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Research Article

Consistent Approximations of the Zeno Behaviour in Affine-Type Switched Dynamic Systems

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This paper proposes a new theoretic approach to a specific interaction of continuous and discrete dynamics in switched control systems known as a Zeno behaviour. We study executions of switched control systems with affine structure that admit infinitely many discrete transitions on a finite time interval. Although the real world processes do not present the corresponding behaviour, mathematical models of many engineering systems may be Zeno due to the used formal abstraction. We propose two useful approximative approaches to the Zeno dynamics, namely, an analytic technique and a variational description of this phenomenon. A generic trajectory associated with the Zeno dynamics can finally be characterized as a result of a specific projection or/and an optimization procedure applied to the original dynamic model. The obtained analytic and variational techniques provide an effective methodology for constructive approximations of the general Zeno-type behaviour. We also discuss shortly some possible applications of the proposed approximation schemes.

1. Introduction

The study of switched and general hybrid dynamic systems has gained lots of interest in recent years (see, e.g., [1–11]). These mathematical abstractions are practically motivated by many significant engineering applications, in which digital devices interact with an analog environment. On the other side, the analytic tools and results related to the switched control constitute a valuable part of the modern systems theory. The control design technique based on the switched systems methodology is nowadays a mature adequate approach to the synthesis of controllers for several nonlinear interconnected dynamic systems. We refer to [2, 4, 5, 7, 10, 12–17] for the basic facts and some interesting real world applications of switched and hybrid dynamic models.

However, in the face of a recent progress in switched systems theory, there are numbers of fundamental properties of these systems that have not been investigated in sufficient detail. Among others, these include a formal description of the so-called Zeno executions (see [2, 4, 10, 18–22]). A behaviour of a system is called a Zeno dynamics, if it takes infinitely many discrete transitions on a finite time interval. The real world engineering and physical systems are, of course, non-Zeno, but a complex switched-type mathematical model of a physical process may be Zeno, due to the usual modelling overabstractions.

Since the above-mentioned abstraction provides an inevitable theoretic basis for a concrete control design procedure, understanding when it leads to a Zeno effect in a switched system is essential, for an adequate numerical implementation of the designed control algorithms. Let us also note that a Zeno-like switched dynamic behaviour can cause imprecise computer simulations and imply the corresponding calculating and modelling errors. It is explainable that the main computer tools and numerical packages developed for switched systems (see, e.g., [19, 22]) get stuck when a large number of discrete transitions take place within a short time interval. A possible improvement of the mentioned simulation technique can be realized on a basis of some adequate extensions of the available theory of the Zeno effect including some approximative approaches. Recall that the Zeno hybrid automata have been usually examined from the point of view of theoretical computer science [2, 23] such that the necessary investigations of the continuous part of a system under consideration and the discrete-continuous interplay are quite underrated. The general Zeno hybrid executions under some specific assumptions have been deeply investigated in [18, 20–22]. On the other hand, an analytic approach based on the modern variational analysis and consistent approximation techniques have not been sufficiently advanced to the Zeno switched systems setting and to the corresponding control design procedures.

Usually, one avoids the examination of a Zeno effect in a work on switched control design by some specific additional assumptions (see, e.g., [5, 12, 13, 17, 24, 25]). The aim of our contribution is to give a constructive characterization of the Zeno executions in a specific case of affine switched systems. We propose an analytic technique that eliminates the original Zeno effect from the consideration. These approximations are consistent in the sense of the resulting trajectories. Theoretic results obtained in our paper can be useful, for instance, when designing switched controllers and simulating the correspondingly sophisticated dynamic behaviour.

The remainder of the paper is organized as follows: Section 2 contains some necessary formal concepts and facts related to the class of systems under consideration. Section 3 includes the main analytic results and the corresponding formal proofs. We propose here a novel approximative approach to the Zeno dynamics. In Section 4 we discuss a specific case of a Zeno behaviour determined by a switched manifold (in the state space) and extend our approximating scheme to this common situation. In this connection we also mention shortly the sliding mode control processes. Section 5 is devoted to the novel variational description of the Zeno effect in affine switched systems. We establish a similarity between the approximative Zeno dynamics and the conventional optimal control methodology. Section 6 summarizes our paper.

2. Zeno Executions in Affine Switched Systems

This paper concerns affine switched control systems (affine with respect to the control inputs) described by the following concept (see also [13, 24, 25]).

Definition 1. An affine switched system (ASS) is 7-tuple

$$\{\mathcal{Q},\mathcal{X},U,\mathcal{A},\mathcal{B},\mathcal{U},\Psi\}, \tag{1}$$

where

(i) $\mathcal{Q}$ is a finite set of indices;

(ii) $\mathcal{X}=\{\mathcal{X}_q\}$, $q\in\mathcal{Q}$, is a family of state spaces such that $\mathcal{X}_q\subseteq\mathbb{R}^n$;

(iii) $U\subseteq\mathbb{R}^m$ is a set of admissible control input values (called control set);

(iv) $\mathcal{A}=\{a_q(\cdot)\}$, $\mathcal{B}=\{b_q(\cdot)\}$, $q\in\mathcal{Q}$ are families of $\mathbb{C}^\infty$-functions

$$\begin{aligned} a_q &: \mathcal{R}\longrightarrow\mathbb{R}^n,\\ b_q &: \mathcal{R}\longrightarrow\mathbb{R}^{n\times m} \end{aligned} \tag{2}$$

(real analytic functions) determined on an open set $\mathcal{R}\subseteq\mathbb{R}^n$;

(v) $\mathcal{U}$ is the set of admissible control functions;

(vi) $\Psi\subset\Xi:=\{(x,x')\mid x\in\mathcal{X}_q, x'\in\mathcal{X}_{q'},q,q'\in\mathcal{Q}\}$ is a transitions' set.

We say that a location switching from q to q' occurs at a switching time from the given (finite) time interval $[0,t_f]$. Consider a specific ASS from Definition 1 with $r\in\mathbb{N}\cup\{\infty\}$ switching times

$$0=t_0<t_1<\cdots<t_{r-1}<t_r\le t_f. \tag{3}$$

Denote $\tau^r:=\{t_i\}_{i=0}^r$ for a given $r\in\mathbb{N}\cup\{\infty\}$. In general, the above finite or infinite sequence of switching times $\{t_i\}$ is not an a priory given sequence. A switched control system remains in location $q_i\in\mathcal{Q}$ for all $t\in[t_{i-1},t_i)$, $i=1,\dots,r$. We additionally assume that the class of admissible controls is the set of bounded measurable functions

$$\mathcal{U}:=\left\{v(\cdot)\in\mathbb{L}_m^\infty\left(0,t_f\right)\mid v(t)\in U\right\}, \tag{4}$$

where U is assumed to be compact. Here $\mathbb{L}_m^\infty(0,t_f)$ denotes the Lebesgue space of all m-valued bounded measurable functions. A "continuous trajectory" of an ASS under consideration is determined as follows.

