Applied Mathematics Letters 24 (2011) 724–730

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Equivariant Hopf bifurcation for functional differential equations of mixed type ☆

Shangjiang Guo

College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, People's Republic of China

ARTICLE INFO

Article history:
Received 10 July 2009
Received in revised form 20 December 2010
Accepted 22 December 2010

Keywords:
Lyapunov–Schmidt reduction
Equivariant Hopf bifurcation
Functional differential equations of mixed type
Lie group

ABSTRACT

In this paper we employ the Lyapunov–Schmidt procedure to set up equivariant Hopf bifurcation theory of functional differential equations of mixed type. In the process we derive criteria for the existence and direction of branches of bifurcating periodic solutions in terms of the original system, avoiding the process of center manifold reduction.

1. Introduction

In this paper, we consider the following parameterized functional differential equations of mixed type (MFDE)

$$\dot{x}(\xi) = L(\alpha)x_\xi + f(\alpha, x_\xi), \tag{1}$$

where $\alpha \in \mathbb{R}$, x is a continuous $\mathbb{R}^n$-valued function and for any $\xi \in \mathbb{R}$, the state $x_\xi \in \mathscr{C} := C([\tau_{\min}, \tau_{\max}], \mathbb{C}^n)$ is defined by $x_\xi(\theta) = x(\xi + \theta)$. We allow $\tau_{\min} \le 0$ and $\tau_{\max} \ge 0$, hence the operators $L(\alpha)$ and $f(\alpha, \cdot)$ may depend on advanced and retarded arguments simultaneously. Furthermore, we assume that $L(\alpha): \mathscr{C} \to \mathbb{R}^n$ is a linear operator, and $f: \mathscr{C} \to \mathbb{R}^n$ is a smooth enough nonlinear operator satisfying $f(\alpha, 0) = 0$. We say that (1) is Γ-equivariant if there exists a representation ϱ of a group Γ such that

$$f(\alpha, \varrho(\gamma)\phi) = \varrho(\gamma)f(\alpha, \phi), \qquad L(\alpha)\varrho(\gamma)\phi = \varrho(\gamma)L(\alpha)\phi, \tag{2}$$

for $(\alpha, \gamma, \phi) \in \mathbb{R} \times \Gamma \times \mathscr{C}$, where $\varrho(\gamma)\phi \in \mathscr{C}$ is given by $(\varrho(\gamma)\phi)(s) = \varrho(\gamma)\phi(s)$ for $s \in [-\tau, 0]$. Recall that a representation ϱ of a group Γ is a group homomorphism $\varrho : \Gamma \to GL(n, \mathbb{R})$. Condition (2) is equivalent to saying that system (1) is invariant under the transformation $(x, t) \to (\varrho(\gamma)x, t)$ in the sense that $x(t)$ is a solution of (1) if and only if $\varrho(\gamma)x(t)$ is a solution. Throughout this paper we always assume that Γ is a compact Lie group and system (1) is Γ-equivariant.

Historically, the primary motivation for the study of MFDE comes from the study of lattice differential equations (LDEs), which are systems of differential equations indexed by points on an (infinite) spatial lattice. In addition, MFDE plays a major role in a number of applications from economic theory.

The linearization around the equilibrium 0 is

$$\dot{x}(\xi) = L(\alpha)x_\xi. \tag{3}$$

☆ This work supported in part by the NSFC (Grant No. 10971057), by the Program for NCET (No. NCET-07-0264) and the Key Project (Grant No. [2009]41) of Chinese Ministry of Education, and by the Hunan Provincial Natural Science Foundation (Grant No. 10JJ1001).

E-mail address: shangjguo@hnu.cn.

0893-9659/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2010.12.017

The ill-posedness of the initial value problem of (3) prevents us from the construction of its semigroup and invariant manifolds for MFDE (1) as well. This drawback has long limited our understanding of the full nonlinear system (1). It was only during the last decade that significant theoretical progress has been made. Recently, a center manifold approach was developed to capture all solutions of (1) that remain sufficiently close to the equilibrium 0. It was shown that the dimension and linear structure on the center manifold are entirely determined by the holomorphic characteristic matrix $\Delta : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$ explicitly given by

$$\Delta(\alpha, \lambda) = \lambda \mathrm{Id}_n - L(\alpha)\exp\{\lambda(\cdot)\}. \tag{4}$$

The presence of the exponential functions with opposite signs in their arguments will in general cause the characteristic equation $\det \Delta(\alpha, \lambda) = 0$ to have an infinite set of roots both to the right and left of the imaginary axis, which is completely different from the spectrum of semigroup generators. The asymptotic location of eigenvalues for (3) was analyzed in early work by Bellman and Cooke [1]. An important observation is that any vertical strip $\{z \in \mathbb{C} : v_1 < \mathrm{Re}\, z < v_2\}$ contains only finitely many roots.

Suppose that a pair of roots of the characteristic equation $\det \Delta(\alpha, \lambda) = 0$ crosses the imaginary axis at a certain parameter value α_0. Under suitable conditions the Hopf bifurcation theorem can be lifted to the infinite dimensional setting of (1) and hence one may conclude the existence of a branch of periodic solutions to (1) bifurcating from the trivial equilibrium $x = 0$ for $\alpha \sim \alpha_0$. In [2] this approach was used to analyze an economic optimal control problem involving delays. This problem was proposed by Rustichini in order to simplify a model describing the dynamics of a capital market [3], whilst still retaining the periodic orbits that are compulsory for any such model. The existence of these periodic orbits was established by numerically analyzing the resulting characteristic equation and looking for root-crossings through the imaginary axis.

On the other hand, the presence of symmetry may cause purely imaginary eigenvalues to arise with higher multiplicities which cause the bifurcation problem to become more complicated, see for instance [4]. The most common approach to study bifurcation problems in function differential equations involves the computation of (normal forms of) reduced bifurcation equations on center manifolds. However, as stated before, major difficulties that need to be overcome in the construction of center manifolds for MFDE are the absence of a semiflow and the ill-posedness of the natural initial value problem. This precludes the direct application of the ideas developed by Faria and Magalhães [5] for retarded functional differential equations.

