Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2009, Article ID 293539, 11 pages
doi:10.1155/2009/293539

Research Article

On a Class of Ky Fan-Type Inequalities

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Received 19 August 2009; Accepted 15 December 2009

Recommended by Sever Silvestru Dragomir

We study one class of Ky Fan-type inequalities, which has ties with the original Ky Fan inequality. Our result extends the known ones.

1. Introduction

Let $M_{n,r}(\mathbf{x};\mathbf{q})$ be the generalized weighted means: $M_{n,r}(\mathbf{x};\mathbf{q}) = \left(\sum_{i=1}^{n} q_i x_i^r\right)^{1/r}$, where $M_{n,0}(\mathbf{x};\mathbf{q})$ denotes the limit of $M_{n,r}(\mathbf{x};\mathbf{q})$ as $r \to 0^+$. Here $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{q} = (q_1, \dots, q_n)$ with $q_i > 0$ $(1 \le i \le n)$ satisfying $\sum_{i=1}^{n} q_i = 1$. In this paper, we always assume $0 < x_1 \le x_2 \le \cdots \le x_n$. To any given $\mathbf{x}$ and $t \ge 0$, we set $\mathbf{x}' = (1 - x_1, \dots, 1 - x_n)$, $\mathbf{x}_t = (x_1 + t, \dots, x_n + t)$.

We define $A_n(\mathbf{x};\mathbf{q}) = M_{n,1}(\mathbf{x};\mathbf{q}), G_n(\mathbf{x};\mathbf{q}) = M_{n,0}(\mathbf{x};\mathbf{q}), H_n(\mathbf{x};\mathbf{q}) = M_{n,-1}(\mathbf{x};\mathbf{q})$, and we shall write $M_{n,r}$ for $M_{n,r}(\mathbf{x};\mathbf{q})$, $M_{n,r,t}$ for $M_{n,r}(\mathbf{x}_t;\mathbf{q})$, and $M'_{n,r}$ for $M_{n,r}(\mathbf{x}';\mathbf{q})$ if $x_n < 1$ and similarly for other means when there is no risk of confusion. We further denote $\sigma_n = \sum_{i=1}^{n} q_i (x_i - A_n)^2$.

When $x_n < 1$, we define

$$\Delta_{r,s,\alpha,t} = \frac{\left(M'^{\alpha}_{n,r,t} - M'^{\alpha}_{n,s,t}\right)/\alpha}{\left(M^{\alpha}_{n,r,t} - M^{\alpha}_{n,s,t}\right)/\alpha}, \tag{1.1}$$

where we set $M^0_{n,r}/0 = \ln M_{n,r}$ and we shall write $\Delta_{r,s,\alpha}$ for $\Delta_{r,s,\alpha,0}$ and $\Delta_{r,s}$ for $\Delta_{r,s,1}$. In order to include the case of equality for various inequalities in our discussions, for any given inequality, we define $0/0$ to be the number which makes the inequality an equality. The author [1, Theorem 2.1] has shown the following (in fact, only the case $\alpha = 1$ is shown there but one can easily extend the result to all $\alpha \le 2$ following the method there).

Theorem 1.1. *For $r > s$ and $\alpha \leq 2$, the following inequalities are equivalent:*

$$\frac{r-s}{2x_1^{2-\alpha}}\sigma_n \geq \frac{\left(M_{n,r}^{\alpha} - M_{n,s}^{\alpha}\right)}{\alpha} \geq \frac{r-s}{2x_n^{2-\alpha}}\sigma_n, \tag{1.2}$$

$$\left(\frac{x_n}{1-x_n}\right)^{2-\alpha} \geq \Delta_{r,s,\alpha} \geq \left(\frac{x_1}{1-x_1}\right)^{2-\alpha}, \tag{1.3}$$

where in (1.3) *one requires $x_n < 1$.*

In fact, one can further show that (see [2]) the two inequalities in Theorem 1.1 are equivalent to

$$\left(\frac{x_n}{t+x_1}\right)^{2-\alpha} \geq \Delta_{r,s,\alpha,t} \geq \left(\frac{x_1}{t+x_n}\right)^{2-\alpha} \tag{1.4}$$

being valid for all $t \geq 0$. We point out here that when inequality (1.2) holds for some r, s, one can often expect for a better result than (1.4), namely,

$$\left(\frac{x_n}{t+x_n}\right)^{2-\alpha} \geq \Delta_{r,s,\alpha,t} \geq \left(\frac{x_1}{t+x_1}\right)^{2-\alpha}. \tag{1.5}$$

We note that inequality (1.2) does not hold for all pairs r, s (see [1]). Cartwright and Field [3] first proved the validity of (1.2) for $r = 1, s = 0, \alpha = 1$. For other extensions and refinements of (1.2), see [2, 4–8]. When $\alpha = 1$, inequality (1.3) is commonly referred as the additive Ky Fan's inequality. We refer the reader to the survey article [9] and the references therein for an account of Ky Fan's inequality.

