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Approximation methods for solving fractional optimal control problems

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Abstract In this review paper, approximation methods for the free final time of fractional optimal control problems (FOCPs) are displayed. The considered problems mainly include the fractional differential equations (FDEs) with fractional derivatives (FDs). In this way, the considered tools and techniques mainly include the necessary optimal conditions in the form of two-point boundary value (TPBV) problem of fractional order. The Legendre operational, Ritz method and the Jacobi, Bernoulli and Legendre polynomials are extended as numerical methods for FOCPs accordingly. At the same time, the techniques for improving the accuracy and computation and storage are also introduced.

Keywords Fractional optimal control problem · Fractional differential equation · Fractional derivative · Fractional two-point boundary value problem · Numerical approximation

Mathematics Subject Classification 49M05 · 49M25 · 65K99

1 Introduction

Fractional derivatives (FDs), which are as old as the classical one, did not attract much attention until the recent decades. Nowadays, many scientists in different fields of engineering, physics and mathematics have been interested in studying the FDs. Maybe that is why FDs provide accurate models of many engineering systems such as in physics (Barkai et al.

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2000; Zaslavsky 2002; Klages et al. 2008; Magin et al. 2008), materials (Diethelm and Freed 1999b), control theory (Podlubny 1999), biology (Magin 2006) and finance (Raberto et al. 2002), and so on. In physics, fractional derivatives are adopted to model anomalous diffusion (Metzler and Klafter 2000, 2004; Li and Zhao 2011a), where particles spread differently than predicted by the classical Brownian motion model. Fractional kinetic equations have been proved to be particularly useful in the context of anomalous slow diffusion (sub-diffusion) (Metzler and Klafter 2000). Therefore, presentation of an applicable numerical approach for solving practical fractional problems has considerable importance. We refer the interested readers in fractional calculus to the books (Kilbas et al. 2006; Diethelm and Walz 1997; Diethelm 1997; Sabatier et al. 2007), where in, many aspects of the subject of FDs have been studied and great efforts have been made to find robust and stable numerical and analytical techniques for solving FDEs. At present, there are several numerical methods to solve FDEs, such as the finite difference method, the finite element method and the spectral method which are relatively rare (Roop 2006, 2008; Zheng et al. 2010a, b; Lin and Liu 2007; Lin and Xu 2007; Li and Zhao 2011a; Meerschaert and Tadjeran 2006; Tadjeran and Meerschaert 2007; Liu et al. 2008; Ford and Simpson 2001; Wang et al. 2010; Soradi Zeid et al. 2017; Canuto et al. 2012; Boyd 2001; Trefethen 2000) in literature.

The optimal control problem refers to the minimization of a performance index subject to dynamic constraints on the state variable and the control variable. If the FDEs are used as the dynamic constraints, this leads to the FOCP. Fractional optimal control theory is a very new area in mathematics. An FOCP can be defined with respect to different definitions of FDs, but the most important types of these types of derivatives are the Caputo and Riemann–Liouville. In this paper, we just review almost all the existing approximation methods for the FOCPs. If some important references have been omitted, we do apologize for those omissions. With the emerging number of the applications of FOCPs, the solution of these kinds of problems has become an important topic for researchers. In the recent decade, many different analytic methods have been introduced to solve the nonlinear problems, such as the homotopy analysis method (HAM) (Liao 2004), the homotopy perturbation method (HPM) (He 1999a, 2000a), the variational iteration method (He 1999b, 2000b), the Adomians decomposition method (ADM) (Adomian 1994; Siddiqui et al. 2010; Wazwaz and El-Sayed 2001) and several modifications to the ADM by numerous authors that can be seen in Jin and Liu (2005), Hosseini and Nasabzadeh (2007), Hasan and Zhu (2008). While there are very few simple cases in which the analytical solutions of FOCPs are available, therefore, developing efficient and reliable numerical methods for these problems is of great importance. Ozdemir et al. (2009) solved two-dimensional FOCPs in polar coordinates by separation variables method. Fractional diffusion problems in three-dimensional and in the spherical coordinates are discussed precisely in Povstenko (2008) and Qi and Liu (2010). Some numerical methods for solving some types of FOCPs can be seen in Bhrawy et al. (2015a, b), Doha et al. (2015), Alipour et al. (2013), Akbarian and Keyanpour (2013), Jarad et al. (2012), Agrawal (2007, 2008), Agrawal et al. (2012), Yousefi et al. (2011), Lotfi et al. (2011), Lotfi and Yousefi (2013), Pooseh et al. (2013a), Baleanu et al. (2009), Tricaud and Chen (2010) and the references cited therein.

We recall that, the approaches for numerical solutions of optimal control problems (OCPs) may be divided into two major classes: indirect methods and the direct methods. The indirect methods are based on the Pontryagin maximum principle (PMP) and require the numerical solution of boundary value problems that result from the necessary conditions of optimal control (Pontryagin et al. 1962). Direct optimization methods transcribe the (infinite-dimensional) continuous problem to a finite-dimensional nonlinear programming problem (NLP) through some parameterization of the state and/or control variables. In the direct methods, initial guesses have to be provided only for physically intuitive quantities such as the

states and possibly controls. The indirect schemes are based on optimizing then discretizing the main OCPs, meanwhile the direct methods are based on discretizing then optimizing the main OCPs. One of the best properties of the indirect schemes is the high credit of the obtained approximate solution of the main OCPs. This specific property is based on satisfying the first order of necessary conditions that originated from the calculus of variation and the PMP.

In this paper, after imposing PMP on the considered FOCPs, we obtain a fractional two-point boundary value problem (TPBVP) such that by solving this system of FDEs, one approximates the solution of the original fractional problem. These systems of FDEs, which are the necessary (and also are sufficient in several special cases) conditions for optimal solutions of FOCPs, originate from the PMP and have considerable importance in optimal control and calculus of variation. For solving TPBVPs, we can use many ideas. One of the well-known methods is integrating from the mentioned problems in an appropriate interval such that the boundary conditions may be imposed. After this procedure, one can use highly accurate Gauss quadrature rules or operational matrices of integration. However, Gauss quadrature rules (Samadi and Tohidi 2012) have a high order of accuracy for smooth data, but using them may give rise to ill-conditioned algebraic systems. On the other hand, by using operational matrices of integration (specially for orthogonal functions and polynomials) we may reach sparse algebraic systems, but the order of accuracy is decreased usually. In practice, if we deal with a complex nonlinear problem, it is better to use Gauss quadrature rules because operational matrices of integration have low accuracy in these cases (Samadi and Tohidi 2012). Otherwise, if we deal with simple (for instance, polynomial forms) nonlinear problems, one may use operational matrices of integration. Using operational matrices of integration goes back to the last four decades. Typical examples of such matrices are related to the Walsh functions (Corrington 1973), Chebyshev polynomials (Elnagar 1997), Bernstein polynomials (Doha et al. 2011), Legendre polynomials (Chang and Wang 1983), block-pulse functions (Hsu and Chang 1989), Laguerre polynomials (Hwang and Shih 1982), Fourier series (Paraskevopoulos et al. 1985), Hermite polynomials (Kekkeris and Paraskevopoulos 1988), Bessel functions (Paraskevopoulos et al. 1990) and Muntz–Legendre polynomials (Ejlali and Hosseini 2016). Notice that all operational matrices make up a class of similarities, that is, given one, we obtain all the others by the sandwich matrix.

Another popular way to solve a TPBVP is direct solving. In other words, we do not integrate the mentioned TPBVP and solve it directly by any idea in the field of approximation theory such as collocation, Galerkin, etc. Some other new direct solvers do not use the classical collocation or Galerkin schemes and are based upon completeness of bases (for instance, Fourier Toutounian et al. 2013a) and operational matrices of differentiation. These new approaches are very applicable and fast for solving high order linear delay (in both cases of neutral and difference) Fredholm integro-differential equations. Collocation and Galerkin techniques have a wide range of applications for solving linear and nonlinear differential equations (including hyperbolic partial differential equations), Fredholm–Volterra integro-differential difference delay equations and their systems, and one can refer to the works (Dascioglu 2009; Gulsu et al. 2011; Sezer et al. 2008; Yalcinbas et al. 2011) for collocation approaches. It should be noted that any direct solver has more efficiency with regard to the methods that deal with the integral forms (especially, operational matrices of integration). For instance, one can refer to Toutounian et al. (2013b).

