MATHEMATICS

GENERALIZED FREE PRODUCTS OF BOOLEAN ALGEBRAS WITH AN AMALGAMATED SUBALGEBRA

BY

PH. DWINGER AND F. M. YAQUB

(Communicated by Prof. H. D. KLOOSTERMAN at the meeting of December 22, 1962)

Introduction

Let B_γ, $\gamma \in \Gamma$, be a family of Boolean algebras and let each B_γ, contain a subalgebra isomorphic to a Boolean algebra A. The *generalized free product C of the family B_γ, $\gamma \in \Gamma$, with the amalgamated subalgebra A* can be loosely described as follows. C is the largest Boolean algebra containing every B_γ as a subalgebra and C is generated by this family. Moreover every two B_γ intersect in a subalgebra which is isomorphic to A. However it should be emphasized that there is a special monomorphism between A and every B_γ, and the amalgamation should be in accordance with these monomorphisms. The case of generalized free products of groups with an amalgamated subgroup has been investigated by SCHREIER and H. NEUMANN [5]. (Also cf. [4].) JONŚSON has defined the concept of generalized free products for universal algebras ([2], [3]). The generalized free product of a family of groups with an amalgamated subgroup always exists, and in the present paper we will show that this result also holds for Boolean algebras.

There are two approaches possible to this problem. The first one is the algebraic one, analogous to the case of groups. We consider the free product of the family B_γ, $\gamma \in \Gamma$, and we obtain C by identifying any two elements in the free product which are images of the same element of A under the given monomorphisms. Then, one has to show that this identification leads to a factor algebra which contains every B_γ as a subalgebra, and moreover that every two B_γ intersect precisely in A, in other words that the identification does not identify too much.

The second approach is based on the well-known duality that exists between Boolean algebras and Boolean spaces. This topological treatment seems to be of some interest in itself and is rather brief and we feel therefore justified to present this as well.

In section 1 we will give the precise formulation of the problem. In section 2 we will treat the problem algebraically whereas in section 3 we will consider the topological aspect of the amalagamation problem.

1. The definition of the generalized free product of a family of Boolean algebras with an amalgamated subalgebra will be given in a way which is analogous to the definition of free products (cf. [6]).

Let B_γ, $\gamma \in \Gamma$ be a family of Boolean algebras and let A be a Boolean algebra such that for every $\gamma \in \Gamma$ there exists a monomorphism $h_\gamma : A \to B_\gamma$.

The generalized free product of the family B_γ, $\gamma \in \Gamma$ *with the amalgamated subalgebra* A is a Boolean algebra C satisfying the following conditions:

(1) For every $\gamma \in \Gamma$ there exists a monomorphism $f_\gamma : B_\gamma \to C$ such that for every pair $\gamma', \gamma'' \in \Gamma$, $f_{\gamma'} h_{\gamma'} = f_{\gamma''} h_{\gamma''}$.

(2) $\bigcup_{\gamma \in \Gamma} f_\gamma(B_\gamma)$ generates C.

(3) If D is a Boolean algebra and if g_γ, $\gamma \in \Gamma$ is a family of homomorphisms $g_\gamma : B_\gamma \to D$ such that for every pair $\gamma', \gamma'' \in \Gamma$, $g_{\gamma'} h_{\gamma'} = g_{\gamma''} h_{\gamma''}$, then there exists a homomorphism $g : C \to D$ such that $g f_\gamma = g_\gamma$ for every $\gamma \in \Gamma$.

We observe that condition (3) formulates the requirement that C is as "large" as possible. On the other hand we have not required that every two $f_\gamma(B_\gamma)$ intersect precisely in A (in accordance with the monomorphisms h_γ) but we will show that this property is a consequence of the conditions (1) and (3). Finally it remains to show that C is unique, and also that the map g of condition (3) is unique. The remaining part of this section is devoted to the proof of these facts. The existence of C will be proved in the next section.

Lemma 1. Let A be a Boolean algebra and let B be a proper subalgebra of A. Then there exist two distinct prime ideals I_1 and I_2 of A such that $I_1 \cap B = I_2 \cap B$.