In this paper, we present a treatment of generic codimension-one Hopf bifurcation for equivariant MFDEs on the basis of equivariant Lyapunov–Schmidt reduction, following the spirit of the treatment of Golubitsky and Stewart [6,4] in the case of equivariant ODEs. Moreover, our results generalize those obtained by Hupkes and Verduyn Lunel [7]. In the process we obtain explicit expressions in terms of the original system that determine the monotonicity of the period and Hopf bifurcation direction of branches of bifurcating symmetric periodic solutions. With these expressions at our disposal, the study of equivariant Hopf bifurcation in explicit examples can be performed without having to resort to lengthy computations associated to center manifold reduction.

2. Main results

We first focus our attention on the state space $\mathscr{C}$ and define a closed and densely defined operator $\mathcal{A}_\alpha : \mathrm{Dom}(\mathcal{A}_\alpha) \subset \mathscr{C} \to \mathscr{C}$, via

$$\begin{aligned} &\mathrm{Dom}(\mathcal{A}_\alpha) = \{\varphi \in \mathscr{C}^1 \mid \varphi'(0) = L(\alpha)\varphi\}, \\ &\mathcal{A}_\alpha \varphi = \varphi', \end{aligned} \tag{5}$$

where $\mathscr{C}^1 = C^1([\tau_{\min}, \tau_{\max}], \mathbb{C}^n)$. Note that the closedness of $\mathcal{A}_\alpha$ can be easily established using the fact that differentiation is a closed operation, together with the continuity of $L(\alpha)$. The density of the domain $\mathrm{Dom}(\mathcal{A}_\alpha)$ follows from the density of C^1-smooth functions in $\mathscr{C}$, together with the fact that for any $\varepsilon > 0$ and any neighbourhood of zero, one can modify an arbitrary C^1 function φ in such a way that $\varphi'(0)$ can be set at will, while $\varphi(0)$ remains unchanged and $\|\varphi\|$ changes by at most ε. $\mathscr{C}$ is indeed a state space for the homogeneous equation (3) in some sense, even though one cannot view this equation as an initial value problem (see, for example, [7]).

The spectrum of $\mathcal{A}_\alpha$, denoted by $\sigma(\mathcal{A}_\alpha)$, is the point spectrum (see [7] for a detailed proof). Moreover, λ is an eigenvalue of $\mathcal{A}_\alpha$, i.e., $\lambda \in \sigma(\mathcal{A}_\alpha)$, if and only if λ satisfies that $\det \Delta(\alpha, \lambda) = 0$, where the characteristic matrix $\Delta(\alpha, \lambda)$ is given by (4). Moreover, $\phi \in \mathscr{C}$ is an eigenvector of $\mathcal{A}_\alpha$ associated with the eigenvalue λ if and only if $\phi(\theta) = e^{\lambda\theta} b$ for $\theta \in [-\tau, 0]$ and some vector $b \in \mathbb{C}^n$ such that $\Delta(\alpha, \lambda)b = 0$. Assume that $\det \Delta(0, \lambda) = 0$ has a pair of purely imaginary roots at $\lambda = \pm \mathrm{i}\beta_0$. The symmetry group Γ often causes purely imaginary roots to be multiple. So, we always assume that

(H1) The characteristic equation $\det \Delta(0, \cdot) = 0$ has a pair of purely imaginary roots at $\pm \mathrm{i}\beta_0$, each of multiplicity m, and no other root belongs to $\mathrm{i}\beta_0\mathbb{Z}$.

In studying the bifurcation problem we wish to consider how the eigenvalues of $\mathcal{A}_\alpha$ cross the imaginary axis at $\alpha = 0$ and to describe the structure of the associated eigenspace $E_{\alpha,\lambda}$. We consider the following nontrivial restrictions on the corresponding imaginary eigenspace of $\mathcal{A}_0$.

(H2) $E_{0,\pm i\beta_0}$ is Γ-simple.

Thus, we make use of the implicit function theorem and Lemma 1.5 in Page 265 of [4] and obtain the following results about the multiplicity of this eigenvalue and its associated eigenvectors of $\mathcal{A}_\alpha$.

Theorem 2.1. *Under conditions* (H1)–(H2) *, for sufficiently small α, infinitesimal generator $\mathcal{A}_\alpha$ has one pair of complex conjugate eigenvalues $\sigma(\alpha) \pm i\rho(\alpha)$, each of multiplicity m. Moreover, σ and ρ are smooth functions of α and satisfy that $\sigma(0) = 0$ and $\rho(0) = \beta_0$.*

In view of (H1) , the purely imaginary eigenvalues of $\mathcal{A}_0$ has high multiplicity, so the standard Hopf bifurcation theorem can not be applied directly. So, we first develop the equivariant Lyapunov–Schmidt reduction for (1) to consider the existence of periodic solutions. Let $\omega_0 = 2\pi/\beta_0$, and $\mathscr{C}_{\omega_0}$ (respectively, $\mathscr{C}^1_{\omega_0}$) be the set of continuous (respectively, differentiable) n-dimensional ω_0-periodic mappings. If we denote

$$\|x\|_0 = \max_{1\le i\le n} \max_{t\in[0,\omega_0]} \{|x_i(t)|\}$$

for $x = (x_1, x_2, \ldots, x_n)^T \in \mathscr{C}_{\omega_0}$, and $\|x\|_1 = \max\{\|x\|_0, \|\dot{x}\|_0\}$ for $x \in \mathscr{C}^1_{\omega_0}$, then $\mathscr{C}_{\omega_0}$ and $\mathscr{C}^1_{\omega_0}$ are Banach spaces when they are endowed with the norms $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively. It is easy to see that $\mathscr{C}_{\omega_0}$ is a Banach representation of the group $\Gamma \times \mathbb{S}^1$ with the action given by

$$(\gamma, \theta)u(t) = \varrho(\gamma)u(t + \theta), \quad \text{for } (\gamma, \theta) \in \Gamma \times \mathbb{S}^1.$$