Because of the nonlocality and weak singularity of the fractional operators, the computational cost and storage of numerical methods for FDEs are more expensive as compared with the corresponding methods for classical equations. In order to overcome these difficulties, some techniques were adopted to reduce the computational cost and storage of the derived

methods, such as short memory principal (Diethelm and Freed 1999a; Deng 2007b), Richard extrapolation (Diethelm and Walz 1997; Yuste 2006; Tadjeran and Meerschaert 2007) to get high order methods and ADI methods for two- and three-dimensional problems that convert high-dimensional problems to separate one-dimensional ones (Meerschaert et al. 2006; Liu et al. 2008), and (Ford and Simpson 2001). A general formulation for FOCPs was extended in Agrawal (2004, 2006), Almedia and Torres (2011), where the necessary conditions of optimization are achieved for FOCPs with the Caputo and Riemann–Liouville derivatives. To some extent, some of the approximation methods for FOCPs can be seen in Alipour et al. (2013), Duan et al. (2012), Jin and Liu (2005), Rajaram and Najafi (2009), Wu and Baleanu (2013).

The present paper is organized as follows: In Sect. 2, we introduce several definitions for different types of fractional derivatives and the notations used in the numerical methods for solving FOCPs. In Sect. 3, some techniques are introduced to approximate the Riemann–Liouville and Caputo fractional derivatives. The existing methods for the FOCPs are surveyed in Sect. 4. The case with the delay FDEs are displayed in Sect. 5. Conclusions and remarks are included in the last section.

2 Preliminaries

In this section, we will introduce the definitions of fractional derivatives and notations used to describe the numerical schemes. There are several different ways to define the fractional derivatives, and the most commonly used fractional derivatives are the Grunwald–Letnikov derivative, the Riemann–Liouville derivative and the Caputo derivative (Podlubny 1999; Samko et al. 1993; Li and Deng 2007; Li et al. 2009; Li and Zhao 2011a). We just introduce the definitions as follows.

Definition 1 The fractional integral (or the Riemann–Liouville integral) with order $\alpha > 0$ of the given function $f(t)$ is defined as

$$I^{\alpha}_{t_0,t} = I^{-\alpha}_{t_0,t} = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s) \mathrm{d}s, \tag{1}$$

where $\Gamma(.)$ is Euler's gamma function.

Definition 2 The left and right Grunwald–Letnikov derivatives with order $\alpha > 0$ of the given function $f(t)$ are defined respectively as

$${}_{GL}D^{\alpha}_{t_0,t} f(t) = \lim_{\Delta h \to 0} h^{-\alpha} \sum_{j=0}^{N} (-1)^j \binom{\alpha}{j} f(t - jh), \tag{2}$$

where $Nh = t - t_0$ and

$${}_{GL}D^{\alpha}_{t,t_f} f(t) = \lim_{\Delta h \to 0} h^{-\alpha} \sum_{j=0}^{N} (-1)^j \binom{\alpha}{j} f(t + jh), \tag{3}$$

in which $Nh = t_f - t$.

Definition 3 The left and right Riemann–Liouville derivatives with order $\alpha > 0$ of the given function $f(t)$ are defined as

$${}_{RL}D^{\alpha}_{t_0,t} f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} [D^{-(n-\alpha)}_{t_0,t} f(t)] = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^{t} (t-s)^{n-\alpha-1} f(s) \mathrm{d}s, \tag{4}$$

and

$$ {}_{RL}D^{\alpha}_{t,t_f} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_t^{t_f} (s-t)^{n-\alpha-1} f(s) \mathrm{d}s, \tag{5} $$

respectively, where n is a non-negative integer and $n-1 \leq \alpha < n$.

Definition 4 The left and right Caputo derivatives with order $\alpha > 0$ of the given function $f(t)$ are defined as

$$ {}_{C}D^{\alpha}_{t_0,t} f(t) = D^{-(n-\alpha)}_{t_0,t}[f^{(n)}(t)] = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s) \mathrm{d}s, \tag{6} $$

and

$$ {}_{C}D^{\alpha}_{t,t_f} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^{t_f} (s-t)^{n-\alpha-1} f^{(n)}(s) \mathrm{d}s, \tag{7} $$

respectively, where n is a non-negative integer and $n-1 \leq \alpha < n$.

Remark 1 Generally speaking, the above definitions of fractional derivatives are not equivalent, the differences and relations are discussed in detail in Samko et al. (1993), Podlubny (1999), Kilbas et al. (2006), Li and Deng (2007), Li et al. (2009, 2011a), and we just list one case as follows:

$$ \begin{aligned} {}_{RL}D^{\alpha}_{t_0,t} f(t) &= {}_{C}D^{\alpha}_{t_0,t} f(t) + \sum_{k=0}^{n-1} \frac{f^{(k)}(t_0)(t-t_0)^{k-\alpha}}{\Gamma(k+1-\alpha)} \\ {}_{RL}D^{\alpha}_{t,t_f} f(t) &= {}_{C}D^{\alpha}_{t_0,t} f(t) + \sum_{k=0}^{n-1} \frac{f^{(k)}(t_f)(t_f-t)^{k-\alpha}}{\Gamma(k+1-\alpha)}, \end{aligned} \tag{8} $$

where $f \in C^{n-1}[t_0, t]$ and $f^{(n)}$ is integrable on $[t_0, t]$. When $0 < \alpha < 1$, we have

$$ \begin{aligned} {}_{RL}D^{\alpha}_{t_0,t} f(t) &= {}_{C}D^{\alpha}_{t_0,t} f(t) + \frac{f(t_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\ {}_{RL}D^{\alpha}_{t,t_f} f(t) &= {}_{C}D^{\alpha}_{t_0,t} f(t) + \frac{f(t_f)(t_f-t)^{-\alpha}}{\Gamma(1-\alpha)}. \end{aligned} \tag{9} $$

Furthermore, if $f \in C^n[t_0, t]$, then ${}_{GL}D^{\alpha}_{t_0,t} f(t) = {}_{RL}D^{\alpha}_{t_0,t} f(t)$. As to the fractional integrability and differentiability of a considered function, the reader can refer to Li and Zhao (2011b).

Lemma 1 *Suppose that* $f(t) \in AC^n([t_0, t_f])$ (*absolutely continuous functions*), $\alpha > 0$ *and* $n = \lceil \alpha \rceil + 1$. *Then,*

$$ I^{\alpha}_{t_0,t} \, {}_{C}D^{\alpha}_{t_0,t} f(t) = f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(t_0)}{j} (t-t_0)^j \tag{10} $$

$$ I^{\alpha}_{t,t_f} \, {}_{C}D^{\alpha}_{t,t_f} f(t) = f(t) - \sum_{j=0}^{n-1} \frac{(-1)^j f^{(j)}(t_f)}{j} (t_f-t)^j. \tag{11} $$

If $0 < \alpha < 1$, *then*

$$ I^{\alpha}_{t_0,t} \, {}_{C}D^{\alpha}_{t_0,t} f(t) = f(t) - f(t_0) \tag{12} $$

$$ I^{\alpha}_{t,t_f} \, {}_{C}D^{\alpha}_{t,t_f} f(t) = f(t) - f(t_f). \tag{13} $$

Lemma 2 *If $\alpha > 0$ and $f(t)$ be a continuous function on $[t_0, t_f]$, then*

$$ {}_C D^{\alpha}_{t_0,t} I^{\alpha}_{t_0,t} f(t) = f(t), \quad {}_C D^{\alpha}_{t,t_f} I^{\alpha}_{t,t_f} f(t) = f(t). \tag{14} $$

Theorem 1 (Generalized Taylors formula) *Let ${}_C D^{i\alpha}_{t_0,t} f(t) \in C(]t_0, t_f])$ for $i = 0, 1, \ldots N+1$ and $0 < \alpha \leq 1$. Then,*

$$ f(t) = \sum_{k=0}^{N} \frac{({}_C D^{k\alpha}_{t_0,t} f)(t_0)}{\Gamma(k\alpha + 1)} (t - t_0)^{k\alpha} + R_{N,t_0}(t), \tag{15} $$

where

$$ R_{N,t_0}(t) = \frac{({}_C D^{(N+1)\alpha}_{t_0,t} f)(\xi)}{\Gamma((N+1)\alpha + 1)} (t - t_0)^{(N+1)\alpha}, $$

with $t_0 \leq \xi \leq t$, $\forall t \in [t_0, t_f]$, and ${}_C D^{N\alpha}_{t_0,t} := {}_C D^{\alpha}_{t_0,t}\, {}_C D^{\alpha}_{t_0,t} \cdots {}_C D^{\alpha}_{t_0,t}$ (N-times).

