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*Research Article*

# Exponential Synchronization Control of Discontinuous Nonautonomous Networks and Autonomous Coupled Networks

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This paper concerns complex delayed neural networks with discontinuous activations. Based on the framework of differential inclusion theory, we design two novel controllers by regulating a parameter $\sigma\,(0 \leq \sigma < 1)$ which covers both discontinuous and continuous controllers. Then, we investigate a nonautonomous cellular neural network system and autonomous neural network with linear coupling, respectively. By choosing a time-dependent Lyapunov-Krasovskii functional candidate and suitable controllers, some criteria are studied to guarantee the exponential synchronization of the complex delayed dynamical network. Finally, two numerical examples are given to illustrate our theoretical analysis.

## 1. Introduction

In the past few decades, the dynamical behavior of synchronization phenomena has attracted much attentions because of its potential practical application in general complex networks [1], pattern recognition [2], secure communication [3], combinational optimization [4], biological systems [5], and so on. Up to now, several types of synchronization of complex neural networks have been studied such as asymptotic synchronization [6], finite-time synchronization [7], and exponential synchronization [8–10]. The synchronization phenomena of a complex dynamical network are said to be an important issue in our theoretical analysis and experimental application.

In real world, there are a large number of nodes in the real-world complex networks. Cao et al. in [11, 12] studied the global synchronization of coupled delayed neural networks with constant and hybrid coupling. The authors in [13] designed a coupling term by $D(x_j(t-\tau(t))) - x_i(t-\tau(t))$ and realized the exponential synchronization for complex dynamical networks with sampled data. After that, some literatures are interested in the synchronization for neural networks with the coupling term $D(x_j(t-\tau(t))) - x_i(t-\tau(t))$; for example, in [14, 15], the authors investigated the synchronization of coupled networks with hybrid coupling, which were composed of constant coupling and a single coupling delay. By this distance, a new unloading method is obtained in global convergence for complete regular coupling configuration. Generally, the coupling structure is designed by a graph which can be unconnected, directed, and undirected.

As we know, many valid control techniques have been extensively applied in the engineering field, such as impulsive control [16], intermittent control [17], feedback control [18], and adaptive control [19]. In recent years, many researchers receive the results on synchronization stabilization of complex chaotic systems and coupled dynamical networks by pinning a suitable control, and most of the existing controllers were designed in the form of $-k\,\mathrm{sign}\,(e(t))\,|e(t)|^{\sigma}(0 \leq \sigma < 1)$; we can see that the controller is continuous if $0 < \sigma < 1$ and the controller is discontinuous if $\sigma = 0$, where $e(t)$ is the synchronization error with control strength $k$. However, few literatures discuss the two types of switching controllers concurrently, and the two categories are discussed separately or only in the field of Lipschitz conditions. Because there still have been a lot of difficulties in overcoming the exponential synchronization problem when the activation functions are discontinuous but the controllers are not. However, to the best of our knowledge, few papers focus on the synchronization issue of complex networks with nonlinear coupling function, and there are two kinds of controllers such as continuous case and discontinuous case when the activation functions are still discontinuous.

The neural network system of this paper is a general nonautonomous neural network system with discontinuous activations, and we also consider the corresponding autonomous system in this paper. The main contributions are as follows:

(1) In the existing exponential synchronization research, the neuron activation functions were restricted to be continuous and bounded, and the assumptions of the system were complex. So this paper consider a more general neural network model and simpler conditions for gaining the exponential synchronization goal

(2) It is the first time that the exponential synchronization control of the nonautonomous systems with discontinuous activation and the autonomous system with linear coupling function is considered. The algorithm in this paper is optimized, where sufficient conditions formulated by the Lyapunov function are established to gain the exponential synchronization. The theoretical results can also be used in a wider scope

(3) Novel analytical techniques are proposed, and strict mathematical proofs are given for the global exponential synchronization of the discontinuous neural network with coupled and time delays. We design novel discontinuous controllers and continuous controllers in this paper. When both neuron functions and controllers are discontinuous, there is still a lack of complete theory of synchronization

(4) The technique skill and control algorithm are different from those in previous papers (e.g., [20]). We introduce some novel tools such as exponential synchronization theorem, differential inclusion in the sense of Filippov, and generalized Lyapunov approach under a 1-norm framework, and the methods proposed in this paper can be extended to investigate the synchronization of neural network systems

The structure of this paper is outlined as follows. In the next section, we design the model and introduce some basic preliminary lemmas and definitions. In Section 3, we design a continuous controller to realize the exponential synchronization of the nonautonomous network system with discontinuous activations and describe a nonlinear coupling function to guarantee the synchronization issue of the time-delayed discontinuous neural network by considering a discontinuous controller. In Section 4, we give two numerical examples to illustrate our theoretical results. Finally, we conclude this paper in Section 5.

*Notation 1.* Let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space, and let the superscript $T$ denote the transposition. Let $x=(x_1,x_2,\dots,x_n)^T$ and $y=(y_1,y_2,\dots,y_n)^T$; by $x>0$ $(x\geq 0)$, we mean that $x_i>0(x_i\geq 0)$ for all $i=1,2,\dots,n$. $\langle x, y\rangle=\sum_{i=1}^n x_i y_i$, $\langle\cdot,\cdot\rangle$ denotes the inner product. If $x\in\mathbb{R}$, $\|x\|$ denotes the vector norm of $x$, while $\|x\|_1=\sum_{i=1}^n|x_i|$ . Given the real matrix $A=(a_{ij})_{n\times n}$, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ represent the maximal and minimal eigenvalues of $A$, respectively. Let diag $(\cdots)$ denote the block diagonal matrix, and let sign $(\cdot)$ denote the sign function.

