

A LATTICE FOR PERSISTENCE

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ABSTRACT. The intrinsic connection between lattice theory and topology is fairly well established. For instance, the collection of open subsets of a topological subspace always forms a distributive lattice. Persistent homology has been one of the most prominent areas of research in computational topology in the past 20 years. In this paper we will introduce an alternative interpretation of persistence based on the study of the order structure of its correspondent lattice. Its algorithmic construction leads to two operations on homology groups which describe an input diagram of spaces as a complete Heyting algebra, which is a generalization of a Boolean algebra. We investigate some of the properties of this lattice, the algorithmic implications of it, and some possible applications.

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INTRODUCTION

25 Persistent (co)homology is one of the central objects of study in applied and computational
 26 topology [16]. Numerous extensions have been proposed to the original formulation including zig-
 27 zag persistence [10] and multidimensional persistence [9], whereas the original persistence looks
 28 at a filtration (i.e., an increasing sequence of spaces). Zig-zag persistence extended the theory
 29 and showed that the direction of the maps does not matter, using tools from quiver theory. In
 30 multidimensional persistence, multifiltrations are considered. In this paper, we also look at the
 31 problem of persistence in more general diagrams of spaces using tools from lattice theory. There is
 32 another key difference in this work however. Rather than try to find a decomposition of the diagram
 33 of spaces into indecomposables, we concentrate on pairs of spaces within diagrams addressing the
 34 more difficult problem of indecomposables in the sequel paper.

35 Lattice theory is the study of order structures. The deep connections between topology and
 36 lattice theory has been known since the work of Stone [21], showing a duality between Boolean
 37 algebras and certain compact and Hausdorff topological spaces, called appropriately *Stone spaces*.
 38 In the first section of this paper we present the basic concepts of lattice theory. These preliminaries
 39 mostly refer to classical results on distributive lattices and Heyting algebras, and can be skipped by
 40 the reader that is familiar with the subject. A study of lattice theory and, in general, of universal
 41 algebra, can be found in [5], [6], [18] and [19].

42 A description of the topological background follows in the second section, reviewing the main
 43 concepts and results of Persistent Homology and suggesting several examples that are a motivation
 44 to this study. Good reviews on topological data analysis are given in [7] and [36], on persistent
 45 homology are given in [32] and [35], and on zig-zag persistence are given in [10], [8] and [28].

46 In the following section we describe the order structure of our input diagram of spaces by a
 47 partial order induced by certain maps between vector spaces, and show that this order provides a
 48 lattice structure. We construct the meet and join operations using the natural concepts of limits
 49 and colimits of linear maps, and show that this construction stabilizes. We shall see that the
 50 constructed lattice is a complete Heyting algebra, one of the algebraic objects of biggest interest in
 51 topos theory.

52 From the latter results we discuss connections with persistent homology, and give a different
 53 perspective on several aspects of this theory. In particular, we look at diagrams of spaces and
 54 retrieve general laws both based on concrete examples (like standard or zig-zag persistence) and
 55 on the interpretation of laws derived from the lattice theoretic analysis. Finally we introduce a few
 56 algorithmic applications which we will develop further in a subsequent paper.

1. PRELIMINARIES

58 A *lattice* is a partially-ordered set (or poset) expressed by (L, \leq) for which all pairs of elements
 59 have an infimum and a supremum, denoted by \wedge and \vee , respectively, commonly known as the

60 *meet* and *join* operations. The lattice properties correspond to the minimal structure that a poset
 61 must have to be seen as an algebraic structure. Such algebraic structure $(L; \wedge, \vee)$ is given by two
 62 operations \wedge and \vee satisfying:

63 L1. *associativity*: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$,

64 L2. *idempotency*: $x \wedge x = x = x \vee x$,

65 L3. *commutativity*: $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$

66 L4. *absorption*: $x \wedge (x \vee y) = x = x \vee (x \wedge y)$.

67 The equivalence between this algebraic perspective of a lattice L and its ordered perspective is
 68 given by the following equivalence: for all $x, y \in L$, $x \leq y$ iff $x \wedge y = x$ iff $x \vee y = y$. At that
 69 stage the order and the algebraic structures hold the same information over different perspectives.
 70 If every subset of a lattice L has a supremum and an infimum, L is named a *complete lattice*.
 71 All finite lattices are complete. A partial order is named *total order* if every pair of elements is
 72 related, that is, for all $x, y \in A$, $x \leq y$ or $y \leq x$. On the other hand, an *antitotal order* is a
 73 partial order for which no two elements are related. Examples of lattices include the power set of
 74 a set ordered by subset inclusion, or the collection of all partitions of a set ordered by refinement.
 75 Every lattice can be determined by a unique undirected graph for which the vertices are the lattice
 76 elements and the edges correspond to the partial order: the *Hasse diagram* of the lattice. With
 77 additional constraints on the operations we get different types of lattices. In particular, a lattice L
 78 is *distributive* if, for all $x, y, z \in S$, it satisfies one of the following equivalent equalities:

79 (d1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;

80 (d2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;

81 (d3) $(x \vee y) \wedge (x \vee z) \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$.

82 The lattice of subsets of a set ordered by inclusion is a distributive lattice. The lattice of normal
 83 subgroups of a group as well as the lattice of subspaces of a vector space are not distributive (cf.
 84 [5]). A lattice L is distributive if and only if for all $x, y, z \in L$, $x \wedge y = x \wedge z$ and $x \vee y = x \vee z$
 85 imply $y = z$ ([5]). A *Boolean algebra* is a distributive lattice with a unary operation \neg and nullary
 86 operations 0 and 1 such that for all elements $a \in A$, $a \vee 0 = a$ and $a \wedge 1 = a$ as $a \vee \neg a = 1$ and
 87 $a \wedge \neg a = 0$. While the power of a set with intersection and union is a Boolean algebra, total orders
 88 are examples of distributive lattices that are not Boolean algebras in general. A bounded lattice L
 89 is a *Heyting algebra* if, for all $a, b \in L$ there is a greatest element $x \in L$ such that $a \wedge x \leq b$. This
 90 element is the *relative pseudo-complement* of a with respect to b denoted by $a \Rightarrow b$. Examples of
 91 Heyting algebras are the open sets of a topological space, as well as all the finite nonempty total
 92 orders (that are bounded and complete). Furthermore, every complete distributive lattice L is a
 93 Heyting algebra with the implication operation given by $x \Rightarrow y = \bigvee \{x \in L \mid x \wedge a \leq b\}$.

94 **Contributions 1.0.1.** *Universal algebra and lattice theory, in particular, are transversal dis-*
 95 *ciplines of Mathematics and have proven to be of interest to the study of any algebraic*
 96 *structure. In the following sections we will describe the construction of a lattice completing*
 97 *a given commutative diagram of homology groups. We will show that this lattice is complete*
 98 *and distributive, thus constituting a complete Heyting algebra. Despite the nice algebraic*
 99 *properties that hold in this structure as a consequence of being such an algebra, it does not*
 100 *constitute a Boolean algebra.*

101

2. PROBLEM STATEMENT

102 We assume a basic familiarity with algebraic topological notions such as (co)homology, simplicial
 103 complexes, filtrations, etc. For an overview, we recommend the references [20] for algebraic topology,
 104 as well as [15] and [36] for applied/computational topology. We motivate our constructions with
 105 the examples in the following paragraphs.

Consider persistent homology, presented in [16]. Let \mathbb{X} be a space and $f : \mathbb{X} \rightarrow \mathbb{R}$ a real function. The object of study of persistent homology is a filtration of \mathbb{X} , i.e., a monotonically non-decreasing sequence

$$\emptyset = \mathbb{X}_0 \subseteq \mathbb{X}_1 \subseteq \mathbb{X}_2 \subseteq \dots \subseteq \mathbb{X}_{N-1} \subseteq \mathbb{X}_N = \mathbb{X}$$

To simplify the exposition, we assume that this is a discrete finite filtration of tame spaces. Taking the homology of each of the associated chain complexes, we obtain

$$H_*(\mathbb{X}_0) \rightarrow H_*(\mathbb{X}_1) \rightarrow H_*(\mathbb{X}_2) \rightarrow \dots \rightarrow H_*(\mathbb{X}_{N-1}) \rightarrow H_*(\mathbb{X}_N)$$

We take homology over a field k – therefore the resulting homology groups are vector spaces and the induced maps are linear maps. In [16], the (i, j) -persistent homology groups of the filtration are defined as

$$H_*^{i,j}(\mathbb{X}) = \text{im}(H_*(\mathbb{X}_i) \rightarrow H_*(\mathbb{X}_j))$$

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This motivates the idea for the construction of a totally ordered lattice. To see this, let us consider the set of the homology groups with a partial order induced by the indexes of the spaces in the filtration. We can define two lattice operations \wedge and \vee as follows:

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$$\begin{aligned} H_*(\mathbb{X}_i) \vee H_*(\mathbb{X}_j) &= H_*(\mathbb{X}_{\max(i,j)}) \\ H_*(\mathbb{X}_i) \wedge H_*(\mathbb{X}_j) &= H_*(\mathbb{X}_{\min(i,j)}) \end{aligned}$$

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With these operations we get a finite total order and, thus, a complete Heyting algebra (see this discussion in the following section). The definition of persistent homology groups can then be rewritten as follows:

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Definition 2.0.2. For any two elements $H_*(\mathbb{X}_i)$ and $H_*(\mathbb{X}_j)$, the rank of the persistent homology classes is

$$\text{im}(H_*(\mathbb{X}_i \wedge \mathbb{X}_j) \rightarrow H_*(\mathbb{X}_i \vee \mathbb{X}_j)).$$

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The case of a filtration, where a total order exists, does not have a very interesting underlying order structure. Let us now look at the case where we have more than one parameter. We define a *diagram* to be a directed acyclic graph of vector spaces (vertices) and linear maps between them (edges). This is known as multidimensional persistence and has been studied in [9] and [11]. We shall start by looking at a bifiltration, i.e., a filtration on two dimensions (or parameters). Observe that, for related elements of the filtration, these operations coincide with the ones defined above for the standard persistence case. However, when we consider incomparable elements, the meet and join operations are given by the rectangles they determine. Adjusting our definitions from above we can define the lattice operations in a natural way by setting:

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$$\begin{aligned} H_*(\mathbb{X}_{i,j}) \vee H_*(\mathbb{X}_{k,\ell}) &= H_*(\mathbb{X}_{\max(i,k), \max(j,\ell)}) \\ H_*(\mathbb{X}_{i,j}) \wedge H_*(\mathbb{X}_{k,\ell}) &= H_*(\mathbb{X}_{\min(i,k), \min(j,\ell)}) \end{aligned}$$

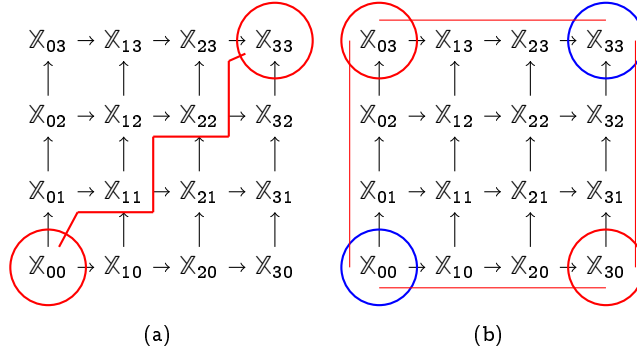


FIGURE 1. The lattice operations in the case of a bifiltration. (a) If the two elements are comparable, by the commutativity of the diagram we can choose any path to find the persistent homology groups. (b) If the elements are incomparable, we can find the smallest and largest elements where they become comparable. In both cases we recover the rank invariant of [9]

127 Consider the bifiltration of dimensions 4×4 from Figure 1. The Hasse diagram of the corre-
 128 spondent underlying algebra is presented in Figure 2. In that diagram, $X_{01} \leq X_{31}$ and clearly,
 129 $X_{01} \wedge X_{31} = X_{01}$ while $X_{01} \vee X_{31} = X_{31}$. On the other hand, X_{02} and X_{11} are unrelated with
 130 $X_{02} \wedge X_{11} = X_{01}$ while $X_{02} \vee X_{11} = X_{12}$. Note that, by the commutativity of the diagram, any two
 131 elements which have the same meet and join define the same rectangle in the bifiltration, determined
 132 by the properties in the Hasse diagrams represented in Figure 2. By the assumed commutativity of
 133 the diagram of spaces, any path through the rectangle has equal rank and so the map of the meet
 134 to join gives the rank invariant of Definition 2.0.2.

135 Both of these cases are highly-structured. Consider the case of a more general diagram of
 136 homology groups in Figure 3. While we can embed this diagram in a multifiltration, by augmenting
 137 the diagram with 0 and unions of space, however the result is not very informative. The defined
 138 lattice operations can bring a complementary knowledge to this study. This is the motivation for
 139 the construction we present in this paper. Since we deal with homology over a field, we look to
 140 analyze more general but commutative diagrams of vector spaces.