We introduce the inner product $\langle\cdot,\cdot\rangle : \mathscr{C}_{\omega_0} \times \mathscr{C}_{\omega_0} \to \mathbb{R}$ defined by $\langle v, u\rangle = \frac{1}{\omega_0}\int_0^{\omega_0} \bar{v}^T(t)u(t)dt$ for $u, v \in \mathscr{C}_{\omega_0}$. Let $\beta \in (-1, 1), x(\xi) = u(t), t = (1+\beta)\xi$. Then Eq. (1) can be rewritten as

$$(1+\beta)\dot{u}(t) = L(\alpha)u_{t,\beta} + f(\alpha, u_{t,\beta}),$$

where $u_{t,\beta}(\theta) = u(t + (1+\beta)\theta)$ for $\theta \in [\tau_{\min}, \tau_{\max}]$. Define $F : \mathscr{C}^1_{\omega_0} \times \mathbb{R}^2 \to \mathscr{C}_{\omega_0}$ by

$$F(u, \alpha, \beta) = -(1+\beta)\dot{u}(t) + f(\alpha, u_{t,\beta}). \tag{6}$$

By varying the newly introduced small variable β, one keeps track not only of solutions of (1) with period ω_0 but also of solutions with nearby period. In fact, solutions to $F(u, \alpha, \beta) = 0$ correspond to $\frac{\omega_0}{1+\beta}$-periodic solutions of (1). It follows that the Γ-equivariance of L and f that F is $\Gamma \times \mathbb{S}^1$-equivariant. The operator $\mathcal{L}u = -\dot{u} + L(0)u_t$ is the linearization of F at the origin. Obviously, the elements of Ker$\mathcal{L}$ correspond to solutions of the linear system $\dot{u} = L(0)u_t$ satisfying $u(t) = u(t + \omega_0)$. Let $\mathcal{L}^*$ be the adjoint operator of $\mathcal{L}$, satisfying $\langle v, \mathcal{L}u\rangle = \langle \mathcal{L}^*v, u\rangle$ for all $u, v \in \mathscr{C}^1_{\omega_0}$. It follows from (H1) that Ker$\mathcal{L} \cong$ Ker$\Delta(0, \pm i\beta_0)$ and Ker$\mathcal{L}^* \cong$ Ker$\Delta^*(0, \pm i\beta_0)$, both of which are $2m$-dimensional. Furthermore, we have.

Lemma 2.1. *Spaces* Ker$\mathcal{L}$, Range$\mathcal{L}$, *and* $\mathscr{Q} = (\text{Ker}\mathcal{L}^*)^\perp \bigcap \mathscr{C}^1_{\omega_0}$ *are $\Gamma \times \mathbb{S}^1$-invariant subspaces of $\mathscr{C}_{\omega_0}$. Moreover, $\mathscr{C}_{\omega_0} =$* Ker$\mathcal{L} \oplus$ Range$\mathcal{L}$ *and* $\mathscr{C}^1_{\omega_0} = \text{Ker}\mathcal{L} \oplus \mathscr{Q}$.

Let P and $I - P$ denote the projection operators defined by $P: \mathscr{C}_{\omega_0} \to$ Range$\mathcal{L}$ and $I - P: \mathscr{C}_{\omega_0} \to$ Ker$\mathcal{L}$. Obviously, P and $I - P$ are $\Gamma \times \mathbb{S}^1$-equivariant. Thus, $F(u, \alpha, \beta) = 0$ is equivalent to the following system:

$$\begin{aligned} &PF(v + w, \alpha, \beta) = 0, \\ &(I - P)F(v + w, \alpha, \beta) = 0. \end{aligned} \tag{7}$$

Here we have written $u \in \mathscr{C}_{\omega_0}$ in the form $u = v + w$, with $v = (I - P)u \in$ Ker$\mathcal{L}$ and $w = Pu \in \mathscr{Q}$. Near the critical point $(u, \alpha, \beta) = (0, 0, 0)$, the implicit function theorem implies that the first equation of (7) can be solved for $w = W(v, \alpha, \beta)$, where W : Ker$\mathcal{L} \times \mathbb{R}^2 \to \mathscr{Q}$ is a continuously differentiable $\mathbb{S}^1$-equivariant map satisfying $W(0, 0, 0) = 0$. Substituting $w = W(v, \alpha, \beta)$ into the second equation of (7), we have

$$\mathcal{B}(v, \alpha, \beta) \equiv (I - P)F(v + W(v, \alpha, \beta), \alpha, \beta) = 0. \tag{8}$$

Thus, we reduce our Hopf bifurcation problem to the problem of finding zeros of the map $\mathcal{B}$: Ker$\mathcal{L} \times \mathbb{R}^2 \to$ Ker$\mathcal{L}$. We refer to $\mathcal{B}$ as the bifurcation map of the system (1). It follows from the $\Gamma \times \mathbb{S}^1$- equivariance of F and W that the bifurcation map $\mathcal{B}$ is also $\Gamma \times \mathbb{S}^1$-equivariant. Moreover, $\mathcal{B}(0, 0, 0) = 0$ and $\mathcal{B}_v(0, 0, 0) = 0$.

Finding periodic solutions to (1) rests on prescribing in advance the symmetry of the solution we seek. This can often be used to select a subspace on which the eigenvalues are simple. In addition, we should take temporal phase-shift symmetries in terms of the circle group $\mathbb{S}^1$ into account as well as spatial symmetries. Here, we place emphasis on two-dimensional fixed-point subspaces and assume that

(H3) $\dim \text{Fix}(\Sigma, E_{0,\pm i\beta_0}) = 2$ for some subgroup Σ of $\Gamma \times \mathbb{S}^1$.
(H4) $\sigma'(0) \neq 0$.