Finally, let $g(t)$ be the continuous function, and we define that

$$\begin{aligned} g^{\max}&=\sup_{t\in\mathbb{R}}|g(t)|,\\ g^{\min}&=\inf_{t\in\mathbb{R}}|g(t)|. \end{aligned}\tag{1}$$

## 2. Preliminaries

In this section, we give some definitions and preliminary lemmas. The main references are the framework of Filippov, set valued maps, differential inclusion, and so on [21–26]. Firstly, we consider the discontinuous function $f$ to introduce the solution of the system, and we denote the closure of the convex hull of $X$ as $K[X]$; we can expand the property of the Filippov solution to the system.

By the discussions in Section 1, in this paper, we consider the following general nonautonomous neural network system with time-varying delays and discontinuous right-hand sides:

$$\begin{aligned} \frac{dx_i(t)}{dt}&=-a_i(t)x_i(t)+\sum_{j=1}^n b_{ij}(t)f_j(x_j(t))\\ &\quad+\sum_{j=1}^n c_{ij}(t)f_j(x_j(t-\tau_{ij}(t)))+I_i(t),\quad i=1,2,\dots,n, \end{aligned}\tag{2}$$

where $x_i(t)$ corresponds to the state vector of the $i$th unit at time $t$, $a_i(t)>0$ denotes the self-inhibition with which the $i$th neuron will reset its potential to the resting state in isolations when disconnected from the network and inputs, $b_{ij}(t)$ and $c_{ij}(t)$ represent the connection strength and the delayed connection strength of the $j$th neuron on the $i$th neuron, respectively, $f_j(x_j(t))$ represents the activation function and the time-delayed activation function of $j$th neuron, $I_i(t)$ is a constant external input vector, $\tau_{ij}(t)$ corresponds to the transmission delay of the $i$th unit along the axon of the $j$th unit at time $t$ and is a continuously differentiable function, and there exist $\tau=$

$\max_{1\le i,j\le n}\{\max_{t\in[0,\omega]}\tau ij(t)\}\ge 0$ and a negative constant $\tau_{ij}^D$ satisfying

$$0\le \tau_{ij}(t)\le \tau,$$
$$\dot{\tau}_{ij}(t)\le \tau_{ij}^D<1. \tag{3}$$

Moreover, we obtain an autonomous system when coefficients are reduced to constants corresponding to model (2) as follows:

$$\frac{dx_i(t)}{dt}=-a_ix_i(t)+\sum_{j=1}^n b_{ij}f_j(x(t))+\sum_{j=1}^n c_{ij}f_j\left(x_j\left(t-\tau_{ij}(t)\right)\right)+I_i,\quad i=1,2,\dots,n. \tag{4}$$

Equivalently, the differential equation system can be transformed into the following matrix format:

$$\frac{dx(t)}{dt}=-Ax(t)+Bf(x(t))+Cf(x(t-\tau(t)))+I, \tag{5}$$

where $A=\operatorname{diag}(a_1,a_2,\dots,a_n)$, $B=(b_{ij})_{n\times n}$, and $C=(c_{ij})_{n\times n}$.

To establish our main results, we assume the following basic conditions for the neuron activations in model (2) or (4):

*Assumption 1.* For every $j=1,2,\dots,n$, $f_j$ is continuous except for a countable set of isolate jump discontinuous points $\rho_k$, where there exist finite right and left limits, and in every compact set of $R$, it has only a finite number of jump discontinuous points.

*Definition 1.* A vector function $x=(x_1,x_2,\dots,x_n)^T:[-\tau,T)\to\mathbb{R}^n$, $T\in(0,+\infty]$, is a state solution of the discontinuous system (2) on $[-\tau,T)$ if

(1) $x$ is continuous on $[-\tau,T)$ and absolutely continuous on any compact interval of $[0,T)$

(2) there exists a measurable function $\gamma_j(t)\in K[f_j(x(t))]$ for a.e. $t\in[-\tau,T)$ and

$$\frac{dx_i(t)}{dt}=-a_i(t)x_i(t)+\sum_{j=1}^n b_{ij}(t)\gamma_j(t)+\sum_{j=1}^n c_{ij}(t)\gamma_j\left(t-\tau_{ij}(t)\right)+I_i(t),\quad t\in[0,T). \tag{6}$$

Any function $\gamma=(\gamma_1,\gamma_2,\dots,\gamma_n)^T$ satisfying (6) is called an output solution associated with the state $x=(x_1,x_2,\dots,x_n)^T$; then, in the sense of Filippov, we point out that the state $x$ is a solution of (2) for a.e. $t\in[0,T)$ and we obtain the following differential inclusion:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}\in -a_i(t)x_i(t)+\sum_{j=1}^n b_{ij}(t)K\left[f_j\left(x_j(t)\right)\right]+\sum_{j=1}^n c_{ij}(t)K\left[f_j\left(x_j\left(t-\tau_{ij}(t)\right)\right)\right]+I_i(t),t\in[0,T). \tag{7}$$

*Definition 2.* The network is said to achieve global exponential synchronization if there exist some constants $\lambda>0$, $T>0$, and $M_0>0$ such that for any initial values $\phi_i(s)$ $(i=1,2,\dots,n)$,

$$\left\|x_j(t)-x_i(t)\right\|\le M_0e^{-\lambda t} \tag{8}$$

hold for all $t>T$ and for any $i,j=1,2,\dots,n$.

