A Novel Method for Harmonic Geometric Transformation Model Based on Wavelet Collocation

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Abstract

Geometric distortion may occur in the data acquisition phase in information systems, and it can be characterized by some geometric transformation models. Once the distorted image is approximated by a certain geometric transformation model, we can apply its inverse transformation for the geometric restoration to remove the distortion. Harmonic model is a very important one, which can cover other linear and nonlinear geometric models. However, its implementation is very complicated, because it can not be described by any fixed functions in mathematics. In fact, it is represented by partial differential equation with a given boundary condition. In this paper, a novel wavelet-based method is presented to handle the harmonic model. Our approach has two main advantages, the shape of an image is arbitrary and the program code is independent of the boundary. The performances are evaluated by experiments.

1. Introduction

Geometric distortion may be produced in the data acquisition phase in pattern recognition systems, and it can be characterized by some geometric transformation models [1, 2]. Harmonic model is a very important one, which can cover other linear and nonlinear geometric models. An example can be found in Fig. 1, where the image of Canadian flag is distorted as shown in Fig. 1(b). Note that the shapes of the flag are changed, which are so complex that they can not be described by any fixed models, i.e. they can not be represented by any fixed functions in mathematics. In fact, these models are characterized by partial differential equation. Unlike the fixed mathematical formulas, solving the partial differential equation is sophisticated.

Let Ω be the region of the elastic plane, where the image is located, and Γ be its boundary. Suppose the functions of the transformation of boundary Γ are $u = f(x_1, x_2)$, and $v = g(x_1, x_2)$. The harmonic transformation

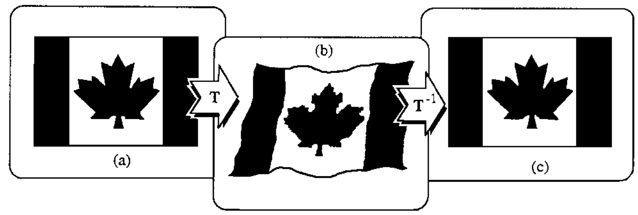

Figure 1. Harmonic distortion and restoration

$$T : (x_1, x_2) \rightarrow (u, v)$$

satisfies the partial differential equation:

$$\begin{cases} \Delta u(x_1, x_2) = 0, & (x_1, x_2) \in \Omega \\ u|_\Gamma = f(x_1, x_2), & (x_1, x_2) \in \Gamma, \end{cases} \quad (1)$$

$$\begin{cases} \Delta v(x_1, x_2) = 0, & (x_1, x_2) \in \Omega \\ v|_\Gamma = g(x_1, x_2), & (x_1, x_2) \in \Gamma, \end{cases} \quad (2)$$

where Δ is Laplace's operator:

$$\Delta : \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

Accordingly, the task of the restoration is solving the above harmonic equation. We return to the previous example as shown in Fig. 1, the distorted image in Fig. 1(b) can be approximated by harmonic transformation. As the corresponding harmonic equation is solved, and its inverse transformation is utilized, the restored image can be obtained in Fig. 1(c). Therefore, solving the harmonic equation, Eqs. (1) and (2), plays a key role in the geometric restoration.

This paper proposes a novel approach based on wavelet analysis to handle the harmonic transformation.

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2. Integral Equation - Wavelet Collocation (IEWC) Approach

A novel approach based on the integral equation and wavelets, called Integral Equation -Wavelet Collocation (IEWC), is presented in this section.

Phase 1: First, the *partial differential equation (PDE)* is changed into the form of integral equation and integral representation on boundary Γ, which are called *boundary integral equation (BIE)* and *boundary integral representation (BIR)* respectively. There are two ways to do so, namely, direct method and indirect method. In this paper, the *indirect method* is utilized [3, 4]. Mathematically, the process for solving u can be written below:

$$PDE: \quad \begin{cases} \Delta u(x_1,x_2)=0, & \forall(x_1,x_2)\in\Omega \\ u|_\Gamma = f(x_1,x_2), & \forall(x_1,x_2)\in\Gamma, \end{cases}$$

$$\Downarrow$$

$$BIE: \quad \int_\Gamma \omega(y) log|x-y| ds_y = f(x), \quad \forall(x_1,x_2)\in\Gamma$$

$$BIR: \quad u(x) = \int_\Gamma \omega(y)\ldots\ldots ds_y, \quad \forall(x_1,x_2)\in\omega$$

The first shortcoming, which arises in the finite element method, can be overcome by this way.