Definition 2. An admissible trajectory associated with an ASS is an absolutely continuous function $x(\cdot)$ such that

(i) $x_i(\cdot)=x(\cdot)|_{(t_{i-1},t_i)}$ is an absolutely continuous function on (t_{i-1},t_i) with $x_i(t)\in\mathcal{X}_i$ continuously prolongable to $[t_{i-1},t_i]$, $t_i\in\tau_r$, $i=1,\dots,r$;

(ii) $\dot{x}_i(t)=a_{q_i}(x_i(t))+b_{q_i}(x_i(t))u_i(t)$ for almost all times $t\in[t_{i-1},t_i]$, where $u_i(\cdot)$ is a restriction of an admissible control function $u(\cdot)\in\mathcal{U}$ on the time interval $[t_{i-1},t_i]$ and $q_i\in\mathcal{Q}$.

Note that under the above assumptions (smoothness of the families $\mathcal{A}$, $\mathcal{B}$ and boundedness of the admissible controls) the trajectories $x(\cdot)$ from Definition 2 have bounded derivatives almost everywhere. We refer to [3, 4, 6, 10, 17] for some alternative concepts of switched, hybrid, and general interconnected dynamic systems. Evidently, every τ_r from Definition 2 determines a finite or infinite partitioning of $[0,t_f]$. The adjoint time intervals $[t_{i-1},t_i)$ and the corresponding locations q_i from $\mathcal{Q}$ are assumed to be a priori unknown. Let $x(\cdot)$ be a trajectory of an ASS associated with an admissible τ_r for a given (finite) number $r\in\mathbb{N}$. This trajectory is an absolutely continuous solution of the following initial value problem:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r}\beta_{[t_{i-1},t_i)}(t)\left(a_{q_i}(x(t))+b_{q_i}(x(t))u(t)\right),\\ x(0) &= x_0, \end{aligned} \tag{5}$$

where $\beta_{[t_{i-1},t_i)}(\cdot)$ is a characteristic function of the interval $[t_{i-1},t_i)$ for $i=1,\dots,r$ and $u(\cdot)\in\mathcal{U}$. Note that, in contrast to the general hybrid systems, the switching times from a sequence τ^r do not depend on the state vector. The complete (switched-type) control input $\rho^r(\cdot)$ associated with the ASS under consideration can now be expressed as follows:

$$\rho^r(\cdot) := \{\beta(\cdot), u(\cdot)\},$$
$$\beta(\cdot) := \left\{\beta_{[t_0,t_1)}(\cdot), \ldots, \beta_{[t_{r-1},t_r)}(\cdot)\right\}, \tag{6}$$

where $r \in \mathbb{N}$, $u(\cdot) \in \mathcal{U}$ and the set Γ of all admissible sequences $\beta(\cdot)$ is characterized by the generic conditions

$$\beta(t) \subset \{0,1\}^r \quad \forall t \in [0, t_f],$$
$$\sum_{i=1}^{r} \beta_{[t_{i-1},t_i)}(t) = 1. \tag{7}$$

Let us note that $\beta(\cdot) \subset \mathbb{L}^1(0,t_f)$, where $\mathbb{L}^1(0,t_f)$ is the standard Lebesgue space of all scalar absolutely integrable functions.

Definition 3. For an admissible control input $\rho^r(\cdot)$, where

$$\beta(\cdot) \subset \mathbb{L}^1(0,t_f),$$
$$u(\cdot) \in \mathcal{U}, \tag{8}$$

an execution of the ASS is defined as a collection $(\{q_i\}, \tau^r, x^r(\cdot))$, where $q_i \in \mathcal{Q}$ and $x^r(\cdot)$ is a solution to (5). Here τ^r is the corresponding sequence of switching times.

Let us assume that every admissible control $\rho^r(\cdot)$ generates a unique execution $(\{q_i\}, \tau^r, x^r(\cdot))$ of the given ASS. Following [18, 20–22] we now introduce a formal concept of a Zeno execution.

Definition 4. An execution $(\{q_i\}^Z, \tau^Z, x^Z(\cdot))$ of an ASS is called Zeno execution if the corresponding switching sequence $\tau^Z := \{t_i^Z\}_{i=0}^{\infty}$ is an infinite sequence such that

$$\lim_{r\to\infty} t_r^Z = \lim_{r\to\infty} \sum_{i=0}^{r} \left(t_i^Z - t_{i-1}^Z\right) = t_f < \infty. \tag{9}$$

An admissible input $\rho^Z(\cdot) := \{\beta^Z(\cdot), u^Z(\cdot)\}$, where

$$\beta^Z(\cdot) := \left\{\beta^Z_{[t_0,t_1)}(\cdot), \ldots, \beta^Z_{[t_{r-1},t_r)}(\cdot), \ldots\right\} \subset \mathbb{L}^1(0,t_f),$$
$$u^Z(\cdot) \in \mathcal{U}^Z \subseteq \mathcal{U}, \tag{10}$$

which generates the Zeno execution $(\{q_i\}^Z, \tau^Z, x^Z(\cdot))$ is called a Zeno input.

Due to the specific character of the characteristic functions $\beta^Z_{[t_{i-1},t_i)}(\cdot)$ (elements of the sequence $\beta^Z(\cdot)$) and taking into account Definition 2, we can express $x^Z(\cdot)$ as a solution to the initial value problem (5) with $r = \infty$. By $\mathcal{U}^Z \subseteq \mathcal{U}$ we denote here a set of the "conventional" control inputs $u^Z(\cdot)$ that imply a Zeno execution in the given ASS.

The existence of the Zeno dynamics for various classes of switched and hybrid systems is deeply discussed in [22]. Note that the Zeno behaviour is a typical effect that occurs in a switched and hybrid dynamic system. It appears in mathematical models of many real engineering systems, such as water tank model and mechanics of a bouncing ball, in aircraft conflict resolution model and in the dynamic models for safety control. We refer to [2, 10, 18–22] for some further switched-type mathematical models that admit Zeno executions. Let us also mention the celebrated Fuller's problem that constitutes a generic example of the Zeno behaviour (see [26]).

3. Consistent Approximations of the Zeno Dynamics

Let us introduce some necessary mathematical concepts and facts. We next use the notation $\mathbb{C}_n(0,t_f)$ for the Banach space of all n-valued continuous functions on $[0,t_f]$ equipped with the usual maximum-norm $\|\cdot\|_{\mathbb{C}_n(0,t_f)}$.