141 **Problem 2.0.3.** *Given a commutative diagram of vector spaces and linear maps between*
 142 *them, we construct an order structure that completes it into a lattice, study its algebraic*
 143 *properties and develop algorithms based on this.*

144 **Remark 2.0.4.** *Quiver theory is also concerned with diagrams of vector spaces and linear*
 145 *maps. However, a key difference is that the diagrams in quiver theory are generally not*
 146 *required to be commutative.*

147 **Remark 2.0.5.** *We concentrate on the persistence between two elements rather than decom-*
 148 *position of the entire diagram. While we believe the constructions in this paper can aid this*
 149 *decomposition, it does not immediately follow. As such, any reference to a diagram should be*

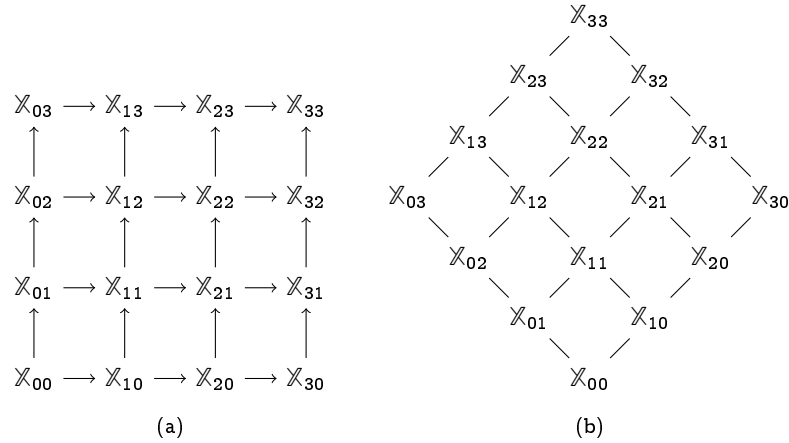


FIGURE 2. The diagram of a bifiltration of dimensions 4×4 (a) and the Hasse diagram of the correspondent underlying Heyting algebra (b).

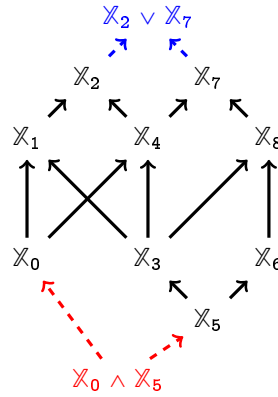


FIGURE 3. General commutative diagrams of spaces and linear maps between them.

150 *understood as referring to the input collection of vector spaces and linear maps, correspond-*
 151 *ing to the partial Hasse diagram of the underlying lattice structure, rather than a persistence*
 152 *diagram.*

153

3. LATTICE STRUCTURE

154 Here we introduce how to retrieve the order information from a diagram of vector spaces and
 155 linear maps, and construct the lattice operations determined by that order, where the elements
 156 are vector spaces. The linear maps between them will define the relations between those vector
 157 spaces and limit concepts like equalizers and coequalizers (roughly, an equalizer is a solution set of

158 equations while a coequalizer is a generalization of a quotient by an equivalence relation) will serve
 159 us to define biggest and least elements.

160 **3.1. The Lattice Operations.** Consider a diagram of vector spaces and linear maps and assume
 161 one unique component. The underlying ordered structure is a poset defined as follows:

162 **Definition 3.1.1.** For all vector spaces A and B of a given diagram \mathcal{D} ,

$$A \leq B \text{ if there exists a linear map } f : A \rightarrow B.$$

163 The partial order \leq is, thus, the set of ordered pairs correspondent to the linear maps in
 164 the commutative diagram of spaces given as input. The identity map ensures the reflexivity
 165 of the relation: for all vector spaces A the identity map id_A provides the endorelation $\subseteq A$.
 166 Transitivity is given by the fact that the composition of linear maps is a linear map and by
 167 the assumption that all diagrams are commutative. Antisymmetry is given by the fact that
 168 $A \leq B$ implies $A \rightsquigarrow B$, that is, A and B are equal up to isomorphism: in detail, having
 169 the identity morphisms and usual composition of linear maps, the existence of linear maps
 170 $f : A \rightarrow B$ and $g : B \rightarrow A$ imply that $g \circ f = \text{id}_A$ and that $f \circ g = \text{id}_B$, as required. This
 171 partial order does not yet have to constitute a lattice but will be completed into one, due to
 172 the following constructions. The extension of the partial order \leq will be noted by the same
 173 symbol, being a part of that bigger partial order.

174 **Remark 3.1.2.** We consider the object under study to be a commutative diagram of vector
 175 spaces and linear maps. As vector spaces are determined up to isomorphism by rank, the
 176 equivalence deserves some additional comments. As described above, the reverse maps exist
 177 in the case of isomorphisms. This further ensures that the poset structure is well-defined since
 178 we cannot arbitrarily reverse the direction of the arrows (as is often the case in representation
 179 theory, where the direction of arrows often does not matter). If we were to reverse an arrow
 180 with a non-unique (but equal rank) map, it is clear that the composition will not commute
 181 with identity unless the map is an isomorphism. Likewise, for equivalence we not only require
 182 the vector spaces to be isomorphic (of the same rank) but also that there exists a composition
 183 of maps in the diagram (possibly including inverses) for which an isomorphism exists. Note
 184 that this does not imply that all the maps must be isomorphisms.

185 In the following paragraphs we will describe the construction of the operations \wedge and \vee over a
 186 given diagram \mathcal{D} of vector spaces and linear maps. The construction of these lattice operations is
 187 based on the concept of direct sum, and the categorical concepts of *limit* and *colimit*. In particular,
 188 it is based in the generalized notions of *equalizer* and *coequalizer* that we describe right away. See
 189 the details of some of these constructions in Appendix B. As we assume that all diagrams of vector
 190 spaces commute, the categorical concepts of *equalizer* and *coequalizer* can be adapted to the
 191 framework of this paper in the following way:

192 **Definition 3.1.3.** Given a pair of vector spaces A and C with two linear maps $f, g : A \Rightarrow B$
 193 between them:

194 (i) the equalizer of f and g is a pair (E, e) where E is a vector space (usually called
 195 kernel set of the equalizer) and $e : E \rightarrow A$ is a linear map such that $fe = ge$, for any

196 other vector space E' and linear map $e' : E' \rightarrow A$ there exists a unique linear map
 197 $\phi : E' \rightarrow E$.
 198 (ii) the coequalizer of f and g is a pair (H, h) where H is a vector space (usually called
 199 the quotient set of the coequalizer) and $h : A \rightarrow H$ is a linear map such that, for any
 200 other vector space H' and linear map $h' : A \rightarrow H'$ there exists a unique linear map
 201 $\phi : H \rightarrow H'$.

202 **Remark 3.1.4.** *The intuitive idea of looking at the equalizer of two maps f and g as the*
 203 *solution set of the equation $f(x) = g(x)$ in the appropriate domain, is extended to a solution*
 204 *set of several equations. Indeed, any system of equations can be seen as one unique (matrix)*
 205 *equation with all the equations that it is constituted being considered as vectors in this matrix.*
 206 *Dual remarks hold for coequalizers of more than two maps.*

207 *The (co)equalizer is sometimes identified with the kernel set (quotient set). Both the*
 208 *concepts of equalizer and coequalizer can be generalized to comprehend the equalization of*
 209 *more than two maps which corresponds to a solution set of several equations. Given vector*
 210 *spaces A, B, C and D , with linear maps $f_A : A \rightarrow C$, $f_B : B \rightarrow C$, $g_A : D \rightarrow A$ and $g_B : D \rightarrow B$*
 211 *we can express these relations by the linear maps $f : A \oplus B \rightarrow C$ and $g : D \rightarrow A \oplus B$ without*
 212 *loss of information. If $\mathcal{F} = \{f, g, h, \dots\}$ its equalizer may be written as $\text{eq}(f, g, h, \dots)$ while*
 213 *its coequalizer is written as $\text{coeq}(f, g, h, \dots)$. For the sake of intuition, the kernel set can*
 214 *be thought of as the space of solutions of all the equations determined by the linear maps*
 215 *that are equalized, while the quotient set of a coequalizer can be thought of as the space of*
 216 *constraints that an equation must satisfy, as the space of obstructions, regarding the equations*
 217 *determined by the considered linear maps. Indeed, for modules over a commutative ring, the*
 218 *equalizer of f and g is $\ker(f - g)$ while their coequalizer is $\text{coker}(f - g) = B/\text{im}(f - g)$. This*
 219 *and other topics are discussed in detail in the appendix of this paper.*

220 **Definition 3.1.5.** *A vector space is a source if it is no codomain of any map, and dually it*
 221 *is a target if it is no domain of any map (corresponding to the categorical concepts of initial*
 222 *element and terminal element, respectively. Moreover, we call common source of a collection*
 223 *of spaces D_i in the given diagram \mathcal{D} , a space $D \in \mathcal{D}$ mapping in \mathcal{D} to each of the spaces D_i .*
 224 *Dually, we call common target of the collection D_i to a space $D \in \mathcal{D}$ such that each D_i maps*
 225 *to D .*

226 **Remark 3.1.6.** *Given vector spaces X, Y, Z and W in a diagram \mathcal{D} ,*

- 227 (i) *if Z is a common target of X and Y then Z is a target of $X \oplus Y$;*
 228 (ii) *if W is a common source of X and Y then W is a source of $X \oplus Y$.*

229 *While (i) follows from the fact that the direct sum is the coproduct in the category of vector*
 230 *spaces and linear maps, to see (ii) consider the inclusion maps $i_X : X \rightarrow X \oplus Y$ and $i_Y : Y \rightarrow$*
 231 *$X \oplus Y$. To see (ii) consider the inclusion maps $i_X : X \rightarrow X \oplus Y$ and $i_Y : Y \rightarrow X \oplus Y$. Due to*
 232 *the hypothesis, there exist maps $f : W \rightarrow X$ and $g : W \rightarrow Y$. Thus, the compositions $i_X \circ f$*
 233 *and $i_Y \circ g$ ensure the inequality $W \leq X \oplus Y$. Moreover,*

- 234 (iii) *if Z is a common target of X and Y , the limit of all linear maps from X and Y to Z*
 235 *is a subalgebra of $X \oplus Y$;*

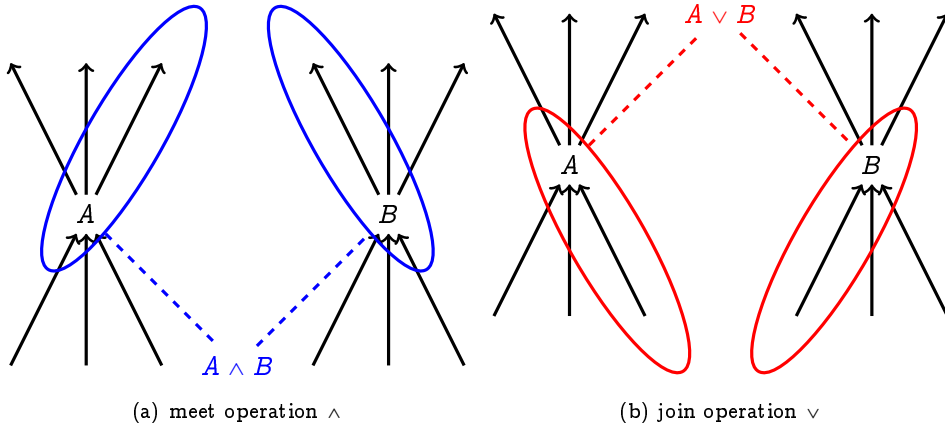


FIGURE 4. Intuition of the defined lattice operations meet, \wedge and join, \vee . (a) Given two elements, A and B , the meet is defined by looking at all the spaces which A and B map into to compare them. (b) For the join, we use the dual construction and compare A and B using all the spaces which map into A and B .

236 (iv) if W is a common source of X and Y , the colimit of all linear maps from W to X
 237 and Y is a quotient algebra of $X \oplus Y$.
 238 both of them constituting vector spaces.

239 **Definition 3.1.7.** Let A and B be vector spaces and I and J be arbitrary sets. Consider the
 240 family of linear maps from $A \oplus B$ to all vector spaces with common sources A and B , i.e.,

$$\mathbb{F}_k = \{f_i : A \oplus B \rightarrow X_k \mid \text{for all vector spaces } X_k \geq A, B \text{ and } i \in I\}$$

241 and, dually, the family of linear maps from all vector spaces with common targets A and B
 242 to $A \oplus B$, i.e.,

$$\mathbb{G}_k = \{g_i : Y_k \rightarrow A \oplus B \mid \text{for all vector spaces } Y_k \leq A, B \text{ and } i \in I\}.$$

243 Define $A \wedge B$ to be the kernel set \mathcal{E} of the equalizer of the linear maps of the family \mathbb{F}_k ,
 244 $\text{eq}(\bigoplus_{k \in J} \mathbb{F}_k)$, and $A \vee B$ to be the quotient set \mathcal{C} of the coequalizer of the linear maps of the
 245 family \mathbb{G}_k , $\text{coeq}(\bigoplus_{k \in J} \mathbb{G}_k)$. These operations are well defined due to Remark 3.1.6.

246 **Remark 3.1.8.** Intuitively, whenever A and B are vector spaces we construct $A \vee B$ as the
 247 limit of all vector spaces that have maps coming in from both A and B by "gathering" together
 248 all those maps to all vector spaces C_i with common sources A and B : in particular, this limit
 249 is the equalizer of such maps. Dually, we construct $A \wedge B$ as the colimit of all the linear
 250 maps from a vector space D_j to common targets A and B . This intuition is represented
 251 in Figure 4. Hence, $A \wedge B$ is the limit of the $\{A, B\}$ -cone and $A \vee B$ is the colimit of the
 252 $\{A, B\}$ -cocone. Recall that (co)complete categories are the ones where the (co)limit of any
 253 diagram $F : I \rightarrow D$ exists. The category of vector spaces is both complete and cocomplete.