Similarly, v can also be obtained in the same way, which will be omitted to save the space. In the remainder of this paper, we only discuss u.

Phase 2: In order to solve the BIE efficiently and to get rid of the second defect in the finite element method, the *boundary measure formula (BMF)* is used [5]. It changes the boundary integral equation and boundary integral representation into an integral equation and an integral representation on the whole plane R^2 rather than the special boundary Γ. They are called *plane integral equation (PIE)* and *plane integral representation (PIR)* respectively. In this way, when Γ is changed, the program code will need not be modified at all. In mathematics, this process can be presented in the following:

$$BIE: \quad \int_\Gamma \omega(y) log|x-y| ds_y = f(x), \quad \forall(x_1,x_2)\in\Gamma$$

$$BIR: \quad u(x) = \int_\Gamma \omega(y)\ldots\ldots ds_y, \quad \forall(x_1,x_2)\in\omega$$

$$\Downarrow$$

$$PIE: \int_{R^2} \omega(y)||\partial\Omega|| log|x-y| ds_y = f(x),\ \forall(x_1,x_2)\in\Gamma$$

$$PIR: u(x) = \int_{R^2} \omega(y)||\partial\Omega||\ldots\ldots ds_y,\ \forall(x_1,x_2)\in\omega$$

Phase 3: Then, *wavelet collocation* technique is used to solve the PIE. In the integral equation, the integrand has a discontinuity across boundary Γ. Hence, there is a kind of singularity in it. Fortunately, wavelets have a good property to approximate this kind of singularity [6]. The mathematical representation can be illustrated below:

$$PIE: \int_{R^2} \omega(y)||\partial\Omega|| log|x-y| ds_y = f(x),\ \forall(x_1,x_2)\in\Gamma$$

$$PIR: u(x) = \int_{R^2} \omega(y)||\partial\Omega||\ldots ds_y,\ \forall(x_1,x_2)\in\omega$$

$$\Downarrow$$

$$\Sigma_{p,q} h^j_{(p,q)} \int_{R^2_y} \phi^j_p(y_1)\phi^j_q(y_2) log|x_k-y| dy - f(x_k) = 0,$$

$$\forall(x_1,x_2)\in\Gamma$$

$$u^j(x) = \Sigma_{p,q} h^j_{(p,q)} \int_{R^2_y} \phi^j_p(y_1)\phi^j_q(y_2)\ldots dy,$$

$$\forall(x_1,x_2)\in\omega$$

$$\Downarrow$$

Solutions

The principal advantages of our method are as follows:

- The algorithm is divided into two parts, integral equation and integral representation. After solving the plane integral equation, we can choose the pixels to be transformed in the domain arbitrarily and use plane integral representation to evaluate their new coordinates. Therefore, only the pixels on the pattern are transformed to the new coordinate space.

- The program code is independent of the domain considered, i.e. the program code will need not be changed for the different kind of boundaries. It benefits from the boundary measure formula. In fact, we do not need the function of boundary Γ at all. What we really need are the original coordinates of the pixels at the boundary and the coordinates of the new ones.

3. Algorithm and Experiments

Experiments have been conducted to evaluate the performances of the new approach. In this section, the wavelet-based algorithm of the IEWC approach is presented followed by several experiments.

Algorithm based on boundary measure and wavelet

- **Step-1** Choose the collocation points $\{x_k\}_{k=1}^{|\Lambda|}$ on Γ, and evaluate matrix K and $\overline{f}$.

- **Step-2** Solve equation $K\overline{h} = \overline{f}$ with least square method to obtain coefficients $\{h^j_{(p,q)}\}_{(p,q)\in\Lambda}$.