Definition 5. Consider an ASS given by system (5) and assume that all the conditions of Section 2 are satisfied. One says that this ASS possesses a strong approximability property (with respect to the admissible control inputs) if there exists a $\mathbb{L}^1(0,t_f) \times \mathbb{L}^\infty_m(0,t_f)$-weakly convergent sequence

$$\rho^r(\cdot) := \chi^r(\cdot) \times \{v^r(\cdot)\}, \quad r = 1, \ldots, \infty \tag{11}$$

of functions $\chi^r(\cdot) \subset \mathbb{L}^1(0,t_f)$, $\{v^r(\cdot)\} \subset \mathbb{L}^\infty_m(0,t_f)$ such that the initial value problem (5) with the control input $\rho^r(\cdot)$ has a unique solution $x^r(\cdot)$ for every $r \in \mathbb{N}$ and $\{x^r(\cdot)\}$ is a $\mathbb{C}_n(0,t_f)$-convergent (uniformly convergent) sequence.

We now formulate the main analytic result that guarantees the strong approximability property for the ASS given affine systems (5) (see [13, 24, 25] for the additional mathematical details).

Theorem 6. *Let all the assumptions from Section 2 be satisfied and let*

$$\left(\{q_i\}^Z, \tau^Z, x^Z(\cdot)\right) \tag{12}$$

be a Zeno execution generated by a Zeno input $\rho^Z(\cdot)$. Consider the initial value problem (5) associated with a $\mathbb{L}^1(0,t_f) \times \mathbb{L}^\infty_m(0,t_f)$-weakly convergent sequence $\{\rho^r(\cdot)\}$ of control inputs $\rho^r(\cdot) = \chi^r(\cdot) \times \{v^r(\cdot)\}$, where

$$\chi^r(\cdot) \subset \mathbb{L}^1(0,t_f),$$
$$\{v^r(\cdot)\} \subset \mathbb{L}^\infty_m(0,t_f) \tag{13}$$

and $\rho^Z(\cdot)$ is a $\mathbb{L}^1(0,t_f) \times \mathbb{L}^\infty_m(0,t_f)$-weak limit of $\{\rho^r(\cdot)\}$. Then, for all $r \in \mathbb{N}$ the corresponding initial value problem (5) has a unique absolutely continuous solution $x^r(\cdot)$ determined on $[0,t_f]$ and

$$\lim_{r\to\infty} \left\|x^r(\cdot) - x^Z(\cdot)\right\|_{\mathbb{C}_n(0,t_f)} = 0. \tag{14}$$

Proof. $\mathbb{L}^{\infty}_m(0,t_f)$-weak convergence implies the $\mathbb{L}^{\infty}_m(0,t_f)$-weak* convergence (see, e.g., [27]). Therefore, $\{\rho^r(\cdot)\}$ also converges $\mathbb{L}^1(0,t_f)\times\mathbb{L}^{\infty}_m(0,t_f)$-weakly* to the Zeno input $\rho^Z(\cdot)$. From the equivalent definition of the $\mathbb{L}^{\infty}_m(0,t_f)$-weak* convergence (see [27]) we next deduce the following:

$$\lim_{r\to\infty}\int_0^{t_f} v^r(t)\,w(t)\,dt=\int_0^{t_f} u^Z(t)\,w(t)\,dt \quad \forall w(\cdot)\in\mathbb{L}^1_m(0,t_f). \tag{15}$$

Here $u^Z(\cdot)\in\mathcal{U}^Z$ is the corresponding "conventional" part of the given Zeno input $\rho^Z(\cdot)$. Using the basic properties of the Lebesgue spaces ($\mathbb{L}^p$-spaces) on the finite-measure sets, we obtain

$$\left(\mathbb{L}^1_m(0,t_f)\right)'=\mathbb{L}^{\infty}_m(0,t_f)\subset\mathbb{L}^1_m(0,t_f). \tag{16}$$

By $(\mathbb{L}^1_m(0,t_f))'$ we denote here the topologically dual space to $\mathbb{L}^1_m(0,t_f)$. Consequently, (15) is also true for all $w(\cdot)$ from $(\mathbb{L}^1_m(0,t_f))'$. This fact implies the $\mathbb{L}^1(0,t_f)\times\mathbb{L}^1_m(0,t_f)$-weak convergence of the given sequence of control inputs

$$\begin{aligned}\rho^r(\cdot)&\subset\mathbb{L}^1(0,t_f)\times\mathbb{L}^{\infty}_m(0,t_f)\\&\subset\mathbb{L}^1(0,t_f)\times\mathbb{L}^1_m(0,t_f)\end{aligned} \tag{17}$$

to the same function

$$\begin{aligned}\rho^Z(\cdot)&\subset\mathbb{L}^1(0,t_f)\times\mathbb{L}^{\infty}_m(0,t_f)\\&\subset\mathbb{L}^1(0,t_f)\times\mathbb{L}^1_m(0,t_f).\end{aligned} \tag{18}$$

The expected result $\lim_{r\to\infty}\|x^r(\cdot)-x^Z(\cdot)\|_{\mathbb{C}_n(0,t_f)}=0$ now follows from the main result of [24, Theorem 3.3, p. 5]. The proof is completed. □

Theorem 6 provides an analytic basis for the constructive approximations of the Zeno executions. This approximation tools can not only be useful for a possible simplified analysis of the Zeno involved ASSs but also be used on the systems modelling phase. Note that the real world engineering processes do not present an exact Zeno behaviour. On the other side the corresponding mathematical models of some engineering interconnected systems may be Zeno due to the associated formal abstraction. This situation evidently implies a specific modelling conflict. The proposed approximations of the Zeno dynamics in ASSs can help to restore the adequateness of the abstract dynamic model under consideration.

Assume that all the conditions of Theorem 6 are fulfilled. Since $\rho^Z(\cdot)$ is bounded and $\rho^r(\cdot)$ is weakly convergent, $\rho^r(\cdot)$ is a uniformly integrable sequence. This is a simple consequence of the celebrated Dunford-Pettis Theorem (see, e.g., [28]). Therefore one can choose an (existing) subsequence $v^r(\cdot)\in\mathbb{L}^{\infty}_m(0,t_f)$ of the approximating sequence $\rho^r(\cdot)$ such that

$$\sup_{v\in\{v^r(\cdot)\}}\int_{\Omega} v(t)\,dt<\epsilon,\quad \epsilon>0 \tag{19}$$

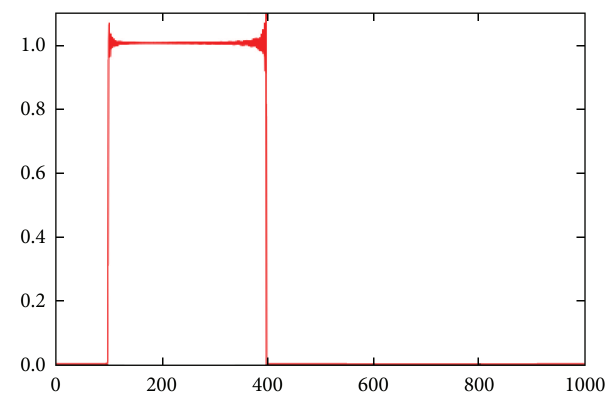

Figure 1: Approximation of an element $\beta^Z_{[t_{i-1},t_i)}(\cdot)$ of a Zeno input $\rho^Z(\cdot)$ by Fourier series.