Another type of the fractional derivative is the Riesz derivative (Samko et al. 1993). The Riesz derivative has several kinds of forms (Podlubny et al. 2009) that are used in the FDEs (Zhuang et al. 2009; Yang et al. 2009a, b, 2010a, b).

In order to better illustrate the numerical methods, some notations are introduced as follows: for single-variable function $f(t)$ defined on $[t_0, t_f]$, define $\Delta t = (t_f - t_0)/N$ to be the uniform time step, N is a positive integer. The temporal grid points t_k is defined by $t_k = t_0 + k\Delta t$, $k = 0, \ldots, N$. We denote f_k to be the approximation of $f(t_k)$.

3 Approximation to the fractional derivatives

In the present section, some numerical methods are introduced to approximate the fractional derivatives of the function $f(t), t \in [t_0, t_f]$ where $\Delta t, t_j$ and N are defined as the preceding section.

3.1 Approximation to the Riemann–Liouville derivative

For a wide class of functions, both the Grunwald–Letnikov derivative and the Riemann–Liouville derivative are equivalent, especially for applications. Therefore, the Riemann–Liouville definition is suitable for the problem formulation, while the Grunwald–Letnikov definition is utilized to obtain the numerical solution (Metzler and Klafter 2000). In this section, some techniques and methods for approximating the Riemann–Liouville derivative are introduced and analyzed.

In applications, the fractional derivative of order $\alpha (0 < \alpha \leq 2)$ is of great interest, we just list some methods to approximate the fractional derivatives in such cases. By the definition of the Grunwald–Letnikov derivative, it is natural to use (2) and (3) to approximate the left and right Riemann–Liouville derivatives. Denote by $w^{\alpha}_j = (-1)^j \binom{\alpha}{j}$, then the left and right Riemann–Liouville derivative operators ${}_{RL} D^{\alpha}_{a,t}$ and ${}_{RL} D^{\alpha}_{t,b}$ can be approximated respectively by:

$$ {}_{RL} D^{\alpha}_{t_0,t} f(t_k) \approx {}_{RL} \tilde{D}^{\alpha}_{t_0,t} f(t_k) = \Delta t^{-\alpha} \sum_{j=0}^{k} w^{\alpha}_j f(t_{k-j}) \tag{16} $$

and

$$ {}_{RL}D^{\alpha}_{t,t_f} f(t_k) \approx {}_{RL}\tilde{D}^{\alpha}_{t,t_f} f(t_k) = \Delta t^{-\alpha} \sum_{j=0}^{n_T-k} w^{\alpha}_j f(t_{k+j}). \tag{17} $$

Both (16) and (17) are also called the standard Grunwald–Letnikov formulae, which may contribute to the unstable numerical schemes in solving FDEs (Meerschaert and Tadjeran 2004) for $1 < \alpha \leq 2$, while the shifted Grunwald–Letnikov formula may be more useful for constructing the stable numerical schemes. The right and left shifted Grunwald–Letnikov formulae (one shift), which are used to approximate the left and right Riemann–Liouville derivatives, are defined respectively by:

$$ {}_{RL}D^{\alpha}_{t_0,t} f(t_k) \approx {}_{RL}\tilde{D}^{\alpha}_{t_0,t} f(t_k) = \Delta t^{-\alpha} \sum_{j=0}^{k+1} w^{\alpha}_j f(t_{k-j+1}) \tag{18} $$

and

$$ {}_{RL}D^{\alpha}_{t,t_f} f(t_k) \approx {}_{RL}\tilde{D}^{\alpha}_{t,t_f} f(t_k) = \Delta t^{-\alpha} \sum_{j=0}^{n_T-k+1} w^{\alpha}_j f(t_{k+j-1}). \tag{19} $$

The approximations (18) and (19) also have one-order convergence, which are useful to construct the stable numerical schemes for the FPDEs.

Another efficient way to approximate the Riemann–Liouville derivative of order $\alpha(0 < \alpha \leq 1)$ is the $L1$ scheme (Oldham and Spanier 1974; Langlands and Henry 2005). The $L2$ scheme and its modification $L2C$ scheme (Oldham and Spanier 1974; Lynch et al. 2003) are suitable to discretize the Riemann–Liouville derivative with order $\alpha(1 < \alpha \leq 2)$.

On the other hand, the Riemann–Liouville derivatives are expandable in a power series involving integer order derivatives only. If $f(.)$ be an analytic function, then (see Kilbas et al. 2006):

$$ {}_{RL}D^{\alpha}_{t_0,t} f(t) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{(t-t_0)^{k-\alpha}}{\Gamma(k+1-\alpha)} x^{(k)}(t), \tag{20} $$

where

$$ \binom{\alpha}{k} = \frac{(-1)^{k-1}\alpha\Gamma(k-\alpha)}{\Gamma(1-\alpha)\Gamma(k+1)}. $$

The obvious disadvantage of using (20) in numerical computations is that in order to have a small error, one has to sum a large number of terms and thus the function has to possess higher-order derivatives, which is not suitable for optimal control. To address this problem, a second approach was carried out in Atanackovic and Stankovic (2008), Pooseh et al. (2013b), where a good approximation is obtained without the requirement of such higher-order smoothness on the admissible functions. The method can be explained, for left derivatives, in the following way. Let $\alpha \in (0, 1)$ and $f(.) \in C^2[t_0, t_f]$. Then,

$$ \begin{aligned} {}_{RL}D^{\alpha}_{t_0,t} f(t) \simeq\ & A(\alpha, N)(t-t_0)^{-\alpha} f(t) + B(\alpha, N)(t-t_0)^{1-\alpha} \dot{f}(t) \\ & - \sum_{p=2}^{N} C(\alpha, N)(t-t_0)^{1-p-\alpha} V_p(t) - \frac{f(t_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}, \end{aligned} \tag{21} $$

where $N \geq 2$ and $V_p(t)$ is defined as the solution of the system

$$\begin{cases} \dot{V}_p(t) = (1-p)(t-t_0)^{p-2} f(t), \\ V_p(t_0) = 0, \quad p = 2, 3, \ldots, N, \end{cases}$$

and

$$\begin{aligned} {}_{RL}D^{\alpha}_{t,t_f} f(t) \simeq\ & A(\alpha, N)(t_f - t)^{-\alpha} f(t) - B(\alpha, N)(t_f - t)^{1-\alpha} \dot{f}(t) \\ & + \sum_{p=2}^{N} C(\alpha, N)(t_f - t)^{1-p-\alpha} W_p(t) - \frac{f(t_f)(t_f - t)^{-\alpha}}{\Gamma(1-\alpha)}, \end{aligned} \tag{22}$$

where $W_p(t)$ is the solution of the differential equation

$$\begin{cases} \dot{W}_p(t) = -(1-p)(t_f - t)^{p-2} f(t), \\ W_p(t_f) = 0, \quad p = 2, 3, \ldots, N, \end{cases}$$

and $A(\alpha, N)$, $B(\alpha, N)$, $C(\alpha, p)$ are defined by:

$$\begin{aligned} A(\alpha, N) &= \frac{1}{\Gamma(1-\alpha)} \left[1 + \sum_{p=2}^{N} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha)(p-1)!} \right], \\ B(\alpha, N) &= \frac{1}{\Gamma(2-\alpha)} \left[1 + \sum_{p=2}^{N} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!} \right], \\ C(\alpha, p) &= \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha-1)} \frac{\Gamma(p-1+\alpha)}{(p-1)!}, \quad p = 2, 3, \ldots, N, \end{aligned}$$

and the error is bounded by

$$|E_{tr}(t)| \leq \max_{\tau \in [t_0, t]} |\ddot{f}(\tau)| \frac{exp((1-\alpha)^2 + 1 - \alpha)}{\Gamma(2-\alpha)(1-\alpha)N^{1-\alpha}} (t - t_0)^{2-\alpha}. \tag{23}$$

See Pooseh et al. (2013b) for proofs and other details. Using (8), a similar formula can be deduced for the Caputo fractional derivative.