- **Step-3** Choose a point x in the domain needed to be transformed to new coordinate, and calculate the coefficients $\{Q_{(p,q)}(x)\}_{(p,q)\in\Lambda}$.
- **Step-4** Obtain the approximation $u^j(x)$.

The harmonic transformation $T : (x, y) \to (u, v)$ satisfies the following equations, which is recalled below:

$$\begin{cases} \Delta u(x_1, x_2) = 0, & (x_1, x_2) \in \Omega \\ u|_\Gamma = f(x_1, x_2), & (x_1, x_2) \in \Gamma \end{cases}$$

$$\begin{cases} \Delta v(x_1, x_2) = 0, & (x_1, x_2) \in \Omega \\ v|_\Gamma = g(x_1, x_2), & (x_1, x_2) \in \Gamma \end{cases}$$

Experiment 1

$$T_1 : (x_1, x_2) \to (u, v), \quad \Omega : 0 \leq x_1, x_2 \leq 0.5$$

$$f(x_1, x_2) = 1.2x_1 + 0.2x_2$$

$$g(x_1, x_2) = x_2 - c * sin(4\pi x_1 - 1), \quad c = 0.05$$

The result is shown in Fig. 2, where the shape of a flag has been changed. Please note that the up-side and bottom-side of the new shape are neither quadratic shape nor cubic one, they can not be obtained by any fixed geometric transformation models.

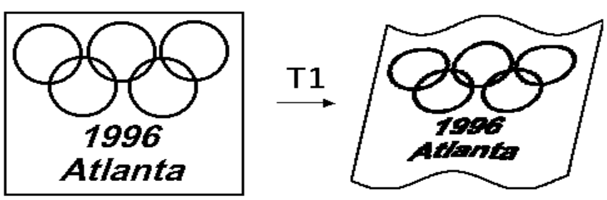

Figure 2. Experiment 1 - The shape of a flag is changed by a harmonic transformation

Experiment 2 (nonlinear-analytic)

$$T_2 : (x_1, x_2) \to (u, v), \quad \Omega : 0 \leq x_1, x_2 \leq 0.5$$

$$f(x_1, x_2) = e^{2x_2}(sin(2x_1) - cos(2x_1))$$

$$g(x_1, x_2) = e^{2x_2}(sin(2x_1) + cos(2x_1))$$

In fact, the solutions $u(x_1, x_2), v(x_1, x_2)$ in this example satisfy the Cauchy-Riemann condition

$$\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_2},$$

$$\frac{\partial u}{\partial x_2} = -\frac{\partial v}{\partial x_1}.$$

In this example, the transformation is conformal. As well known, conformal mapping is very important because it has many properties such as the angles preserving. The result is shown in Fig. 3.

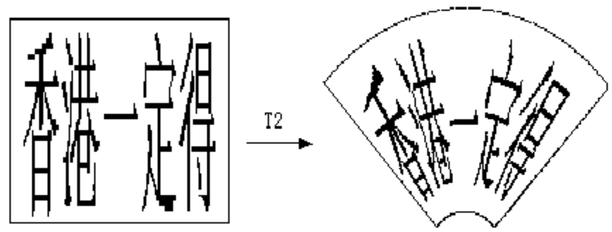

Figure 3. Experiment 2 - The shapes of Chinese words "Hong Kong for sure" are changed by an analytic (conformal) transform

4. Experimental Applications

Geometric transformation models can be widely used in many aspects, such as computer graphics, computer vision, robot vision, signal processing, image processing and pattern recognition, etc. In this paper, we emphasize their application in two fields:

(1) Distortion — the forward transformation is used to generate different geometric shapes, such as the generation of graphs in computer graphics and produce fonts of characters in pattern recognition.

(2) Restoration — the inverse transformation can be used to remove the distortions, therefore, it can be employed for the normalization of geometric shapes.