In this paper, we will focus on the special case $\alpha = 0$ of (1.2), which has ties with the following result of Ky Fan that initiated the study of the whole subject.

Theorem 1.2 (see [10, page 5]). *For $x_i \in (0, 1/2]$, $\Delta_{1,0,0} \leq 1$, with equality holding if and only if $x_1 = \cdots = x_n$.*

A nice result of Wang and Chen [11] determines all the pairs r, s with $r > s$ such that $\Delta_{r,s,0} \leq 1$ is satisfied when $x_i \in (0, 1/2]$. Their result is contained in the following.

Theorem 1.3. *For $r > s, x_i \in (0, 1/2]$, $\Delta_{r,s,0} \leq 1$ holds if and only if $|r+s| \leq 3, 2^s/s \geq 2^r/r$ when $s > 0$, $s2^s \leq r2^r$ when $r < 0$.*

We note here that Theorem 1.2 follows from the left-hand side inequality of (1.3) for the case $r = 1, s = 0,$ and $\alpha = 0$, which in turn is a consequence of the above mentioned result of Cartwright and Field. In fact, we have the following result which is contained implicitly in [12].

Theorem 1.4. *If either side of inequality* (1.2) *holds for $r, s, \alpha \leq 2$, then the same side inequality of* (1.2) *also holds for r, s and any $\beta \leq \alpha$. Moreover, the above assertion also holds when applied to* (1.3) *or* (1.4).

On combining the above result with the result of Cartwright and Field we see that (1.2) holds for $r = 1, s = \alpha = 0$ and consequently (1.3) holds for $r = 1, s = \alpha = 0$ in virtue of Theorem 1.1.

Now, it is natural to be motivated by the result of Wang and Chen, in view of the discussions above, to ask whether one can determine all the pairs r, s with $r > s$ such that either one of the inequalities (1.2)–(1.4) holds for $\alpha = 0$. It is our goal in this paper to investigate such a problem. Before we proceed, we would like to summarize the known results in this area. On taking $l = 2, t = 1$ in [5, Proposition 2.3], we deduce with the help of Theorem 1.4 that (1.2) holds for $-1 \le s \le 1, s \le r \le 1 + s, \alpha = 0$. On the other hand, [5, Corollary 3.2] combined with Theorem 1.4 implies that (1.3) holds for $\alpha = 0, 0 \le s \le r \le 1$ and $r - 1 \le s \le 1, 0 \le r \le 2$. We also observe that if (1.2) holds for $r > s$ and $s > s'$, then it also holds for $r > s'$. As (1.2) and (1.3) are equivalent, we conclude that when $\alpha = 0$, (1.2) holds for any $r > s, 0 \le r \le 2, -1 \le s \le 1$.

2. The Main Theorem

Lemma 2.1. *Let $r > s$, $I_1 = (0, 1], I_2 = [1, +\infty)$ and let E denote the region $E = \{(q_1, q_2) \mid q_1 \ge 0, q_2 \ge 0, q_1 + q_2 = 1\}$. Define*

$$D_{r,s}(t; q_1, q_2) = t^{r-1} - t^{s-1} + (r - s)(1 - t)\left(q_1 + q_2 t^r\right)\left(q_1 + q_2 t^s\right). \tag{2.1}$$

Then for $s \ge 0$, $D_{r,s}(t; q_1, q_2) \le 0$ holds for all $(t, q_1, q_2) \in I_1 \times E$ if and only if $s \le 1$ and $r + s \le 3$ and $D_{r,s}(t; q_1, q_2) \le 0$ holds for all $(t, q_1, q_2) \in I_2 \times E$ if and only if $r \le 2$ and $r + s \le 3$.

For $s < 0$, if $r \le 0$, then $D_{r,s}(t; q_1, q_2) \le 0$ holds for all $(t, q_1, q_2) \in I_1 \times E$ if and only if $-1 \le r \le 0$ and $-3 \le r + s \le 0$ and $D_{r,s}(t; q_1, q_2) \le 0$ holds for all $(t, q_1, q_2) \in I_2 \times E$ if and only if $s \ge -2$ and $-3 \le r + s \le 0$.

For $s < 0 < r$, $D_{r,s}(t; q_1, q_2) \le 0$ holds for all $(t, q_1, q_2) \in I_2 \times E$ if and only if $r \le 2$ and $r + s \ge 0$ or $s \ge -2$ and $r + s \le 0$.