for every measurable set $\Omega\subset[0,t_f]$ with measure mes$\{\Omega\}<\delta$, $\delta=\delta(\epsilon)>0$. For example, $\{v^r(\cdot)\}$ can be chosen as a sequence of step-functions. A concrete approximation $\chi^r(t)$, $t\in[t_{i-1},t_i)$, $t_i\in\tau^r$, of elements $\beta^Z_{[t_{i-1},t_i)}(\cdot)$ of a Zeno input $\rho^Z(\cdot)$ can be found from a suitable class of numerically tractable approximations. In the context of Theorem 6 one can consider, for example, the Fourier series:

$$\chi^r(t)\big|_{t\in[t_{i-1},t_i)}=\sum_{l=1}^{r}\frac{2}{l\pi}\left(\cos\frac{l\pi t_{i-1}}{t_i}-\cos l\pi\right)\sin\frac{l\pi t}{t_i},\quad t\in[0,t_f]. \tag{20}$$

The Fourier series converges (as $r\to\infty$) in the sense of $\mathbb{L}^2(0,t_f)$-norm to a characteristic function $\beta_{[t_{i-1},t_i)}(\cdot)$ of the chosen time interval $[t_{i-1},t_i)$. This fact is a simple consequence of the Parseval equality (see, e.g., [27, 29]). The given $\mathbb{L}^2(0,t_f)$-convergence implies $\mathbb{L}^1(0,t_f)$-convergence of (20). Let us now assume that $\tau^r\to\tau^Z$ as $r\to\infty$ (the simple point wise convergence). This construction implies $\mathbb{L}^1(0,t_f)$-convergence of the sequence $\chi^r(\cdot)$ to $\beta^Z(\cdot)$ (see Figure 1).

Evidently, the component $\beta^Z(\cdot)$ of a Zeno input $\rho^Z(\cdot)$ takes its values in the set of vertices ver(T_r) of the following r-simplex:

$$T_r:=\left\{\beta\in\mathbb{R}^r\mid\beta_i\geq 0,\sum_{i=1}^{r}\beta_i=1\right\}. \tag{21}$$

$\mathbb{L}^1(0,t_f)$-weakly convergent component $\chi^r(\cdot)$ of an approximating (Zeno input) sequence $\rho^r(\cdot)=\chi^r(\cdot)\times\{v^r(\cdot)\}$ can also be defined as a function that takes its values from the above simplex $\chi^k(t)\in T_r$. In that case $\chi^r(\cdot)$ possesses the additional useful properties summarized in the following theorem.

Theorem 7. *Assume that all the conditions of Theorem 6 are satisfied. Let $\chi^r(\cdot)$ be $\mathbb{L}^1(0,t_f)$-weakly convergent component of the approximating sequence $\rho^r(\cdot)$ and let $\chi^r(t)\subset T_r$ for almost all $t\in[0,t_f]$. Then $\chi^r(\cdot)$ also converges strongly to $\beta^Z(\cdot)$.*

Proof. Since T_r is a convex polyhedron, the tangent cone to this simplex at every vertex is a pointed cone (does not contain a complete line). Therefore, for an element $e \in \text{ver}(T_r)$ there exists a number $c > 0$ and a unit vector $\vartheta \in \mathbb{R}^r$ such that for all $d \in \mathbb{R}^r$ we have

$$\|d - e\|_{\mathbb{R}^r} \leq c \langle \vartheta, d - e \rangle_{\mathbb{R}^r}. \tag{22}$$

By $\|\cdot\|_{\mathbb{R}^r}$ and $\langle\cdot,\cdot\rangle_{\mathbb{R}^r}$ we denote here the norm and the scalar product in $\mathbb{R}^r$. The above estimation and $\mathbb{L}^1(0,t_f)$-weak convergence of $\chi^r(\cdot)$ imply the following:

$$\begin{aligned} &\left\|\chi^r(\cdot) - \beta^Z(\cdot)\right\|_{\mathbb{L}^1(0,t_f)} \\ &\quad \leq c \int_0^{t_f} \left\langle \vartheta(t), \chi^r(t) - \beta^Z(t) \right\rangle_{\mathbb{R}^r} dt \leq \epsilon(r), \end{aligned} \tag{23}$$

where $\lim_{r\to\infty}\epsilon(r) = 0$. This completes the proof. □

Evidently, Theorem 7 brings out the best choice of the element $\chi^r(\cdot)$ of an approximating sequence $\rho^r(\cdot)$ associated with a Zeno execution.

4. The Zeno-Type Dynamics Determined by a Switching Manifold

Let us consider a smooth manifold

$$\mathcal{S} := \{y \in \mathbb{R}^n \mid h(y) = 0\}, \tag{24}$$

with

$$s := \dim(\mathcal{S}) < n. \tag{25}$$

Here $h : \mathbb{R}^n \to \mathbb{R}^s$, $s \in \mathbb{N}$, is a continuously differentiable function. Following [30, 31] we call $\mathcal{S}$ an invariant manifold associated with a dynamic system of the form

$$\begin{aligned} &\dot{y}(t) = \phi(y(t)), \quad t \in \mathbb{R}_+, \\ &y(0) = y_0 \end{aligned} \tag{26}$$

if $y(t) \in \mathcal{S}$ for all $t \geq \hat{t} \in \mathbb{R}_+$. Here $\phi(\cdot)$ is an appropriate smooth function. The next abstract result gives a general invariance criterion.

Theorem 8. *A smooth manifold $\mathcal{S}$ is invariant for system (26) iff $\phi(y)$ belongs to the tangent space* $\text{Tan}_{\mathcal{S}}$ *of $\mathcal{S}$ for all $y \in \mathbb{R}^n$.*

The proof of Theorem 8 is based on the extended Lyapunov-type technique (see [30] for details). Recall that the tangent space $\text{Tan}_{\mathcal{S}}$ of $\mathcal{S}$ takes the form

$$\text{Tan}_{\mathcal{S}} = \{\zeta \in \mathbb{R}^n \mid Dh(y)\zeta = 0\}, \tag{27}$$

where $Dh(y)$ denotes a derivative of $h(y)$.

We now consider a specific case of ASSs such that the switching times t_i, $i = 1,\ldots,r$, are determined by a smooth manifold $\mathcal{S}$:

$$\tau^r := \{t_i, i = 1,\ldots,r \mid h(x^r(t_i)) = 0\}, \tag{28}$$

where $x(\cdot)$ is a solution of (5). This modelling approach corresponds to some particular mathematical models of hybrid control processes. We refer to [20] for relevant examples and further formal details.