Proof. The map $I \to I \cap B$, where I is a prime ideal of A, maps the family of all prime ideals of A onto the family of all prime ideals of B. It is well-known that this map is one-one if and only if $A = B$ [1].

We are now able to prove that the conditions (1) and (3) imply that every two $f_\gamma(B_\gamma)$ intersect precisely in A.

Lemma 2. Suppose that the family of Boolean algebras B_γ, $\gamma \in \Gamma$ and the Boolean algebras A and C satisfy the conditions (1) and (3). Let γ' and γ'' be any pair of elements of Γ, $\gamma' \neq \gamma''$, then

$$f_{\gamma'}(B_{\gamma'}) \cap f_{\gamma''}(B_{\gamma''}) = f_{\gamma'} h_{\gamma'}(A) = f_{\gamma''} h_{\gamma''}(A).$$

Proof. Suppose that there would be a pair γ', γ'', $\gamma' \neq \gamma''$ for which the lemma would not hold. Then $f_{\gamma'} h_{\gamma'}(A)$ would be a proper subalgebra of $f_{\gamma'}(B_{\gamma'}) \cap f_{\gamma''}(B_{\gamma''})$. Then according to Lemma 1 there would exist an element $a \in f_{\gamma'}(B_{\gamma'}) \cap f_{\gamma''}(B_{\gamma''})$, $a \notin f_{\gamma'} h_{\gamma'}(A)$ and two prime ideals I_1 and I_2 of $f_{\gamma'}(B_{\gamma'}) \cap f_{\gamma''}(B_{\gamma''})$ such that $I_1 \cap f_{\gamma'} h_{\gamma'}(A) = I_2 \cap f_{\gamma'} h_{\gamma'}(A)$. (We observe that by condition (1) $f_{\gamma'} h_{\gamma'}(A) = f_{\gamma''} h_{\gamma''}(A)$) and such that $a \in I_1$ and $a \notin I_2$. Now we can extend I_1 to a prime ideal $I_1{}^*$ of $f_{\gamma'}(B_{\gamma'})$. Thus

$$I_1{}^* \cap (f_{\gamma'}(B_{\gamma'}) \cap f_{\gamma''}(B_{\gamma''})) = I_1.$$

Similarly, we extend I_2 to a prime ideal I_2^* of $f_{\gamma''}(B_{\gamma''})$ and again

$$I_2^* \cap (f_{\gamma'}(B_{\gamma'}) \cap f_{\gamma''}(B_{\gamma''})) = I_2.$$

Now I_1 and I_2 induce two maps of $f_{\gamma'}(B_{\gamma'})$ and $f_{\gamma''}(B_{\gamma''})$ respectively, onto the two elements Boolean algebra 2 which clearly coincide of $f_{\gamma'}h_{\gamma'}(A)$. But these two maps cannot be extended to all of C, because $a/I_1^* = 0$ and $a/I_2^* = 1$. This completes the proof of Lemma 2.

Lemma 3. Suppose that the family of Boolean algebras B_γ, $\gamma \in \Gamma$ and the Boolean algebras A and C satisfy the conditions (1), (2) and (3) then the map g is always unique.

Proof. Suppose there would be two homomorphic extensions g and g'. Let $V = \{a : g(a) = g'(a),\ a \in C\}$. Clearly, V is a subalgebra of C containing $\bigcup_{\gamma \in \Gamma} f_\gamma(B_\gamma)$ but $\bigcup_{\gamma \in \Gamma} f_\gamma(B_\gamma)$ generates C thus $V = C$ and thus $g = g'$.

Lemma 4. There is at most (up to isomorphisms) one algebra C satisfying the conditions (1), (2) and (3).

Proof. Suppose that C and C' both satisfy the conditions (1), (2) and (3). We denote the monomorphic maps of the B_γ to C' by f'_γ. According to (3) there exists a homomorphic map $g : C \to C'$ such that $gf_\gamma = f_\gamma'$ for all $\gamma \in \Gamma$, and a homomorphic map $g' : C' \to C$ such that $g'f_\gamma' = f_\gamma$. The map $g'g$ has the property that $g'gf_\gamma = f_\gamma$ for all $\gamma \in \Gamma$, and thus according to (3) and to Lemma 4 we have $g'g = 1$. Similarly $gg' = 1$. It follows that both g and g' are isomorphic maps and $g' = g^{-1}$.