**Lemma 1** *(see [10]). If $V(x)$: $\mathbb{R}^n\to\mathbb{R}$ is C-regular and $x(t)$: $[0,+\infty)\to\mathbb{R}^n$ is absolutely continuous on any compact subinterval of $[0,+\infty)$. Then, $x(t)$ and $V(x(t))$: $[0,+\infty)\to\mathbb{R}$ are differentiable for almost all $t\in[0,+\infty)$ and*

$$\frac{dV(x(t))}{dt}=\left\langle \varsigma(t)\frac{dx(t)}{dt}\right\rangle,\quad \forall\varsigma(t)\in\partial V(x(t)). \tag{9}$$

**Lemma 2** *(see [11, 12]). Given an undirected graph F with the adjacency matrix $C=[c_{ij}]$ and Laplacian matrix L, equality*

$$x^TLx=\frac{1}{2}\sum_{i,j=1}^n c_{ij}\left(x_i-x_j\right)^2 \tag{10}$$

*holds for arbitrary $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$.*

*Let* $\mathbb{F}(x)\triangleq K[f(x)]=(K[f_1(x),Kf_2(x)],\dots,K[f_n(x)])$, *where* $K[f_i(x)]=[\min\{f_i(x^-),f_i(x^+)\},\max\{f_i(x^-),f_i(x^+)\}]$. *Then, we assume the neuron activation functions in (2) or (4) to satisfy the following condition:*

*Assumption 2.* For $x,y\in\mathbb{R}$, there exist nonnegative constants $\alpha$ and $\beta$ such that

$$\|\mathbb{F}[f(x)-f(y)]\|=\sup_{\xi\in\mathbb{F}[f(x)-f(y)]}\|\xi\|\le a\|x-y\|+\beta. \tag{11}$$

## 3. Main Results

In this section, the discontinuous controller and continuous controller are designed; then, we divide this section into two parts to derive the global exponential synchronization conditions of discontinuous nonautonomous networks and autonomous coupled networks, respectively.

*3.1. Exponential Synchronization with the Continuous Controller.* Firstly, we consider the nonautonomous neural network model (6) as the driver system, and the controlled response system can be described as follows:

$$\frac{dy_i(t)}{dt} = -a_i(t)y_i(t) + \sum_{j=1}^{n} b_{ij}(t)f_j\left(y_j(t)\right) + \sum_{j=1}^{n} c_{ij}(t)f_j\left(y_i\left(t-\tau_{ij}(t)\right)\right) + I_i(t) + u_i(t), \quad i = 1, 2, \ldots, n, \tag{12}$$

where $u_i(t)$ is the controller to be designed for realizing the synchronization of the driver response system. The other parameters are the same as those in model (6).

Our first goal is to drive the response network model (12) to synchronize with the nonautonomous network model (6) with continuous controllers. To this end, choose the parameter $0 < \sigma < 1$, and the continuous controller $u_i(t)$ is given by

$$u_i(t) = -k_1\left(y_i(t) - x_i(t)\right) - k_2 \operatorname{sign}\left(y_i(t) - x_i(t)\right)\left|y_i(t) - x_i(t)\right|^{\sigma}. \tag{13}$$

Then, by subtracting (6) from (12), let $e_i(t) = y_i(t) - x_i(t)$. In view of Assumption 1 and Definition 1, by differential inclusions and set valued maps, we can see that there exists a measurable function $\xi_j(t) \in K[f_j(y_j(t))]$ for a.e. $t \in [0, T]$ and we can obtain the synchronization error system as follows:

$$\frac{de_i(t)}{dt} = -a_i(t)e_i(t) + \sum_{j=1}^{n} b_{ij}(t)\Gamma_j(t) + \sum_{j=1}^{n} c_{ij}(t)\Gamma_j\left(t-\tau_{ij}(t)\right)) - k_1 e_i(t) - k_2 \operatorname{sign}\left(e_i(t)\right)\left|e_i(t)\right|^{\sigma}, \tag{14}$$

where $\Gamma_j(t) = \xi_j(t) - \gamma_j(t)$ and $\Gamma_j(t-\tau_{ij}(t)) = \xi_j(t-\tau_{ij}(t)) - \gamma_j(t-\tau_{ij}(t))$.

Then, we give the following theorem to derive the response network system (6) with $0 < \sigma < 1$ synchronizing with the driver network system (2). Before doing this, we give a further condition on the discontinuous activation function $f_j$ as follows:

**Theorem 1.** *If Assumptions 1 and 2 hold, the nonautonomous discontinuous neural networks achieve global exponential synchronization under the continuous switching controller (13) with $0 < \sigma < 1$; if there exist positive $\zeta_1, \zeta_2, \ldots, \zeta_n$ and a very small positive constant $\varepsilon > 0$, for $i = 1, 2, \ldots, n$, the following conditions are satisfied:*

$$\lim_{t\to+\infty} \sup Q_i(t) < 0, \tag{15}$$

*where*

$$Q_i(t) = \zeta_i b_{ii}(t) + \sum_{j=1, j\neq i}^{n} \zeta_j\left|b_{ij}(t)\right| + \sum_{j=1}^{n} \zeta_j e^{\varepsilon\tau} \frac{\left|c_{ij}\left(\varphi_{ij}^{-1}(t)\right)\right|}{1-\dot{\tau}_{ij}\left(\varphi_{ij}^{-1}(t)\right)}. \tag{16}$$

*Proof 1.* Consider the following candidate Lyapunov function:

$$V(t) = e^{\varepsilon t}\sum_{i=1}^{n} \varsigma_i\left|e_i(t)\right| + \sum_{i,j=1}^{n} \varsigma_i \times \int_{t-\tau_{ij}(t)}^{t} \frac{\left|c_{ij}\left(\varphi_{ij}^{-1}(s)\right)\right|}{1-\dot{\tau}_{ij}\left(\varphi_{ij}^{-1}(s)\right)}\left|\Gamma_j(s)e^{\varepsilon(s+\tau)}\right| ds, \tag{17}$$

where $\varphi_{ij}^{-1}$ is the inverse function of $\varphi_{ij}(t) = t - \tau_{ij}(t)$.