Proof. When $s \ge 0$, in order for $D_{r,s}(t; q_1, q_2) \le 0$ to hold for all $(t, q_1, q_2) \in I_1 \times E$, one just needs to check the case $q_1 = 1, q_2 = 0$. In this case we can rewrite $D_{r,s}(t; 1, 0)$ as

$$f(t) = t^{r-1} - t^{s-1} + (r - s)(1 - t). \tag{2.2}$$

Note that $f(1) = f'(1) = 0$; hence in order for $f(t) \le 0$ to hold for all $0 < t \le 1$, it is necessary that $f''(1) \le 0$. Note that $f''(t) = (r-1)(r-2)t^{r-3} - (s-1)(s-2)t^{s-3}$ and from this one checks easily that $f''(1) \le 0$ is equivalent to $r + s \le 3$. On the other hand, on taking $t \to 0^+$, we see that one needs to have $s \le 1$ in order for $f(t) \le 0$ to hold for all $0 < t \le 1$. Now, it also follows from $s \le 1$ that $t^{3-s} f''(t) = (r-1)(r-2)t^{r-s} - (s-1)(s-2) \le \max\{(r-1)(r-2) - (s-1)(s-2), -(s-1)(s-2)\} \le 0$. Hence one deduces via Taylor expansion of $f(t)$ at 1 that $f(t) \le 0$ for all $0 < t \le 1$.

Similarly, when $s \ge 0$, in order for $D_{r,s}(t; q_1, q_2) \le 0$ to hold for all $(t, q_1, q_2) \in I_2 \times E$, one just needs to check $f(t) \le 0$ for $t \ge 1$. As $f(1) = f'(1) = 0$, certainly it is necessary to have $f''(1) \le 0$ and $\lim_{t \to +\infty} f(t) \le 0$. These imply that $r \le 2$ and $r + s \le 3$ and one checks easily that these conditions are also sufficient.

As a consequence of the above discussion, one can deduce the assertion of the lemma for the case $s < 0$ and $r \le 0$ by noting that $D_{r,s}(t; q_1, q_2) = t^{r+s} D_{-s,-r}(t; q_2, q_1)$.

It remains to treat the case $r > 0 > s$. We let $g(q) = (1-q+qt^r)(1-q+qt^s)$, and note that $g''(q) = 2(t^r-1)(t^s-1) \le 0$. It follows from this that in order for $D_{r,s}(t;q_1,q_2) \le 0$ to hold for $(t,q_1,q_2) \in I_2 \times E$, it suffices to check the cases $q_2 = 0, 1$. When $r+s \ge 0$, we only need to check the case $q_2 = 0$ and in this case one can discuss similarly to the case $s \ge 0$ above to conclude the assertion of the lemma. We just point out here that as $s < 0 < r \le 2$, we have $r+s < 2$. When $r+s \le 0$, it suffices to check the case $q_2 = 1$ and in this case one uses the relation $D_{r,s}(t;0,1) = t^{r+s}D_{-s,-r}(t;1,0)$ to convert this to the previous case that has been discussed. □

Theorem 2.2. *Let $r > s$. The right-hand side inequality of* (1.2) *holds for $\alpha = 0$ when $0 \le s \le 1, r+s \le 3$ or $s < 0, -1 \le r \le 0$ and $-3 \le r+s \le 0$. The left-hand side inequality of* (1.2) *holds for $\alpha = 0$ when $-2 \le s \le 0, -3 \le r+s \le 0$.*

Proof. To prove the first assertion of the theorem, we may assume $r > 2$ or $-1 \le r \le 0$ in view of our discussion in the last paragraph of Section 1 and for the case $r > 2$, we define

$$g_n(\mathbf{q},\mathbf{x}) = \ln M_{n,r} - \ln M_{n,s} - \frac{r-s}{2x_n^2}\sigma_n. \tag{2.3}$$

Similar to the proof of Theorem 5.1 [2], it suffices to show that $\partial g_n/\partial x_1 \le 0$. Calculation shows that

$$\frac{1}{q_1}\frac{\partial g_n}{\partial x_1} = \frac{x_1^{r-1}}{M_{n,r}^r} - \frac{x_1^{s-1}}{M_{n,s}^s} - \frac{r-s}{x_n^2}(x_1 - A_n) := f_n(\mathbf{q},\mathbf{x}). \tag{2.4}$$

We now show by induction on n that $f_n(\mathbf{q},\mathbf{x}) \le 0$. When $n = 1$, there is nothing to prove. When $n = 2$, this becomes

$$\frac{1}{q_2}f_2(\mathbf{q},\mathbf{x}) = \frac{x_2^{r+s-1}D_{r,s}(x_1/x_2;q_2,q_1)}{M_{2,r}^r M_{2,s}^s} \le 0, \tag{2.5}$$

by Lemma 2.1.