2. In this section we will prove that C exists and is therefore, according to Lemma 4, uniquely determined.

Theorem 1. The generalized free product of a family of Boolean algebras with an amalgamated subalgebra exists and is unique (up to isomorphisms).

Proof. We use the notation of section 1. Let C' be the free product of the family B_γ, $\gamma \in \Gamma$. We shall from now on identify the Boolean algebras B_γ with their isomorphic copies in C'. We consider the elements u of C' of the form $u = h_{\gamma'}(x) \dotplus h_{\gamma''}(x)$ for all pairs $\gamma', \gamma'' \in \Gamma$ and for all $x \in A$, where $\dotplus$ denotes symmetric difference. Let I be the ideal of C' generated by the elements u. We claim that $B_\gamma \cap I = (0)$ for all $\gamma \in \Gamma$. Let γ_0 be some fixed γ, then we will show that $B_{\gamma_0} \cap I = (0)$. It is clear that I is also generated by elements u of the form $u = h_{\gamma_0}(x) \dotplus h_\gamma(x)$ for all $\gamma \in \Gamma$ and $x \in A$. Now suppose that $a \in B_{\gamma_0} \cap I$, $a \neq 0$ thus $a \leqslant \sum_{i=1}^{n} u_i$ where each u_i is a generator of I. Thus $a \leqslant \sum_{i=1}^{n} (h_{\gamma_0}(x_i) \dotplus h_{\gamma_i}(x_i))$ for some $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$, and $x_1, x_2, \ldots, x_n \in A$. Let $p : B_{\gamma_0} \to 2$ be a homomorphic map such that $p(a) = 1$, and let for every i, $q_i : h_{\gamma_i}(A) \to 2$ be a homomorphic map defined by $q_i h_{\gamma_i}(x) = p h_{\gamma_0}(x)$ for all $x \in A$. Observe that for every i, the kernel of q_i is a prime ideal of $h_{\gamma_i}(A)$ and we extend this prime ideal to a prime ideal Q_i of B_{γ_i}. Let $q_i^* : B_{\gamma_i} \to B_{\gamma_i}/Q_i$ be the canonical map, then q_i^* is for each i an extension of q_i. Since C' is the free product of the B_γ, there exists a homomorphic map $r : C' \to 2$ which is an extension of p and of each q_i^*. Now, for every i we have $rh_{\gamma_0}(x_i) = = ph_{\gamma_0}(x_i) = q_i h_{\gamma_i}(x_i)$ and $rh_{\gamma_i}(\bar{x}_i) = q_i^* h_{\gamma_i}(\bar{x}_i) = q_i h_{\gamma_i}(\bar{x}_i)$. Thus it follows that $[rh_{\gamma_0}(x_i)]\, [rh_{\gamma_i}(\bar{x}_i)] = 0$. Similarly, $[rh_{\gamma_0}(\bar{x}_i)]\, [rh_{\gamma_i}(x_i)] = 0$ for each i. Now $a \leqslant \sum_{i=1}^{n} (h_{\gamma_0}(x_i) \dotplus h_{\gamma_i}(x_i))$. Thus it follows that $r(a) = 0$ which contradicts the fact that $r(a) = p(a) = 1$. This proves that $B_{\gamma_0} \cap I = (0)$.

Now let $C = C'/I$ and let $f : C' \to C = C'/I$ be the canonical map and let for every $\gamma \in \Gamma$, f_γ be the restriction of f to B_γ. Since $B_\gamma \cap I = (0)$ for every $\gamma \in \Gamma$, it follows that each f_γ is a monomorphism. Moreover since for every pair $\gamma', \gamma'' \in \Gamma$ and for each $x \in A$ $h_{\gamma'}(x) \dotplus h_{\gamma''}(x) \in I$, we have that $f_{\gamma'}(h_{\gamma'}(x)) = f_{\gamma''}(h_{\gamma''}(x))$. We have therefore completed the proof that condition (1) holds.