3.2 Approximation to the Caputo derivative

Since the Riemann–Liouville derivative and the Caputo derivative have the relationship as (8), the $L1$, $L2$ and $L2C$ method can be extended to approximate the Caputo derivative directly. In Odibat (2009), a computational algorithm for approximating the Caputo derivative was also developed, and the convergence order is $O(\Delta t^2)$ for all $0 < \alpha \leq 2$. Another difference method of order two was derived in Sousa (2010) for $1 < \alpha \leq 2$. In Diethelm et al. (2005), numerical algorithms for solving the fractional order integral, the Caputo derivative and the differential equations with the Caputo derivative were provided and discussed in detail. Other methods that are used to approximate the Caputo derivative, refer to Odibat (2006), Murio (2006), Schmidt and Gaul (2006).

In Podlubny (2000), Podlubny et al. (2009), matrix approach is adopted to solve FDEs. This method is based on triangular strip matrix approach to discretize the differentiation and integration operators with arbitrary order. This technique can obtain all the numerical solutions at the mesh grids at once, avoiding the traditional step-by-step method by moving from the previous time layer to the next one to get the numerical solutions. The matrix

approach is quite simple to put into implementation. This method can also be extended to the cases of nonlinear problems, see Podlubny (2000), Podlubny et al. (2009) for more details.

4 Numerical methods for FOCPs

In this section we review the approximation methods for a constrained dynamic optimization problem of a fractional order system with free final time. Now, suppose that α be a real number in $(0, 1)$, and $F, G : [t_0, t_f] \times \mathbb{R}^2 \to \mathbb{R}$ are two continuously differentiable functions. A general form of FOCPs can be introduced as follows:

$$\min J(u) = \int_{t_0}^{t_f} F(t, x(t), u(t))\mathrm{d}t \tag{24}$$

subject to the fractional dynamic control system

$$A\dot{x}(t) + B\ {}_C D^{\alpha}_{t_0,t} x(t) = G(t, x(t), u(t)), \tag{25}$$

and the initial condition

$$x(t_0) = x_0, \tag{26}$$

where $x(t)$ is the state variable, $u(t)$ is the control variable, $(A, B) \neq 0$ and x_0 is a given constant. Two situations are considered: $x(t_f)$ fixed or free. Sufficient and necessary conditions to obtain solutions for this problem were studied in Pooseh et al. (2014). The aim is to find a control vector $u^*(t)$ such that the cost functional (24) is minimized, while the dynamic equality constraint is satisfied. There have already been several methods for the numerical solutions of (24)–(26). Here we proceed through different approaches.

4.1 Indirect methods for FOCPs

To obtain the necessary conditions, define the following Hamiltonian function:

$$H(x(t), u(t), \lambda(t), t) = F(t, x(t), u(t)) + \lambda^T(t) G(t, x(t), u(t)), \tag{27}$$

where $\lambda \in \mathbb{R}^n$ is the vector of the Lagrange multiplier. According to discussions in Pooseh et al. (2013c), if (x, u) be a minimum solution of (24)–(26), then there exists a $\lambda(t)$ which (x, u, λ) satisfies the following nonlinear TPBVP (see He 1999b):

$$\begin{aligned} A\dot{\lambda}(t) - B\ {}_{RL} D^{\alpha}_{t,t_f} \lambda(t) &= -\frac{\partial H}{\partial x} \\ A\dot{x}(t) + B\ {}_C D^{\alpha}_{t_0,t} x(t) &= \frac{\partial H}{\partial \lambda} \\ \frac{\partial H}{\partial u} &= 0, \quad t \in [t_0, t_f] \\ x(t_0) = x_0, \quad \lambda(t_f) &= 0. \end{aligned} \tag{28}$$

where H denotes the Hamiltonian and is defined in the form of (27). It should be mentioned that in practice, we obtain u in terms of λ and x from the condition $\frac{\partial H}{\partial u} = 0$. Therefore, the above-mentioned system can be rewritten in the following form:

$$\begin{aligned} A\dot{\lambda}(t) - B\ {}_{RL} D^{\alpha}_{t,t_f} \lambda(t) &= M(t, x(t), \lambda(t)) \\ A\dot{x}(t) + B\ {}_C D^{\alpha}_{t_0,t} x(t) &= N(t, x(t), \lambda(t)) \\ x(t_0) = x_0, \quad \lambda(t_f) &= 0 \end{aligned} \tag{29}$$

where $M(t, x(t), \lambda(t))$ and $N(t, x(t), \lambda(t))$ are known functions in terms of x and λ. As it was pointed out in Pooseh et al. (2013c), the above-mentioned fractional system contains necessary conditions for optimality of solutions of (24)–(26). If $F(t, x, u)$ and $G(t, x, u)$ be two convex functions in terms of x and u, then (29) contains necessary and sufficient conditions for optimal solutions x^* and u^*. It should be recalled that we should approximate $x(t)$ and $\lambda(t)$ in (29). Therefore, one may obtain an approximate optimal solution of (24)–(26).

Now, we outline the numerical schemes of fractional TPBVP (29) by proposing the research methods. It should be noted that, in all of the research works (Yuzbasi 2009, 2011, 2012; Yuzbasi et al. 2011, 2012), the authors used operational matrix of differentiation even for nonlinear problems. Moreover, all of the considered nonlinear problems have polynomial forms. The authors in Tohidi and Nik (2014) proposed an indirect numerical scheme to approximate the solutions of (29) by the truncated Bessel series as follows:

$$\begin{aligned} x(t) \approx x_N(t) &= \sum_{n=0}^{N} X_n J_n(t), \\ \lambda(t) \approx \lambda_N(t) &= \sum_{n=0}^{N} \Lambda_n J_n(t), \end{aligned} \tag{30}$$

where X_n and Λ_n, $n = 0, 1, \ldots, N$, are the unknown Bessel coefficients to be determined and $J_n(t)$, $n = 0, 1, \ldots, N$, are the Bessel polynomials of first kind that defined by Yuzbasi (2009):

$$J_n(t) = \sum_{k=0}^{[\frac{N-n}{2}]} \frac{(-1)^k}{k!(k+n)!} \left(\frac{t}{2}\right)^{2k+n}, \quad 0 \le t < \infty, \tag{31}$$

where N is the order of approximation. This representation is more clear than the matrix forms that were used in Yuzbasi (2009, 2011, 2012), Yuzbasi et al. (2011, 2012). They construct the associated system in a manner which needs less computational time. For this purpose, they solve the following system of $2N$ algebraic equations:

$$\begin{aligned} A\dot{\lambda}_N(t_k) - B\left({}_{RL}D^{\alpha}_{t_k,t_f}\right)\lambda_N(t_k) &= M(t_k, x_N(t_k), \lambda_N(t_k)), \ k = 0, 1, \ldots, N-1 \\ A\dot{x}_N(t_j) + B\left({}_{C}D^{\alpha}_{t_0,t_j}\right)x_N(t_j) &= N(t_j, x_N(t_j), \lambda_N(t_j)), \ j = 1, 2, \ldots, N \\ x_N(t_0) = x_0, \quad \lambda_N(t_f) &= 0. \end{aligned} \tag{32}$$

For solving this system one can apply the Newton algorithm in many softwares such as MAPLE. This procedure may be done by $fsolve$ command in MAPLE software. After solving the above-mentioned system of nonlinear algebraic equations, the unknown Bessel coefficients X_n and Λ_n ($n = 0, 1, \ldots, N$) will be extracted and replaced in (30). The accuracy of this collocation approach is $E_N(t_q) \le 10^{-k_q}$ (k_q is a positive integer), $q = 0, 1, 2, \ldots$.

