kant [8] did for his orthogonal drawings: Rather than computing explicit x-coordinates, assign each vertical line segment to a list C of columns. We can then add columns freely as needed between other columns, and compute the final x-coordinates by traversing C in order. This gives coordinates in an $O(n) \times O(n)$-grid.

4.2 Rectangle-of-influence drawings

A planar straight-line drawing of a graph is called a *(weak, closed) rectangle-of-influence drawing* (or *RI-drawing* for short) if for any edge (u, v) the *rectangle* $R(u, v)$ *defined by* u, v is *empty*, i.e., contains no other points of vertices of the graph. It, however, may contain parts of other edges. Here, $R(u, v)$ is the minimum axis-aligned rectangle that contains the points of u and v; it degenerates into a line segment if u or v are on a horizontal or vertical line. The following result is known:

Theorem 3 ([1]) *Let G be a 4-connected plane triangulation and let e be one edge of the outer face. Then $G - e$ has a (weak, closed) rectangle-of-influence drawing.*

We re-prove this result using the $(3, 1)$-canonical ordering. We note here that the drawing created is exactly the same as in [1]; but our description is significantly easier to understand since we separate finding the next vertex set from placing it in the drawing.

Proof: Let the outer face be $\{u_1, u_2, u_3\}$, chosen such that $e = (u_1, u_2)$. Find a $(3, 1)$-canonical ordering $V_1 \cup \cdots \cup V_L$ of G. We now build the RI-drawing of $G - e$ by drawing $G_k - e$ for $k = 1, \ldots, L$. By construction $e = (u_1, u_2)$ is an edge on the outer face of G_k, and we can hence enumerate the outer face of G_k as $c_1^k, \ldots, c_{\ell_k}^k$ with $c_1^k = u_1$ and $c_{\ell_k}^k = u_2$. We maintain the invariant that in the RI-drawing of G_k

$$x(c_1^k) < x(c_2^k) < \cdots < x(c_{\ell_k}^k) \quad \text{and} \quad y(c_1^k) > y(c_2^k) > \cdots > y(c_{\ell_k}^k).$$

Such a drawing is easily created for $G_1 - e$, since G_1 is a triangle and so $G_1 - e$ is a path $u_1 - z - u_2$, where z is the third vertex of the interior face at (u_1, u_2). Now assume G_k is drawn and consider adding either a singleton or a

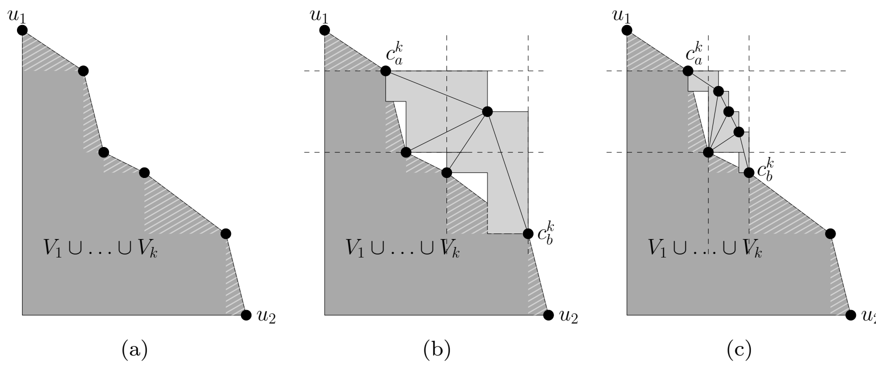

Figure 4: (a) The invariant for RI-drawings. Hatched regions contain no points due to the RI-drawing. (Middle and right) Adding a singleton and a fan. The light gray region contains the new rectangles of influence.

fan V_{k+1}. Let a be the smallest and b be the largest index such that c_a^k and c_b^k are adjacent to a vertex in V_{k+1}. We cannot have $a = b - 1$, else vertices c_a^k and c_b^k would be adjacent on the outer-face of G_k and hence a chord of graph G_{k+1}, contradicting 3-connectivity of G_{k+1}. So $b \geq a + 2$. If V_{k+1} is a singleton z, then define

$$x(z) = \frac{1}{2}\left(x(c_{b-1}^k) + x(c_b^k)\right)$$

and

$$y(z) = \frac{1}{2}\left(y(c_a^k) + y(c_{a+1}^k)\right).$$

See also Figure 4(b). By $a \leq b - 2$ adding this new point satisfies the invariant. All rectangles $R(z, c_j^k)$ are empty for $a \leq j \leq b$, because they do not intersect the drawing of G_k except in rectangles $R(c_a^k, c_{a+1}^k)$ and $R(c_{b-1}^k, c_b^k)$. So we have the desired RI-drawing.