The proof that condition (2) holds is immediate and is left to the reader. It remains to show that condition (3) holds. Thus suppose that D is a Boolean algebra and that $g_\gamma : B_\gamma \to D$ is a family of homomorphic maps such that for every pair $\gamma', \gamma'' \in \Gamma$, $g_{\gamma'} h_{\gamma'} = g_{\gamma''} h_{\gamma''}$. We must show that there exists a homomorphic map $g : C \to D$ such that $g f_\gamma = g_\gamma$ for every $\gamma \in \Gamma$. Now, C' is the free product of the B_γ and we can therefore extend the maps g_γ to a homomorphic map $g^* : C' \to D$. Let J be the kernel of g^*. We shall show that $I \subset J$.

Indeed, suppose that $z \in I$, then $z \leqslant \sum_{i=1}^{n} (h_{\gamma_i'}(x_i) \dotplus h_{\gamma_i''}(x_i))$ for $x_1, x_2, \ldots, x_n \in A$ and $\gamma_i', \gamma_i'' \in \Gamma$ for $i = 1, 2, \ldots, n$. Now $g^*(h_{\gamma_i'}(x_i) h_{\gamma_i''}(\bar{x}_i)) = = g_{\gamma_i'}(h_{\gamma_i'}(x_i)\, g_{\gamma_i''}(h_{\gamma_i''}(\bar{x}_i)) = 0$ because $g_{\gamma_i'} h_{\gamma_i'} = g_{\gamma_i''} h_{\gamma_i''}$.

Similarly $g^*(h_{\gamma_i'}(\bar{x}_i) h_{\gamma_i''}(x_i) = 0$ and it follows that $g^*(z) = 0$ and thus $I \subset J$. We now define a map $g : C \to D$ by $g(f(x)) = g^*(x)$ for all $x \in C'$. This map is well defined and is a homomorphic map because $I \subset J$. Now we have for every $\gamma \in \Gamma$, $g f_\gamma(x) = g f(x) = g^*(x) = g_\gamma(x)$ for all $x \in B_\gamma$. Thus $g f_\gamma = g_\gamma$ for all $\gamma \in \Gamma$. This completes the proof that condition (3) holds and the proof of the theorem is complete.

3. In this section we will consider the topological aspects of the amalgamation problem. We recall that there exists a complete duality between Boolean algebras and their dual spaces (cf. [1] and [6]). The crucial element in this duality is, that homomorphic maps and continuous maps are dual. Thus with a homomorphic map of a Boolean algebra B_1 to a Boolean algebra B_2 there corresponds a continuous map of $S(B_2)$ to $S(B_1)$. (We will always denote the dual space of a Boolean algebra B by $S(B)$; a space which is the dual space of a Boolean algebra is called a Boolean space). In particular, with an epimorphism there corresponds

a homeomorphic map "into", and with a monomorphism there corresponds a continuous map "onto".

We illustrate this situation with an example that will be useful in the sequel. Let B be a Boolean algebra and let B_1 and B_2 be subalgebras of B such that B_1 is set-theoretically included in B_2. The topological situation is then as follows. There exists continuous maps, $f_1 : S(B) \to S(B_1)$, $f_2 : S(B) \to S(B_2)$ and $f_{12} : S(B_2) \to S(B_1)$ (all maps are "onto") such that $f_1 = f_{12} f_2$.

Now suppose that B is a Boolean algebra and that B_γ, $\gamma \in \Gamma$ is a family of subalgebras of B, and suppose that $\bigcup_{\gamma \in \Gamma} B_\gamma$ generates B. Then the topological situation can be described as follows. There exists a family of continuous maps $f_\gamma : S(B) \to S(B_\gamma)$ such that the following hold. If X is a Boolean space, and if there exists a continuous map $g : S(B) \to X$ and a family of continuous maps $g_\gamma : X \to S(B_\gamma)$ such that $f_\gamma = g_\gamma g$ for every $\gamma \in \Gamma$, then g is a homeomorphism (all the aforementioned continuous maps are "onto"). We also recall that (cf. [1]) a subalgebra C of a Boolean algebra B coincides with B if and only if the corresponding continuous map of $S(B)$ onto $S(C)$ is a homeomorphic map. Finally, the following well-known fact will also be used. If B_γ, $\gamma \in \Gamma$ is a family of Boolean algebras, then the dual space of the free product of the family B_γ, $\gamma \in \Gamma$ is the topological product of the dual spaces $S(B_\gamma)$ (cf. [6]).