254 Thus, we can generalize this to an arbitrary set of vector spaces $\{A_0, A_1, \dots, A_i, \dots\}$ in the
 255 sense of complete lattices (discussed later in Section 3.2). The definitions for \wedge and \vee have a
 256 constructive nature that will show to be useful when we later describe the computation of the
 257 operations. To resume, given a diagram of vector spaces and linear maps \mathcal{D} , and arbitrary
 258 vector spaces X and Y in \mathcal{D} we call meet of spaces X, Y to the limit in \mathcal{D} of all linear maps
 259 from $X \oplus Y$ to common targets of X and Y , i.e.,

$$X \wedge Y = \lim\{X \rightarrow Z \leftarrow Y : Z \text{ common target of } X \text{ and } Y\}$$

260 Dually, we call join of spaces X, Y to the colimit in \mathcal{D} of all linear maps from common
 261 sources of X and Y to $X \oplus Y$, i.e.,

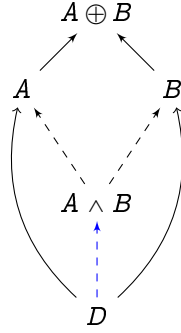
$$X \vee Y = \text{colim}\{X \leftarrow Z \rightarrow Y : Z \text{ common source of } X \text{ and } Y\}$$

262 **Remark 3.1.9.** Regarding the algorithmic implementation of equalizers and coequalizers,
 263 we refer to [32] where, given linear maps f and g , the authors discuss the computation
 264 of $\ker(f - g)$ and $\text{coker}(f - g)$ that correspond to the computation of pullbacks and push
 265 outs, respectively. As shown above, under the assumptions of this paper, these correspond
 266 to equalizers and coequalizers. Furthermore, when considering families of linear maps $\mathcal{F} =$
 267 $(f_i)_{i \in I}$ and $\mathcal{G} = (g_j)_{j \in I}$ of more than two maps, the equalizer of \mathcal{F} is $\bigcap_{i, j \in I} \ker(f_i - f_j)$ and the
 268 coequalizer of \mathcal{G} is $B / \bigcup_{i, j \in I} \text{im}(g_i - g_j)$. In fact, any such solution set of multiple equations
 269 can be seen as the solution set of one equation and thus we can reduce the computation to
 270 one kernel, Dual remarks hold for the computation of the coequalizer.

271 **3.2. The Lattice Proofs.** In the following result we will show that the elements of a commutative
 272 diagram of vector spaces together with the operations \vee and \wedge defined above determine a lattice.
 273 We will refer to it as the *persistence lattice* of a given diagram of vector spaces and linear maps,
 274 i.e., the completion of that diagram into a lattice structure using the lattice operations \vee and
 275 \wedge . We shall also show the stability of the lattice operations defined above, and show that these
 276 determine a complete lattice.

277 **Theorem 3.2.1.** Let \mathcal{D} be a diagram of spaces and maps between them. Consider the partially
 278 ordered set $\mathcal{P} = (\mathcal{D}^*; \leq)$, with the operations \vee and \wedge defined as above, where $*$ is the closure
 279 of \mathcal{P} relative to these operations. Then \mathcal{P} constitutes a lattice.

280 *Proof.* Let us see that $A \wedge B$ is the biggest lower bound of the set $\{A, B\}$. Due to Remark 3.1.6
 281 we need only to see that given another vector space D such that $D \leq A, B$, then there exists a
 282 linear map from D to $A \wedge B$, i.e., $D \leq A \wedge B$. Let us consider the following diagram:



283

284 The compositions of either with the maps from A and B to some common target C ($A \oplus B$,
 285 for instance) commute by assumption. Due to the construction of $A \wedge B$ as a limit, we get that
 286 $D \leq A \wedge B$ by universality. Hence, $A \wedge B$ is the greatest lower bound (the biggest subalgebra)
 287 regarding all the other subalgebras of $A \oplus B$ that are maps from $A \oplus B$ to the vector spaces above
 288 both A and B . The proof that $A \vee B$ is the least upper bound (the finest partition) of the set
 289 $\{A, B\}$ is analogous and derives from the universality of its construction as a colimit. \square

290 **Theorem 3.2.2.** *Given vector spaces A and B , the construction of $A \wedge B$ and $A \vee B$ stabilizes.*

291 *Proof.* In the following proof we will show that the skew lattice construction stabilizes, i.e., when-
 292 ever we are given vector spaces A and B and

- 293 (1) we first construct $A \wedge B$ from $A, B \leq A \oplus B$,
- 294 (2) then we construct $A \vee B$ from $A \wedge B \leq A, B$,
- 295 (3) then we again construct $(A \wedge B)'$ from $A, B \leq A \vee B$,

296 we can ensure that $(A \wedge B)' = A \wedge B$. The dual result follows analogously.

297 **Case 1: Sources.** In this case, we assume that the elements are two sources and that there
 298 exists an element above both of them. We denote the elements A, B and C , respectively. We
 299 are then able to define $M = A \wedge B$ that is constituted by elements (a, b) of $A \oplus B$ such that
 300 $(f, 0)(a, b) = (g, 0)(a, b)$, where f and g map to C . Since there is now an element below A and
 301 B , we can define $J = A \vee B$ as all the quotient space of $A \oplus B$. Define $M \rightarrow A \oplus B$ where the
 302 map is (k, ℓ) . Therefore we now have $A \oplus B \rightarrow A \oplus B / \langle \langle (k(x), \ell(x)) \mid x \in M \rangle \rangle$. Call these maps
 303 v and w . What remains to show is that the elements which satisfy $(v, 0)(a, b) = (0, w)(a, b)$ are
 304 the same as above. Now if $(f, 0)(a, b) = (g, 0)(a, b) \neq (0, 0)$, by commutivity and universality,
 305 $(v, 0)(a, b) = (0, w)(a, b) \neq (0, 0)$. However, if $(f, 0)(a, b) = (g, 0)(a, b) = (0, 0)$, then there exists
 306 an element $m \in M$ such that $m \mapsto (a, b)$ which implies that $(v, 0)(a, b) = (0, w)(a, b)$, since this is
 307 precisely the relation in the definition. Since M can only get smaller with additional constraints,
 308 it follows that the resulting M has stabilized.

309 **Case 2: Targets.** In this case, we assume that the elements are two sources and that there
 310 exists an element below them. We denote the elements A, B and C respectively. We define
 311 $J = A \vee B$, constituted by the quotient $A \oplus B / \langle \langle (f(x), g(x)) \mid x \in C \rangle \rangle$. Denote this map (k, ℓ) . Based
 312 on this we define the $M = A \wedge B$ as the subspace such that $(k, 0)(a, b) = (0, \ell)(a, b)$. Denote the
 313 map from this space to the direct sum as (v, w) . Now we need to show $A \oplus B / \langle \langle (f(c), g(c)) \mid c \in$
 314 $C \rangle \rangle = A \oplus B / \langle \langle (v(m), w(m)) \mid m \in M \rangle \rangle$. By universality it follows that there exists an $m \in M$ such

315 that $c \mapsto m$ and hence $f(c) = v(m)$ and $g(c) = w(m)$. It follows that $f(c)\theta g(c)$ is equivalent to
 316 $v(m)\theta w(m)$. If we do not want to use universality, if $(f, g)(c) \neq (0, 0)$, there must be an element
 317 in J such that $k(f(c)) = \ell(g(c)) = j$. Hence we conclude that there is an element $c \mapsto m$. If
 318 $(f, g)(c) = (x, 0)$, then by the quotient $k(f(c)) = 0$ and again there must be an element $m \mapsto (x, 0)$.
 319 Finally if $(f, g)(c) = (0, 0)$, there is no element other than 0 such that $k(f(c)) = \ell(g(c))$ and hence
 320 $c \mapsto (0, 0) \in M$.

321

□

322 **Theorem 3.2.3.** *Persistence lattices are complete, i.e., both of the lattice operations extend*
 323 *to arbitrary joins $\bigvee_i D_i$ and meets $\bigwedge_i D_i$ (note that both $\bigvee_i D_i$ and $\bigwedge_i D_i$ might not be in*
 324 *\mathcal{D}).*

325 *Proof.* Consider a subset S of the underlying set of spaces of the given persistence lattice \mathcal{P} . Take
 326 their direct sum $X = \bigoplus_{\ell} \{A_{\ell} \in S\}$. To see that the arbitrary set S has a general meet just consider
 327 $\bigwedge S$ to be the limit of all the maps from all vector spaces $A_{\ell} \in S$ to a common vector space $\bigoplus_k C_k$
 328 such that $A, B \leq C_k$, for each k , i.e.,

$$\bigwedge S = \{x \in X : f_i(x) = f_j(x), \text{ for all } f_i, f_j \in \bigcup_k \text{Hom}(X, C_k)\}.$$

329 This is the kernel set determined by the parcels of the direct sum X that satisfy the system of
 330 equations determined by the considered maps, i.e.,

$$\bigwedge_{\ell} A_{\ell} = \{x \in \bigoplus_{\ell} A_{\ell} : f_{A_i A_j}(x) = f_{A_u A_v}(x)\}.$$

331 Dually, $\bigvee S$ is the colimit of the union of all maps from a common vector space $\bigoplus_k D_k$ all vector
 332 spaces $A_i \in S$ such that $D_k \leq A, B$, for each $k \in I$. Hence,

$$\bigvee_{\ell} A_{\ell} = (\bigoplus_{\ell} A_{\ell}) / \langle (f_i(x), f_j(x)) \mid x \in \bigoplus_k D_k \rangle$$

333 which is the quotient of the product of the vector spaces A_{ℓ} by the equivalence generated by the
 334 union of respective equivalences, i.e.,

$$\bigvee_{\ell} A_{\ell} = (\bigoplus_{\ell} A_{\ell}) / \langle \bigcup \theta_{A_i A_j} \rangle.$$

335

□

336 **Remark 3.2.4.** *According to our definition of \wedge and join,*

337 (i) *the \bigwedge of spaces X_i is the limit in \mathcal{P} of all linear maps from $\bigoplus_{i \in I} X$ to common targets*
 338 *of X_i , i.e.,*

$$\bigwedge_{i \in I} X_i = \lim \{X_i \rightarrow Z : Z \text{ common target of } X_i\}$$

339 (i) *the \bigvee of spaces X_i is the colimit in \mathcal{P} of all linear maps from common sources of X_i*
 340 *to $\bigoplus_{i \in I} X$, i.e.,*

$$\bigvee_{i \in I} X_i = \text{colim} \{X_i \leftarrow Z : Z \text{ common source of } X_i\}$$

341 **Remark 3.2.5.** *Completeness is a very important property in the study of ordered structures.*
 342 *The open sets of a topological space, ordered by inclusion, are examples of such structures*
 343 *where \vee is given by the union of open sets and \wedge by the interior of the intersection. In the*
 344 *last section we will see an algorithm application for this particular lattice property. We will*
 345 *refer to it as the largest injective by then.*

346 **3.3. The Lattice Properties.** In the following we describe some of the most relevant character-
 347 istics of the lattice that we have described in the earlier section. We shall see that, besides the
 348 algebraic properties due to its lattice nature, it is also modular and distributive.

349 **Remark 3.3.1.** *Let us first have a look at the properties of the operations \wedge and \vee of the*
 350 *persistence lattice \mathcal{H} constructed above over an input poset. The identity map implies that*
 351 *$A \wedge A = A$ and $A \vee A = A$. This algebraic property follows from the order structure of the*
 352 *correspondent persistence lattice. The equivalence between the algebraic structure and the*
 353 *order structure of the underlying algebra ensures that a linear map $f : A \rightarrow B$ exists iff*
 354 *$A = A \wedge B$ iff $A \vee B = B$. Moreover, the following lattice identities hold:*

$$A \wedge (A \vee B) = A = A \vee (A \wedge B) = A.$$

355 The following result will enlighten this theory with a nice relation between the lattice operations
 356 and the direct sum. This property is not frequently used in the study of lattice properties but will
 357 permit us to show the distributivity of a persistence lattice in the next paragraphs.

358 **Theorem 3.3.2.** *Let A and B be vector spaces. Then,*

$$A \wedge B \rightarrow A \oplus B \rightarrow A \vee B \text{ is a short exact sequence.}$$

359 *Proof.* First observe that the limit map $f : A \wedge B \rightarrow A \oplus B$ is injective and the colimit map
 360 $g : A \oplus B \rightarrow A \vee B$ is surjective (cf. [26]). We thus need to show that $\text{im } f = \ker g$ to prove the
 361 isomorphism

$$A \vee B \cong A \oplus B / f(A \wedge B).$$

362 If $y \in \text{im } f$ then there exists $x \in A \wedge B$ mapping to y such that $g_i(x) = g_j(x)$ for all $g_k : A \oplus B \rightarrow$
 363 $A \vee B$ and thus $y \in \ker g$. On the other hand, if $x \in \ker g$, then $g|_A(x|_A) = g|_B(x|_B)$ implying there
 364 exists an element in $x \in A \wedge B$ which maps to y .