Assumption (H4) is the transversality condition analogous to those of the standard Hopf bifurcation theorem. Now, we can present our main results about equivariant Hopf bifurcation.

Theorem 2.2. *Under conditions* (H1)–(H4)*, in every neighbourhood of* $(x = 0, \alpha = 0)$ *system* (1) *has a bifurcation of periodic solutions whose spatio-temporal symmetry can be completely characterized by* Σ.

Proof. We consider the restriction mapping $\widetilde{\mathcal{B}} : \mathrm{Fix}(\Sigma, \mathrm{Ker}\mathcal{L}) \times \mathbb{R}^2 \to \mathrm{Ker}\mathcal{L}$ of $\mathcal{B} : \mathrm{Ker}\mathcal{L} \times \mathbb{R}^2 \to \mathrm{Ker}\mathcal{L}$ on $\mathrm{Fix}(\Sigma, \mathrm{Ker}\mathcal{L}) \times \mathbb{R}^2$, i.e., $\widetilde{\mathcal{B}}(v, \alpha, \beta) = (I - P)F(v + W(v, \alpha, \beta), \alpha, \beta)$ for $v \in \mathrm{Fix}(\Sigma, \mathrm{Ker}\mathcal{L})$, $\alpha \in \mathbb{R}$, and $\beta \in \mathbb{R}$. Clearly, $\widetilde{\mathcal{B}}$ is also $\Gamma \times \mathbb{S}^1$-equivariant, and satisfies that $\widetilde{\mathcal{B}}(0, 0, 0) = 0$ and $\widetilde{\mathcal{B}}_v(0, 0, 0) = 0$. Moreover, it is easy to see that $\mathrm{Range}\widetilde{\mathcal{B}} \subseteq \mathrm{Fix}(\Sigma, \mathrm{Ker}\mathcal{L})$. Namely, $\widetilde{\mathcal{B}}$ maps $\mathrm{Fix}(\Sigma, \mathrm{Ker}\mathcal{L}) \times \mathbb{R}^2$ to $\mathrm{Fix}(\Sigma, \mathrm{Ker}\mathcal{L})$. Therefore, we only need to consider the existence of nontrivial zeroes of $\widetilde{\mathcal{B}}$.

Without loss of generality, assume that $\mathrm{Fix}(\Sigma, \mathrm{Ker}\mathcal{L}) = \mathrm{span}\{q, \overline{q}\}$, where $q(\theta) = Ae^{i\beta_0\theta}$ and $A \in \mathbb{C}^n$ satisfies $\Delta(0, i\beta)A = 0$. As stated in Theorem 2.1, for sufficiently small α, the infinitesimal generator $\mathcal{A}_\alpha$ has one pair of complex conjugate eigenvalues $\lambda(\alpha)$, each of multiplicity m. Moreover, there exists a C^1-continuous function $A(\alpha)$ such that $A(0) = A$ and $\Delta(\alpha, \lambda(\alpha))A(\alpha) \equiv 0$ for all sufficiently small α, we differentiate it with respect to α at $\alpha = 0$ and obtain

$$[\Delta_\alpha(0, i\beta_0) + \lambda'(0)\Delta_\lambda(0, i\beta_0)]A + \Delta(0, i\beta_0)A'(0) = 0. \tag{9}$$

In addition, there exists $B \in \mathbb{C}^n$ such that $\overline{B}^T\Delta(0, i\beta_0) = 0$ and $p = Be^{i\beta_0(\cdot)} \in \mathrm{Fix}(\Sigma, \mathrm{Ker}\mathcal{L}^*) = \mathrm{Fix}(\Sigma, \mathrm{Ker}\mathcal{L})^*$. Thus, multiplying both sides of (9) by $\overline{B}^T$ gives us

$$\overline{B}^T\Delta_\alpha(0, i\beta_0)A + \lambda'(0)\overline{B}^T\Delta_\lambda(0, i\beta_0)A = 0. \tag{10}$$

In fact, we can normalize $B \in \mathbb{C}^n$ such that $\overline{B}^T\Delta_\lambda(0, i\beta_0)A = 1$. Thus, it follows from (10) that $\lambda'(0) = -\overline{B}^T\Delta_\alpha(0, i\beta_0)A$.

For each $\phi \in \mathrm{Fix}(\Sigma, \mathrm{Ker}\mathcal{L})$, $\phi = zq + \overline{zq}$, where $z = \langle p, \phi\rangle$. Let

$$g(z, \alpha, \beta) := \langle p, \widetilde{\mathcal{B}}(zq + \overline{zq}, \alpha, \beta)\rangle.$$

Thus, we only need to consider the existence of nontrivial solutions to $g(z, \alpha, \beta) = 0$. It follows that $g_z(0, 0, 0) = 0$ and $g_{\overline{z}}(0, 0, 0) = 0$. It is easy to see that $g(z, \alpha, \beta)$ is $\mathbb{S}^1$-equivariant. Using similar arguments to that in [4], we can find two functions $\Re, \Im : \mathbb{R}^3 \to \mathbb{R}$ such that

$$g(z, \alpha, \beta) = \Re(|z|^2, \alpha, \beta)z + \Im(|z|^2, \alpha, \beta)iz. \tag{11}$$

It follows from $g_z(0, 0, 0) = 0$ that $\Re(0, 0, 0) = 0$ and $\Im(0, 0, 0) = 0$. Let $z = re^{i\theta}$. Then solving g is equivalent to either solving $r = 0$ or $\Re(r^2, \alpha, \beta) = 0$ and $\Im(r^2, \alpha, \beta) = 0$. In view of the implicitly defined function $W(v, \alpha, \beta)$, which vanishes through first order in $v = zq + \overline{zq}$, we have

$$F(v + W(v, \alpha, \beta), \alpha, \beta) = -(1 + \beta)\dot{v}(t) + L(\alpha)v_{t,\beta} + O(|z|^2).$$