A method that is presented in Sweilam et al. (2013) is based on a spectral method by using Chebyshev polynomials in which the fractional derivative is described in the Caputo sense. The first approach follows the paradigm optimize first, then discretize and relies on the approximation of the necessary optimality conditions (29). In the second approach, the state equation is discretized first using the Clenshaw and Curtis scheme (Clenshaw and Curtis 1960) for the numerical integration of nonsingular functions followed by the Rayleigh–Ritz method to evaluate both the state and control variables. Indeed, they provide numerical approximations of the left CFD and the right RLFD using Chebyshev polynomials. They

choose the grid points to be the Chebyshev–Gauss–Lobatto points associated with the interval $[0, L]$; that is,

$$t_r = \frac{L}{2} - \frac{L}{2} cos\left(\frac{\pi r}{N}\right), \quad r = 0, 1, \ldots, N. \tag{33}$$

Also, they introduced an approximation x_N of the function x with respect to the shifted Chebyshev polynomials as follows:

$$x_N(t) = \sum_{n=0}^{N_1} a_n T_n^*(t); \quad a_n = \frac{2}{N} \sum_{r=0}^{N_2} x(t_r) T_n^*(t_r), \tag{34}$$

in which, $T^*(t)$ is the shifted Chebyshev polynomials that are define in Sweilam et al. (2013). Now, an approximation of the fractional derivative of order α in the Caputo sense of the function x at t_s is given by:

$${}_C D_{t_0,t}^{\alpha} x_N(t_s) \cong \sum_{r=0}^{N} x(t_r) d_{s,r}^{\alpha}, \quad \alpha > 0, \tag{35}$$

where $s, r = 0, 1, \ldots, N$ and

$$d_{s,r}^{\alpha} = \frac{4\theta_r}{N} \sum_{n=\lceil \alpha \rceil}^{N} \sum_{j=0}^{N} \sum_{k=\lceil \alpha \rceil}^{n} \frac{n\theta_n}{a_j} \frac{(-1)^{n-k}(n+k-1)!\Gamma(k-\alpha+1/2) T_n^*(t_r) T_j^*(t_s)}{L^{\alpha}\Gamma(k+1/2)(n-k)!\Gamma(k-\alpha-j+1)\Gamma(k-\alpha+j+1)}, \tag{36}$$

with $\theta_0 = \theta_N = 1/2$, $\theta_i = 1$ for all $i = 1, 2, \ldots, N-1$. Also, they approximate the Right RLFD as follows:

$${}_{RL} D_{t,t_f}^{\alpha} x(t) = \frac{x(t_f)}{\Gamma(1-\alpha)} (t_f - t)^{-\alpha} + \frac{J(t; x)}{\Gamma(1-\alpha)}, \tag{37}$$

where

$$J(t; x) = \int_t^{t_f} (\tau - t)^{-\alpha} x'(\tau) \mathrm{d}\tau, \quad 0 < t < t_f, \tag{38}$$

and approximate $x(t), 0 \leq t \leq t_f$, by a sum of shifted Chebyshev polynomials according to

$$x(t) \approx P_N(t) = \sum_{k=0}^{N_2} a_k T_k^* \left(\frac{2t}{t_f} - 1\right), \quad a_k = \frac{2}{N} \sum_{j=0}^{N_2} x(t_j) T_k^* \left(\frac{2t_j}{t_f} - 1\right), \tag{39}$$

and $t_j = (t_f/2) - (t_f/2)cos(\pi j/N)$, $j = 0, 1, \ldots, N$. Moreover, ${}_{RL} D_{t,t_f}^{\alpha} x(t)$ can be approximated by means of

$${}_{RL} D_{t,t_f}^{\alpha} x(t) \approx \frac{x(t_f)}{\Gamma(1-\alpha)} (t_f - t)^{-\alpha} + \frac{J(t; P_N)}{\Gamma(1-\alpha)}. \tag{40}$$

The choice of Chebyshev base has several advantages: it avoids the Runge phenomenon naturally with its equipped nodes, and in the algorithms with temporal evolution, which demand strong processing, the polynomial calculations can be done by fast Fourier transform, including the derivatives. Now, by using Chebyshev expansion, they get an approximate solution of the coupled system (29) under the specified boundary conditions. But please note that the nodes for the Gauss–Legendre Quadrature are not immediately obtainable. There is, as in Chebyshev's case, a short analytical expression to determine it. In addition, in only a few

pathological cases, the difference between the Gauss quadrature and the Clenshaw–Curtis quadrature is significant in relation to other elements of the numerical process (Trefethen 2008).

The numerical methods for Eq. (29) can be extended to other types of FOCPs. In Doha et al. (2015) a direct numerical method for solving a general class of FOCPs based on the Jacobi polynomials is presented, for $(A, B) = (0, 1)$. The authors of Doha et al. (2015), Agrawal (2004) approximate $x(t)$ and $\lambda(t)$ as

$$x(t) = \sum_{j=1}^{N} c_j P_j(t), \quad \lambda(t) = \sum_{j=1}^{N} d_j P_j(t), \tag{41}$$

where $P_j(t)$, $j = 1, \ldots, N$, are the shifted Legendre polynomials which satisfy the orthonormality conditions, c_j and d_j, $j = 1, \ldots, N$, are polynomial coefficients and N is the number of polynomials selected. Indeed, the method is based upon the Legendre orthonormal polynomial basis. In this case, the operational matrix of the fractional Caputo derivative is constructed in Doha et al. (2015), Lotfi et al. (2011), Bhrawy et al. (2015), Sweilam and Al-Ajami (2015) as follows:

$$D^{\alpha} = \begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1(m+1)} \\ \vdots & \vdots & \ldots & \vdots \\ D_{(m+1)1} & D_{(m+1)2} & \cdots & D_{(m+1)(m+1)} \end{bmatrix} \tag{42}$$

where

$$D_{ij} = \hat{B}_{i-1\,j-1}, \quad 1 \le i, j \le m+1, \tag{43}$$

and

$$\hat{B}_{ij} = \sqrt{(2i+1)(2j+1)} \sum_{\lceil \alpha \rceil}^{i} \sum_{l=0}^{j} \frac{(-1)^{i+k+j+l}(i+k)!(j+l)!}{(i-k)!k!\Gamma(k+1-\alpha)(j-l)!(l!)^2(l+k-\alpha+1)}.$$

By this method, the given optimization problem reduces to the problem of solving a system of algebraic equations. By solving this system, we achieve the solution of the FOCP. In this technique, at first, they expand the unknown function in terms of the modified Jacobi polynomials and then, derive a compact form of fractional derivative of the unknown function in terms of the Jacobi polynomials. Also, they show that for large enough values of m, this method will converge to the exact solutions. The convergence of the method is extensively discussed in Lotfi et al. (2013).

Note that it is not necessary to select orthonormal polynomials as the basis functions. Orthonormal polynomials are selected here because they lead to numerically stable sparse matrices, and in many cases the properties of the polynomials can be used to generate the desired matrices efficiently. It is not necessary to select the shifted Legendre orthonormal polynomials only. Other orthonormal polynomials can also be selected for this task. However, this may require some modifications in the formulation so that one can take advantage of the properties of the orthonormal polynomials.