Suppose now $n \ge 3$; in order to show $f_n(\mathbf{q},\mathbf{x}) \le 0$, we may assume that $0 < x_1 < x_n$ are being fixed and it suffices to show that the maximum value of $f_n(\mathbf{q},\mathbf{x})$ is non-positive on the region $R_n \times S_{n-2}$, where $R_n = \{(q_1,q_2,\ldots,q_n) : 0 \le q_i \le 1, 1 \le i \le n, \sum_{i=1}^n q_i = 1\}$ and $S_{n-2} = \{(x_2,\ldots,x_{n-1}) : x_i \in [x_1,x_n], 2 \le i \le n-1\}$.

Let $(\mathbf{q}',\mathbf{x}')$ be a point of $R_n \times S_{n-2}$ in which the absolute maximum of f_n is reached. If $x_i' = x_{i+1}'$ for some $1 \le i \le n-1$, by combining x_i' with x_{i+1}' and q_i' with q_{i+1}', we are back to the case of $n-1$ variables with different weights. If $q_i' = 1$ for some i, then we have

$$\begin{aligned} \frac{x_1^{r-1}}{M_{n,r}^r} - \frac{x_1^{s-1}}{M_{n,s}^s} - \frac{r-s}{x_n^2}(x_1 - A_n) &= \frac{x_1^{r-1}}{x_i^r} - \frac{x_1^{s-1}}{x_i^s} - \frac{r-s}{x_n^2}(x_1 - x_i) \\ &\le \frac{x_1^{r-1}}{x_i^r} - \frac{x_1^{s-1}}{x_i^s} - \frac{r-s}{x_i^2}(x_1 - x_i) \\ &= \frac{1}{x_i}D_{r,s}\left(\frac{x_1}{x_i};1,0\right) \le 0, \end{aligned} \tag{2.6}$$

by Lemma 2.1. If $q_i' = 0$ for some $1 < i < n$, we are back to the case of $n-1$ variables. If $q_n' = 0$, then we may assume that $q_{n-1}' \neq 0$ and note that we have $M_{n,r} = M_{n-1,r}, M_{n,s} = M_{n-1,s}, A_n = A_{n-1}$ and that

$$\frac{x_1^{r-1}}{M_{n,r}^r} - \frac{x_1^{s-1}}{M_{n,s}^s} - \frac{r-s}{x_n^2}(x_1 - A_n) \leq \frac{x_1^{r-1}}{M_{n-1,r}^r} - \frac{x_1^{s-1}}{M_{n-1,s}^s} - \frac{r-s}{x_{n-1}^2}(x_1 - A_{n-1}), \tag{2.7}$$

and we are again back to the case $n-1$. If $q_1' = 0$, then similarly we may assume that $q_2' \neq 0$ and if we can show that (again with $M_{n,r} = M_{n-1,r}, M_{n,s} = M_{n-1,s}$ and $A_n = A_{n-1}$ here)

$$\frac{x_1^{r-1}}{M_{n,r}^r} - \frac{x_1^{s-1}}{M_{n,s}^s} - \frac{r-s}{x_n^2}(x_1 - A_n) \leq \frac{x_2^{r-1}}{M_{n-1,r}^r} - \frac{x_2^{s-1}}{M_{n-1,s}^s} - \frac{r-s}{x_n^2}(x_2 - A_{n-1}), \tag{2.8}$$

then we are back to the case of $n-1$ variables. Note that the above inequality will follow if the function

$$x \longmapsto \frac{x^{r-1}}{M_{n,r}^r} - \frac{x^{s-1}}{M_{n,s}^s} - \frac{r-s}{x_n^2}x \tag{2.9}$$

is an increasing function for $0 < x \leq M_{n,r}$ (in fact, one only needs this for $0 < x \leq x_2$) and its derivative is

$$\frac{(r-1)x^{r-2}}{M_{n,r}^r} + \frac{(1-s)x^{s-2}}{M_{n,s}^s} - \frac{r-s}{x_n^2} \geq \frac{(r-1)x^{r-2}}{x_n^r} + \frac{(1-s)x^{s-2}}{x_n^s} - \frac{r-s}{x_n^2} := h(x), \tag{2.10}$$

with the inequality holding for the case $r > 2$ (note that together with $r+s \leq 3$, this implies that $s < 1$). It also follows from $r+s \leq 3$ that $h'(x) = 0$ has no root in $(0, x_n)$. One then deduces from $h(x_n) = 0$ and $\lim_{x \to 0^+} h(x) = +\infty$ that $h(x) \geq 0$ for $0 < x \leq x_n$.

So from now on it remains to consider the case $q_i' \neq 0, 1, x_i' \neq x_j'$ for $1 \leq i, j \leq n, i \neq j$ and this implies that $(\mathbf{q}', \mathbf{x}')$ is an interior point of $R_n \times S_{n-2}$. We will now show that this cannot happen.