Therefore,

$$\begin{aligned} g_\alpha(z, 0, 0) &= \langle p, F_\alpha(v, 0, 0)\rangle \\ &= \langle p, L'(0)v_t\rangle + O(|z|^2) \\ &= \langle p, L'(0)q_t\rangle z + \langle p, L'(0)\overline{q}_t\rangle\overline{z} + O(|z|^2) \\ &= z\overline{B}^T L'(0)(Ae^{i\beta_0(\cdot)}) + O(|z|^2) \\ &= z\lambda'(0) + O(|z|^2). \end{aligned}$$

In addition,

$$\begin{aligned} g_\beta(z, 0, 0) &= \langle p, F_\beta(v, 0, 0)\rangle \\ &= \langle p, -\dot{v}(t) + i\beta_0 L(0)(\theta v(t + \theta))\rangle + O(|z|^2) \\ &= \langle p, -i\beta_0 q(t) + i\beta_0 L(0)(\theta q(t + \theta))\rangle z + \langle p, -i\beta_0\overline{q}(t) + i\beta_0 L(0)(\theta\overline{q}(t + \theta))\rangle\overline{z} + O(|z|^2) \\ &= -i\beta_0 z + i\beta_0 L(0)(\theta Ae^{i\beta_0\theta})z + O(|z|^2) \\ &= -i\beta_0 z + O(|z|^2). \end{aligned}$$

Therefore, $G_\alpha(0, 0, 0) = \lambda'(0)$ and $G_\beta(0, 0, 0) = -\beta_0$. So the Jacobian determinant of the real and imaginary part of function g with respect to α and β is

$$\det\begin{bmatrix} \mathrm{Re}\{g_\alpha(0, 0, 0)\} & \mathrm{Re}\{g_\beta(0, 0, 0)\} \\ \mathrm{Im}\{g_\alpha(0, 0, 0)\} & \mathrm{Im}\{g_\beta(0, 0, 0)\} \end{bmatrix} = -\beta_0\,\mathrm{Re}\{\lambda'(0)\}.$$

Thus, under condition (H4), the above Jacobi determinant is nonzero. The implicit function theorem implies that there exists a unique function $\alpha = \alpha(r^2)$ and $\beta = \beta(r^2)$ satisfying $\alpha(0) = 0$ and $\beta(0) = 0$ such that

$$\Re(r^2, \alpha(r^2), \beta(r^2)) \equiv 0, \qquad \Im(r^2, \alpha(r^2), \beta(r^2)) \equiv 0 \tag{12}$$

for all sufficient small r. Therefore, $g(z, \alpha(|z|^2), \beta(|z|^2)) \equiv 0$ for z sufficiently near 0. Therefore, system (1) has a bifurcation of periodic solutions whose spatio-temporal symmetry can be completely characterized by Σ. This completes the proof of Theorem 2.2. □

Remark 2.1. Theorem 2.2 implies that a Hopf bifurcation for (1) occurs at $\alpha = 0$. Namely, in every neighbourhood of $(x = 0, \alpha = 0)$ there is a branch of Σ-symmetric periodic solutions $x(t, \alpha)$ with $x(t, \alpha) \to 0$ as $\alpha \to 0$. The period ω_α of $x(t, \alpha)$ satisfies that $\omega_\alpha \to \omega_0$ as $\alpha \to 0$. Moreover, Γ-equivariance implies that there are $(\Gamma \times \mathbb{S}^1)/\Sigma$ different periodic solutions, which have isotopy subgroups conjugate to Σ in $\Gamma \times \mathbb{S}^1$.

In what follows, we consider the bifurcation direction. Assuming sufficient smoothness of f, we write

$$f(0, \varphi) = \frac{1}{2}\mathcal{E}(\varphi, \varphi) + \frac{1}{6}\mathcal{F}(\varphi, \varphi, \varphi) + o(\|\varphi\|^3). \tag{13}$$

Write $W(zq + \overline{zq}, 0, 0)$ and $g(z, 0, 0)$ as

$$W(zq + \overline{zq}, 0, 0) = \sum_{s+l\geq 2} \frac{1}{s!l!} W_{sl} z^s \overline{z}^l \qquad g(z, 0, 0) = \sum_{s+l\geq 2} \frac{1}{s!l!} g_{sl} z^s \overline{z}^l.$$

It follows from (11) that $g_{21} = \Re_1(0, 0, 0) + i\Im_1(0, 0, 0)$, where $\Re_1(u, \alpha, \beta) = \Re_u(u, \alpha, \beta)$ and $\Im_1(u, \alpha, \beta) = \Im_u(u, \alpha, \beta)$. Therefore, $\Re_1(0, 0, 0) = \mathrm{Re}\{g_{21}\}$ and $\Im_1(0, 0, 0) = \mathrm{Im}\{g_{21}\}$. From (12), we can calculate the derivatives of $\alpha(r^2)$ and $\beta(r^2)$ and evaluate at $r = 0$:

$$\alpha'(0) = -\frac{\mathrm{Re}\{g_{21}\}}{\mathrm{Re}\{\lambda'(0)\}}, \qquad \beta'(0) = -\frac{\mathrm{Im}\{\lambda'(0)\overline{g_{21}}\}}{\mathrm{Re}\{\lambda'(0)\}}.$$

The bifurcation direction is determined by sign $\alpha'(0)$, and the monotonicity of period of bifurcating closed invariant curve depends on sign $\beta'(0)$. Using a similar argument as that in [6], we have

$$g_{21} = \langle p, \mathcal{F}(q, q, \overline{q})\rangle + 2\langle p, \mathcal{E}(q, W_{11})\rangle + \langle p, \mathcal{E}(\overline{q}, W_{20})\rangle.$$

We still need to compute W_{11} and W_{20}. In fact, it follows that

$$W_{20} = -\mathcal{L}^{-1} P\mathcal{E}(q, q),$$
$$W_{11} = -\mathcal{L}^{-1} P\mathcal{E}(q, \overline{q}).$$