Note that the function $|e_i(t)|$ is locally continuous (Lipschitz) in $e_i$ on $R$; then, we can see that $V(e(t))$ is regular. According to the definition of Clarke's generalized gradient of the absolute value function $|e_i(t)|$ at $e_i(t)$, we obtain that there exist $\partial(|e_i(t)|) = K[\operatorname{sign}(e_i(t))] = 1$ if $e_i(t) < 0$, $\partial(|e_i(t)|) = K[\operatorname{sign}(e_i(t))] = -1$ if $e_i(t) > 0$, and $\partial(|e_i(t)|) = K[\operatorname{sign}(e_i(t))] = [-1, 1]$ if $e_i(t) = 0$. For any $\vartheta_i(t) \in K[\operatorname{sign}(e_i(t))]$, we have $\vartheta_i(t) = \operatorname{sign}(e_i(t))$, if $e_i(t) \neq 0$; $\vartheta_i(t)$ can arbitrarily be selected in $[-1, 1]$, if $e_i(t) = 0$.

Then, by Lemma 1 and calculating the time derivative of $V(t)$, we obtain that

$$\begin{aligned}
\frac{dV(t)}{dt} &= \varepsilon e^{\varepsilon t}\sum_{i=1}^{n} \varsigma_i\left|e_i(t)\right| + e^{\varepsilon t}\sum_{i=1}^{n} \varsigma_i \operatorname{sign}\left(e_i(t)\right) \cdot \\
&\quad \cdot \left\{-a_i(t)e_i(t) + \sum_{j=1}^{n} b_{ij}(t)\Gamma_j(t) + \sum_{j=1}^{n} c_{ij}(t)\Gamma_j\left(t-\tau_{ij}(t)\right) - k_1\left|e_i(t)\right| - k_2 \operatorname{sign}\left(e_i(t)\right)\left|e_i(t)\right|^{\sigma}\right\} \\
&\quad + \sum_{i,j=1}^{n} \varsigma_i \frac{\left|c_{ij}\left(\varphi_{ij}^{-1}(t)\right)\right|}{1-\dot{\tau}_{ij}\left(\varphi_{ij}^{-1}(t)\right)}\left|\Gamma_j(t)e^{\varepsilon(t+\tau)}\right| \\
&\quad - \sum_{i,j=1}^{n} \varsigma_i\left|c_{ij}(t)\right|\left|\Gamma_j\left(t-\tau_{ij}(t)\right)\right|\left|e^{\varepsilon\left(t+\tau-\tau_{ij}(t)\right)}\right| \\
&\leq -\sum_{i=1}^{n} \varsigma_i e^{\varepsilon t}\left(k_1 + a_i(t) - \varepsilon\right)\left|e_i(t)\right| + \sum_{i=1}^{n} \varsigma_i e^{\varepsilon t} b_{ii}(t)\left|\Gamma_j(t)\right| \\
&\quad + \sum_{i=1}^{n}\sum_{j\neq i}^{n} \varsigma_i e^{\varepsilon t}\left|b_{ij}(t)\right|\left|\Gamma_j(t)\right|
\end{aligned}$$

$$+\sum_{i,j=1}^{n}\varsigma_i e^{\varepsilon(t+\tau)}\frac{\left|c_{ij}\left(\varphi_{ij}^{-1}(t)\right)\right|}{1-\dot{\tau}_{ij}\left(\varphi_{ij}^{-1}(t)\right)}\left|\Gamma_j(t)\right|-k_2|e_i(t)|^{\sigma}$$
$$=-\sum_{i=1}^{n}\varsigma_i e^{\varepsilon t}(k_1+a_i(t)-\varepsilon)|e_i(t)|$$
$$+e^{\varepsilon t}\sum_{i=1}^{n}\left\{\varsigma_i b_{ii}(t)+\sum_{j=1,j\neq i}^{n}\varsigma_i\left|b_{ij}(t)\right|\right.$$
$$\left.+\sum_{j=1}^{n}\varsigma_j e^{\varepsilon t}\frac{\left|c_{ij}\left(\varphi_{ij}^{-1}(t)\right)\right|}{1-\dot{\tau}_{ij}\left(\varphi_{ij}^{-1}(t)\right)}\right\}\left|\Gamma_j(t)\right|-k_2|e_i(t)|^{\sigma}$$
$$\leq-\sum_{i=1}^{n}\varsigma_i e^{\varepsilon t}\left(k_1+a_i^{\min}-\varepsilon\right)|e_i(t)|+e^{\varepsilon t}\sum_{i=1}^{n}Q_i(t)|\Gamma_i(t)|$$
$$-k_2|e_i(t)|^{\sigma}. \tag{18}$$

By the assumption of the theorem, $\varepsilon$ can be a very small positive constant, and we can see that there exist positive constants $\theta_i$ and $t_0\geq 0$ such that if $t\geq t_0$,

$$Q_i(t)\leq\theta_i\leq 0. \tag{19}$$

Then, let $\theta_0=\min\{-\theta_1,-\theta_2,\ldots,-\theta_n\}$, and we deduce that

$$\dot{V}(t)\leq-\min_{1\leq i\leq n}\left\{\varsigma_i\left(k_i+a_i^{\min}-\varepsilon\right)\right\}e^{\varepsilon t}\sum_{i=1}^{n}|e_i(t)|$$
$$-\theta_0 e^{\varepsilon t}\sum_{i=1}^{n}|\Gamma_i(t)|-k_2|e_i(t)|^{\sigma}\leq 0, \tag{20}$$

which implies that

$$\sum_{i=1}^{n}|e_i(t)|\leq\frac{V(e_0,0)}{\min\{\zeta_1,\zeta_2,\ldots,\zeta_n\}}e^{-\varepsilon t}. \tag{21}$$

By Definition 2, the synchronization error $e(t)$ converges to zero. That is to say, the nonautonomous discontinuous and delayed neural networks (2) and (4) can achieve the global exponential synchronization under the continuous switching controller (13). The proof is completed.