A developed numerical procedure based on the Bernstein polynomials approximation was presented in Nemati et al. (2016). They construct a new fractional operational matrix applicable in the Ritz method to estimate the fractional and integer order derivatives of the basis. The presented methods are similar to Lotfi et al. (2011) with the fractional Caputo

derivative approximated by the Bernstein polynomials, which can be written as:

$$D^{\alpha} = \begin{bmatrix} b_{00} & b_{01} & \cdots & b_{0m} \\ \vdots & \vdots & \cdots & \vdots \\ b_{m0} & b_{m1} & \cdots & b_{mm} \end{bmatrix} \tag{44}$$

where

$$\begin{aligned} b_{ij} = & \sum_{k=0}^{min(m-[\alpha]-1,m-i)} \sum_{q=0}^{m} \sum_{r=0}^{m-q} \sum_{l=0}^{min(j,q)} (-1)^{2m-i-k-r+j} \\ & \times \frac{(2l+1)}{\binom{m}{j}} \binom{m}{i} \binom{m-i}{k} \binom{m+l+1}{m-j} \binom{m-l}{m-j} \binom{m+l+1}{m-q} \binom{m-l}{m-q} \binom{m-q}{r} \\ & \times \frac{(m-k)!}{(2m-\alpha-r-k+1)\Gamma(m-k-\alpha+1)}, \qquad 0 \le i, j \le m. \end{aligned}$$

This approximation method is computationally consistent and, moreover, has a good flexibility in the sense of satisfying the initial and boundary conditions of the optimal control problems. The established method provided an upper bound error for Caputo fractional operational matrix of Bernstein polynomials in terms of Gram determinant that tends to zero.

In Yousefi et al. (2011) the Legendre multiwavelet basis with the aid of a collocation method has been applied to give the approximate solution for the FOCPs. In this approach, a function $x(t)$ defined over $[t_0, t_f]$ may be expanded as

$$x(t) = \sum_{i=0}^{2^k-1} \sum_{j=0}^{M} c_{ij} \psi_{ij} = C^T \Psi(t), \tag{45}$$

where k can be assumed to be any positive integer,

$$\psi_{nm}(t) = \begin{cases} \sqrt{2m+1}\frac{2^{k/2}}{\sqrt{t_f-t_0}} P_m(\frac{2^k(t-t_0)}{t_f-t_0} - n), & \frac{n(t_f-t_0)}{2^k} + t_0 \le t < \frac{(n+1)(t_f-t_0)}{2^k} + t_0 \\ 0, & \text{otherwise}, \end{cases} \tag{46}$$

where $m = 0, 1, \ldots, M-1, n = 0, 1, \ldots, 2^k-1$, $P_m(t)$ are the well-known shifted Legendre polynomials of order m and c_{ij} can be calculated from Heydari et al. (2016):

$$c_{ij} = \int_{t_0}^{t_f} \psi_{ij}(t) x(t) \mathrm{d}t, \tag{47}$$

with $i = 0, 1, \ldots, 2^k - 1$, $j = 0, 1, \ldots M$. Using the Legendre multiwavelet basis they get two degrees of freedom which increase the accuracy of the method. One of these parameters is the dilation argument k and the other is m corresponds to the number of elements of the basis in every subinterval $[\frac{n(t_f-t_0)}{2^k} + t_0, \frac{(n+1)(t_f-t_0)}{2^k} + t_0]$. They approximate $x(t)$, $u(t)$ and $\lambda(t)$ in the form of truncated series by multiwavelets (46) as follows:

$$
\begin{aligned}
x(t) &= \sum_{i=0}^{2^k-1} \sum_{j=0}^{M} (t - t_0) c x_{ij} \psi_{ij}(t) + x_0, \\
u(t) &= \sum_{i=0}^{2^k-1} \sum_{j=0}^{M} c u_{ij} \psi_{ij}(t), \\
\lambda(t) &= \sum_{i=0}^{2^k-1} \sum_{j=0}^{M} (t - t_f) c \lambda_{ij} \psi_{ij}(t).
\end{aligned} \tag{48}
$$

This approximation provides greater flexibility in which to impose the initial conditions of the system (29). These properties together with the collocation method are then utilized to reduce original problem to the solution of an algebraic system. As we can see in the presented numerical result of Yousefi et al. (2011), with only a small number of Legendre multiwavelets we can obtain satisfactory results.

Another approach is based on the fact that equation (29) for $(A, B) = (0, 1)$, is equivalent to the following Volterra integral equations (Soradi Zeid et al. 2016):

$$
\begin{aligned}
\lambda(t) &= -\frac{1}{\Gamma(\alpha)} \int_t^{t_f} (\tau - t)^{\alpha-1} M(\tau, x(\tau), \lambda(\tau)) \mathrm{d}\tau, \\
x(t) &= x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \tau)^{\alpha-1} N(\tau, x(\tau), \lambda(\tau)) \mathrm{d}\tau,
\end{aligned} \tag{49}
$$

will be called the forward Volterra integral equation and the backward Volterra integral equation, respectively. There are a number of schemes for the numerical solution of the initial value problem (29) (or (49)) in the literature such as, the class of the fractional multistep method (Garrappa 2009) and the fractional Adams method (Li and Tao 2009; Odibat and Momani 2008).

To obtain the numerical solution of equations (49), in Agrawal (2008) the authors divide the time domain $[t_0, t_f]$ into N equal intervals and let $h = \frac{1}{N}$. Then, the value of $\lambda(t)$ at node t_i is given as

$$
\lambda_i(t) = -\frac{1}{\Gamma(\alpha)} \int_{ih}^{t_f} (\tau - ih)^{\alpha-1} M(\tau, x(\tau), \lambda(\tau)) \mathrm{d}\tau \tag{50}
$$

and the value of $x(t)$ at node t_i can be given by

$$
x_i = x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{ih} (ih - \tau)^{\alpha-1} N(\tau, x(\tau), \lambda(\tau)) \mathrm{d}\tau. \tag{51}
$$

After some manipulations, Eqs. (50) and (51) reduce to a set of $2N$ linear simultaneous equations in terms of $2N$ unknowns, which can be solved using a direct scheme such as the Gaussian elimination method or an iterative scheme such as the conjugate gradient method (Doha et al. 2015). In a similar way, in Ozdemir et al. (2009a), an FOCP of a distributed system was investigated in cylindrical coordinates in which the fractional time derivative was defined in the Riemann–Liouville sense. The performance index of a FOCP is considered as a function of state and control variables and system dynamics are given as a partial fractional differential equation (PFDE). The method of separation of variables is used to find the solution of the problem. Therefore, the PFDE was decomposed into fractional ordinary and Bessel differential equations. Eigen-functions are used to eliminate the terms containing space parameters and to define the problem in terms of a set of generalized state and control

variables and for numerical computations, Grunwald–Letnikov approach is used (Ozdemir et al. 2009b). The solutions of this approach are converged when the time discretization was increased and as order of fractional derivative approached 1.

In this way, in Sabouri et al. (2016) proposed an artificial neural network with unknown weights to approximate the solution of FOCPs. By using perceptron neural networks ability in approximating a nonlinear function, they propose approximating functions to estimate control, state and co-state functions which they satisfy the initial or boundary conditions:

$$\begin{cases} x_K(t,\psi_x) = A(t) + B(t)K(t,\psi_x) \\ u_K(t,\psi_u) = C(t) + D(t)K(t,\psi_u) \\ \lambda_K(t,\psi_\lambda) = F(t) + G(t)K(t,\psi_\lambda), \end{cases} \tag{52}$$

where $A(t)$, $B(t)$, $C(t)$, $D(t)$, $F(t)$ and $G(t)$ are real single variable functions such that the approximations of x_K, u_K and λ_K satisfy the initial or final conditions. For example, if $x(0) = 0$ then we must choose $A(t)$ and $B(t)$ such that $x_K(0,\psi_x) = 0$, thus we can choose $A(t) = 0$ and $B(t) = t$. Also, ψ_x, ψ_u and ψ_λ are the corresponding weight vectors containing the weights of $x(t)$, $u(t)$ and $\lambda(t)$, respectively. Then, by substituting approximation (52) in (49), we get:

$$\begin{aligned} \lambda_K(t,\psi_\lambda) &= -\frac{1}{\Gamma(\alpha)}\int_t^{t_f}(\tau - t)^{\alpha-1}M_K(\tau, x_K(t,\psi_x), \lambda_K(t,\psi_\lambda))\mathrm{d}\tau, \\ x_K(t,\psi_x) &= x_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t - \tau)^{\alpha-1}N_K(\tau, x_K(t,\psi_x), \lambda_K(t,\psi_\lambda))\mathrm{d}\tau, \end{aligned} \tag{53}$$