In order to evaluate function W_{20}, we must solve the following differential equations

$$\dot{W}_{20} - L(0)W_{20} = P\mathcal{E}(q, q). \tag{14}$$

Note that $\mathcal{B}(q, q) = \mathcal{B}(Ae^{i\beta_0(\cdot)}, Ae^{i\beta_0(\cdot)})e^{2i\beta_0 t}$ and $\mathcal{E}(q, q) = \mathcal{E}(Ae^{i\beta_0(\cdot)}, Ae^{i\beta_0(\cdot)})e^{2i\beta_0 t}$. So, $g_{20} = \langle p, \mathcal{E}(q, q)\rangle = 0$. Namely, $\mathcal{E}(q, q) \in \mathrm{Range}\mathcal{L}$. Hence, the projection P on $\mathcal{E}(q, q)$ acts as the identity, and (14) is an inhomogeneous difference equations with constant coefficients. Thus, there is a particular solution of (14) of the form $W_{20}^*(t) = D_2 e^{2i\beta_0 t}$. Substituting W_{20}^* into (14) and comparing the coefficients, we obtain

$$D_2 = \Delta^{-1}(0, 2i\beta_0)\mathcal{E}(Ae^{i\beta_0(\cdot)}, Ae^{i\beta_0(\cdot)}). \tag{15}$$

In addition, W_{20}^* is orthogonal to p, so it belongs to Range$\mathcal{L}$. Thus $W_{20}(0, 0, 0)$ is equal to W_{20}^* with D_2 determined by (15). Similarly, we have

$$W_{02} = \overline{D}_2 e^{-2i\beta_0 t}, \qquad W_{11} = D_0,$$

where $D_0 = \Delta^{-1}(0, 0)\mathcal{E}(Ae^{i\beta_0(\cdot)}, \overline{A}e^{-i\beta_0(\cdot)})$. Therefore,

$$g_{21} = \overline{B}^T \mathcal{F}(Ae^{i\beta_0(\cdot)}, Ae^{i\beta_0(\cdot)}, \overline{A}e^{-i\beta_0(\cdot)}) + 2\overline{B}^T \mathcal{E}(Ae^{i\beta_0(\cdot)}, D_0) + \overline{B}^T \mathcal{E}(\overline{A}e^{-i\beta_0(\cdot)}, D_2).$$

We summarize the above discussion as follows.

Theorem 2.3. *In addition to conditions* (H1)–(H4)*, assume that* $L(\alpha)$ *and* $f(\alpha, \cdot)$ *are sufficiently smooth. Then there exists a branch of* Σ*-symmetric periodic solutions, parameterized by* α*, bifurcating from the trivial solution* $x = 0$ *of* (1)*. Moreover,*

(i) $\mathrm{Re}\{\lambda'(0)\}\mathrm{Re}\{g_{21}\}$ *determines the direction of the bifurcation: the bifurcation is supercritical (respectively, subcritical), i.e. the bifurcating periodic solutions exist for* $\alpha > 0$ *(respectively,* < 0*), if* $\mathrm{Re}\{\lambda'(0)\}\mathrm{Re}\{g_{21}\} < 0$ *(respectively,* > 0*). and*
(ii) $\mathrm{Re}\{\lambda'(0)\}\mathrm{Im}\{\lambda'(0)\overline{g_{21}}\}$ *determines the period of the bifurcating periodic solutions along the branch: the period is greater than (respectively, smaller than)* ω_0 *if it is positive (respectively, negative).*

3. Application to a ring network

To illustrate the results presented above, we consider a ring network consisting of 3 identical elements with nearest-neighbour coupling:

$$\dot{u}_j(t) = -u_j(t) + 2u_j(t-1) + ag(u_{j+1}(t+1) - u_{j+1}(t-1)) + ag(u_{j-1}(t+1) - u_{j-1}(t-1)), \tag{16}$$

where $j \pmod 3$, $g \in C^3(\mathbb{R}; \mathbb{R})$ with $g(0) = g''(0) = 0$ and $g'(0) = 1$, and $a \in \mathbb{R}$ is the bifurcation parameter. Define the action of the dihedral group $\mathbb{D}_3$ on $\mathbb{R}^3$ by

$$(\rho \cdot u)_j = u_{j+1} \quad \text{and} \quad (\kappa \cdot u)_j = u_{2-j} \tag{17}$$

for all $j \pmod 3$ and $u \in \mathbb{R}^3$. It is easy to see that system (16) is $\mathbb{D}_3$-equivariant. The holomorphic characteristic matrix of the linearized system of (16) around the equilibrium 0 is given by

$$\Delta(a, \lambda) = \begin{bmatrix} \lambda + 1 - 2e^{-\lambda} & a(e^{-\lambda} - e^{\lambda}) & a(e^{-\lambda} - e^{\lambda}) \\ a(e^{-\lambda} - e^{\lambda}) & \lambda + 1 - 2e^{-\lambda} & a(e^{-\lambda} - e^{\lambda}) \\ a(e^{-\lambda} - e^{\lambda}) & a(e^{-\lambda} - e^{\lambda}) & \lambda + 1 - 2e^{-\lambda} \end{bmatrix}.$$

Obviously, $\det \Delta(a, \lambda) = [\lambda + 1 - 2ae^{\lambda} - 2(1-a)e^{-\lambda}][\lambda + 1 + ae^{\lambda} - (2+a)e^{-\lambda}]^2$. Let $\mathcal{A}(a)$ be the linear operator associated with the linearization of (16) about the equilibrium 0. It can be shown that $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}(a)$ if and only if $\det \Delta(a, \lambda) = 0$, that is, either $[\lambda + 1 - 2ae^{\lambda} - 2(1-a)e^{-\lambda} = 0$ or $\lambda + 1 + ae^{\lambda} - (2+a)e^{-\lambda}]^2 = 0$.