*Remark 1.* Unlike the previous studies, a great difference in our model is that we permit the neuron activation to be discontinuous and unbounded. One can see that the nonlinear function $f$ in this paper may not satisfy the Lipschitz condition any more. There are few results on the synchronization problem if the activations are discontinuous and the controllers are continuous. Our studies extend the previous researches.

*3.2. Exponential Synchronization with the Discontinuous Controller.* In this part, we describe the following corresponding $N$-coupled time-delayed neural networks of (5):

$$\frac{dz_i(t)}{dt}=-Az_i(t)+Bf(z_i(t))+Cf(z_i(t-\tau))+I(t)$$
$$+m\sum_{j=1}^{N}d_{ij}\Phi\varphi(z_j-z_i), \tag{22}$$

where $z_i(t)=(z_{i1}(t),z_{i2}(t),\ldots,z_{in}(t))^T\in\mathbb{R}^n$ $(i=1,2,\ldots,N)$ denotes the state variable of the $i$th neuron at time $t$, $m$ is the coupling strength, $\Phi=\text{diag}(\phi_1,\phi_2,\ldots,\phi_n)$ with $\phi_l>0$, $l=1,2,\ldots,n$, $\varphi(s)$ is the coupling function, $D=[d_{ij}]$ denotes the adjacency matrix of subsystems, where the corresponding Laplacian matrix is represented as $L$, and all of them are applicable to undirected weighted networks.

Moreover, in order to realize exponential synchronization, a suitable coupling function is important to improve the network performance. Our goal is to derive the coupled time-delayed neural networks with discontinuous controllers synchronizing with the isolated neural network (5). To this end, in this paper, we consider the following coupled neural networks:

$$\frac{dz_i(t)}{dt}=-Az_i(t)+Bf(z_i(t))+Cf(z_i(t-\tau))+I(t)$$
$$+m\sum_{j=1}^{N}d_{ij}\Phi\varphi(z_i-z_i)+v_i(t), \tag{23}$$

where $D=(d_{ij})_{N\times N}\in\mathbb{R}^{N\times N}$ with $d_{ij}>0$ $(i\neq j)$ and $d_{ij}=0$ $(i, j=1,2,\ldots,N)$ and $v_i(t)$ is the control algorithm vector similar to (13) when $\sigma=0$ for the strongly connected network topology which is given as follows:

$$v_i(t)=-k_1(z_i(t)-x(t))-k_2\ \text{sign}\ (z_i(t)-x(t)), \tag{24}$$

where $k_1$ and $k_2$ are the gain coefficients to be determined. We can see that the controller $v_i(t)$ is discontinuous when $\sigma=0$.

Then, we choose the discontinuous controller with $\sigma=0$, and we define the linear coupling function $\varphi:\mathbb{R}^n\to\mathbb{R}^n$ as

$$\varphi(s)=s. \tag{25}$$

Then, the coupled time-delayed complex network can be described as follows:

$$\frac{dz_i(t)}{dt}=-Az_i(t)+Bf(z_i(t))+Cf(z_i(t-\tau))+I(t)$$
$$+m\sum_{j=1,j\neq i}^{N}d_{ij}\Phi z_j(t)+v_i(t), \tag{26}$$

where $\Phi=\text{diag}(\phi_1,\phi_2,\ldots,\phi_n)$ with $\phi_l>0$, $l=1,2,\ldots,n$.

Similarly, let $w_i(t)=z_i(t)-x(t)$, and we choose the novel discontinuous switching controller (24) and the linear function (25). Also, by differential inclusions and set valued maps, when $i = 1, 2, \ldots, N$, we can obtain the error dynamical system as follows:

$$\begin{aligned}\frac{dw_i(t)}{dt} &= -Aw_i(t) + B\tilde{\gamma}_i(t) + C\tilde{\gamma}_i(t-\tau)) \\ &\quad + m\sum_{j=1,j\neq i}^{N} d_{ij}\Phi w_j(t) - k_i w_i(t) - k_2 \mathrm{SIGN}(w_i(t)),\end{aligned} \tag{27}$$

where $\mathrm{SIGN}(w_i(t)) = (\mathrm{SIGN}(w_i(t)), \mathrm{SIGN}(w_{i2}(t)), \ldots, \mathrm{SIGN}(w_{\mathrm{in}}(t)))^T$ with $\mathrm{SIGN}(s) = -1$ if $s < 0$, $\mathrm{SIGN}(s) = [-1, 1]$ if $s = 0$, and $\mathrm{SIGN}(s) = 1$ if $s > 0$ and $\tilde{\gamma}_i(t) = (\tilde{\gamma}_{i1}(t), \tilde{\gamma}_{i2}(t), \ldots, \tilde{\gamma}_{\mathrm{in}}(t))^T = (\xi_{i1}(t) - \gamma_{i1}(t), \xi_{i2}(t) - \gamma_{i2}(t), \ldots, \xi_{\mathrm{in}}(t) - \gamma_{\mathrm{in}}(t))^T$.

**Theorem 2.** *If Assumptions 1 and 2 hold, we give the further condition:*

*Assumption 3.* $\min/(1 \le k \le n)\{k_1 + a_k - \sum_{l=1}^{n} a|b_{kl}| - \sum_{l=1}^{n} a|c_{kl}|\} > 0$ and $\min/(1 \le k \le n)\{k_2 - \sum_{l=1}^{n}\beta|b_{kl}| - \sum_{l=1}^{n}\beta|c_{kl}|\} > 0$.

Then, by choosing the coupling function (12), the coupled networks (26), and the isolated model (5), the exponential synchronization under the discontinuous controller (24) with $\sigma = 0$ can be realized.