We will now formulate the definition of section 1 topologically.

Let Y_γ, $\gamma \in \Gamma$ be a family of Boolean spaces and let X be a Boolean space such that for every $\gamma \in \Gamma$ there exists a continuous map (which is "onto") $h_\gamma : Y_\gamma \to X$. *The generalized topological product of the family Y_γ, $\gamma \in \Gamma$ with the co-amalgamated Boolean quotientspace X is a Boolean space Z satisfying the following conditions:*

(1′) For every $\gamma \in \Gamma$ there exists a continuous map "onto", $f_\gamma : Z \to Y_\gamma$ such that for every pair $\gamma', \gamma'' \in \Gamma$ $h_{\gamma'} f_{\gamma'} = h_{\gamma''} f_{\gamma''}$.

(2′) If V is a Boolean space such that there exists continuous maps "onto" $f^* : Z \to V$ and $f_\gamma^* : V \to Y_\gamma$ for every $\gamma \in \Gamma$ such that $f_\gamma = f_\gamma^* f^*$ for every $\gamma \in \Gamma$ then f^* is a homeomorphism.

(3′) If U is a Boolean space and if g_γ, $\gamma \in \Gamma$, is a family of continuous maps $g_\gamma : U \to Y_\gamma$ such that for every pair $\gamma', \gamma'' \in \Gamma$, $h_{\gamma'} g_{\gamma'} = h_{\gamma''} g_{\gamma''}$, then there exists a continuous map $g : U \to Z$ such that $f_\gamma g = g_\gamma$ for every $\gamma \in \Gamma$.

We will now formulate Theorem 1 topologically and also give a topological proof of this theorem.

Theorem 1′. The generalized topological product of a family of Boolean spaces with a co-amalgamated Boolean quotient space exists, and is unique (up to homeomorphisms).

Proof. We use the notation of the definition given above. Let T be the topological product of the Boolean spaces Y_γ. Let for every $\gamma \in \Gamma$,

f_γ denote the projection map of T onto Y_γ. We now define a subspace Z of T by $Z=\{p : h_{\gamma'}f_{\gamma'}(p)=h_{\gamma''}f_{\gamma''}(p)$ for every pair $\gamma', \gamma'' \in \Gamma\}$.

Let f_γ also denote the restriction of f_γ to Z. We will show that Z is the desired space. First we show that Z is a closed subspace of T and therefore a Boolean space. Indeed, suppose $p \notin Z$, then there exists a pair $\gamma', \gamma'' \in \Gamma$ such that $h_{\gamma'}f_{\gamma'}(p) \neq h_{\gamma''}f_{\gamma''}(p)$. Call $p'=h_{\gamma'}f_{\gamma'}(p)$ and $p''=h_{\gamma''}f_{\gamma''}(p)$. X is a Hausdorff space, thus there exist open subsets K and L of X such that $p' \in K$ and $p'' \in L$ and $K \cap L=\phi$. Let $K^*=f_{\gamma'}^{-1}(h_{\gamma'}^{-1}(K))$ and $L^*=f_{\gamma''}^{-1}(h_{\gamma''}^{-1}(L))$. Then K^* and L^* are open subsets of T. Now $K^* \cap L^*$ is open and $p \in K^* \cap L^*$ but $K^* \cap L^* \cap Z=\phi$. Indeed, let q be an arbitrary point of $K^* \cap L^*$ then $h_{\gamma'}f_{\gamma'}(q) \in K$ and $h_{\gamma''}f_{\gamma''}(q) \in L$ and thus $h_{\gamma'}f_{\gamma'}(q) \neq h_{\gamma''}f_{\gamma''}(q)$ thus $q \notin Z$. It follows that p is contained in an open subset of T that does not intersect Z in a non void-subset, and this shows that Z is a closed subspace of T.