To solve (53) they introduce the following squared residual error functions:

$$\begin{aligned} R_\lambda(\Psi,t) &= [\lambda_K(t,\psi_\lambda) + \frac{1}{\Gamma(\alpha)}\int_t^{t_f}(\tau - t)^{\alpha-1}M_K(\tau, x_K(t,\psi_x), \lambda_K(t,\psi_\lambda))\mathrm{d}\tau]^2, \\ R_x(\Psi,t) &= [x_K(t,\psi_x) - x_0 - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t - \tau)^{\alpha-1}N_K(\tau, x_K(t,\psi_x), \lambda_K(t,\psi_\lambda))\mathrm{d}\tau]^2, \end{aligned} \tag{54}$$

where $\Psi = (\psi_x, \psi_u, \psi_\lambda)$ is a vector containing all weights of three approximator functions (52). To solve (54), divide the interval $[t_0, t_f]$ into m subinterval and calculate the integrals in any subintervals, by using any numerical integration technique such as Simpson's rule. Then, the following unconstrained optimization problem is introduced:

$$\min_{\Psi} R(\Psi) = \sum_{i=1}^{m}\Big[R_\lambda(\Psi,t_i) + R_x(\Psi,t_i)\Big], \tag{55}$$

which can be solved by any classical mathematical optimization algorithm such as Quasi-Newton methods that we use in this paper. Suppose that $\Psi^* = (\psi_x^*, \psi_u^*, \psi_\lambda^*)$ is the optimal solution of optimization problem (55). Since the neural networks are universal approximators, the obtained weights are convergent to the optimal values. This concept is illustrated in numerical examples by plotting the convergence of the weights. Substituting these optimal weights into the corresponding approximate functions x_K, u_K and λ_K in (52), the following final approximated solution of FOCP (24)-(26) will be obtained:

$$\begin{cases} x_K(t,\psi_x^*) = A(t) + B(t)K(t,\psi_x^*) \\ u_K(t,\psi_u^*) = C(t) + D(t)K(t,\psi_u^*) \\ \lambda_K(t,\psi_\lambda^*) = F(t) + G(t)K(t,\psi_\lambda^*). \end{cases} \tag{56}$$

We mention that to attain more accurate solutions, we can use more neurons or use any heuristic optimization algorithm. Since the neural networks are universal approximators, the obtained weights are convergent to the optimal values.

In recent years, the authors presented a new iterative formula to solve a class of FOCPs. The variational iteration method (VIM) is used to solve the resulting fractional differential equations (29) with $(A, B) = (0, 1)$. In order to illustrate the basic concepts of the VIM, we consider the following general FDE system:

$$ {}_C D^{\alpha}_{t_0,t} x(t) + L[x(t)] + N[x(t)] = f(t), \tag{57} $$

where ${}_C D^{\alpha}_{t_0,t} x(t)$ is the Caputo derivative, L is a linear operator, N is a nonlinear operator and $f(t)$ is a given continuous function. The VIM presents a correction functional for equation (57) in the following form:

$$ x_{n+1}(t) = x_n(t) + \int_0^t \lambda(\tau)\Big[{}_C D^{\alpha}_{t_0,t} x_n(\tau) - L[\tilde{x_n}(\tau)] - N[\tilde{x_n}(\tau)] - f(\tau)\Big] \mathrm{d}\tau, \quad t > 0, \quad \alpha > 0, \tag{58} $$

where λ is the Lagrange multiplier which can be identified explicitly based on fractional variational theory. The terms $L[\tilde{x_n}]$ and $N[\tilde{x_n}]$ are restricted variations, that is, $\delta L[\tilde{x_n}] = \delta N[\tilde{x_n}] = 0$. The approximations $x_n(t)$ of the solution $x(t)$ can be achieved using any zeroth approximation $x_0(t)$. Consequently, the solution is given as $x(t) = \lim_{n\to\infty} x_n(t)$. But in practice, especially for nonlinear equations, it is difficult to calculate this limit and it can only be approximated. Consequently, an approximation solution can be obtained by considering a large value for n, and the solution is given as $x(t) \approx x_n(t)$.

Authors in Alizadeh and Effati (2016) use the VIM for solving Eq. (29) and then, they have:

$$ \begin{aligned} \lambda_{n+1}(t) &= \lambda_n(t) - {}_t I^{\alpha}_{t_f}\Big[{}_{RL} D^{\alpha}_{t,t_f} \lambda_n(t) - M(t, x_n(t), \lambda_n(t))\Big] \\ x_{n+1}(t) &= x_n(t) - {}_{t_0} I^{\alpha}_{t}\Big[{}_C D^{\alpha}_{t_0,t} x_n(t) - N(t, x_n(t), \lambda_n(t))\Big]. \end{aligned} \tag{59} $$

To start the iteration process, we select the initial approximation of $\lambda_n(t)$. The approximation solutions of equations (59) must hold for the initial conditions, the boundary conditions and the transversality boundary conditions, that is, $x(t_0) = x_0$, $\lambda(t_f) = 0$. At each iteration the approximate solutions of $x_n(t)$ and $\lambda_n(t)$ deduced from equations (59) are utilized in the next iteration, and thus the solutions of $x_n(t)$ and $\lambda_n(t)$ must be approximate at the same time. The authors show that this technique rapidly provides the convergent successive approximations of the exact solution and the solutions approach the classical solutions of the problem as the order of the FDs approaches 1.

Authors of Tang et al. (2015) provide a unified framework and develop integral fractional pseudospectral methods for solving FOCPs. As a generalization of conventional pseudospectral integration matrices, fractional pseudospectral integration matrices (FPIMs) and their efficient and stable computation are the keys to their approximation. In order to achieve this goal, they use a special and smart method to compute FPIMs. The essential idea is to transform the fractional integral of Lagrange interpolating polynomials through a change of variables into their Jacobi-weight integral, which can be calculated exactly using the Jacobi–Gauss quadrature. This, together with the stable barycentric representation of Lagrange interpolating polynomials and the explicit barycentric weights for the Gauss-, flipped Radau-, and Radau-type points corresponding to the Jacobi polynomials, leads to an exact, efficient,

and stable scheme to compute FPIMs even at millions of Jacobi-type points. The pseudostate space in Biswas and Sen (2011), proposed a direct numerical technique based on the Grunwald–Letnikov approximation that is used to solve the resulting equations in (29).

In Biswas and Sen (2014) a discrete method was proposed based on the Grunwald–Letnikov approximation (16) and (17). They divided the entire time domain into N equal domains, and labeled the time at node j by $t_j = jh$ where $h = \frac{t_f - t_0}{N}$. By using this approach, Eq. (29) can be written as

$$\begin{aligned} -h^{-\alpha} \sum_{j=0}^{N-m} w_j^{\alpha} \lambda_{m+j} &= M(mh, x_m(t), \lambda_m(t)), \quad m = N-1, N-2, \ldots, 0 \\ h^{-\alpha} \sum_{j=0}^{m} w_j^{\alpha} x_{m-j} &= N(mh, x_m(t), \lambda_m(t)), \quad m = 1, 2, \ldots, N \end{aligned} \tag{60}$$

where x_i and λ_i are the numerical approximations of $x(t)$ and $\lambda(t)$ at node i and w_j^{α} are defined as before. These equations with the boundary conditions $x(t_0) = x_0$ and $\lambda(t_f) = 0$ form $2N$ equations in terms of $2N$ unknowns and they can be solved using any linear equation solver (Biswas and Sen 2009).