*Proof 2.* Define a candidate Lyapunov function as follows:

$$\begin{aligned}V(t) = V(w(t)) &= e^{\varepsilon t}\sum_{i=1}^{N}\|w_i(t)\|_1 \\ &\quad + \sum_{i=1}^{N}\sum_{k,l=1}^{n}\int_{t-\tau}^{t} e^{\varepsilon(s+\tau)}|c_{kl}||\tilde{\gamma}_{il}(s)|ds,\end{aligned} \tag{28}$$

where $\|w_i(t)\|_1 = \sum_{k=1}^{n}|w_{ik}(t)|$. Similar to Proof 1, we denote

$$\begin{aligned}\frac{dV(t)}{dt} &= V(e(t)) = \varepsilon e^{\varepsilon t}\sum_{i=1}^{N}\sum_{k=1}^{n}|w_{ik}(t)| \\ &\quad + e^{\varepsilon t}\sum_{i=1}^{N}\sum_{k=1}^{n}\frac{dw_{ik}(t)}{dt}\cdot\vartheta_{ik}(t) \\ &\quad + \sum_{i=1}^{N}\sum_{k,l=1}^{n} e^{\varepsilon(t+\tau)}|c_{kl}||\tilde{\gamma}_{il}(t)| - \sum_{i=l}^{N}\sum_{k,l=1}^{n} e^{\varepsilon t}|c_{kl}||\tilde{\gamma}_{il}(t-\tau)| \\ &= \varepsilon e^{\varepsilon t}\sum_{i=1}^{N}\sum_{k=1}^{n}|w_{ik}(t)| + e^{\varepsilon t}\sum_{i=1}^{N}\sum_{k=1}^{n}\mathrm{sign}(w_{ik}(t))\cdot \\ &\quad \cdot\left\{-a_k w_{ik}(t) + \sum_{l=1}^{n} b_{kl}\tilde{\gamma}_{il}(t) + \sum_{l=1}^{n} c_{kl}\tilde{\gamma}_{il}(t-\tau)\right. \\ &\quad \left. + m\sum_{j=1,j\neq i}^{N} d_{ij}\phi_k w_{jk}(t) - k_1 w_{ik}(t) - k_2 \text{ sign }(w_{ik}(t))\right\} \\ &\quad + \sum_{i=1}^{N}\sum_{k,l=1}^{n} e^{\varepsilon(t+\tau)}|c_{kl}||\tilde{\gamma}_{il}(t)| - \sum_{i=l}^{N}\sum_{k,l=1}^{n} e^{\varepsilon t}|c_{kl}||\tilde{\gamma}_{il}(t-\tau)| \\ &\le \varepsilon e^{\varepsilon t}\sum_{i=l}^{N}\sum_{k=1}^{n}|w_{ik}(t)| + e^{\varepsilon t}\sum_{i=1}^{N}\sum_{k=1}^{n}(-a_k|w_{ik}(t)| \\ &\quad + \sum_{l=1}^{n}|b_{kl}||\tilde{\gamma}_{il}(t)|| \text{ sign }(w_{ik}(t))| + \sum_{l=1}^{n} e^{\varepsilon\tau}|c_{kl}||\tilde{\gamma}_{il}(t)|) \\ &\quad + m\sum_{j=1,j\neq i}^{N} d_{ij}\phi_k|w_{jk}(t)| - k_1|w_{ik}(t)| \\ &\quad - k_2|\text{sign }(w_{ik}(t))| \le -e^{\varepsilon t}\sum_{i=l}^{N}\sum_{k=1}^{n} \\ &\quad \cdot\left(k_1 + a_k - \varepsilon \quad - \sum_{l=1}^{n} a|b_{kl}| - \sum_{l=1}^{n} e^{\varepsilon t} a|c_{kl}|\right)|w_{ik}(t)| \\ &\quad + m\sum_{j=1,j\neq i}^{N} d_{ij}\phi_k|w_{jk}(t)| - e^{\varepsilon t}\sum_{i=l}^{N}\sum_{k=1}^{n} \\ &\quad \cdot\left(k_2 - \sum_{l=1}^{n}\beta|b_{kl}| - \sum_{l=1}^{n}\beta e^{\varepsilon t}|c_{kl}|\right)|\text{sign }(w_{ik}(t))|.\end{aligned} \tag{29}$$

By Lemma 2 and the property of adjacency matrix $D$, we deduce that

$$\begin{aligned}m\sum_{i=l}^{N}\sum_{j=1}^{N} d_{ij}\Phi w_j(t) &\le m\sum_{k=1}^{n}\phi k\left[\sum_{i=1}^{N}\sum_{j=1}^{N} d_{ij}|w_{jk}(t)|\right] \\ &= -m\sum_{k=1}^{n}\phi_k\sum_{i=l}^{N}\sum_{j=1,j\neq i}^{N} d_{ij}\left|w_{jk}^T - w_{jk}^T\right| \le 0.\end{aligned} \tag{30}$$

Then, from (30), we deduce that

$$\frac{dV(t)}{dt} \le -e^{\varepsilon t}\sum_{i=1}^{N}\chi_1|w_{ik}(t)| - e^{\varepsilon t}\sum_{i=l}^{N}\chi_2|\text{sign }(w_{ik}(t))|, \tag{31}$$

where $\chi_1 = \min_{1<k<n}\{k_1 + a_k - \varepsilon - \sum_{l=1}^{n} a|b_{kl}| - \sum_{l=1}^{n} e^{\varepsilon\tau} a|c_{kl}|\}$ and $\chi_2 = \min_{1<k<n}\{k_2 - \sum_{l=1}^{n}\beta|b_{kl}| - \sum_{l=1}^{n}\beta e^{\varepsilon\tau}|c_{kl}|\}$. By the assumption of the theorem, there must exist a small enough positive $l = 1$ constant $\varepsilon$, such that $\chi_1 > 0$ and $\chi_2 > 0$, which implies

$$\frac{dV(t)}{dt} \le 0, \quad \text{for a.e. } t \ge 0, \tag{32}$$

which yields $V(w(t)) \le V(w(0))$, meaning that $V(w(t))$ is bounded; then, we have

$$\sum_{i=1}^{N}\|w_i(t)\|_1 \le V(w_0, 0)e^{-\varepsilon t}. \tag{33}$$

By Definition 2, the synchronization error $w(t)$ converges to zero. That is to say, the coupled discontinuous and delayed neural networks (26) can be globally exponentially synchronized with the isolated model (5) under the discontinuous switching controller (24). The proof is completed.