Theorem 9. *Assume that all the conditions of Theorem 6 are satisfied and the switching times of an ASS are determined by (28), where $\mathcal{S}$ is a smooth manifold. Let $(\{q_i\}^Z, \tau^Z, x^Z(\cdot))$ be a Zeno execution generated by a Zeno input $\rho^Z(\cdot)$. Let $\rho^r(\cdot) = \chi^r(\cdot) \times \{v^r(\cdot)\}$ be an approximating sequence from Theorem 6. Then there exist projected dynamic processes $y(\cdot)$ given by the following initial value problem:*

$$\begin{aligned} &\dot{y}(t) = \sum_{i=1}^{\infty} \beta^Z_{[t_{i-1},t_i)}(t) \\ &\qquad \cdot \text{Pr}_{\text{Tan}_{\mathcal{S}}}\left[a_{q_i}(y(\cdot)) + b_{q_i}(y(\cdot))\hat{u}(\cdot)\right](t), \\ &y(0) = y_0 \in \mathcal{S}, \end{aligned} \tag{29}$$

where $\hat{u}(\cdot) \in \mathcal{U}$ is a strong limit of a subsequence of $\{v^r(\cdot)\}$ and $\text{Pr}_{\text{Tan}_{\mathcal{S}}}[x]$ *is a projection of a vector $x \in \mathbb{R}^n$ on the manifold $\mathcal{S}$. Moreover, $\mathcal{S}$ is an invariant manifold for (29) and*

$$\lim_{t\to t_f} \left\|x^Z(t) - y(t)\right\|_{\mathbb{R}^n} = 0. \tag{30}$$

Proof. From the assumptions of Section 2 (smoothness of the families $\mathcal{A}$, $\mathcal{B}$ and boundedness of the admissible controls) it follows that the trajectories $x(\cdot)$ of the ASS under consideration have bounded derivatives for almost all $t \in [0,t_f]$. This fact and the basic Definition 4 imply

$$\lim_{t\to t_f} \text{dist}\left[x^Z(t), \mathcal{S}\right] = 0, \tag{31}$$

where $\text{dist}[x,\mathcal{S}]$ denotes the Euclidean distance between a vector $x \in \mathbb{R}^n$ and $\mathcal{S}$. Consider $\mathbb{L}^1(0,t_f) \times \mathbb{L}^\infty_m(0,t_f)$-weakly convergent sequence $\{\rho^r(\cdot)\}$ of controls from Theorem 6 such that

$$\lim_{r\to\infty} \left\|x^r(\cdot) - x^Z(\cdot)\right\|_{\mathbb{C}_n(0,t_f)} = 0, \tag{32}$$

where $\{x^r(\cdot)\}$ is a sequence of solutions to (5) generated by $\{\rho^r(\cdot)\}$. From (31) and (32) we next deduce

$$\begin{aligned} &\lim_{r\to\infty}\lim_{t\to t_f} \left\|x^r(t) - x^Z(\cdot)\right\|_{\mathbb{R}^n} = 0, \\ &\lim_{r\to\infty}\lim_{t\to t_f} \left\|x^r(t) - \text{Pr}_{\text{Tan}_{\mathcal{S}}}(x^r(\cdot))(t)\right\|_{\mathbb{R}^n} = 0. \end{aligned} \tag{33}$$

Here $\text{Pr}_{\text{Tan}_{\mathcal{S}}}(\cdot)(\cdot)$ is a projection on the tangent space $\text{Tan}_{\mathcal{S}}$. Moreover, the above relations (33) imply

$$\lim_{r\to\infty}\lim_{t\to t_f} \left\|x^Z(t) - \text{Pr}_{\text{Tan}_{\mathcal{S}}}(x^r(\cdot))(t)\right\|_{\mathbb{R}^n} = 0. \tag{34}$$

Since the mappings $\|\cdot\|_{\mathbb{R}^n}$ and $\text{Pr}_{\text{Tan}_{\mathcal{S}}}(\cdot)$ are continuous, we next obtain

$$\lim_{t\to t_f} \left\|x^Z(t) - y(t)\right\|_{\mathbb{R}^n} = 0, \tag{35}$$

where $y(\cdot)$ is $\mathbb{C}_n(0,t_f)$-limit of $\mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}(\cdot)$ if $r \to \infty$,

$$y(t) := \lim_{r\to\infty} \mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}(x^r(\cdot))(t) \quad \forall t \in [0,t_f]. \tag{36}$$

The continuity and linearity of the projection operator $\mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}(\cdot)$ on the subspace $\mathrm{Tan}_{\mathcal{S}}$ imply

$$\dot{y}(t) = \lim_{r\to\infty} \mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}(\dot{x}^r(\cdot))(t) \tag{37}$$

for almost all $t \in [0,t_f]$. The obtained relation can be rewritten as follows:

$$\begin{aligned}
\dot{y}(t) &= \lim_{r\to\infty} \mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}\left[\sum_{i=1}^{r}\beta_{[t_{i-1},t_i)}(\cdot)\right.\\
&\quad\left.\cdot\left(a_{q_i}(x^r(\cdot)) + b_{q_i}(x^r(\cdot))u(\cdot)\right)\right](t)\\
&= \sum_{i=1}^{\infty}\beta^{Z}_{[t_{i-1},t_i)}(t)\\
&\quad\cdot\lim_{r\to\infty}\mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}\left[\left(a_{q_i}(x^r(\cdot)) + b_{q_i}(x^r(\cdot))u(\cdot)\right)\right](t)\\
&= \sum_{i=1}^{\infty}\beta^{Z}_{[t_{i-1},t_i)}(t)\\
&\quad\cdot\mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}\left[\lim_{r\to\infty}\left(a_{q_i}(x^r(\cdot)) + b_{q_i}(x^r(\cdot))u(\cdot)\right)\right](t).
\end{aligned} \tag{38}$$

The smoothness of the families $\mathcal{A}$, $\mathcal{B}$ from Definition 1 implies the projected differential equation

$$\begin{aligned}
&\dot{y}(t)\\
&= \sum_{i=1}^{\infty}\beta^{Z}_{[t_{i-1},t_i)}(t)\,\mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}\left[a_{q_i}(y(\cdot)) + b_{q_i}(y(\cdot))\hat{u}(\cdot)\right](t)
\end{aligned} \tag{39}$$

determined on the tangent space $\mathrm{Tan}_{\mathcal{S}}$. We refer to [32] for the necessary concepts and facts from the theory of projected dynamic systems defined by ordinary differential equations. Here $\hat{u}(\cdot) \in \mathcal{U}$ is $\mathbb{L}^{\infty}_m(0,t_f)$-limit (strong limit) of a subsequence of $\{v^r(\cdot)\}$. Note that every $\mathbb{L}^{\infty}_m(0,t_f)$-weakly convergent sequence $\{v^r(\cdot)\}$ has a strongly convergent subsequence. The inclusion $\hat{u}(\cdot) \in \mathcal{U}$ is a consequence of the closeness of the set $\mathcal{U}$ of admissible controls. Using the basic property (idempotency $\mathrm{Pr}^2_{\mathrm{Tan}_{\mathcal{S}}} = \mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}$) of the projection operator, we finally obtain

$$\mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}[\dot{y}(\cdot)](t) = \dot{y}(t) \tag{40}$$

and $\dot{y}(t) \in \mathrm{Tan}_{\mathcal{S}}$. By construction, $\mathcal{S}$ is an invariant manifold for system (39). The proof is completed. □

Note that Theorem 9 has a natural geometrical interpretation. The Zeno trajectory related to a Zeno execution with switching rules given by (28) converges to a smooth dynamic process determined on $\mathcal{S}$.