Firstly, $\lambda + 1 - 2ae^{\lambda} - 2(1-a)e^{-\lambda}$ has a pair of purely imaginary zeros $\pm \mathrm{i}\beta_0$ if $1 = 2\cos\beta_0$ and $\beta_0 = 2(2a-1)\sin\beta_0$. This results in a family of bifurcation values $a_{1,k} \in \mathbb{R}$, where $a_{1,k} = (\beta_k + 2\sin\beta_k)/(4\sin\beta_k)$ for $k \in \mathbb{N}$, and $\{\beta_k\}_{k=1}^{\infty}$ is a strictly increasing sequences of positive numbers satisfying $\cos\beta_k = \frac{1}{2}$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty}\beta_k = \infty$. Obviously, $\beta_1 = \frac{\pi}{3}$. Moreover, if $\lambda(a)$ is a smooth curve of zeros of $\lambda + 1 - 2ae^{\lambda} - 2(1-a)e^{-\lambda}$ with $\lambda(a_{1,k}) = \mathrm{i}\beta_k$, it is easy to see that

$$\lambda'(a_{1,k}) = 4\mathrm{i}h_{1,k}^{-1}\sin\beta_k \tag{18}$$

and hence that $\mathrm{Re}\{\lambda'(a_{1,k})\} < 0$, where $h_{1,k} = 1 - 2a_{1,k}e^{\mathrm{i}\beta_k} + 2(1-a_{1,k})e^{-\mathrm{i}\beta_k}$. Therefore, $\mathcal{A}(a_{1,k})$ has a pair of purely imaginary eigenvalues $\pm\mathrm{i}\beta_k$ with the associated eigenspace E_0 spanned by the eigenvectors $e^{\mathrm{i}\beta_k(\cdot)}v_0$ and $e^{-\mathrm{i}\beta_k(\cdot)}v_0$, where $v_0 = (1, 1, 1)^T$. In view of (17), it follows from the action of $\mathbb{D}_3 \times \mathbb{S}^1$ on E^0 that the $\mathbb{D}_3 \times \mathbb{S}^1$-action on $\mathbb{C}$ is given by $\rho \cdot z = z$, $\kappa \cdot z = z$, and $\theta \cdot z = e^{\mathrm{i}\beta_k\theta}z$ for $z \in \mathbb{C}$. Obviously, the maximal isotropy subgroup is $\mathbb{D}_3$, which corresponds to a Hopf bifurcation in which $\mathbb{D}_3$ symmetry is preserved. Thus, all elements are synchronous (i.e., have the same waveform and move in phase). Namely, the state $(u_1(t), u_2(t), u_3(t))$ of system (16) satisfies $u_j(t) = u(t)$ for all $j = 1, 2, 3$, where $u(t)$ is the periodic solution to the following system

$$\dot{u}(t) = -u(t) + 2u(t-1) + 2ag(u(t+1) - u(t-1)). \tag{19}$$

Similarly, $\lambda + 1 + ae^{\lambda} - (2+a)e^{-\lambda}$ has a pair of purely imaginary zeros $\pm\mathrm{i}\beta_0$ if $1 = 2\cos\beta_0$ and $\beta_0 = -2(a+1)\sin\beta_0$. This results in a family of bifurcation values $a_{2,k} \in \mathbb{R}$, where $a_{2,k} = -(\beta_k + 2\sin\beta_k)/(2\sin\beta_k)$ for $k \in \mathbb{N}$, and $\{\beta_k\}_{k=1}^{\infty}$ is defined as above. Moreover, if $\lambda(a)$ is a smooth curve of zeros of $\lambda + 1 + ae^{\lambda} - (2+a)e^{-\lambda}$ with $\lambda(a_{2,k}) = \mathrm{i}\beta_k$, it is easy to see that

$$\lambda'(a_{2,k}) = -2\mathrm{i}h_{2,k}^{-1}\sin\beta_k \tag{20}$$

and hence that $\mathrm{Re}\{\lambda'(a_{2,k})\} > 0$, where $h_{2,k} = 1 + a_{2,k}e^{\mathrm{i}\beta_k} + (2 + a_{2,k})e^{-\mathrm{i}\beta_k}$. Therefore, $\mathcal{A}(a_{2,k})$ has a pair of purely imaginary eigenvalues $\pm\mathrm{i}\beta_k$ with the associated eigenspace $E_0 = \mathrm{span}\{e^{\mathrm{i}\beta_k(\cdot)}v_1, e^{\mathrm{i}\beta_k(\cdot)}\overline{v_1}, e^{-\mathrm{i}\beta_k(\cdot)}v_1, e^{-\mathrm{i}\beta_k(\cdot)}\overline{v_1}\}$, where $v_1 = (1, e^{2\pi\mathrm{i}/3}, e^{-2\pi\mathrm{i}/3})$. It follows from [8] that

$$\begin{aligned} \mathrm{Fix}(\Sigma_\kappa^+) &= \mathrm{span}\{w_1\cos(\beta_k t), w_1\sin(\beta_k t)\}, \\ \mathrm{Fix}(\Sigma_\kappa^-) &= \mathrm{span}\{w_2\cos(\beta_k t), w_2\sin(\beta_k t)\}, \\ \mathrm{Fix}(\Sigma_\rho^+) &= \mathrm{span}\{\mathrm{Re}(\overline{v_1}e^{\mathrm{i}\beta_k t}), \mathrm{Im}(\overline{v_1}e^{\mathrm{i}\beta_k t})\}, \\ \mathrm{Fix}(\Sigma_\rho^-) &= \mathrm{span}\{\mathrm{Re}(v_1e^{\mathrm{i}\beta_k t}), \mathrm{Im}(v_1e^{\mathrm{i}\beta_k t})\}, \end{aligned} \tag{21}$$