365 □

366 **Theorem 3.3.3.** *Persistence lattices are distributive.*

367 *Proof.* Let A, B and X be vector spaces such that $X \vee A = X \vee B$ and $X \wedge A = X \wedge B$ in order
 368 to show that $A \cong B$. Consider the following commutative diagram of spaces:

$$\begin{array}{ccccc}
& & X \vee A = X \vee B & & \\
& & \uparrow & & \uparrow \\
& & f & & s \\
& & \nearrow & & \nwarrow \\
A & & & B & & X \\
& & \nwarrow & & \nearrow \\
& & g & & t \\
& & \downarrow & & \downarrow \\
& & X \wedge A = X \wedge B & &
\end{array}$$

369

The result will follow from the definition of distributivity for the lattice operations, the Five Lemma and exactness of the sequence (cf. Theorem 3.3.2)

$$0 \rightarrow Y \wedge Z \xrightarrow{f} Y \oplus Z \xrightarrow{g} Y \vee Z \rightarrow 0$$

370 Consider the the following diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & A \wedge X & \longrightarrow & A \oplus X & \longrightarrow & A \vee X & \longrightarrow & 0 \\
\uparrow \cong & & \uparrow \cong & & & & \uparrow \cong & & \uparrow \cong \\
0 & \longrightarrow & B \wedge X & \longrightarrow & B \oplus X & \longrightarrow & B \vee X & \longrightarrow & 0
\end{array}$$

371

372 The first and last isomorphism are trivial, while the other isomorphisms follow by assumption. The
373 existence of the linear map $f : A \oplus B \rightarrow B \oplus X$ is ensured by the fact that we are dealing with
374 vector spaces, assuming the commutativity of the diagram. Therefore, by the Five Lemma, we
375 conclude that $A \oplus X \cong B \oplus X$ and hence $A \cong B$, concluding the proof.

376

□

377 The distributive property is of great interest in the study of order structures. With it we are able
378 to retrieve a rich structure satisfying many interesting identities. The next result follows directly
379 from the distributivity of persistence lattices.

380 **Corollary 3.3.4.** *The persistence lattice intervals $[A \wedge B, B]$ and $[A, A \vee B]$ are isomorphic due
381 to the maps $f : [A \wedge B, B] \rightarrow [A, A \vee B]$, defined by $X \mapsto X \vee A$, and $g : [A, A \vee B] \rightarrow [A \wedge B, B]$,
382 defined by $Y \mapsto Y \wedge B$.*

383 **Remark 3.3.5.** *Due to Dilworth's results on poset decompositions, there exists an antitotal
384 order of vector spaces S and a partition of the order in A into a family F of total orders of
385 vector spaces such that the number of total orders in the partition equals the cardinality of
386 S and, thus, S is the largest antitotal order in the order, and F must be the smallest family
387 of total orders into which the order can be partitioned. Dually, the size of the largest total
388 order of vector spaces in a finite poset of vector spaces as such equals the smallest number of
389 antitotal orders of vector spaces into which the order may be partitioned.*

390 **Theorem 3.3.6.** *Persistence lattices are discrete, finite and bounded.*

391 *Proof.* In the following we will give an upper bound for the number of elements of a persistence
 392 lattice of a given diagram of spaces. The finiteness of the lattice implies that it is discrete and
 393 complete. Thus, it follows that it is a bounded lattice. Indeed, an upper bound for the number of
 394 elements of the persistence lattice correspondent to a diagram with $|V| = n$ is given by

$$\sum_i \binom{n}{i} 2^{i-1} \leq 2^n \cdot 2^n = 2^{2n}.$$

395 To see the above bound consider a string of V_i 's. Since the operations are commutative and
 396 associative, we will need to only consider all combinations of nodes which are included in the
 397 string. To get an element of the lattice, we must also consider the two operations. For a string of
 398 length of m , this implies $m - 1$ operations. Since we have two operations this implies there are
 399 $2^{(m-1)}$ operations on the string. Since $m < n$, we can bound the sum by 2^{2n} , implying that we
 400 add a finite number of elements. \square

401 **Remark 3.3.7.** *This is a very loose bound intended only to illustrate finiteness. In practice,*
 402 *there will be far fewer elements due to distributivity and even fewer elements of interest.*

403 **Theorem 3.3.8.** *Persistence lattices constitute complete Heyting algebras.*

404 *Proof.* Recall that nonempty finite distributive lattices are bounded and complete, thus forming
 405 Heyting algebras. Hence, this result follows from Theorems 3.3.3, 3.2.3 and 3.3.6. \square

406 **Remark 3.3.9.** *Whenever A and B are vector spaces in a diagram, there exists a vector space*
 407 *X that is maximal in the sense of $X \wedge A \leq B$, i.e., the implication operation is given by the*
 408 *colimit*

$$A \Rightarrow B = \bigvee \{ X_i \in L \mid \bigoplus_i (X_i \wedge A) \rightarrow B \}.$$

409 *Observe that the case of standard persistence we have that*

$$A \Rightarrow B = \begin{cases} B, & \text{if } B \leq A \\ 1, & \text{if } A \leq B \end{cases}.$$

410 *The study of the interpretation of the implication operation in the framework of other general*
 411 *models of persistence, as zig-zag or multidimensional persistence, is a matter of further*
 412 *research.*

413 **Remark 3.3.10.** *Persistence lattices \mathcal{P} are not Boolean algebras. To see this just consider the*
 414 *standard persistence case that is represented by a total order, or the total order $\{C, B, D\}$ in*
 415 *the above bifiltration and observe that there is no $X \in L$ such that $B \wedge X = D$ and $B \vee X = C$.*
 416 *Hence, B also doesn't have a complement in \mathcal{P} .*

417 **Remark 3.3.11.** *The results of this section permit us to discuss several directions of future*
 418 *work that can contribute with further information on the order and algebraic properties of*
 419 *this structure and motivate the construction of new algorithms. A topos is essentially a*
 420 *category that "behaves" like a category of sheaves of sets on a topological space, while sheaves*
 421 *of sets are functors designed to track locally defined data attached to the open sets of a*
 422 *topological space and transpose it to a global perspective using a certain "gluing property".*
 423 *Topos theory has important applications in algebraic geometry and logic (cf. [23] and [21]),*

424 and has recently been used to construct the foundations of quantum theory (cf. [14]). The
 425 category of sheaves on a Heyting algebra is a topos (cf. [1]). Whenever skew lattices, a
 426 noncommutative variation of lattices, satisfy a certain distributivity, they constitute sheaves
 427 over distributive lattices (and over Heyting algebras in particular (cf. [2])). The study of such
 428 algebras, developed by the second author of this paper in [25], might be of great interest to the
 429 research on the properties of persistence lattices and their interpretation in the framework
 430 of persistent homology. Furthermore, complete Heyting algebras are of great importance to
 431 study of frames and locales that form the foundation of pointless topology, leading to the
 432 categorification of some ideas of general topology (cf. [21]).

433 **Remark 3.3.12.** A natural and well studied relationship between lattice theory and topology
 434 is described by the duality theory [12]. These dualities are of great interest to the study of
 435 algebraic and topological problems taking advantage of the categorical equivalence between
 436 respective structures (cf. [17]). In the case of complete Heyting algebras, the Esakia duality
 437 permits the correspondence of such algebras to dual spaces, called Esakia spaces that are com-
 438 pact topological spaces equipped with a partial order, satisfying a certain separation property
 439 that will imply them to be Hausdorff and zero dimensional (cf. [4]). These spaces are a
 440 particular case of Priestley spaces that are homeomorphic to the spectrum of a ring (cf.[3]).
 441 We are interested in the study of such topological spaces and correspondent ring.

442 4. ALGORITHMS AND APPLICATIONS

443 We now give some interpretations of both the order structure and the algebraic structure of the
 444 lattice in the framework of persistent homology.

445 **4.1. Interpretations Under Persistence.** We saw that in the case of standard persistence, we
 446 have a total order where A and B are related and thus (L1) tells us that, $\mathbb{X}_m \wedge \mathbb{X}_n = \mathbb{X}_m$, the
 447 domain of the map f connecting \mathbb{X}_m and \mathbb{X}_n , while $\mathbb{X}_m \vee \mathbb{X}_n = \mathbb{X}_n$, its codomain. On the other
 448 hand, to analyze the multidimensional case we saw that using

$$\mathbb{X}_{n,m} \wedge \mathbb{X}_{p,q} = \mathbb{X}_{\min\{n,p\}, \max\{m,q\}} \text{ and } \mathbb{X}_{n,m} \vee \mathbb{X}_{p,q} = \mathbb{X}_{\max\{n,p\}, \min\{m,q\}}.$$

449 for the meet and join respectively we recover the rank invariant. We will return to the bifiltration
 450 case but first discuss its connections with zig-zag persistence. In the case of zig-zag persistence, we
 451 get the following diagram:

$$\begin{array}{ccccccc} & & H(\mathbb{X}_1) & & H(\mathbb{X}_3) & & H(\mathbb{X}_5) & & H(\mathbb{X}_7) & & \\ & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \\ H(\mathbb{X}_0) & & & & H(\mathbb{X}_2) & & H(\mathbb{X}_4) & & H(\mathbb{X}_6) & & H(\mathbb{X}_8) \end{array}$$

452
 453 Without loss of generality, if we assume that we have an alternating zig-zag as above, we see that
 454 we have a partial order: the odds are strictly greater than the even indexed spaces. This is not an
 455 interesting partial order as most elements are incomparable. In [8] and [29] it was noted that using
 456 unions and relative homology, the above could be extended to a case where all elements become
 457 comparable with possible dimension shifts. The resulting zig-zag can be extended into a Möbius
 458 strip through exact squares. By exactness any two elements can be compared by considering unions
 459 and relative homologies as shown in Figure 5.

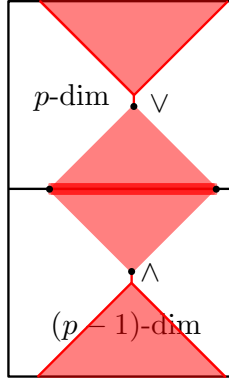


FIGURE 5. Here we show a possible choice of meet and join for zig-zag persistence based on the Möbius strip construction of [8].

460 Using a special case of our construction, using pullbacks and pushouts as limits and colimits,
 461 the authors in [31], developed a parallelized algorithm for computing zig-zag persistence.

To compare two general elements define

$$H_*(\mathbb{X}_i) \wedge H_*(\mathbb{X}_j) = \begin{cases} K \rightarrow H_*(\mathbb{X}_i) \oplus H_*(\mathbb{X}_j) \rightrightarrows H_*(\mathbb{X}_{i+1}) & j = i + 2 \\ H_*(\mathbb{X}_i) \wedge H_*(\mathbb{X}_{i+2}) \wedge \cdots \wedge H_*(\mathbb{X}_j) \end{cases}$$

and

$$H_*(\mathbb{X}_i) \vee H_*(\mathbb{X}_j) = \begin{cases} H_*(\mathbb{X}_{i+1}) \rightrightarrows H_*(\mathbb{X}_i) \oplus H_*(\mathbb{X}_j) \rightarrow P & j = i + 2 \\ H_*(\mathbb{X}_i) \vee H_*(\mathbb{X}_{i+2}) \vee \cdots \vee H_*(\mathbb{X}_j) \end{cases}$$

462 With this definition it is not difficult to verify the following results

- 463 (1) The rank of $H_*(\mathbb{X}_i) \wedge H(\mathbb{X}_j) \rightarrow H_*(\mathbb{X}_i) \vee H(\mathbb{X}_j)$ is equal to the rank in the original zig-zag
 464 definition.
 465 (2) The structure can be built up iteratively, comparing all elements two steps away then three
 466 steps away and so on, leading to the parallelized algorithm.

467 **Remark 4.1.1.** In [31], an additional trick was used so that only the meets had to be computed.

4.2. Largest Injective. For the first application, we consider the computation of the largest injective of a diagram. In principle, we are looking for something which persists over an entire diagram. While satisfying the properties of the underlying lattice structure, the largest injective must fulfill to be in the following images

$$\text{im} (H_*(\mathbb{X}_i) \wedge H_*(\mathbb{X}_j) \rightarrow H_*(\mathbb{X}_i) \vee H_*(\mathbb{X}_j)) \quad \forall i, j$$

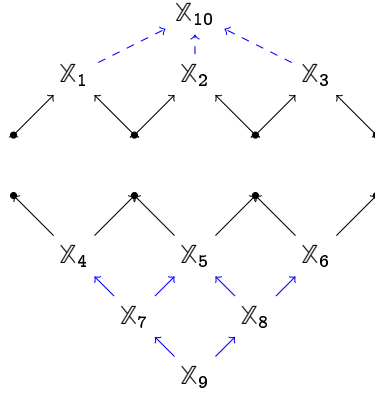
By completeness, it follows that this can be written as

$$\text{im} \left(\bigwedge_i H_*(\mathbb{X}_j) \rightarrow \bigvee_i H_*(\mathbb{X}_i) \right)$$

Using the order structure, we can rewrite the above as

$$\text{im} \left(\bigwedge_{i \in \text{sources}} H_*(X_i) \rightarrow \bigvee_{j \in \text{targets}} H_*(X_j) \right).$$

468 Recall that *sources* are all the elements in original diagram which are not the codomain of any
 469 maps and *targets* are the elements which are not the domain of any maps. Assuming we have
 470 n sources, m targets and the longest total order in the diagram is k assuming an $O(1)$ time to
 471 compute a \vee or \wedge of two elements, we have a run time of $O(n + m + k)$. On a parallel machine,
 472 the operations can be computed independently and using associativity, we can construct the total
 473 meet/join using a binary tree scheme, giving a run time of $O(k + \log(\max(n, m)))$.