We define

$$p(x) = -\frac{x_1^{r-1}x^r}{M_{n,r}^{2r}} + \frac{x_1^{s-1}x^s}{M_{n,s}^{2s}} + \frac{(r-s)x}{x_n^2} - \lambda. \tag{2.11}$$

Note here in the definition of $p(x)$ that $M_{n,r}$ and $M_{n,s}$ are not functions of x, they take values at some point $(\mathbf{q}, \mathbf{x})$ to be specified, and λ is also a constant to be specified.

As $(\mathbf{q}', \mathbf{x}')$ is an interior point of $R_n \times S_{n-2}$, we may use the Lagrange multiplier method to obtain a real number λ so that at $(\mathbf{q}', \mathbf{x}')$,

$$\frac{\partial f_n}{\partial q_i} = \lambda \frac{\partial}{\partial q_i}\left(\sum_{i=1}^{n} q_i - 1\right), \quad \frac{1}{q_j}\frac{\partial f_n}{\partial x_j} = 0 \tag{2.12}$$

for all $1 \leq i \leq n$ and $2 \leq j \leq n-1$.

By (2.12), a computation shows that each x'_i $(1 \le i \le n)$ is a root of $p(x) = 0$ (where $M_{n,r}, M_{n,s}$ take their values at $(\mathbf{q}', \mathbf{x}')$) and each x'_i $(2 \le i \le n-1)$ is a root of $p'(x) = 0$. Now $n \ge 3$ implies $p(x_2) = 0$. As $p(x_1) = p(x_2) = p(x_n) = 0$, it follows from Rolle's Theorem that there must be two numbers $x_1 < a < x_2 < b < x_n$ such that $p'(a) = p'(x_2) = p'(b) = 0$. However, it is easy to see that $p'(x) = 0$ has at most two positive roots and this contradiction implies the first assertion of the theorem for the case $0 \le s \le 1$.

Now to show the right-hand side inequality hold of (1.2) for the case $s < 0, -1 \le r \le 0$ and $-3 \le r + s \le 0$, once again it suffices to show that the function $g_n(\mathbf{q}, \mathbf{x})$ defined above is nonnegative for any integer $n \ge 1$. We note that when $n = 1$, this is obvious and when $n = 2$, this follows again from $\partial g_2/\partial x_1 \le 0$ by Lemma 2.1.

Suppose now $n \ge 3$; in order to show $g_n(\mathbf{q}, \mathbf{x}) \ge 0$, we may assume that $0 < x_1 < x_n$ are being fixed and it suffices to show the minimum value of $g_n(\mathbf{q}, \mathbf{x})$ is nonnegative on the region $R_n \times S_{n-2}$, where R_n and S_{n-2} are defined as above.

Let $(\mathbf{q}', \mathbf{x}')$ be a point of $R_n \times S_{n-2}$ in which the absolute minimum of g_n is reached. Note that $\sigma_n = M_{n,2}^2 - A_n^2$; thus if $x'_i = x'_{i+1}$ for some $1 \le i \le n-1$, by combining x'_i with x'_{i+1} and q'_i with q'_{i+1}, we are back to the case of $n-1$ variables with different weights. Similarly, if $q'_i = 1$ for some i, then we are back to the case $n = 1$. If $q'_i = 0$ for some $1 \le i < n$, we are back to the case of $n-1$ variables. If $q'_n = 0$, then we may assume that $q'_{n-1} \ne 0$ and note that we have $M_{n,r} = M_{n-1,r}, M_{n,s} = M_{n-1,s}, M_{n,2} = M_{n-1,2}, A_n = A_{n-1}$ and that

$$\begin{aligned}
\ln M_{n,r} - \ln M_{n,s} - \frac{r-s}{2x_n^2}\sigma_n &= \ln M_{n,r} - \ln M_{n,s} - \frac{r-s}{2x_n^2}\left(M_{n,2}^2 - A_n^2\right) \\
&\ge \ln M_{n-1,r} - \ln M_{n-1,s} - \frac{r-s}{2x_{n-1}^2}\left(M_{n-1,2}^2 - A_{n-1}^2\right) \\
&= \ln M_{n-1,r} - \ln M_{n-1,s} - \frac{r-s}{2x_{n-1}^2}\sigma_{n-1},
\end{aligned} \tag{2.13}$$

and we are again back to the case of $n-1$ variables.

So from now on it remains to consider the case $q'_i \ne 0, 1, x'_i \ne x'_j$ for $1 \le i, j \le n, i \ne j$, and this implies that $(\mathbf{q}', \mathbf{x}')$ is an interior point of $R_n \times S_{n-2}$. We will now show that this cannot happen.

We define

$$a(x) = \frac{x^r}{rM_{n,r}^r} - \frac{x^s}{sM_{n,s}^s} - \frac{(r-s)\left(x^2 - 2A_n x\right)}{2x_n^2} - \lambda. \tag{2.14}$$

Here we define $x^0/0 = \ln x$. Also note here in the definition of $a(x)$, that $M_{n,r}, M_{n,s}$, and A_n are not functions of x, they take values at some point $(\mathbf{q}, \mathbf{x})$ to be specified, and λ is also a constant to be specified.