A specific case of the presented dynamic behaviour is constituted by the celebrated sliding mode control. We refer to [8, 31, 33, 34] for formal definitions and basic results. In that specific case we have $s = \dim(\mathcal{S}) \equiv m$ (the dimension of $\mathcal{S}$ is equal to the dimension of the control vector) and the approximating process $y(\cdot)$ can directly be generated by a special control system via so-called "equivalent control" $w_{\mathrm{eq}}(\cdot)$ introduced in [34]. Note that we consider here the case of invertible matrices from the given family $\mathcal{B}$ (see Definition 1) associated with the given dynamic system (5)

$$\begin{aligned}
w_{\mathrm{eq}}(y) := &-\left(\nabla h(y)\sum_{i=1}^{\infty}\beta^{z}_{[t_{i-1},t_i)}(t)\,b_{q_i}(y)\right)^{-1}\\
&\times\left[\nabla h(y)\sum_{i=1}^{\infty}\beta^{z}_{[t_{i-1},t_i)}(t)\,a_{q_i}(y)\right].
\end{aligned} \tag{41}$$

For the Zeno-like chattering behaviour in sliding mode systems see also [33, 35].

Finally note that (29) constitutes in fact the projected Zeno behaviour. We call the dynamic process $y(\cdot)$ from Theorem 9 the projected Zeno dynamics. On the other side the projected Zeno dynamics given by (29) constitutes a natural generalisation of the conventional sliding mode control strategies from [34]. This generalization is characterised by the condition $s = \dim(\mathcal{S}) \le n$ that is more general in comparison to the specific assumption $s = m$ from the classic sliding mode control theory.

5. Approximations of the Zeno Dynamics Involving the Optimal Control Methodology

In this section we deal with a novel alternative representation and approximation approach to a Zeno behaviour determined by a smooth manifold $\mathcal{S}$ as in Section 4. Let us introduce the necessary additional notation: by $F^{q_i}_j(x)$, $j = 1,\ldots,m$, $q_i \in \mathcal{Q}$, we denote the vector-columns of the matrix $b_{q_i}(x)$. Additionally we assume here that the rank of $\mathrm{span}\{F^{q_i}_1(x),\ldots,F^{q_i}_m(x)\}$ is equal to m (the dimension of the control vector). Consider a dynamic process

$$\begin{aligned}
\dot{\delta y}(t) = &\sum_{i=1}^{\infty}\beta^{Z}_{[t_{i-1},t_i)}(t)\\
&\cdot\mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}\left[\left[\frac{\partial a_{q_i}}{\partial y}(y(t)) + \sum_{j=1}^{m}\frac{\partial F^{q_i}_j}{\partial y}(y(t))\,\hat{u}^j(t)\right]\right.\\
&\left.\cdot\,\delta y(t) + b_{q_i}(y(t))\,\delta u(t)\right],
\end{aligned} \tag{42}$$

where $\delta y(0) = 0$ and $y(\cdot)$, $\hat{u}(\cdot)$ are determined by Theorem 9. The dynamic process $\delta y(\cdot)$ is in fact a linearisation of the projected Zeno dynamics $y(\cdot)$. This linearisation is generated by a bounded measurable control input $\delta u(\cdot)$. Note that the linearity of the operator $\mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}$ (projection on a linear subspace) implies

$$\frac{\partial}{\partial y}\mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}(\cdot) = \mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}\left(\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial u}\mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}(\cdot) = \mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}\left(\frac{\partial}{\partial u}\right). \tag{43}$$

Evidently, $y(t) + \delta y(t) \in \mathcal{S}$ for almost all $t \in [0, t_f]$. Let $\Phi(\cdot)$ be the fundamental matrix for (42). Our next result contains an alternative Hamiltonian-based representation for the original Zeno dynamics determined by a Zeno execution $(\{q_i\}^Z, \tau^Z x^Z(\cdot))$.

Theorem 10. *Assume that all the conditions of Theorem 9 are satisfied and*

$$\left(\{q_i\}^Z, \tau^Z, x^Z(\cdot)\right) \tag{44}$$

is a Zeno execution. Let $y(\cdot)$ be a projected Zeno dynamic process given by (28). Then there exists an absolutely continuous function (the "adjoint vector")

$$p(\cdot), p(t) \in \mathbb{R}^n \setminus \{0\} \tag{45}$$

such that $(\hat{u}(\cdot), y(\cdot), p(\cdot))$ are solutions a.e. on $[0, t_f]$ of the following Hamiltonian system:

$$\begin{aligned} \dot{y}(t) &= \frac{\partial H}{\partial p}(t, \hat{u}(t), y(t), p(t)), \\ \dot{p}(t) &= -\frac{\partial H}{\partial y}(t, \hat{u}(t), y(t), p(t)), \\ \frac{\partial H}{\partial u}&(t, \hat{u}(t), y(t), p(t)) = 0, \end{aligned} \tag{46}$$

where

$$H(t,u,y,p) := \left\langle p, \sum_{i=1}^{\infty} \beta^Z_{[t_{i-1},t_i)}(t)\, \mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}\left[a_{q_i}(y) + b_{q_i}(y)u\right]\right\rangle \tag{47}$$

is the pseudo-Hamiltonian associated with the given ASS and $\langle\cdot,\cdot\rangle$ denotes a scalar product in $\mathbb{R}^n$.