474

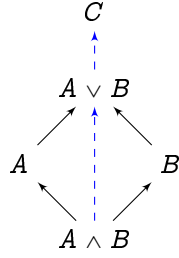
$$\text{im} \left(\bigwedge_{i \in \text{sources}} H_*(X_i) \rightarrow \bigvee_{j \in \text{targets}} H_*(X_j) \right)$$

475 Unfortunately, we cannot always compute the meet or join in constant time as we may need to
 476 compose a linear number of maps. In the future, we will do a more fine grain analysis, but we
 477 note that given that we have a distributive lattice, all maximal total orders are of constant length,
 478 allowing us to bound the time to compute any meet and join by this length.

479 **4.3. Stability of the Lattice.** Here we look at a possible description of stability relating to a
 480 persistence lattice. The general idea is to show that if some local conditions hold, we can infer the
 481 existence of some persistent classes.

482 **Lemma 4.3.1.** *Let A, B, C and D be vector spaces such that $A \wedge B \leq C$ and $D \leq A \vee B$.*
 483 *Then, $A \vee B \leq C$ and $D \leq A \wedge B$.*

484 *Proof.* Assume the existence of a linear map $f : A \wedge B \rightarrow C$. As $A \wedge B$ is a subalgebra of $A \oplus B$
 485 then it is possible to construct linear maps $f_A : A \rightarrow C$ and $f_B : B \rightarrow C$ implying that $A, B \leq C$.
 486 Thus, the universality of $A \vee B$ constructed as a coequalizer implies the existence of a unique linear
 487 map $h : A \vee B \rightarrow C$, i.e., $A \vee B \leq C$.



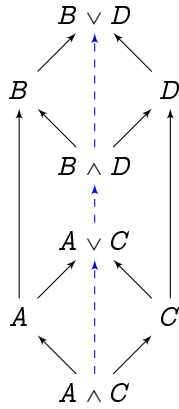
488

489 Dually, the existence of a linear map $g : D \rightarrow A \vee B$ implies that $D \leq A, B$ so that the universality
 490 of $A \wedge B$ as an equalizer implies the existence of a linear map $k : D \rightarrow A \wedge B$, i.e., $D \leq A \wedge B$. \square

491 We can now state the following theorem:

492 **Theorem 4.3.2.** *Let A, B, C and D be vector spaces such that $A \leq B$ and $C \leq D$. Then,*
 493 $A \vee C \leq B \wedge D$.

494 *Proof.* Assume that $A \leq B$ and $C \leq D$ and consider the following diagram:



495

496 As $A \wedge C \leq B \vee D$, Lemma 4.3.1 implies that the map $f : A \wedge C \rightarrow B \vee D$ decomposes into
 497 maps

$$A \wedge C \rightarrow A \vee C \rightarrow B \wedge D \rightarrow B \vee D.$$

498

\square

499 To place this into context, consider $A \rightarrow B$ to be part of one filtration and $C \rightarrow D$ a second
 500 filtration such that they are interleaved. In this case for any class in $A \rightarrow B$, $A \wedge C$, and $B \vee D$
 501 must also be in $C \rightarrow D$. In this case, the idea is that local conditions such as $A \wedge C \rightarrow A \vee C$ and
 502 $B \wedge D \rightarrow B \vee D$, imply something about the persistence between other elements. In above case, if
 503 we assume ϵ -interleaving we can recover such a statement on these local conditions. We now give
 504 a more general statement:

505 **Theorem 4.3.3.** *Let A, B, C and D be vector spaces. Then $(A \wedge C) \vee (B \wedge D) \leq (A \vee C) \wedge (B \vee D)$.*

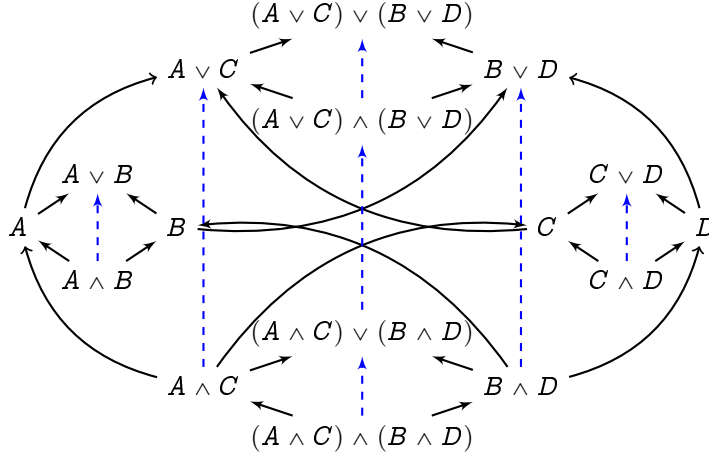


FIGURE 6. Hasse diagram representation of the stabilization theorem for subalgebras of a persistence lattice.

506 *Proof.* Consider the diagram of Figure 6 where $R_1 = A \vee C$, $R_2 = B \vee D$, $P_1 = A \wedge C$ and
 507 $P_2 = B \wedge D$. The existence of the dashed maps is guaranteed by Lemma 4.3.1 and the fact that
 508 $A \wedge B \wedge C \wedge D \leq A \vee B \vee C \vee D$.

509

□

510 Here we do not introduce the notion of metrics or interleaving to give a more substantial result.
 511 However, we believe such a result is possible and we will address it in further work.

512 4.4. Sections. Finally we return to the bifiltration case to highlight the difference between our
 513 construction and the one we presented in Section 2 which yielded the rank invariant. Consider
 514 Figure 7. The rank invariant requires that all the elements of a square have class to contribute to
 515 the rank of the square. However, using our construction, a class will persist between two elements if
 516 and only if there is a sequence of maps in the diagram such that the classes map into each other (or
 517 from each other). In this case we can find persistent sections across incomparable elements yielding
 518 finer grained information than the rank invariant. Furthermore, in highly structured diagrams
 519 such as multifiltrations, additional properties such as associativity have algorithmic consequences
 520 as well.

521

5. DISCUSSION

522 In this paper, we have investigated the properties of a lattice which contains information about
 523 the persistent homology classes in a general commutative diagram of vector spaces. There are still
 524 numerous open questions including:

- 525 • What kind of decompositions exist in the spirit of persistence diagrams for this distributive
 526 lattice, since all maximal total orders are the same length and therefore we can decompose
 527 this lattice into a canonical sequence of antitotal orders?

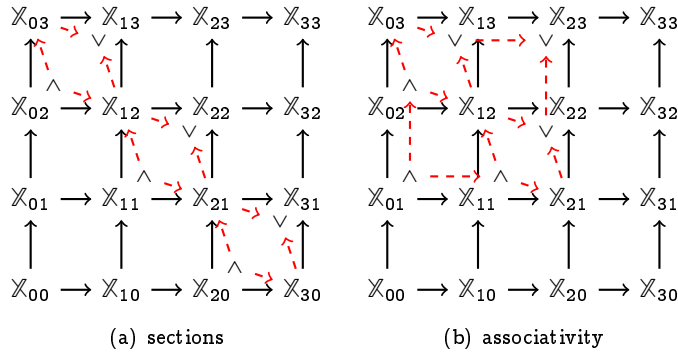


FIGURE 7. While the associativity of the lattice operations in the bifiltration corresponds to the possible paths in the diagram (b), the sections in the lattice can be explained by the diagram (a).

- 528 • What are further algorithmic implications of this structure?
- 529 • What is the correct metric to consider to general commutative diagrams as “close”?
- 530 • In what other contexts do such diagrams appear and what can we say about their structure?

531 We will address some of these questions in a subsequent paper.

REFERENCES

532

533 [1] M. Barr and C. Wells. *Category theory for computing science*, volume 10. Prentice Hall, 1990.

534 [2] A. Bauer, K. Cvetko-Vah, M. Gehrke, S. J. van Gool, and G. Kudryavtseva. A non-commutative priestley duality. *Topology and its Applications*, 2013.

535 [3] G. Bezhanishvili. Bitopological duality for distributive lattices and Heyting algebras. *Mathematical Structures in Computer Science*, 20(3):359–393, 2010.

536 [4] N. Bezhanishvili. *Lattices of intermediate and cylindric modal logics*. Institute for Logic, Language and Com-

537 *putation*, 2006.

538 [5] G. Birkhoff. *Lattice theory*, volume 5. AMS Colloquium Publications, Providence RI, third edition, 1940.

539 [6] S. Burris and H. P. Sankappanavar. *A course in universal algebra*. S. Burris and H. P. Sankappanavar, 1981.

540 [7] G. Carlsson. Topology and data. *Bulletin-American Mathematical Society*, 46(2):1–54, 2009.

541 [8] G. Carlsson, V. De Silva, and D. Morozov. Zigzag persistent homology and real-valued functions. In *Proceedings of the Annual Symposium on Computational Geometry*, pages 247–256, March 2009.

542 [9] G. Carlsson and A. Zomorodian. The theory of multidimensional persistence. *Discrete & Computational Ge-*

543 *ometry*, 42(1):71–93, 2009.

544 [10] G. Carlsson and V. de Silva. Zigzag persistence. *Found. Comput. Math.*, 10(4):367–405, 2010.

545 [11] G. Carlsson, G. Singh and A. Zomorodian. Computing Multidimensional Persistence. *arXiv.org*, July 2009.

546 [12] B. A. Davey and H. A. Priestley. *Introduction to lattices and order*. Cambridge University Press, 2002.

547 [13] R. P. Dilworth. A decomposition theorem for partially ordered sets. *Annals of Mathematics*, 51(1):161–166,

548 1950.

549 [14] A. Döring and C. Isham. A topos foundation for theories of physics: I. formal languages for physics. *Journal of*

550 *Mathematical Physics*, 49, 2008.

551 [15] H. Edelsbrunner and J. L. Harer. *Computational Topology: An Introduction*. American Mathematical Society,

552 2010.

553 [16] H. Edelsbrunner, D. Letscher, and A. Zomorodian. Topological persistence and simplification. *Discrete & Com-*

554 *putational Geometry*, 28(4):511–533, December 2012.

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- 558 [17] J. D. Farley. The automorphism group of a function lattice: A problem of Jónsson and McKenzie. *Algebra*
559 *Universalis*, 36(1):8–45, 1996.
- 560 [18] G. Grätzer. *Lattice theory*. WH Freeman and Co, San Francisco, 1971.
- 561 [19] G. Grätzer. *Universal Algebra*. Springer, second edition, 1979.
- 562 [20] A. Hatcher. *Algebraic Topology*. Hatcher, December 2000.
- 563 [21] P. T. Johnstone. *Stone Spaces*. Cambridge University Press, August 1986.
- 564 [22] S. Lang. *Linear Algebra*. Springer, 3rd edition, 1987.
- 565 [23] S. Lang. *Algebra*, volume 211. Graduate texts in mathematics, 2002.
- 566 [24] W. A. Luxemburg and A. C. Zaanen. *Riesz spaces*. North-Holland Publishing Company, 1971.
- 567 [25] J. Leech M. Kinyon and J. Pita Costa. Distributive skew lattices. *Submitted to Semigroup Forum*, 2013.
- 568 [26] S. Mac Lane. *Categories for the Working Mathematician*. Springer, 1998.
- 569 [27] L. Mirsky. A dual of Dilworth's decomposition theorem. *American Mathematical Monthly*, 78(8):876–877, 1971.
- 570 [28] S. Oudot and D. Sheehy. Zigzag Zoology: Rips Zigzags for Homology Inference. In *Proceedings of the twenty-*
571 *ninth annual symposium on Computational geometry*, pages 387–396, 2012.
- 572 [29] H. Edelsbrunner P. Bendich, S. Cabello. A point calculus for interlevel set homology. *Pattern Recognition*
573 *Letters*, 33(11):1436–1444, 2012.
- 574 [30] A. Polishchuk and L. Positselski. *Quadratic Algebras, Clifford Algebras, and Arithmetic Witt Groups*. American
575 Mathematical Society, 2005.
- 576 [31] P. Škraba and M. Vejdemo-Johansson. Parallel and scalable zig-zag persistent homology. In *NIPS 2012 Workshop*
577 *on Algebraic Topology and Machine Learning*, 2012.
- 578 [32] P. Škraba and M. Vejdemo-Johansson. Persistence modules: Algebra and algorithms. *arXiv*, cs.CG, February
579 2013.
- 580 [33] A. A. Tuganbaev. *Semidistributive Modules and Rings*. Mathematics and Its Applications. Springer, 1998.
- 581 [34] K. Yosida. On the representation of the vector lattice. *Proceedings of the Japan Academy, Series A, Mathematical*
582 *Sciences*, 18(7):339–342, 1942.
- 583 [35] A. Zomorodian, G. Carlsson, A. Collins, and L. Guibas. Persistence Barcodes for Shapes. *International Journal*
584 *of Shape Modeling (2005)*, pages 1–12, January 2013.
- 585 [36] A. J. Zomorodian. *Topology for computing*, volume 16. Cambridge University Press, 2005.