If V_{k+1} is a fan $\{z_1, \ldots, z_f\}$, then $b = a + 2$. For $h = 1, \ldots, f$, define

$$x(z_h) = \frac{h}{f+1}\left(x(c_{b-1}^k) + x(c_b^k)\right)$$

and

$$y(z_h) = \frac{f-h+1}{f+1}\left(y(c_a^k) + x(c_{a+1}^k)\right).$$

See also Figure 4(c). By $a = b-2$ adding these new points satisfies the invariant. All rectangles $R(z_h, c_j^k)$ are empty for $a \leq j \leq b$, because they do not intersect the drawing of G_k except in rectangles $R(c_a^k, c_{a+1}^k)$ and $R(c_{b-1}^k, c_b^k)$. So we have the desired RI-drawing. $\square$

Again, our coordinates are not necessarily integers, but can be made integral in an $n \times n$-grid by rearranging without changing the relative order of

x-coordinates and y-coordinates. This was shown by Liotta et al. [10] to maintain an RI-drawing, and one easily shows that due to the empty rectangles-of-influence it also preserves planarity.

5 Conclusion

We showed the existence of new canonical order for 4-connected triangulations. We used this canonical order to give simplified proofs of some previously known graph drawing results for 4-connected triangulations. Furthermore, we provided a brief survey of canonical orderings for planar graphs and laid the groundwork for their further investigation. Of particular interest to us are the following questions:

- Does every planar c-connected triangulation have an (r, s)-canonical ordering for all $r + s = c$ and reasonable restrictions on vertex sets V_k? The missing case is a $(4, 1)$-canonical ordering for 5-connected triangulations.
- The (r, s)-canonical ordering definition naturally generalizes to planar graphs that are not necessarily triangulated. For the corresponding $(2, 1)$-orderings [8] and $(2, 2)$-orderings [12] it suffices to allow adding *chains*, i.e., induced paths. Are there $(3, 1)$-orderings, $(3, 2)$-orderings and $(4, 1)$-orderings for 4-connected/5-connected planar graphs with some simple subgraphs as vertex sets V_k? Likewise, exploration of (r, s)-canonical orders for non-planar graphs for $r + s \geq 5$ remains completely open.

References

[1] T. C. Biedl, A. Bretscher, and H. Meijer. Rectangle of influence drawings of graphs without filled 3-cycles. In J. Kratochvíl, editor, *7th International Symposium on Graph Drawing, GD'99*, volume 1731 of *Lecture Notes in Computer Science*, pages 359–368. Springer, 1999. doi:10.1007/3-540-46648-7_37.

[2] M. Chrobak and S.-I. Nakano. Minimum-width grid drawings of plane graphs. *Computational Geometry*, 11(1):29 – 54, 1998. doi:10.1016/S0925-7721(98)00016-9.

[3] S. Curran, O. Lee, and X. Yu. Chain decompositions of 4-connected graphs. *SIAM Journal on Discrete Mathematics*, 19(4):848–880, 2005. doi:10.1137/S0895480103434592.

[4] H. de Fraysseix, J. Pach, and R. Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10(1):41–51, 1990. doi:10.1007/BF02122694.

[5] R. Diestel. *Graph Theory, 4th Edition*, volume 173 of *Graduate texts in mathematics*. Springer, 2012.

[6] D. Eppstein, E. Mumford, B. Speckmann, and K. Verbeek. Area-universal and constrained rectangular layouts. *SIAM Journal on Computing*, 41(3):537–564, 2012. doi:10.1137/110834032.

[7] X. He, M.-Y. Kao, and H.-I. Lu. Linear-time succinct encodings of planar graphs via canonical orderings. *SIAM Journal on Discrete Mathematics*, 12(3):317–325, 1999. doi:10.1137/S0895480197325031.

[8] G. Kant. Drawing planar graphs using the canonical ordering. *Algorithmica*, 16(1):4–32, 1996. doi:10.1007/BF02086606.

[9] G. Kant and X. He. Regular edge labeling of 4-connected plane graphs and its applications in graph drawing problems. *Theoretical Computer Science*, 172(1):175 – 193, 1997. doi:10.1016/S0304-3975(95)00257-X.

[10] G. Liotta, A. Lubiw, H. Meijer, and S. Whitesides. The rectangle of influence drawability problem. *Computational Geometry*, 10(1):1 – 22, 1998. doi:10.1016/S0925-7721(97)00018-7.

[11] S. Nagai and S.-I. Nakano. A linear-time algorithm to find independent spanning trees in maximal planar graphs. In U. Brandes and D. Wagner, editors, *26th International Workshop on Graph-Theoretic Concepts in Computer Science, WG 2000*, volume 1928 of *Lecture Notes in Computer Science*, pages 290–301. Springer, 2000. doi:10.1007/3-540-40064-8_27.

[12] S.-I. Nakano, M. Rahman, and T. Nishizeki. A linear-time algorithm for four-partitioning four-connected planar graphs. *Information Processing Letters*, 62(6):315 – 322, 1997. doi:10.1016/S0020-0190(97)00083-5.

[13] J. M. Schmidt. The mondshein sequence. In J. Esparza, P. Fraigniaud, T. Husfeldt, and E. Koutsoupias, editors, *41st International Colloquium on Automata, Languages, and Programming, ICALP 2014*, volume 8572 of *Lecture Notes in Computer Science*, pages 967–978. Springer, 2014. doi:10.1007/978-3-662-43948-7_80.

[14] C. Thomassen. Plane representations of graphs. In *Progress in Graph Theory*, pages 43–69. Academic Press, 1984.

[15] P. Ungar. On diagrams representing maps. *Journal of the London Mathematical Society*, s1-28(3):336–342, 1953. doi:10.1112/jlms/s1-28.3.336.