Proof. Note that the first equation from (46) is a simple consequence of the linear structure of the pseudo-Hamiltonian. Let $\delta u(\cdot) \in \mathbb{L}^{\infty}_m(0, t_f)$ such that $\hat{u}(\cdot) + \delta u(\cdot) \in \mathcal{U}$. Evidently, the dimension of the linear space

$$\left\{\delta y(\cdot) \in \mathbb{L}^{\infty}_n(0, t_f) \mid \Phi(t_f)\int_T^{t_f} \Phi^{-1}(t) \times \sum_{i=1}^{\infty} \beta^Z_{[t_{i-1},t_i)}(t)\, \mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}\left[b_{q_i}(y(t))\,\delta u(t)\,dt\right], \delta u(\cdot) \in \mathbb{L}^{\infty}_m(0, t_f)\right\} \tag{48}$$

is equal to $s = \dim(\mathcal{S}) < n$. Therefore, there exists a vector $\mu \in \mathbb{R}^n$, $\mu \neq 0$, such that

$$\mu^T \Phi(t_f)\Phi^{-1}(t) \sum_{i=1}^{\infty} \beta^Z_{[t_{i-1},t_i)}(t)\, \mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}\left[b_{q_i}(y(t))\right] = 0 \tag{49}$$

for almost all $t \in [0, t_f]$. We now introduce the function $p(\cdot)$ by setting

$$p^T(t) := \mu^T \Phi(t_f)\Phi^{-1}(t). \tag{50}$$

The definition of $p(\cdot)$ implies that this function is a solution to the adjoint system

$$\dot{p}(t) = -p(t)\sum_{i=1}^{\infty} \beta^Z_{[t_{i-1},t_i)}(t) \cdot \mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}\left[\frac{\partial a_{q_i}}{\partial y}(y(t)) + \sum_{j=1}^{m}\frac{\partial F_j^{q_i}}{\partial y}(y(t))\,u_y(t)\right]. \tag{51}$$

Moreover, it satisfies almost everywhere the equality

$$p(t)\sum_{i=1}^{\infty} \beta^Z_{[t_{i-1},t_i)}(t)\, \mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}\left[b_{q_i}(y(t))\right] = 0. \tag{52}$$

We get the second and the third equations from (28). □

Theorem 10 gives a variational characterization of the approximating dynamic process $y(\cdot)$. To put it another way, a given Zeno execution $(\{q_i\}^Z, \tau^Z x^Z(\cdot))$ determined by a smooth manifold $\mathcal{S}$ (discussed in Section 4) can be approximated by the solution of the Hamiltonian-type system (46). Moreover, the obtained condition (52) can be interpreted as a generalized "equivalent control" $\hat{u}(\cdot)$.

Evidently, relations (46) constitute a specific description of the Zeno behaviour using the optimal control methodology (see [30]). It is easy to see that conditions (46) are similar to the generic formalism of the weak Pontryagin Maximum Principle for the classic singular optimal control. A pair $(\hat{u}(\cdot), y(\cdot))$ can be interpreted as an extremal pair associated with an auxiliary optimal control problem involved system (5). In that context one can consider, for example, a terminal functional $\phi(y(t_f))$ such that

$$\begin{aligned} &\frac{\partial \phi}{\partial y}(y(t_f)) = p^T(t_f) = \mu^T \in \mathbb{R}^n, \quad \mu \neq 0, \\ &\mu^T \Phi(t_f)\Phi^{-1}(t) \sum_{i=1}^{\infty} \beta^Z_{[t_{i-1},t_i)}(t)\, \mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}\left[b_{q_i}(y(t))\right] = 0. \end{aligned} \tag{53}$$

Let us now discuss a concrete computational scheme for an extremal control determined by (46). For a nonlinearly ASS of type (5) the last equation from (46) implies the condition

$$\left\langle p(t), \sum_{i=1}^{\infty} \beta^Z_{[t_{i-1},t_i)}(t)\, \mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}\left[F_j^{q_i}(y(t))\right]\right\rangle = 0 \quad \forall j = 1, \ldots, m. \tag{54}$$

Differentiating (54) with respect to t, we get the next equation

$$\xi\left(y\left(t\right),p\left(t\right)\right)+O\left(y\left(t\right),p\left(t\right)\right)\widehat{u}\left(t\right)=0, \tag{55}$$

where $\xi(y,p)$ is the m-dimensional vector with components

$$\xi_j\left(y,p\right) := \left\langle p,\left[\sum_{i=1}^{\infty}\beta^{Z}_{[t_{i-1},t_i)}\left(t\right)\mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}\left(F_j^{q_i}\right),\mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}\left(a_{q_i}\right)\right]\cdot\left(y\right)\right\rangle \tag{56}$$

and $O(y,p)$ is the $m\times m$ matrix with components

$$O_{l,j} := \left\langle p,\left[\sum_{i=1}^{\infty}\beta^{Z}_{[t_{i-1},t_i)}\left(t\right)\mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}\left(F_l^{q_i}\right),\mathrm{Pr}_{\mathrm{Tan}_{\mathcal{S}}}\left(F_j^{q_i}\right)\right]\cdot\left(y\right)\right\rangle. \tag{57}$$

By $[\cdot,\cdot]$ we denote here the Lie brackets:

$$\left[Z_1,Z_2\right]\left(\varsigma\right) := \frac{\partial Z_1}{\partial\varsigma}\left(\varsigma\right)Z_2\left(\varsigma\right)-\frac{\partial Z_2}{\partial\varsigma}\left(\varsigma\right)Z_1\left(\varsigma\right). \tag{58}$$

The constructive relation (55) characterizes an unknown control input $\widehat{u}(\cdot)$ for the projected Zeno dynamics determined by (29). This control design is a direct consequence of the proposed optimal control based description of the Zeno behaviour in system (5).

6. Concluding Remarks

This contribution initiates theoretic investigations devoted to the constructive approximations of the Zeno dynamics in switched systems. The abstract idea and the proposed (constructive) approximative approach are based on some novel analytic techniques and also involve a specific application of the optimal control methodology. For the control systems under consideration we developed a relative simple implementable approach that makes it possible to eliminate the practically nonrealistic exact Zeno behaviour from consideration and study a more realistic approximative non-Zeno model. Our approach can rather be interpreted as a part of the "numerical analysis" of the general Zeno behaviour.

Moreover, we established a natural relation between the Zeno dynamics in affine switched systems and the conventional sliding mode control. This relationship is based on the projected dynamics and involves a natural generalization of the classic sliding mode control theory. On the other side, the involvement of the optimal control techniques implies the obtained variational description of the Zeno effect in the affine switched systems. The Zeno executions can now be characterized by solutions to the specific Hamiltonian system that constitutes a necessary optimality condition for a particular (auxiliary) optimal control problem. The proposed variational representation of the Zeno phenomena incorporates the standard Hamiltonian formalism and some additional extremal conditions. This representation has a theoretic and numeric potential to be applied to concrete control design procedures for some classes of switched Zeno-type systems.

The analytic results discussed in our contribution must be extended by some concrete control applications. Moreover, one can expect to obtain numerically tractable approximation schemes and solution procedures for Zeno executions in some concrete examples of switched systems. Finally let us note that the main mathematical tools and techniques used in our paper have a general analytic nature and can be applied to wide classes of control problems associated with the general types of hybrid and switched dynamic systems.

Competing Interests

The author declares that there are no competing interests.

References

[1] V. Azhmyakov, J. Cabrera Martinez, A. Poznyak, and R. R. Serrezuela, "Optimization of a class of nonlinear switched systems with fixed-levels control inputs," in *Proceedings of the American Control Conference (ACC '15)*, pp. 1770–1775, Chicago, Ill, USA, July 2015.