586

APPENDIX A. BASICS OF LATTICE THEORY

587 **A.1. Orders and lattice structures.** Partial orders are important tools in the study of topology.
588 Moreover, lattices are partially ordered sets (or posets for short) that have just enough structure to
589 be seen as algebraic structures with operations determined by the underlying order structure. In
590 what follows we will provide the basic knowledge on the theory of lattices regarding the needs of
591 this paper. For further reading on lattice theory and, in general, on universal algebra, we suggest
592 [5], [6], [18] and [19].

593 **Definition A.1.1.** *A preorder is a binary relation R that satisfies reflexivity (i.e., for all $x \in A$,*
594 *xRx) and transitivity (i.e., for all $x, y, z \in A$, xRy and yRz implies xRz). A preorder \leq is*
595 *a partial order if, for all $x, y \in A$, $x \leq y$ and $y \leq x$ implies $x = y$ (antisymmetry). A poset*
596 *(P, \leq) is an order structure consisting of a set P and a partial order \leq .*

597 **Example A.1.2.** *Examples of posets are the real numbers ordered by the standard order,*
598 *the natural numbers ordered by divisibility, the set of subspaces of a vector space ordered by*
599 *inclusion, or the vertex set of a directed acyclic graph ordered by reachability.*

600 **Definition A.1.3.** *A partial order is named total order if every pair of elements is related,*
601 *that is, for all $x, y \in A$, $x \leq y$ or $y \leq x$. On the other hand, an antitotal order is a partial order*
602 *for which no two distinct elements are related. For every finite partial order there exists an*
603 *antitotal order S and a partition of the order in A into a family F of total orders such that*

604 the number of total orders in the partition equals the cardinality of S . Thus, S must be the
 605 largest antitotal order in the order, and F must be the smallest family of total orders into
 606 which the order can be partitioned (cf. [13]). Dually, The size of the largest total order in
 607 a partial order (if finite) equals the smallest number of antitotal orders into which the order
 608 may be partitioned (cf. [27]).

609 **Example A.1.4.** The natural numbers form a total order under the usual order, and form a
 610 partial order under divisibility.

611 **Definition A.1.5.** A lattice is a poset for which all pairs of elements have an infimum and
 612 a supremum. Whenever every subset of a lattice L has a supremum and an infimum, L is
 613 named a complete lattice. Every total order is a lattice. Other examples of lattices are the
 614 power set of A ordered by subset inclusion, or the collection of all partitions of A ordered by
 615 refinement. A lattice A can be seen as an algebraic structure $(L; \wedge, \vee)$ with two operations \wedge
 616 and \vee satisfying associativity (i.e., $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$),
 617 idempotence (i.e., $x \wedge x = x = x \vee x$), commutativity (i.e., $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$)
 618 and absorption (i.e., $x \wedge (x \vee y) = x = x \vee (x \wedge y)$). The equivalence between this algebraic
 619 perspective of a lattice L and its ordered perspective is given by the following: for all $x, y \in L$,
 620 $x \leq y$ iff $x \wedge y = x$ iff $x \vee y = y$.

621 **Example A.1.6.** Recall that an equivalence E in a set A is a preorder such that, for all
 622 $x, y \in A$, xEy implies yEx (symmetry). The set $A/E = \{x \in A : xEa : a \in A\}$ is a partition
 623 of A . Conversely, every partition P of A determines an equivalence θ_P of A defined by $x\theta_P y$
 624 iff there exists $X \in P$ such that $x, y \in X$. Thus the notions of equivalence relation and
 625 partition are essentially the same. The axiom of choice guarantees for any partition of a
 626 set X the existence of a subset of X containing exactly one element from each part of the
 627 partition. This implies that given an equivalence relation on a set one can select a canonical
 628 representative element from every equivalence class. Arithmetical equality and geometrical
 629 similarity are examples of well known equivalences. The partition of a set X into nonempty
 630 and non-overlapping subsets, called blocks (or cells), determines a complete lattice for which
 631 the meet operation \wedge is the intersection of blocks.

632 **Example A.1.7.** It is well known that the subspaces of a vector space form a complete lattice
 633 (cf. [5]). In fact, considering the partial order structure to be the subspace relation, whenever
 634 A and B are vector spaces one can define the lattice operations as $A \wedge B = A \cap B$ and
 635 $A \vee B = A \oplus B$. The minimum of this lattice is the trivial subspace $\{0\}$ while the maximum is
 636 the full vector space V . Clearly, $A \cap B = A$ iff $A \subseteq B$ iff $A \oplus B = B$. Furthermore, $A \cap A \equiv A$,
 637 $A \oplus A \equiv A$, $A \cap (B \cap C) \equiv (A \cap B) \cap C$, $A \oplus (B \oplus C) \equiv (A \oplus B) \oplus C$ and $A \cap (A \oplus B) \equiv A \cap A \oplus (A \cap B)$.
 638 If V is a finite dimensional vector space over the field K and $W \leq V$, then there exists $U \leq V$
 639 such that $V = W \oplus U$ ([22]). On the other hand, $U \cap W = \{0\}$ giving us a sense of complement.
 640 This complement is not unique (and thus the lattice cannot be cancellative, or equivalently,
 641 the lattice is not distributive as will later be discussed). A linear lattice is a sublattice of the
 642 equivalences lattice of a set, on which any two elements commute. A typical example can be
 643 found in Geometry: the lattice of subspaces of a vector space is isomorphic to a commuting
 644 equivalences lattice, dened in the vector space seen as a set. If V is a vector space and W is

645 one of its subspaces, we define the equivalence of two vectors $x, y \in V$ as $x \equiv_W y$ if, and only if,
 646 $x - y \in W$, associating to each subspace an equivalence. If W' is another subspace of V , then
 647 the equivalences \equiv_W and $\equiv_{W'}$ commute, describing an isomorphism between the lattice $L(V)$
 648 of all vector subspaces of V and a lattice of commuting equivalences, $(Eq_{com}(V) \cap L(V^2); \cap, \circ)$.
 649 Such lattices are of frequent occurrence, including the lattice of normal subgroups of a group,
 650 or the lattice of ideals of a ring.

651 **Example A.1.8.** A vector lattice (or Riesz space) E is any vector space endowed with a partial
 652 order \leq such that $(E; \leq)$ is a lattice and, for all vectors $x, y, z \in E$ and any scalar $\alpha \geq 0$: $x \leq y$
 653 implies $x + y \leq y + z$, and $x \leq y$ implies $\alpha x \leq \alpha y$. Given a topological space X , its ring of
 654 continuous functions $C(X)$ is a vector lattice. In particular, any finite dimensional Euclidean
 655 space \mathbb{R}^n is a vector lattice. Roughly, vector lattice is a partially ordered real vector space
 656 where the order structure is a lattice. A representation of such an algebraic structure is given
 657 in [34] assuming the Archimedean-unit and describing a representation space using maximal
 658 prime ideals. Being vector spaces, subalgebras are just subspaces that constitute sublattices.
 659 Riesz spaces have wide range of applications, having a great impact in measure theory. A
 660 large discussion on this topic can be found in [24]. A Banach space is any complete normed
 661 vector lattice. Examples of such lattices are C^* algebras, constituting associative algebras
 662 over the complex numbers which are Banach spaces with an involution map.

663 **Definition A.1.9.** A lattice L is complete if every subset S of L has both a greatest lower
 664 bound $\bigwedge S$ and a least upper bound $\bigvee S$ in L . In particular, when S is the empty set, $\bigwedge S$ is
 665 the greatest element of L . Likewise, $\bigvee \emptyset$ yields the least element. Complete lattices constitute
 666 a special class of bounded lattices. Any lattice with arbitrary meets and a biggest element is
 667 complete. This condition and its dual characterize complete lattices.

668 **Example A.1.10.** Examples of complete lattices are abundant: the power set of a given set
 669 ordered by inclusion with arbitrary intersections and unions as meets and joins; the non-
 670 negative integers ordered by divisibility where the operations are given by the least common
 671 multiple and the greatest common divisor; the subgroups of a group, the submodules of a
 672 module or the ideals of a ring ordered by inclusion; the unit interval $[0, 1]$ and the extended
 673 real number line, with the familiar total order and the ordinary suprema and infima. A
 674 totally ordered set with its order topology is compact as a topological space if it is complete
 675 as a lattice (cf. [18]).

676 **Remark A.1.11.** There are several mathematical concepts that can be used to represent
 677 complete lattices being the Dedekind-MacNeille completion one of the most popular ones. It
 678 is used to extend a poset to a complete lattice. By applying it to a complete lattice one can
 679 see that every complete lattice is isomorphic to a complete lattice of sets. When noting that
 680 the image of any closure operator on a complete lattice is again a complete lattice one obtains
 681 another representation: since the identity function is a closure operator too, this shows that
 682 complete lattices are exactly the images of closure operators on complete lattices.

683 **A.2. Boolean algebras and Heyting algebras.** Maybe due to their important role as models
 684 to classical logic (constructive logic), Boolean algebras (Heyting algebras, respectively) are some of

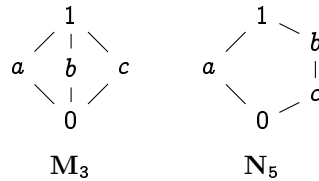
685 the the most well known lattices in Mathematics. We will present these varieties of algebras in the
 686 following paragraphs, discuss their important properties and present some examples.

687 **Definition A.2.1.** *A lattice L is modular if, for all $x, y, z \in S$, $y \leq x$ implies $x \wedge (y \vee z) =$
 688 $y \vee (x \wedge z)$. A lattice L is distributive if, for all $x, y, z \in S$, it satisfies one of the following
 689 equivalent equalities:*

- 690 (d1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;
- 691 (d2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;
- 692 (d3) $(x \vee y) \wedge (x \vee z) \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$.

693 **Example A.2.2.** *The lattice of normal subgroups of a group is modular (cf. [5], V). Other
 694 examples of modular lattices are the elements of any projective geometry or the ideals of
 695 any modular lattice (under set-inclusion) A lattice of subsets of a set is usually called a ring
 696 of sets. Any ring of sets forms a distributive lattice in which the intersection and union
 697 operations correspond to the lattice's meet and join operations, respectively. Conversely,
 698 every distributive lattice is isomorphic to a ring of sets; in the case of finite distributive
 699 lattices, this is Birkhoff's Representation Theorem and the sets may be taken as the lower sets
 700 of a partially ordered set. Every field of sets and so also any σ -algebra also is a ring of sets
 701 (cf. [5]).*

702 **Remark A.2.3.** *Below are the Hasse diagrams of the diamond M_3 and the pentagon N_5 , the
 703 forbidden algebras regarding distributivity in lattices.*



705 *The following are useful characterizations of the distributivity and modularity of a lattice*
 706 *L :*

- 707 (i) *L is modular iff all x, y, z with $y \leq z$ are such that $x \wedge y = x \wedge z$ and $x \vee y = x \vee z$
 708 imply $y = z$;*
- 709 (ii) *L is distributive iff all x, y, z are such that $x \wedge y = x \wedge z$ and $x \vee y = x \vee z$ imply $y = z$;*
- 710 (iii) *L is modular iff if it does not have embedded any copy of the pentagon N_5 ;*
- 711 (iv) *L is distributive iff if it does not have embedded any copy of the diamond M_3 or of
 712 the pentagon N_5 .*

713 **Remark A.2.4.** *The modularity of distributive lattices also determines the diamond isomor-
 714 phism theorem describing the isomorphism between $[a \wedge b, b]$ and $[a, a \vee b]$ using the maps
 715 $f : (a \vee b)/a \rightarrow b/(a \wedge b)$, $x \mapsto x \wedge b$, and $g : b/(a \wedge b) \rightarrow (a \vee b)/a$, $y \mapsto a \vee y$. This result is
 716 equivalent to the 3rd isomorphism theorem in Group Theory, being a particular case of the
 717 Correspondence Theorem established in the domain of Universal Algebra.*

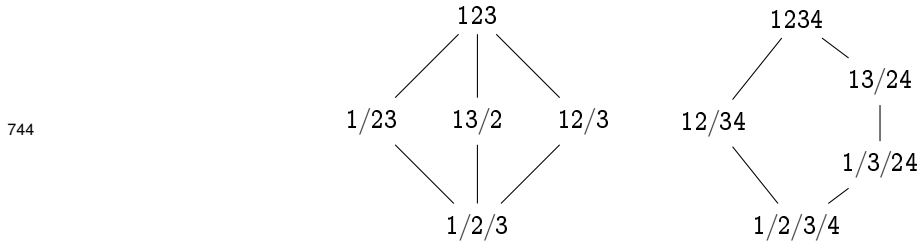
718 **Example A.2.5.** *While every vector lattice, defined in Example A.1.8, is distributive (see [5]),
 719 the subspace lattices defined in Example A.1.7) are modular but not distributive: all indecom-
 720posable triples of vector spaces X, Y and Z but one are distributive; the only nondistributive*

721 indecomposable triple is that of three lines in a plane (cf. [30]). Let V be a 2-dimensional
 722 vector space. Consider the sublattice of the subspace lattice where the bottom element is the
 723 zero space, the top element is V , and the rest of the elements of $\text{Sub}(V)$ are 1-dimensional:
 724 lines through the origin. For 1-dimensional spaces, there is no relation $a \leq b$ unless a and
 725 b coincide. The Hasse diagram of such a lattice is the diamond M_3 above where $1 = V$, the
 726 total space. Observe that for distinct elements a, b, c in the middle level, we have for example
 727 $x \wedge y = 0 = x \wedge z$ (0 is the largest element contained in both a and b), and also for example
 728 $b \vee c = 1$ (1 is the smallest element containing b and c). It follows that $a \wedge (b \vee c) = a \wedge 1 = a$
 729 whereas $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$. The distributive law thus fails.