As $(\mathbf{q}', \mathbf{x}')$ is an interior point of $R_n \times S_{n-2}$, we may use the Lagrange multiplier method to obtain a real number λ so that at $(\mathbf{q}', \mathbf{x}')$,

$$\frac{\partial g_n}{\partial q_i} = \lambda \frac{\partial}{\partial q_i}\left(\sum_{i=1}^{n} q_i - 1\right), \quad \frac{1}{q_j}\frac{\partial g_n}{\partial x_j} = 0 \tag{2.15}$$

for all $1 \le i \le n$ and $2 \le j \le n-1$.

By (2.15), a computation shows that each x_i' $(1 \le i \le n)$ is a root of $a(x) = 0$ (where $M_{n,r}, M_{n,s},$ and A_n take their values at $(\mathbf{q}', \mathbf{x}')$) and each x_i' $(2 \le i \le n-1)$ is a root of $a'(x) = 0$. Now $n \ge 3$ implies $a(x_2) = 0$. As $a(x_1) = a(x_2) = a(x_n) = 0$, it follows from Rolle's Theorem that there must be two numbers $x_1 < c < x_2 < d < x_n$ such that $a'(c) = a'(x_2) = a'(d) = 0$. However, we have

$$a'(x) = \frac{x^{r-1}}{M_{n,r}^r} - \frac{x^{s-1}}{M_{n,s}^s} - \frac{(r-s)(x-A_n)}{x_n^2}. \tag{2.16}$$

It is easy to see that $a'''(x) = 0$ has at most one positive root, which implies that $a'(x) = 0$ has at most three positive roots. As $r \le 0$, it follows from $\lim_{x\to 0^+} a'(x) = -\infty$ and $\lim_{x\to +\infty} a'(x) = -\infty$ that $a'(x) = 0$ has even numbers of roots so that $a'(x) = 0$ can have at most two positive roots. This contradiction now establishes the right-hand side inequality of (1.2) for the case $s < 0, -1 \le r \le 0$, and $-3 \le r + s \le 0$.

One can show the second assertion of the theorem using an argument similar to the above and we shall leave this to the reader. □

3. Further Discussions

As we have pointed out in Section 1 that if either one of the inequalities (1.2)–(1.4) holds for some $r, s, \alpha \le 2$, then one often expects inequality (1.5) to hold as well for the same r, s, α. In view of this, one may ask whether it is feasible to prove so for those pairs $r, s, \alpha = 0$ satisfying Theorem 2.2. We now prove a special case here.

Theorem 3.1. *Let $-3 \le r \le 3, r \ne 0, t \ge 0$, then the following inequality holds:*

$$x_n^2 |\ln G_n - \ln M_{n,r}| \ge (x_n + t)^2 |\ln G_{n,t} - \ln M_{n,r,t}|. \tag{3.1}$$

Proof. We first prove the theorem for the case $-3 \le r < 0$. For this, we may assume that $t > 0$ is fixed and replace r with $-r$ so that $0 < r \le 3$ in what follows. We define

$$f_n(\mathbf{q}, \mathbf{x}) = x_n^2 (\ln G_n - \ln M_{n,-r}) - (x_n + t)^2 (\ln G_{n,t} - \ln M_{n,-r,t}). \tag{3.2}$$

As in the proof of Theorem 2.2, it suffices to show that $\partial f_n / \partial x_1 \le 0$ and calculation shows

$$-\frac{1}{q_1} \frac{\partial f_n}{\partial x_1} = g_n(\mathbf{q}, \mathbf{x}) - g_n(\mathbf{q}, \mathbf{x}_t), \tag{3.3}$$

where

$$g_n(\mathbf{q}, \mathbf{x}) = x_n^2 \left(\frac{\sum_{i=1}^n q_i (x_i^r - x_1^r) / x_i^r}{x_1 \sum_{i=1}^n q_i (x_1 / x_i)^r} \right). \tag{3.4}$$

It is easy to check that

$$\frac{x_n}{x_i} \geq \frac{x_n+t}{x_i+t}, \quad \frac{x_1}{x_i} \leq \frac{x_1+t}{x_i+t}. \tag{3.5}$$