*Remark 2.* In Proof 2, we choose the linear coupling function $\varphi(s) = s$, without the loss of generality, even if the coupling function becomes more complex such as nonlinear function or coupling delay function; many synchronization criteria for delay dependence were derived under these circumstances [20, 27, 28]. In the existing literatures, when the neuron functions were discontinuous, the only thing discussed is a single case for either $\sigma = 0$ or $0 < \sigma < 1$, respectively. When both neuron functions and controllers are discontinuous, there is still no complete conclusion of the issue of synchronization. In this paper, we discuss the exponential synchronization problem of the time-delayed neural network with discontinuous activations under a unified framework of $0 \leq \sigma < 1$.

## 4. Examples and Simulation Experiment

In this section, to show the effectiveness of our proposed method, two numerical examples are introduced to demonstrate its validity.

*Example 1.* We consider the following 2-dimensional nonautonomous complex network system:

$$\begin{cases} \dfrac{dx_1(t)}{dt} = -x_i(t) - (3 + \cos t) f(x_1(t)) \\ \qquad + \left(\dfrac{1}{4} + \dfrac{1}{4} \cos t\right) f(x_2(t)) \\ \qquad + \left(\dfrac{1}{3} + \dfrac{1}{6} \sin t\right) f(x_1(t - \tau_{11}(t))) \\ \qquad + \left(\dfrac{1}{2} + \dfrac{1}{2} \sin t\right) f(x_2(t - \tau_{12}(t))) + 4, \\ \dfrac{dx_2 t}{dt} = -x_2(t) + \cos t\, f(x_1(t)) - (3 + \sin) f(x_2(t)) \\ \qquad + \dfrac{1}{2} \sin t\, f(x_1(t - \tau_{21}(t))) + 3 + \cos t. \end{cases} \tag{34}$$

Therefore, we can see that $a_1^L = a_2^L = 1$, $b_{11}^M = b_{22}^M = -2$, $c_{11}^M = c_{21}^M = 1/2$, $b_{21}^M = c_{12}^M = 1$, $b_{12}^M = 1/2$, and $c_{22}^M = 0$. The discontinuous activation function can be described as $f(s) = s + \text{sign}\,(s)$. Let $\tau_{ij}(t) = 1\ (i, j = 1, 2)$. We choose the switching continuous controller $u_i(t) = -e_i(t) - \text{sign}\,(e_i(t))\,|e_i(t)|^{1/2}$. Then, Figure 1 shows the time evolution of variables $x_1(t)$ and $x_2(t)$ for the driver neural networks (34); moreover, we can see that the exponential synchronization between the driver system (34) and the corresponding response system can be achieved in Figure 1, which is suitable for our results.

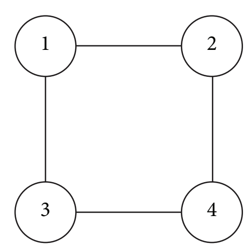
*Example 2.* We consider three-dimensional autonomous coupled complex dynamical networks as follows:

$$\begin{cases} \dfrac{dx_1(t)}{dt} = -x_1(t) - \dfrac{1}{2} f(x_1(t)) + f(x_2(t)) - \dfrac{1}{10} f(x_1(t-1)) \\ \qquad + \dfrac{1}{4} f(x_3(t-1)), \\ \dfrac{dx_2(t)}{dt} = -x_2(t) + \dfrac{1}{3} f(x_2(t)) - \dfrac{1}{5} f(x_3(t)) + \dfrac{1}{4} f(x_2(t-1)), \\ \dfrac{dx_3(t)}{dt} = -x_3(t) + \dfrac{1}{5} f(x_1(t)) - \dfrac{1}{8} f(x_2(t)) + \dfrac{1}{2} f(x_3(t)) \\ \qquad + \dfrac{1}{6} f(x_2(t-1)) + \dfrac{1}{4} f(x_3(t-1)). \end{cases} \tag{35}$$

The discontinuous activation functions are taken as

$$f(s) \begin{cases} 0.1s - 0.5, & s \geq 0, \\ 0.1s + 0.5, & s < 0. \end{cases} \tag{36}$$

Then, let $\alpha = 0.1$ and $\beta = 0.5$, and it is obvious that the conditions (Assumptions 1 and 2) are satisfied. Let the coupling strength be $m = 1$; we choose random switching rules for the coupled networks, and their topologies are illustrated as follows:




(37)

where the adjacency matrix $D$ is easily denoted as

$$D = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \tag{38}$$

Then, we consider the discontinuous controller $v_i = -e_i(t) - 2\,\text{sign}\,(e_i(t))$ with $2k_1 = k_2 = 2$; by substituting the above parameters, we can see that the condition (Assumption 3) holds. We can see that the exponential synchronization between the driver system (35) and the corresponding response system can be depicted in Figure 2, which is suitable for our results.