730 **Example A.2.6.** Also a module M over a ring R can be considered a lattice with operations $+$
 731 and \cdot as \vee and \wedge , respectively. Lattice modularity corresponds to the Jordan-Dedekind total
 732 order condition. Moreover, M is distributive iff for all $m, n \in M$, $(m + n)R = mI + nI$ for
 733 some ideal I (cf. [33]). The condition is easily seen to be necessary. For sufficiency, observe
 734 that distributivity is equivalent to $(m + n)R = (m + n)R \cap mR + (m + n)R \cap nR$ and to prove
 735 this, the argument says: the modular law implies that $(m + n) \cap mR = (mI + nI) \cap mR = mI$
 736 and respectively for nI .

737 **Example A.2.7.** Moreover, the lattice of subgroups of a group ordered by inclusion is a
 738 modular lattice that is not distributive: consider G to be the non-cyclic group of order 4, and
 739 a, b and c the three subgroups of order 2 having two distinct elements. We thus get the copy
 740 of M_3 in the Hasse diagram above (cf. [21]).

741 **Example A.2.8.** Furthermore, the partition lattice, defined in Example A.1.6, is not dis-
 742 tributive for $n > 3$ and is not modular for $n > 4$. In detail just consider the following Hasse
 743 diagrams of the correspondent forbidden algebras:



745 **Definition A.2.9.** A Boolean algebra is a distributive lattice with a unary operation \neg and
 746 nullary operations 0 and 1 such that for all elements $x, y, z \in A$ the following axioms hold:

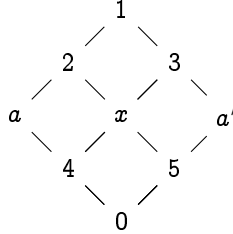
- 747 $L_6.$ $a \vee 0 = a$ and $a \wedge 1 = a$;
 748 $L_7.$ $a \vee \neg a = 1$ and $a \wedge \neg a = 0$.

749 **Example A.2.10.** Examples of Boolean algebras are the power set of any set X ordered by
 750 inclusion, or the divisors D_n of a natural number n bigger than 1 that is not divided by the
 751 square of any prime number.

752 **Remark A.2.11.** The following result permits us to identify a Boolean algebra by observa-
 753 tion of its Hasse diagram. Whenever L is a bounded distributive lattice, the following are
 754 equivalent:

- 755 (i) L is a Boolean algebra;
- 756 (ii) for all $x \in L$ there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$;
- 757 (iii) for all $x, y, z \in L$ such that $x \leq y \leq z$ there exists $w \in L$ such that $y \wedge w = x$ and
- 758 $y \vee w = z$.

759 Due to this it is easy to observe that total orders are not Boolean algebras. The distributive
 760 lattice represented by the Hasse diagram below is not a Boolean algebra: consider the total
 761 order $\{3, x, 4\}$ and observe that there is no $y \in L$ such that $x \wedge y = 4$ and $x \vee y = 3$.



762

763 **Definition A.2.12.** A bounded lattice L is a Heyting algebra if, for all $a, b \in L$ there is a
 764 greatest element $x \in L$ such that $a \wedge x \leq b$. This element is the relative pseudo-complement of
 765 a with respect to b denoted by $a \Rightarrow b$.

766 **Example A.2.13.** Examples of Heyting algebras are the open sets of a topological space, as
 767 well as all the finite nonempty total orders (that are bounded and complete). Furthermore,
 768 every complete distributive lattice L is a Heyting algebra with the implication operation given
 769 by $x \Rightarrow y = \bigvee \{x \in L \mid x \wedge a \leq b\}$.

770

APPENDIX B. ALGEBRAIC CONSTRUCTIONS

771 **B.1. On limits and colimits.** In the following paragraphs of this appendix we shall recall the
 772 categorical nature of products and coproducts of vector spaces. We will also recall the definitions
 773 of equalizer and coequalizer, give some examples, and discussing their relation to pullbacks and
 774 pushouts.

775 **Remark B.1.1.** As any other poset, the set of vector spaces ordered by \leq constitutes a
 776 category, denoted by \mathcal{V} , considering vector spaces as elements and linear maps as morphisms.
 777 It is a subcategory of $\mathbf{R} - \text{mod}$, the category of R -modules and R -module homomorphisms.
 778 Recall that in the category of modules over some ring R , the product is the cartesian product
 779 with addition defined componentwise and distributive multiplication. Thus, the direct product
 780 of vector spaces A and B , noted by $A \times B$, is a vector space when we define the sum and
 781 product by scalar componentwise. It is the biggest vector space that can be projected into A
 782 and B , simultaneously. Recall also that $A \cup B$ is a subspace iff $A = B$. The smallest element
 783 of \mathcal{V} containing $A \cup B$ is $A + B = \{a + b \mid a \in A, b \in B\}$. The direct sum of vector spaces A and B ,
 784 noted by $A \oplus B$, is the smallest vector space which contains the given vector spaces as subspaces
 785 with minimal constraints. The direct product is the categorical product, noted by \square : whenever
 786 A and B are vector spaces, the natural projections $\pi_A : A \times B \rightarrow A$ and $\pi_B : A \times B \rightarrow B$ show
 787 that $A \times B \leq A, B$; on the other hand, whenever D is a vector space such that $D \leq A, B$ with

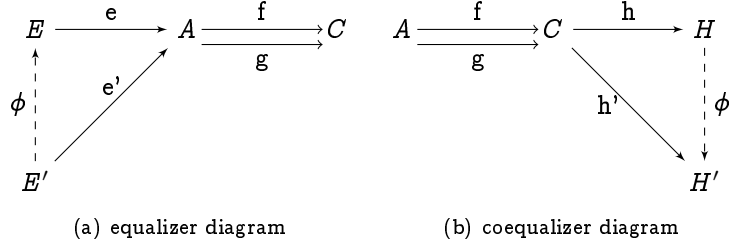


FIGURE 8. Diagram representation of the equalizer and coequalizer of maps f and g between vector spaces A and C on a given diagram \mathcal{D} .

788 maps $f : D \rightarrow A$ and $g : D \rightarrow B$, the map $f \times g : D \rightarrow A \times B$ defined by $f \times g(x) = (f(x), g(x))$
789 is well defined and unique up to isomorphism, ensuring us with the universal property. As
790 well, the direct sum is the categorical coproduct, noted by \sqcup : whenever A and B are vector
791 spaces, the inclusion maps $i_A : A \rightarrow A \oplus B$ and $i_B : B \rightarrow A \oplus B$ show us that $A, B \leq A \oplus B$; on
792 the other hand, whenever C is a vector space such that $A, B \leq C$ with maps $f : A \rightarrow C$ and
793 $g : B \rightarrow C$, the map $f \oplus g : A \oplus B \rightarrow C$ defined by $f \oplus g(x) = f \oplus g(x_A + x_B) = f(x_A) + f(x_B)$
794 is well defined and unique up to isomorphism, ensuring us with the universal property. A
795 biproduct of a finite collection of objects in a category with zero object is both a product and a
796 coproduct. In a preaddictive category the notions of product and coproduct coincide for finite
797 collections of objects. The biproduct generalizes the direct sum of modules. The category of
798 modules over a ring is preaddictive (and also additive). In particular, the category of vector
799 spaces over a field is preaddictive with the trivial vector space as zero object.

800 In the following we are going to discuss in detail the (generalized) categorical concepts of equalizer
801 and coequalizer that we use in this paper to construct the lattice operations. We will also interpret
802 these in the framework of persistence in order to use such ideas to construct the lattice operations.

803 **Definition B.1.2.** Given a pair of vector spaces A and C with two linear maps $f, g : A \Rightarrow B$
804 between them, the equalizer of f and g is a pair (E, e) where E is a vector space (usually
805 called kernel set of the equalizer) and $e : E \rightarrow A$ is a linear map such that $fe = ge$, with the
806 following universal property: for any other vector space E' and linear map $e' : E' \rightarrow A$ such
807 that $fe' = ge'$, there exists a unique linear map $\phi : E' \rightarrow E$ such that $e\phi = e'$ (as represented
808 in the diagram of Figure 8).

809 Dually, the coequalizer of f and g is a pair (H, h) where H is a vector space (usually called
810 the quotient set of the coequalizer) and $h : A \rightarrow H$ is a linear map such that $hf = hg$, with
811 the following universal property: for any other vector space H' and linear map $h' : A \rightarrow H'$
812 in \mathcal{V} such that $h'f = h'g$, there exists a unique morphism $\phi : H \rightarrow H'$ such that $\phi h = h'$ (as
813 represented in the diagram of Figure 8).

814 **Example B.1.3.** In the category of sets, given maps $f, g : X \rightarrow Y$, the equalizer of f and
815 g is the set $\{x \in X \mid f(x) = g(x)\}$ while the coequalizer of f and g is the quotient of Y
816 by the equivalence generated by the set $\{(f(x), g(x)) \mid x \in X\}$, i.e., the smallest equivalence θ

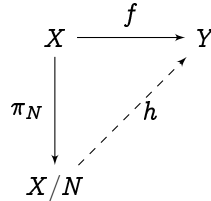
817 such that for every $x \in X$, $f(x)\theta g(x)$ holds. For instance, consider the sets $X = \{a, c, d\}$ and
 818 $Y = \{a, c, d, e\}$, and the maps $f : X \rightarrow Y = \{a \mapsto e, c \mapsto d, d \mapsto c\}$ and $g : X \rightarrow Y = \{a \mapsto$
 819 $d, c \mapsto d, d \mapsto c\}$. The equalizer of f and g is given by the kernel set $\mathcal{E} = \{c, d\}$ and the
 820 injection $eq : E \rightarrow X = \{c \mapsto c, d \mapsto d\}$. On the other hand, the coequalizer of f and g is given
 821 by the quotient set $C = \{\{a\}, \{c\}, \{d, e\}\}$ and the surjection $coeq : Y \rightarrow C = \{a \mapsto \{a\}, c \mapsto$
 822 $\{c\}, d \mapsto \{d, e\}, e \mapsto \{d, e\}\}$.

823 The equalizer of the real functions $f(x, y) = x^2 + y^2$ and $g(x, y) = 4$ is the circumference
 824 $E = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4\}$ together with projection maps.

825 **Example B.1.4.** In the category of groups, the equalizer of homomorphisms $f, g : X \rightarrow Y$
 826 can still be seen as the solution set of an equation determined by $f(x) = g(x)$ while their
 827 coequalizer is the quotient of Y by the normal closure of the set $S = \{f(x)g(x)^{-1} \mid x \in X\}$. In
 828 detail, the elements of Y/N must be equivalence classes y/N such that, for all $y, y' \in Y$,

$$y\theta y' \text{ iff } f.g^{-1} \in N.$$

829 In particular, for abelian groups, the equalizer is the kernel of the morphism $f - g$ while the
 830 coequalizer is the factor group $Y/im(f - g)$, i.e., the cokernel of $f - g$. Moreover, the kernel
 831 of a linear map f is the equalizer of the maps f and 0 constituting a normal subgroup with
 832 the following property: for any normal subgroup $N \subseteq G$, $N \subseteq \ker f$ iff there is a (necessarily
 833 unique) homomorphism $h : X/N \rightarrow Y$ such that $h \circ \pi_N = f$ implying the commutativity of the
 834 diagram below:



835

836 Hence, every group homomorphism factors as a quotient followed by an injective homo-
 837 morphism (every group homomorphism has a kernel). On the other hand, a coequalizer
 838 of a homomorphism $f : X \rightarrow Y$ and the zero homomorphism is the natural surjection
 839 $\pi_{f(X)} : Y \rightarrow Y/f(X)$ on the quotient $Y/f(X)$. More generally, a coequalizer of homomor-
 840 phisms $f, g : X \rightarrow Y$ is a coequalizer of $f - g : X \rightarrow Y$ and the zero homomorphism, that is,
 841 the natural surjection $Y \rightarrow Y/(f - g)(X)$. This holds for the category of vector spaces and
 842 linear maps.

843 **Remark B.1.5.** Now we will show that, for the purposes of this paper, the information
 844 retrieved by pullbacks and pushouts is essential the same than the one obtained by computing
 845 equalizers and coequalizers, respectively. It is well known that (pushouts) pullbacks can be
 846 constructed from (co)equalizers: a pullback is the equalizer of the morphisms $f \circ \pi_1, g \circ \pi_2 :$
 847 $X \times Y \rightarrow Z$ where $X \times Y$ is the binary product of X and Y , and π_1 and π_2 are the natural
 848 projections, showing that pullbacks exist in any category with binary products and equalizers.
 849 In general, we have the following: The equalizer of the family $(f_i)_{i \in I} : A \oplus B \rightarrow C$ is the
 850 pullback of the pair of morphisms $((f_{iA})_{i \in I}, (f_{iB})_{i \in I})$ with $f_{iA} : A \rightarrow C, f_{iB} : B \rightarrow C$ and

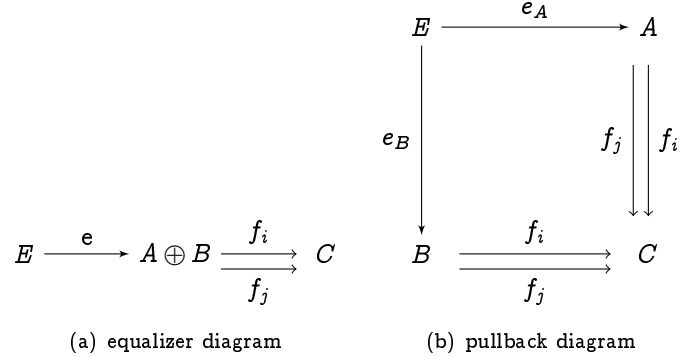


FIGURE 9. Equivalence of the considered equalizer diagrams and pullback diagrams.