In view of (3.5), the inequality $\partial f_n/\partial x_1 \leq 0$ will follow from

$$d_1(x_i) = \frac{x_n^2(x_i^r - x_1^r)}{x_1 x_i^r} - \frac{(x_n+t)^2((x_i+t)^r - (x_1+t)^r)}{(x_1+t)(x_i+t)^r} \geq 0 \tag{3.6}$$

for $x_1 \leq x_i \leq x_n$. We may assume that $x_n > x_1$ here and it is easy to see that $d_1'(x) = 0$ can have at most one root x_0 in between x_1 and x_n. This combined with the observation that $d_1(x_1) = 0, d_1'(x_1) > 0$ implies that $d_1(x)$ reaches its local maximum at x_0 if it exists. Hence we are left to check that $d_1(x_n) \geq 0$. In this case we note that $x_n - x_1 = (x_n+t) - (x_1+t)$ and we rewrite $d_1(x_n)$ as

$$\begin{aligned} d_1(x_n) &= \frac{x_n^2(x_n^r - x_1^r)}{x_1 x_n^r(x_n - x_1)} - \frac{(x_n+t)^2((x_n+t)^r - (x_n+t)^r)}{(x_1+t)(x_n+t)^r((x_n+t)-(x_1+t))} \\ &= e\left(\frac{x_n}{x_1}\right) - e\left(\frac{x_n+t}{x_1+t}\right), \end{aligned} \tag{3.7}$$

where

$$e(x) = \frac{x^r - 1}{x^{r-2}(x-1)}. \tag{3.8}$$

In view of (3.5) again, we just need to show that $e(x)$ is an increasing function for $x > 1$. Note that

$$e'(x) = \frac{x^{r-3}(x^{r+1} - 2x^r + (r-1)x - (r-2))}{(x^{r-2}(x-1))^2}, \tag{3.9}$$

and it is easy to see that the function $x^{r+1} - 2x^r + (r-1)x - (r-2)$ is non-negative for $x \geq 1$ when $0 < r \leq 3$ by considering its Taylor expansion at $x = 1$ and this completes the proof for the assertion of the theorem for the case $-3 \leq r < 0$.

To prove the theorem for the case $0 < r \leq 3$, we may again assume that $t > 0$ is fixed and define

$$u_n(\mathbf{q}, \mathbf{x}) = x_n^2(\ln M_{n,r} - \ln G_n) - (x_n+t)^2(\ln M_{n,r,t} - \ln G_{n,t}). \tag{3.10}$$

Again it suffices to show that $\partial u_n/\partial x_1 \leq 0$ and calculation shows

$$-\frac{1}{q_1}\frac{\partial u_n}{\partial x_1} = v_n(\mathbf{q}, \mathbf{x}) - v_n(\mathbf{q}, \mathbf{x}_t), \tag{3.11}$$

where

$$v_n(\mathbf{q},\mathbf{x}) = x_n^2\left(\frac{\sum_{i=1}^n q_i\left(x_i^r - x_1^r\right)/x_n^r}{x_1\sum_{i=1}^n q_i(x_i/x_n)^r}\right). \tag{3.12}$$

In view of (3.5), the inequality $\partial u_n/\partial x_1 \le 0$ will follow from

$$d_2(x_i) = \frac{x_n^2\left(x_i^r - x_1^r\right)}{x_1 x_n^r} - \frac{(x_n+t)^2\left((x_i+t)^r - (x_1+t)^r\right)}{(x_1+t)(x_n+t)^r} \ge 0 \tag{3.13}$$

for $x_1 \le x_i \le x_n$. We may assume that $x_n > x_1$ here and it is easy to see that $d_2'(x) = 0$ can have at most one root x_0 in between x_1 and x_n. This combined with the observation that $d_2(x_1) = 0, d_2'(x_1) > 0$ implies that $d_1(x)$ reaches its local maximum at x_0 if it exists. Hence we are left to check that $d_2(x_n) \ge 0$. As $d_2(x_n) = d_1(x_n)$, this completes the proof for the remaining case $0 < r \le 3$ of the theorem. □

Now we show that, in general, it is not true that for $-3 \le r \le 3, r \ne 0, t \ge 0$,

$$x_1^2|\ln G_n - \ln M_{n,r}| \le (x_1+t)^2|\ln G_{n,t} - \ln M_{n,r,t}|. \tag{3.14}$$

To proceed, we first look at the following related inequalities (with $r > s$ here):

$$\ln M_{n,r} - \ln M_{n,s} - \frac{(r-s)\sigma_n}{2x_1^2} \le \ln M_{n,r,t} - \ln M_{n,s,t} - \frac{(r-s)\sigma_n}{2(x_1+t)^2}, \tag{3.15}$$

$$\ln M_{n,r} - \ln M_{n,s} - \frac{(r-s)\sigma_n}{2x_n^2} \ge \ln M_{n,r,t} - \ln M_{n,s,t} - \frac{(r-s)\sigma_n}{2(x_n+t)^2}. \tag{3.16}$$