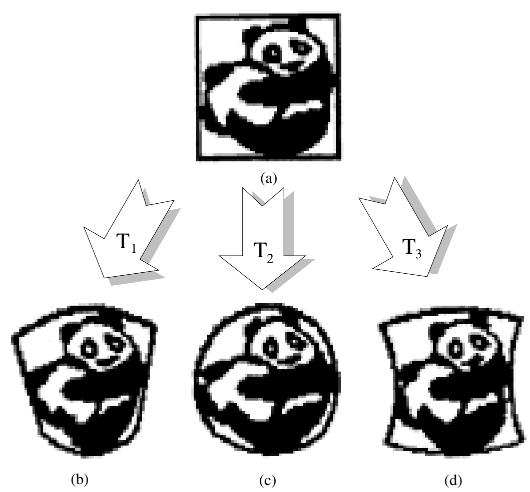

Figure 4. Different shapes of image of "panda" are generated by geometric transformation

4.1 Generation of Different Shapes

In computer graphics and pattern recognition, it is necessary to generate graphs and samples from a seed. For example, from an original figure of "panda" shown in Fig. 4(a), different shapes of "panda", which are illustrated in Figs. 4(b)-(d), are produced by geometric transformations.

4.2 Normalization of Geometric Shapes

Inverse geometric transformation can be used in the normalization process. Once the transformation T, which characterizes a distortion, has been established, its inverse transformation T^{-1} can be obtained, and employed for restoration. For instance, let us return to Fig. 1 in Section 1 (Introduction). A picture of the national flag of Canada is illustrated in Fig. 1(a). When it stands in the wind as shown in Fig. 1(b), the boundary of the converted image is arbitrary. The special shape transformation model such as bilinear, quadratic/bi-quadratic and cubic/bi-cubic will fail in this case. Thus, harmonic transformation has been employed. As a result, a restored image of the distorted flag has been reconstructed, which is exemplified in Fig. 1(c).

Normalization of geometric shapes of image "panda" by geometric transformation can be found in Fig. 5.

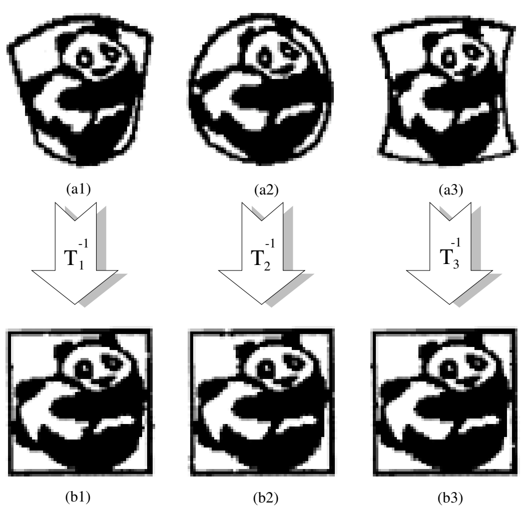

Figure 5. Normalization of geometric shapes of image by geometric transformation

5. Conclusions

In this paper, we have presented a wavelet-based approach for the harmonic transformation. Unlike the finite element method and finite difference method, the pixels needed to be transformed to new coordinates can be chosen arbitrarily. Meanwhile, the program code in our method is independent of the boundary. We need only a set of the original coordinates of the pixels on the boundary of the image as well as their new coordinates in the transformed image. To make the algorithm more efficiency, Daubechies wavelet (scale) functions and a Gauss-type quadrature formula have been used. Different examples have been tested with the anticipated results.

Some further works are still under study. As discussed above, there are singularities along the boundary, which can be treated efficiently by wavelet. More recently, D. Donoho [6] has constructed a new tool called *curvelet* to handle this kind of singularity, which is built from Meyer wavelet basis. In our further work, the curvelet will be utilized. In this way, the boundary measure will be approximated by the curvelet with the same accuracy as we use wavelet. Meantime, compared with wavelet or sinusoid basis, fewer terms will be computed when the curvelet will be used. It will makes our algorithm more efficient [7].

6 Acknowledgment

This work was supported by research grants received from Research Grant Council (RGC) of Hong Kong, and Faculty Research Grant (FRG) of Hong Kong Baptist University.

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