4.2 Direct methods for FOCPs

The authors of Almeida and Torres (2015) replace the operator ${}_C D_{t_0,t}^{\alpha} x(t)$ with the help of approximation (21). With relation (8) they get

$$\begin{aligned} &A\dot{x}(t) + B\Big[A(\alpha, N)(t - t_0)^{-\alpha} x(t) + B(\alpha, N)(t - t_0)^{1-\alpha} \dot{x}(t) \\ &- \sum_{p=2}^{N} C(\alpha, N)(t - t_0)^{1-p-\alpha} V_p(t) - \frac{x(t_0)(t - t_0)^{-\alpha}}{\Gamma(1-\alpha)}\Big] = G(t, x(t), u(t)). \end{aligned} \tag{61}$$

Thus, one has

$$\dot{x}(t) = \frac{G(t, x(t), u(t)) - B\Big[A(\alpha, N)(t - t_0)^{-\alpha} x(t) + \sum_{p=2}^{N} C(\alpha, N)(t - t_0)^{1-p-\alpha} V_p(t) + \frac{x(t_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\Big]}{A + B \times B(\alpha, N)(t - t_0)^{1-\alpha}}. \tag{62}$$

Define the vector $\bar{V(t)} = (V_2(t), V_3(t), \ldots, V_N(t))$ and the new function

$$\bar{G}(t, x, \bar{V}, u) = \frac{G(t, x(t), u(t)) - B\Big[A(\alpha, N)(t - t_0)^{-\alpha} x(t) + \sum_{p=2}^{N} C(\alpha, N)(t - t_0)^{1-p-\alpha} V_p(t) + \frac{x(t_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\Big]}{A + B \times B(\alpha, N)(t - t_0)^{1-\alpha}}. \tag{63}$$

So, they established a new optimal control problem as follows:

$$\min \bar{J}(x, \bar{V}, u) = \int_{t_0}^{t_f} F(t, x(t), u(t)) \mathrm{d}t \tag{64}$$

subject to the dynamic constraints

$$\begin{cases} \dot{x}(t) = \bar{G}(t, x, \bar{V}, u) \\ \dot{V}_p(t) = (1 - p)(t - t_0)^{p-2} x(t), \quad p = 2, 3, \ldots, N, \end{cases} \tag{65}$$

and the initial conditions

$$\begin{cases} x(t_0) = x_0, \\ V_p(t_0) = 0, \quad p = 2, 3, \ldots, N. \end{cases} \tag{66}$$

To solve the problem (64)–(66), one can consider the Hamiltonian function and by the Pontryagin maximum principle, to solve the problem, one should solve a system of ordinary differential equations (ODEs). Instead of this indirect approach, one can apply a direct method, based on an Euler discretization, to obtain a finite-dimensional approximation of the continuous problem (64)–(66). You can see the briefly summarized method in Almeida and Torres (2015).

Notice that $f(t) \in L^2[0, 1]$ may be expanded into Rationalized Haar (RH) functions as

$$f(t) = \sum_{r=0}^{\infty} a_r RH(r, t), \tag{67}$$

where $RH(r, t)$, $r = 1, 2, \ldots$, are the RH functions that be defined on the interval $[0, 1)$ by Marzban and Razaghi (2010) and a_r are given by

$$a_r = 2^i \int_0^1 f(t) RH(r, t) \mathrm{d}t, \quad r = 0, 1, \ldots, \tag{68}$$

with $r = 2^i + j - 1$, $i = 0, 1, 2, \ldots$, $j = 1, 2, \ldots, 2^i$, and $r = 0$ for $i = j = 0$. If, we let $i = 0, 1, 2, \ldots, \alpha$, then the infinite series in (67) is truncated into its first K terms as

$$f(t) \simeq \sum_{r=0}^{K-1} a_r RH(r, t) = P^T \Phi_K(t), \tag{69}$$

where $K = 2^{\alpha+1}, \alpha = 0, 1, 2, \ldots$ and the RH functions coefficient vector P and RH functions vector $\Phi_K(t)$ are defined as:

$$\begin{aligned} P &= [a_0, a_1, \ldots, a_{K-1}]^T, \\ \Phi_K(t) &= [\Phi_0(t), \Phi_1(t), \ldots, \Phi_{K-1}(t)]^T, \end{aligned} \tag{70}$$

where $\Phi_r(t) = RH(r, t), r = 0, 1, \ldots, K-1$. Haar functions are also notable for their rapid convergence for the expansion of functions, and this capability makes them very useful with regard to the Haar functions theory. The authors of Hosseinpour and Nazemi (2015) using these Haar wavelets for solving FOCPs in the Caputo sense. They applied this technique to transform the state and control variables into non-linear programming (NLP) parameters at collocation points. For this purpose, they assume that the fractional derivative of the state variables and control variable can be approximated by Haar wavelets with K collocation points, i.e.,

$$\begin{cases} {}_C D^{\alpha}_{t_0,t} x(t) \approx C^T \Phi(t) \Rightarrow x(t) \approx C^T U^{\alpha}_{K\times K} \Phi(t) + x(t_0), \\ u(t) \approx D^T \Phi(t) \end{cases} \tag{71}$$

where $U^{\alpha}_{K\times K}$ is the fractional operational matrix of integration of order α and

$$\begin{cases} C^T = [c_1, c_2, \ldots, c_K]^T, \\ D^T = [d_1, d_2, \ldots, d_K]^T. \end{cases} \tag{72}$$

When the Haar collocation method is applied in the FOCPs, directly, the NLP variables can be set as the unknown coefficients vector of the fractional derivative of the state variables and control variables. In this way, the FOCPs are transformed into NLP problems in a structured form which is solved by Lingo 11 software (Plant et al. 1997). Also, they show that the proposed orthogonal collocation method leads to rapid convergence as the number of collocation points increases.

5 Numerical methods for delay FOCPs

A special type of FOCP is the delay FOCP (DFOCP). We can often see delays in the transmission of material or information between different parts of the systems. Many papers have been devoted to delayed optimal control problems (DOCPs) and the derivation of necessary optimality conditions. The theory of delay differential equations was introduced in Driver (2012). In Deng et al. (2007), the stability of linear fractional differential equations was analyzed. The relationship between signal delay and fractional dynamics was analyzed in Martins Lima et al. (2008). In 1977, Driver (2012) introduced the theory of delay differential equations that have been used in many real-life phenomena such as communication, power systems, transportation, biological, electronics, manufacturing and chemical (Jamshidi and Wang 1984; Malek-Zavarei and Jamshidi 1987). The delay fractional optimal control problem is an optimal control problem in which the performance index is a delay fractional differential equation, see Witayakiattilerd (2013), Wang et al. (2014), Jarad et al. (2012). In Rosenblueth (1988) optimal control problems with time delay in the calculus of variations were discussed. Fractional variational problems in the presence of delay were studied in Baleanu et al. (2008).

Recently, Safaie et al. (2015) used the Bernstein polynomials as basis function of a numerical technique for solving a fractional optimal control problem with delay in state, while in Safaie and Farahi (2014), the authors introduced a numerical technique for a fractional optimal control problem with both state and control delays. The authors in Dehghan and Keyanpour (2015) present a method based upon the moments problem for solving a class of FOCPs with time delay. The performance index of this problem is considered as a function of both the state and control variables and the dynamics of system is given as an ordinary fractional differential equation with time delay. The FD is described in the Riemann–Liouville sense in which the FD order is $\alpha \in (0, 1]$. The main reason of using this technique is the convexification of a non-linear and non-convex FOCP with time delay in which the non-linearity in the control variable can be expressed as polynomials. The Grunwald–Letnikov formula is used as an approximation for FD in numerical computations of this work. In Bhrawy and Ezz-Eldien (2015), A new numerical approach is constructed for numerically approximating the state and control variables of the fractional optimal control problem with both delay in state and control. The proposed approach is based on the shifted Legendre orthonormal polynomials. The main advantage of this proposed algorithm is, adding few terms of the shifted Legendre orthonormal polynomials, good approximations of the state and control variables were achieved.

6 Conclusions

In this paper, we present a survey on the existing numerical approximations for the FOCPs. Some of the existing methods can be seen as the generalizations of the numerical methods for

classical optimal control problems. From the publications available, these methods mainly focus on the extended polynomial approximations such as the Jacobi, Bernoulli, Bernstein and Legendre polynomials, as numerical methods for FOCPs accordingly. However, very limited work has been done for delay ones.

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