Let $f_n(\mathbf{q},\mathbf{x},t)$ denote the right-hand side expression of (3.15); then (3.15) holds if and only if $\partial f_n(\mathbf{q},\mathbf{x},0)/\partial t \ge 0$. As $\mathbf{x}$ is arbitrary, we can recast this condition as

$$\frac{M_{n,r-1}^{r-1}}{M_{n,r}^r} - \frac{M_{n,s-1}^{s-1}}{M_{n,s}^s} + \frac{(r-s)\sigma_n}{x_1^3} \ge 0. \tag{3.17}$$

Similarly, (3.16) holds if and only if the following inequality holds:

$$\frac{M_{n,r-1}^{r-1}}{M_{n,r}^r} - \frac{M_{n,s-1}^{s-1}}{M_{n,s}^s} + \frac{(r-s)\sigma_n}{x_n^3} \le 0. \tag{3.18}$$

As a first step towards establishing (3.17), we consider the case $n = 2$ here; in this case we let $x_1 = 1 \le t = x_2$ and rewrite the left-hand side of (3.17) as

$$\frac{q_1+q_2t^{r-1}}{q_1+q_2t^r} - \frac{q_1+q_2t^{s-1}}{q_1+q_2t^s} + (r-s)q_1q_2(t-1)^2 = \frac{q_1q_2(1-t)}{(q_1+q_2t^r)(q_1+q_2t^s)}D_{r,s}(t;q_1,q_2) \tag{3.19}$$

with $D_{r,s}(t;q_1,q_2)$ being defined as in Lemma 2.1. Using the same notations as in Lemma 2.1, we see that in order for (3.17) to hold for $n = 2$, we need to have $D_{r,s}(t;q_1,q_2) \le 0$ for $(t,q_1,q_2) \in I_2 \times E$. Similar treatment of (3.18) shows that in order for it to hold in the case $n = 2$, one needs to have $D_{r,s}(t;q_1,q_2) \le 0$ for $(t,q_1,q_2) \in I_1 \times E$.

It follows from the proof of Lemma 2.1 that $D_{r,0}(t;1,0) \le 0$ fails to hold for all $t \ge 1$ when $r > 2$. In another words, there exists $\mathbf{x}, \mathbf{q}$ such that when $r > 2$,

$$\frac{M_{n,r-1}^{r-1}}{M_{n,r}^{r}} - \frac{1}{H_n} + \frac{(r-s)\sigma_n}{x_1^3} < 0 \tag{3.20}$$

holds. Now we return to the inequality (3.14) and we take $r > 0$ there. Just as in the discussion above, one sees that (3.14) is equivalent to

$$\frac{2(\ln M_{n,r} - \ln G_n)}{x_1} + \frac{M_{n,r-1}^{r-1}}{M_{n,r}^{r}} - \frac{1}{H_n} \ge 0. \tag{3.21}$$

This combined with (3.20) now implies that for $r > 2$,

$$\ln M_{n,r} - \ln G_n > \frac{(r-s)\sigma_n}{2x_1^2}. \tag{3.22}$$

However, on taking $t \to +\infty$ on (3.14), we get the above inequality reversed (with $>$ replaced by $\le$) and this leads to a contradiction; hence (3.14) does not hold for $r > 2$ in general.

To end this paper, we note that it is an open problem to determine all the triples (r,s,α) so that inequality (1.2) holds. However, when $\alpha = 0$ with $r = 0$ or $s = 0$, the result given in Theorem 2.2 is best possible. In this case Theorem 2.2 implies that for $|r| \le 3, r \ne 0$,

$$|\ln M_{n,r} - \ln G_n| \ge \frac{|r|}{2x_n^2}\sigma_n. \tag{3.23}$$

We point out here that inequality (3.23) does not hold in general when $|r| > 3$. To see this, it suffices to consider the case $n = 2$ and in this case we can set $0 < x_1 = t \le x_2 = 1$ and consider more generally for $r > s$, the function $g_2(\mathbf{q},\mathbf{x})$ defined in the proof of Theorem 2.2, regarding it as a function $f(t)$ of t. It is easy to check that $f(1) = f'(1) = f''(1) = 0$; hence by the Taylor expansion of $f(t)$ around $t = 1$, we need $f^{(3)}(1) \le 0$ in order for $f(t) \ge 0$ to hold for any $0 < t \le 1$. Calculation shows that

$$f^{(3)}(1) = q_1(r-s)\Big(r+s-3-3(r+s-1)q_1+2(r+s)q_1^2\Big). \tag{3.24}$$

On taking $q_1 \to 0^+$, one sees immediately that we must have $r+s \le 3$ here in order for $f(t) \ge 0$ for all $0 < t \le 1$. On taking $s = 0$, we see that one needs $r \le 3$ in order for (3.23) to hold for positive r. Similarly, one checks easily that in the case $n = 2$, if inequality (3.23) holds for some r, then it also holds for $-r$ by a change of variables $x_i \to 1/x_{2-i+1}$. Hence one needs $r \ge -3$ in order for (3.23) to hold for any negative r.

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