[2] M. S. Branicky, V. S. Borkar, and S. K. Mitter, "A unified framework for hybrid control: model and optimal control theory," *IEEE Transactions on Automatic Control*, vol. 43, no. 1, pp. 31–45, 1998.

[3] C. G. Cassandras and S. Lafortune, *Introduction to Discrete Event Systems*, Springer, New York, NY, USA, 2008.

[4] M. Egerstedt and B. Mishra, Eds., *Hybrid Systems: Computation and Control*, vol. 4981 of *Lecture Notes in Computer Science*, Springer, Berlin, Germany, 2008.

[5] R. Galván-Guerra, V. Azhmyakov, and M. Egerstedt, "Optimization of multiagent systems with increasing state dimensions: hybrid LQ approach," in *Proceedings of the American Control Conference (ACC '11)*, pp. 881–887, San Francisco, Calif, USA, July 2011.

[6] D. Liberzon, *Switching in Systems and Control*, Birkhäuser, Boston, Mass, USA, 2003.

[7] J. Lygeros, *Lecture Notes on Hybrid Systems*, Cambridge University Press, Cambridge, UK, 2003.

[8] A. Poznyak, A. Polyakov, and V. Azhmyakov, *Attractive Ellipsoids in Robust Control*, Birkhäuser, Basel, Switzerland, 2014.

[9] F. Taringoo and P. E. Caines, "The sensitivity of hybrid systems optimal cost functions with respect to switching manifold parameters," in *Hybrid Systems: Computation and Control*, vol. 5469 of *Lecture Notes in Computer Science*, pp. 475–479, Springer, Berlin, Germany, 2009.

[10] A. J. van der Schaft and H. Schumacher, *An Introduction to Hybrid Dynamical Systems*, vol. 251 of *Lecture Notes in Control and Information Sciences*, Springer, Berlin, Germany, 2000.

[11] H. Yu and C. G. Cassandras, "Perturbation analysis of communication networks with feedback control using stochastic hybrid models," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 65, no. 6, pp. 1251–1280, 2006.

[12] V. Azhmyakov, R. Galvan-Guerra, and M. Egerstedt, "Hybrid LQ-optimization using dynamic programming," in *Proceedings*

of the American Control Conference, pp. 3617–3623, St. Louis, Mo, USA, June 2009.

[13] V. Azhmyakov, M. Egerstedt, L. Fridman, and A. Poznyak, "Continuity properties of nonlinear affine control systems: applications to hybrid and sliding mode dynamics," in *Proceedings of the 3rd IFAC Conference on Analysis and Design of Hybrid Systems (ADHS '09)*, pp. 204–209, Zaragoza, Spain, September 2009.

[14] X.-C. Ding, Y. Wardi, and M. Egerstedt, "On-line optimization of switched-mode dynamical systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 9, pp. 2266–2271, 2009.

[15] P. Martin and M. B. Egerstedt, "Hybrid systems tools for compiling controllers for cyber-physical systems," *Discrete Event Dynamic Systems: Theory and Applications*, vol. 22, no. 1, pp. 101–119, 2012.

[16] T. Moor and J. Raisch, "Supervisory control of hybrid systems within a behavioural framework," *Systems and Control Letters*, vol. 38, no. 3, pp. 157–166, 1999.

[17] M. S. Shaikh and P. E. Caines, "On the hybrid optimal control problem: theory and algorithms," *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1587–1603, 2007.

[18] Y. Or and A. D. Ames, "Stability and completion of Zeno equilibria in Lagrangian hybrid systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 6, pp. 1322–1336, 2011.

[19] K. H. Johansson, M. Egerstedt, J. Lygeros, and S. Sastry, "On the regularization of Zeno hybrid automata," *Systems and Control Letters*, vol. 38, no. 3, pp. 141–150, 1999.

[20] A. Lamperski and A. D. Ames, "Sufficient conditions for Zeno behavior in Lagrangian hybrid systems," in *Hybrid Systems: Computation and Control*, vol. 4981 of *Lecture Notes in Computer Science*, pp. 622–625, Springer, Berlin, Germany, 2008.

[21] A. Lamperski and A. D. Ames, "Lyapunov theory for Zeno stability," *IEEE Transactions on Automatic Control*, vol. 58, no. 1, pp. 100–112, 2013.

[22] J. Zhang, K. H. Johansson, J. Lygeros, and S. Sastry, "Zeno hybrid systems," *International Journal of Robust and Nonlinear Control*, vol. 11, no. 5, pp. 435–451, 2001.

[23] R. Alur and D. L. Dill, "Automata for modeling real-time systems," in *Automata, Languages and Programming*, vol. 443 of *Lecture Notes in Computer Science*, pp. 322–335, Springer, Berlin, Germany, 1990.

[24] V. Azhmyakov and M. T. Angulo, "Applications of the strong approximability property to a class of affine switched systems and to relaxed differential equations with affine structure," *International Journal of Systems Science*, vol. 42, no. 11, pp. 1899–1907, 2011.

[25] V. Azhmyakov, M. Egerstedt, L. Fridman, and A. Poznyak, "Approximability of nonlinear affine control systems," *Nonlinear Analysis: Hybrid Systems*, vol. 5, no. 2, pp. 275–288, 2011.

[26] M. I. Zelikin and V. F. Borisov, *Theory of Chattering Control: With Applications to Astronautics, Robotics, Economics and Engineering*, Birkhäuser, Boston, Mass, USA, 1994.

[27] K. Atkinson and W. Han, *Theoretical Numerical Analysis*, Springer, New York, NY, USA, 2009.

[28] H. Attouch, G. Buttazzo, and G. Michaille, *Variational Analysis in Sobolev and BV Spaces: Applications to PDEs and Optimization*, SIAM and MPS, Philadelphia, Pa, USA, 2006.

[29] W. Rudin, *Functional Analysis*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill, New York, NY, USA, 1973.

[30] F. H. Clarke, *Optimization and Nonsmooth Analysis*, vol. 5 of *Classics in Applied Mathematics*, SIAM, Philadelphia, Pa, USA, 2nd edition, 1990.

[31] H. K. Khalil, *Nonlinear Systems*, Prentice Hall, Upper Saddle River, NJ, USA, 2002.

[32] A. Nagurney and D. Zhang, *Projected Dynamical Systems and Variational Inequalities with Applications*, Kluwer Academic Publishers, New York, NY, USA, 1996.

[33] V. Azhmyakov, "On the set-valued approach to optimal control of sliding mode processes," *Journal of the Franklin Institute*, vol. 349, no. 4, pp. 1323–1336, 2012.

[34] V. I. Utkin, *Sliding Modes in Control and Optimization*, Communications and Control Engineering Series, Springer, Berlin, Germany, 1992.

[35] I. Boiko, *Discontinuous Control Systems Frequency-Domain Analysis and Design*, Birkhäuser, New York, NY, USA, 2009.