## 5. Conclusions

In this paper, we investigate the exponential synchronization of a class of complex dynamical networks based on the framework of nonsmooth analysis and novel technique analysis. By adding a continuous switching controller, we

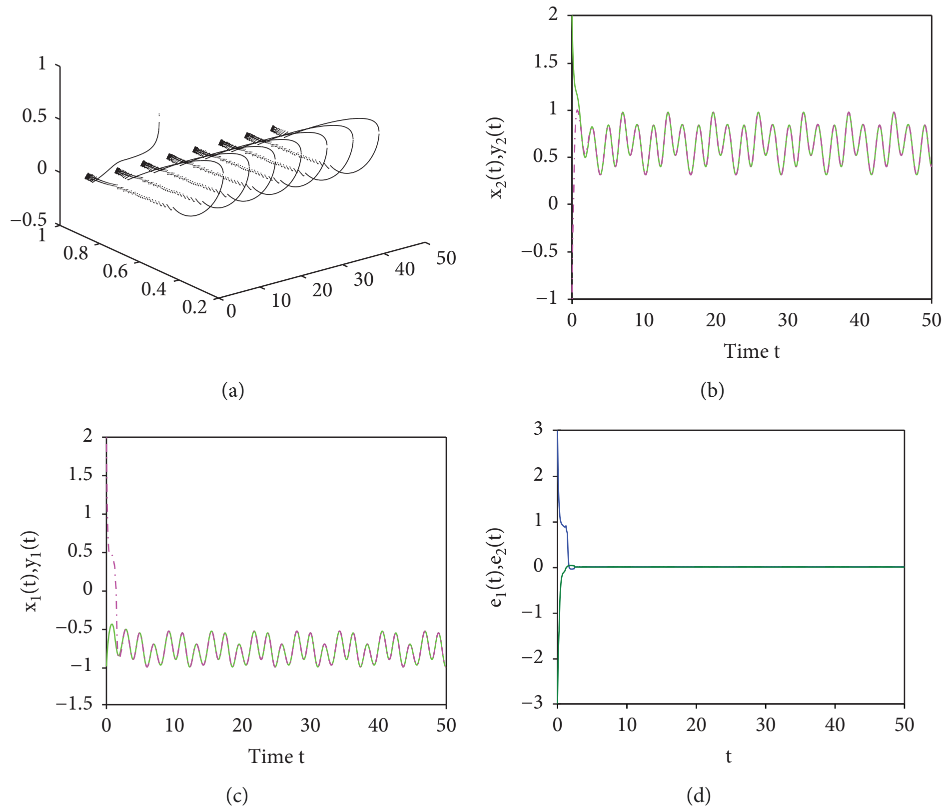



Figure 1: (a) The three-dimensional trajectory of state variables $x_1$ and $x_2$. (b–c) The time evolution for the driver network system and corresponding response system (34). (d) The time response of the synchronization error between the driver system (34) and corresponding response system with the continuous controller.

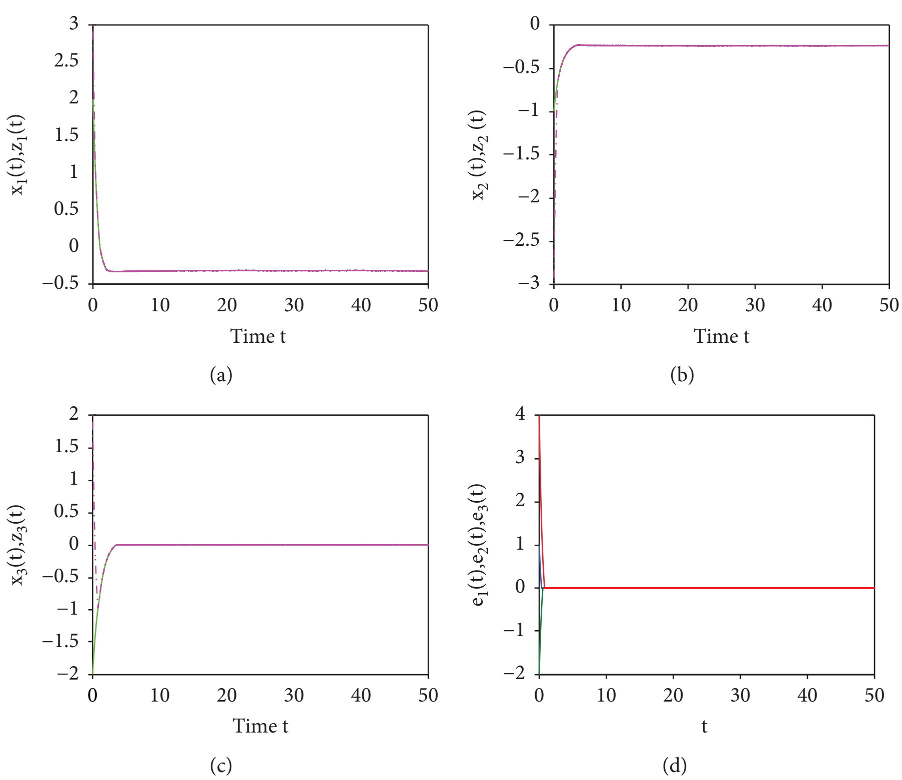



Figure 2: (a–c) The time evolution for the driver network system (35) and corresponding response system. (d) The time response of the synchronization error between the driver system (35) and corresponding response system with the discontinuous controller.

realize the global exponential synchronization of the nonautonomous discontinuous and delayed neural networks. Then, we choose a linear coupling function, and the autonomous complex dynamical network can be globally exponentially synchronized with the isolated model under the discontinuous switching controller, by constructing a C-regular Lyapunov-like function which is time-dependent. However, it is not easy to go beyond the conventional Lyapunov function for achieving the exponential synchronization goal. This paper overcomes the limitation of traditional controllers and proposes some novel discontinuous controllers. Moreover, the results have been verified by the numerical examples and computer simulations. In short, our results are provided with an important application significance in the design of synchronized complex dynamical networks.

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

We declare that there is no conflict of interest regarding the publication of this paper. And data sharing allows researchers to verify the results of the article, replicate the analysis, and conduct secondary analyses.

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