where $w_1 = \mathrm{Re}(v_1)$, $w_2 = \mathrm{Im}(v_1)$, Σ_κ^+ is generated by $(\kappa, 0) \in \mathbb{D}_3 \times \mathbb{S}^1$, Σ_κ^- is generated by $(\kappa, \pi) \in \mathbb{D}_3 \times \mathbb{S}^1$, and $\Sigma_\rho^\pm$ is generated by $(\rho, \pm\frac{2\pi}{3}) \in \mathbb{D}_3 \times \mathbb{S}^1$. Namely, $\Sigma_\kappa^\pm$ and $\Sigma_\rho^\pm$ are maximal isotropy subgroups of $\mathbb{D}_3 \times \mathbb{S}^1$. Firstly, the isotropy subgroups $\Sigma_\rho^\pm$ correspond to phase-locked waves of (16), which take the form

$$u_i(t) = u_{i+1}\left(t \pm \frac{\omega}{3}\right) \tag{22}$$

for all $t \in \mathbb{R}$ and $i \pmod 3$, where $\omega > 0$ is a period of $u(t)$. That is, all elements have identical waveforms but are phase-shifted by $\omega/3$. Next, the isotropy subgroup Σ_κ^+ corresponds to a mirror-reflecting wave of (16), which takes the form

$$x_i(t) = x_{2-i}(t) \tag{23}$$

for all $t \in \mathbb{R}$ and i (mod 3). Thus, the second and third elements behave identical, i.e., they have the same waveform and move in phase. Finally, the isotropy subgroup Σ_κ^- corresponds to a standing wave of (16), which takes the form

$$x_i(t) = x_{2-i}\left(t - \frac{\omega}{2}\right) \tag{24}$$

for all $t \in \mathbb{R}$ and i (mod 3), where $\omega > 0$ is a period of $u(t)$. Namely, the second and third elements have identical waveforms but are phase-shifted by half a period, the first element has different waveform and twice the frequency of the others, i.e., it is half a period out of phase with itself.

We summarize the above discussion as follows.

Theorem 3.1. (i) *Near $a = a_{1,k}$ for each $k \in \mathbb{N}$, there exists a branch of synchronous periodic solutions of period ω near $(2\pi/\beta_k)$ bifurcated from the zero solution of the system.* (ii) *Near $a = a_{2,k}$ for each $k \in \mathbb{N}$, there exist* 8 *branches of asynchronous periodic solutions of period ω near $(2\pi/\beta_k)$ bifurcated from the zero solution of the system and these are two phase-locked waves,* 3 *mirror-reflecting waves, and* 3 *standing waves.*

In what follows, we start with the two phase-locked oscillations mentioned above, which are characterized by $\Sigma_\rho^\pm$. In view of (21), for the vectors A and B defined in Section 2, we choose $A = 3\overline{h_{2,k}}B = \overline{v}_1$ or $A = 3\overline{h_{2,k}}B = v_1$ such that

$$\Delta(a_{2,k}, \mathrm{i}\beta_0)A = 0, \qquad \overline{B}^T \Delta(a_{2,k}, \mathrm{i}\beta_0) = 0, \qquad \overline{B}^T \Delta_\lambda(a_{2,k}, \mathrm{i}\beta_0)A = 1. \tag{25}$$

We have

$$g_{21} = -12\mathrm{i}a_{2,k}g'''(0)h_{2,k}^{-1}\sin^3\beta_k = 6a_{2,k}g'''(0)\sin^2\beta_k\lambda'(a_{2,k})$$

and hence

$$\mathrm{sgn}\{\mathrm{Re}\{g_{21}\}\} = \mathrm{sgn}\{a_{2,k}g'''(0)\}. \tag{26}$$

Similarly, we choose $A = \frac{3}{2}\overline{h_{2,k}}B = w_1$ for the mirror-reflecting waves characterized by Σ_κ^+, and $A = \frac{3}{2}\overline{h_{2,k}}B = w_2$ the standing waves characterized by Σ_κ^-. By a direct computation, we also have $\mathrm{sgn}\{\mathrm{Re}\{g_{21}\}\} = \mathrm{sgn}\{a_{2,k}g'''(0)\}$.

Finally, for the synchronous periodic solution mentioned in Theorem 3.1, we can show that $\mathrm{sgn}\{\mathrm{Re}\{g_{21}\}\} = -\mathrm{sgn}\{a_{1,k}g'''(0)\}$. Thus, applying Theorem 2.3, we have the following results.

Theorem 3.2. *Near $a = a_{j,k}$ for each $j \in \{1, 2\}$ and $k \in \mathbb{N}$, system (16) undergoes a Hopf bifurcation, the bifurcation direction is determined by the sign of $a_{j,k}g'''(0)$. More precisely, if $a_{j,k}g'''(0) < 0$ (or >0) then the Hopf bifurcation is supercritical (respectively, subcritical) and all the bifurcating asynchronous periodic solutions exist for $a > a_{j,k}$ (respectively, $<a_{j,k}$).*

References

[1] R. Bellman, K. Cooke, Differential Difference Equations, Academic Press, New York, 1963.
[2] J. Benhabib, K. Nishimura, The Hopf bifurcation and the existence and stability of closed orbits in multisector models of optimal economic growth, J. Econ. Theory 21 (1979) 421–444.
[3] A. Rustichini, Functional differential equations of mixed type: the linear autonomous case, J. Dynam. Differential Equations 11 (1989) 121–143.
[4] M. Golubitsky, I. Stewart, D.G. Schaeffer, Singularities and Groups in Bifurcation Theory, vol. 2, Springer-Verlag, New York, 1988.
[5] T. Faria, L.T. Magalhães, Normal forms for retarded delay differential equations with parameters and applications to Hopf bifurcation, J. Differential Equations 122 (1995) 181–200.
[6] M. Golubitsky, D.G. Schaeffer, Singularities and Groups in Bifurcation Theory, vol. 1, Springer-Verlag, New York, 1985.
[7] H.J. Hupkes, S.M. Verduyn Lunel, Center manifold theory for functional differential equations of mixed type, J. Dynam. Differential Equations 19 (2007) 497–560.
[8] S. Guo, L. Huang, Hopf bifurcating periodic orbits in a ring of neurons with delays, Physica D 183 (2003) 19–44.