The next step is to show that condition (1′) holds. We already know, because of the definition of Z, that $h_{\gamma'}f_{\gamma'}=h_{\gamma''}f_{\gamma''}$ for every pair $\gamma', \gamma'' \in \Gamma$. It remains to show that all the maps f_γ are "onto". Indeed, suppose $p \in Y_{\gamma_0}$. Let for every $\gamma \in \Gamma$, $\gamma \neq \gamma_0$, q_γ be a point of Y_γ such that $q_\gamma=h_\gamma^{-1}(h_{\gamma_0}(p))$ and let $q_{\gamma_0}=p$. Then, clearly, the point q of Z whose γth coordinate is q_γ will be mapped by f_{γ_0} on p.

We now proceed to prove that condition (2′) holds. It suffices to show that f^* is one-one. Suppose $f^*(p)=f^*(q)$ then by hypothesis $f_\gamma(p)=f_\gamma{}^*(f^*(p))=f_\gamma{}^*(f^*(q))=f_\gamma(q)$ for all $\gamma \in \Gamma$. But the f_γ are projection maps and from this it follows that $p=q$.

Finally, we must show that condition (3′) holds. Let r be an arbitrary point of U and let $p_\gamma=g_\gamma(r)$ for every $\gamma \in \Gamma$. Let p be the point of T whose γth coordinate is $g_\gamma(r)$. It is easy to see that $p \in Z$. Now define the map $g : U \to Z$ by $g(r)=p$. It easily follows that $f_\gamma g=g_\gamma$ for every $\gamma \in \Gamma$. It remains to show that g is continuous. For this, it suffices to show that the intersection of a subbasic set of T with Z has a pre-image under g which is open. Let γ_0 be a fixed γ, and let U be an open subset of Y_{γ_0}. Let $V=f_{\gamma_0}^{-1}(U) \cap Z$. We claim that $g^{-1}(V)=g_{\gamma_0}^{-1}(U)$. Indeed, suppose that $r \in g^{-1}(V)$ thus $g(r) \in V$. But $g(r) \in V \Rightarrow g(r) \in f_{\gamma_0}^{-1}(U) \Rightarrow f_{\gamma_0}(g(r))=g_{\gamma_0}(r) \in$ $\in U \Rightarrow r \in g_{\gamma_0}^{-1}(U)$. Conversely, suppose that $r \in g_{\gamma_0}^{-1}(U)$ and thus $g_{\gamma_0}(r)=$ $=f_{\gamma_0}(g(r)) \in U$. This implies $g(r) \in f_{\gamma_0}^{-1}(U)$ but also $g(r) \in Z$ and thus $g(r) \in V$ and it follows that $r \in g^{-1}(V)$. This completes the proof of (3′) and we have shown that Z satisfies the conditions (1′), (2′) and (3′). The proof of the uniqueness is a routine matter and is left to the reader.

Remark. It is of course also possible to formulate Lemma 2 topologically. The topological proof of the dual of this lemma is easy and is therefore omitted.

Technological University of Delft, The Netherlands
University of California, Davis, United States of America.

REFERENCES

1. Dwinger, Ph., Introduction to Boolean algebras, Physica Verlag, Wurzburg, 1961.
2. Jonsson, B., Can. Journ. Math. Vol. 13, 256–264 (1961).
3. ————, Math. Scand. Vol. **4**, 193–208 (1956).
4. Kurosh, A. G., The Theory of Groups II, New York, 1956.
5. Neumann, H., Amer. Journ. of Math. Vol. **70**, 590–625 (1948).
6. Sikorski, Boolean Algebras, Springer Verlag, 1960.

Added in proof: The authors have been informed (in a letter from Prof. J. Mycielski) that an algebraic proof of the amalgamationproblem was also presented in a seminar in Berkeley in 1962 by Haim Gaifman.