851 $f_i(z) = f_{iA}(x) + f_{iB}(y)$, for all $z = x + y \in A \oplus B$ and all $i \in I$. Dually, the coequalizer of the
 852 family $(g_i)_{i \in I} : D \rightarrow A \oplus B$ is exactly the pushout of the pair of morphisms $((g_{jA})_{j \in J}, (g_{jB})_{j \in J})$
 853 with $g_{jA} : D \rightarrow A$, $g_{jB} : D \rightarrow B$ and $g_j(x) = g_{jA}(x) \oplus g_{jB}(x)$, for all $x \in D$ and all $j \in J$. To
 854 see this in detail, just observe that all maps $f_i : A \oplus B \rightarrow C$ split into maps $f_{iA} : A \rightarrow C$ and
 855 $f_{iB} : B \rightarrow C$ with $f_j(x) = f_{jA}(x) + f_{jB}(x)$, for all $j \in I$. Hence, the diagrams of Figure 9 are
 856 equivalent. Clearly, $e = e_A + e_B$ defined by $e(x) = e_A(x) + e_B(x)$ and thus

$$\{x \in A \oplus B \mid f_i(x) = f_j(x)\} = \{(x, y) \in A \times B \mid f_i(x) = f_j(y)\}.$$

857 The dual result has a similar argument.

858 **Remark B.1.6.** Equalizers (coequalizers) are unique up to isomorphism. The equalizing map
 859 e (coequalizing map h) is always a monomorphism (epimorphism) and, monomorphisms
 860 (epimorphisms) are injective (surjective) maps in the context of vector spaces and linear
 861 maps. Hence, the equalizing map e (coequalizing map h) is an isomorphism iff $f = g$, for all
 862 $f, g \in \text{Hom}(A, C)$ (cf. [26]).

863 **Remark B.1.7.** Sometimes, the equalizer is identified with the object E while the morphism e
 864 can be taken to be the inclusion map of E as a subset of A . Dual remarks hold for coequalizers.
 865 As we assume that all diagrams of vector spaces commute, the categorical concepts of equalizer
 866 and coequalizer can be adapted to the framework of this paper as in Definition .

867 **B.2. A construction for the lattice operations.** In the following we will discuss a natural
 868 generalization of the equalizer and coequalizer constructions where the definition of the lattice
 869 operations exhibited in Section 3 is based.

870 **Remark B.2.1.** Let us first recall that linear maps from common sources are maps from the
 871 direct sum of those sources, and that linear maps to common targets are maps to the direct
 872 sum of those targets. To see this consider the vector spaces A, B, C and D , and the linear
 873 maps $f : A \rightarrow C$ and $g : B \rightarrow C$. Then we can construct $f \oplus g : A \oplus B \rightarrow C$ defining it by
 874 $f \oplus g(z) = f(x) + g(y)$ for all $z = x + y \in A \oplus B$. Moreover, given the linear maps $f : A \rightarrow C$

875 and $h : A \rightarrow B$ we can construct $f \oplus h : A \rightarrow C \oplus B$ by defining it as $f \oplus h(x) = f(x) \oplus h(x)$,
 876 for all $x \in A$. Conversely, any linear map $f : A \oplus B \rightarrow C$ “splits” to maps $f_A : A \rightarrow C$ and
 877 $f_B : B \rightarrow C$ such that $f(x) = f_A(x) + f_B(x)$ for all $x \in A \oplus B$. Dually, a map $g : D \rightarrow A \oplus B$
 878 can also “split” into maps $g_A : D \rightarrow A$ and $g_B : D \rightarrow B$ such that $g(x) = g_A(x) + g_B(x)$ for all
 879 $x \in A \oplus B$ with $g_A(x) \in A$ and $g_B(x) \in B$.

880 **Remark B.2.2.** Whenever A, B, C and D are vector spaces of a given diagram \mathcal{D} ,

- 881 (i) if $A, B \leq C$ then $A \oplus B \leq C$;
 882 (ii) if $D \leq A, B$ then $D \leq A \oplus B$.

883 Indeed, (i) follows from the fact that the direct sum is the coproduct in the category of
 884 vector spaces and linear maps. To see (ii) consider the inclusion maps $i_A : A \rightarrow A \oplus B$ and
 885 $i_B : B \rightarrow A \oplus B$ and observe that, due to the hypothesis, there exist maps $f : D \rightarrow A$ and
 886 $g : D \rightarrow B$. Thus, the compositions $i_A \circ f$ and $i_B \circ g$ ensure the inequality $D \leq A \oplus B$.

887 **Definition B.2.3.** Let A and B be vector spaces and I and J be arbitrary sets. Consider the
 888 family of linear maps from $A \oplus B$ to all vector spaces with common sources A and B , i.e.,

$$\mathbb{F}_k = \{f_i : A \oplus B \rightarrow X_k \mid \text{for all vector spaces } X_k \geq A, B \text{ and } i \in I\}$$

889 and, dually, the family of linear maps from all vector spaces with common targets A and B
 890 to $A \oplus B$, i.e.,

$$\mathbb{G}_k = \{g_i : Y_k \rightarrow A \oplus B \mid \text{for all vector spaces } Y_k \leq A, B \text{ and } i \in I\}.$$

891 Define $A \wedge B$ to be the kernel set \mathcal{E} of the equalizer of the linear maps of the family \mathbb{F}_k ,
 892 $\text{eq}(\bigoplus_{k \in J} \mathbb{F}_k)$, and $A \vee B$ to be the quotient set \mathcal{C} of the coequalizer of the linear maps of the
 893 family \mathbb{G}_k , $\text{coeq}(\bigoplus_{k \in J} \mathbb{G}_k)$. These operations are well defined due to Remark B.2.2. Moreover,
 894 as all considered maps on the construction of the kernel set $A \wedge B$ and the quotient set $A \vee B$
 895 are linear, $A \wedge B$ is a subalgebra of $A \oplus B$ and $A \vee B$ is a quotient algebra of $A \oplus B$. Both of
 896 them constitute vector spaces.

897 **Remark B.2.4.** We shall discuss now the equalizer set and quotient set constituting the meet
 898 and the join, respectively, of vector spaces in a given diagram. Whenever A and B are vector
 899 spaces with common targets C_1 and C_2 , i.e., such that $A, B \leq C_1$ and $A, B \leq C_2$, we can
 900 consider linear maps $f_1, g_1 : A \oplus B \rightarrow C_1$ and $f_2, g_2 : A \oplus B \rightarrow C_2$, and the equalizers $\text{eq}(f_1, g_1)$
 901 and $\text{eq}(f_2, g_2)$ with kernel sets $\mathcal{E}_1 = \{x \in A \mid f_1(x) = g_1(x)\}$ and $\mathcal{E}_2 = \{x \in A \mid f_2(x) = g_2(x)\}$,
 902 respectively. Define $\text{eq}(f_1, g_1) \vee \text{eq}(f_2, g_2)$ to be the pair (\mathcal{E}, e) with kernel set determined by
 903 the union of equations in \mathcal{E}_1 and in \mathcal{E}_2 , i.e.,

$$\mathcal{E}_{1,2} = \{x \in A \oplus B \mid f_k(x) = g_k(x), k \in \{1, 2\}\}$$

904 and corresponding inclusion map $e : \mathcal{E} \hookrightarrow A \oplus B$. This new pair is an equalizer of all
 905 the considered maps from $A \oplus B$ to $\bigoplus_{k \in \{1,2\}} C_k$ (as represented in Figure 10 (a)). Indeed
 906 $\mathcal{E}_{1,2} = \mathcal{E}_1 \cap \mathcal{E}_2$: if we look at \mathcal{E}_k as a set of equations, $\mathcal{E}_{1,2}$ is determined by both the defining
 907 equations in \mathcal{E}_1 and \mathcal{E}_2 , i.e.,

$$\mathcal{E}_1 \cap \mathcal{E}_2 = \{x \in A \oplus B \mid (f_1(x), f_2(x)) = (g_1(x), g_2(x))\} = \{x \in A \oplus B \mid f_1 \oplus f_2(x) = g_1 \oplus g_2(x)\}.$$

908 *Dually, whenever A and B are vector spaces with common sources D_1 and D_2 , i.e., such*
 909 *that $D_1 \leq A, B$ and $D_2 \leq A, B$ we can consider linear maps $f_1, g_1 : D_1 \rightarrow A \oplus B$ and $f_2, g_2 :$*
 910 *$D_2 \rightarrow A \oplus B$. The quotient sets of the coequalizers $\text{coeq}(f_1, g_1)$ and $\text{coeq}(f_2, g_2)$ are quotients*
 911 *of $A \oplus B$ by the equivalences $\theta_1 = \langle \{ (f_1(x), g_1(x)) \mid x \in D_1 \} \rangle$ and $\theta_2 = \langle \{ (f_2(x), g_2(x)) \mid x \in$*
 912 *$D_2 \} \rangle$, respectively. Define $\text{coeq}(f_1, g_1) \vee \text{coeq}(f_2, g_2)$ to be the pair (H, h) with underlying set*
 913 *constituted by the quotient of $A \oplus B$ by the equivalence θ generated by*

$$\langle \theta_1 \cup \theta_2 \rangle = \langle \{ (f_k(x), g_k(x)) \mid x \in D_1 \cap D_2, i \in \{1, 2\} \} \rangle$$

914 *and corresponding linear map $h : A \oplus B \rightarrow A \oplus B/\theta$. This new pair is a coequalizer of all*
 915 *the considered maps from $\bigoplus_{k \in \{1, 2\}} D_k$ to $A \oplus B$ (as represented in Figure 10 (b)). Whenever*
 916 *$D_1 \cap D_2 = \{0\}$, the respective equivalence θ is $\langle (0, 0) \rangle = \{(0, 0)\} = 0$ and thus $C_{1,2} = A \oplus B/0 \cong$*
 917 *$A \oplus B$. Observe that we are generating the equivalence that includes all the possible pairs*
 918 *given by the linear maps to each D_k . In fact, the union of equivalences is not, in general, an*
 919 *equivalence but it is clearly included in the equivalence generated by this union. The quotient*
 920 *by this bigger equivalence θ , generated by the union of all the others, will correspond to*
 921 *the smallest quotient above A and B in the requested conditions. To see this consider the*
 922 *partition semilattice of quotients of A with the meet operation defined as*

$$A/\theta_1 \wedge A/\theta_2 = \{x/\theta_1 \cap x/\theta_2 \mid x \in A\} = A/(\theta_1 \cap \theta_2).$$

923 *Thus, $A/\theta \subseteq A/\theta_1 \wedge A/\theta_2$. In general, whenever θ is the equivalence generated by the union*
 924 *of the equivalences θ_k corresponding to each vector space Y_k above A and B , then*

$$A/\theta \subseteq \bigwedge_{k \in J} (A/\theta_k).$$

925 **Proposition B.2.5.** *Let I be an index set and A, B, C_i and D_j be vector spaces such that*
 926 *$D_j \leq A, B \leq C_i$, for all $i, j \in I$. Consider the families of linear maps $\mathcal{F}_k = \{f_{ik} : A \oplus B \rightarrow C_k\}$,*
 927 *$\mathcal{F}'_k = \{f'_{ik} : D_k \rightarrow A \oplus B\}$, for some $k \leq i, j$. Consider also the equalizers $\text{eq}(\mathcal{F}_k) = (E_k, e_k)$*
 928 *and the coequalizers $\text{coeq}(\mathcal{F}'_k) = (H_k, h_k)$. Then,*

- 929 (i) *the kernel set $E_k = \mathcal{E}((F_k)_{k \in I})$ is the intersection of all the kernel sets corresponding*
 930 *to the equalizers of linear maps of the family $(F_k)_{k \in I}$.*
 931 (ii) *the quotient set $H_k = \mathcal{C}((F'_k)_{k \in I})$ is constituted by the quotient of $A \oplus B$ by the equiva-*
 932 *lence generated by the union of all equivalences corresponding to the family of linear*
 933 *maps from $(F'_k)_{k \in I}$.*

934 *Proof.* Consider the kernel set of the equalizer $\text{eq}((F_k)_{k \in I})$ given by

$$\mathcal{E} = \bigcap_{k \in J} \{ \text{eq}(f_{ik}, f_{jk}) \mid f_{ik}, f_{jk} \in \text{Hom}(A \oplus B, D_k) \}, \text{ that is,}$$

$$\mathcal{E} = \{ x \in A \oplus B \mid f_{ik}(x) = f_{jk}(x), \text{ for some } f_{ik}, f_{jk} \in \bigcup_{k \in J} \text{Hom}(A \oplus B, D_k) \}.$$

935 The corresponding linear map e is the inclusion map $E \hookrightarrow A \oplus B$. Furthermore, the universal
 936 property derives from the conjugation of the universal properties valid to each equalizer $\text{eq}(\mathcal{F}_k)$.

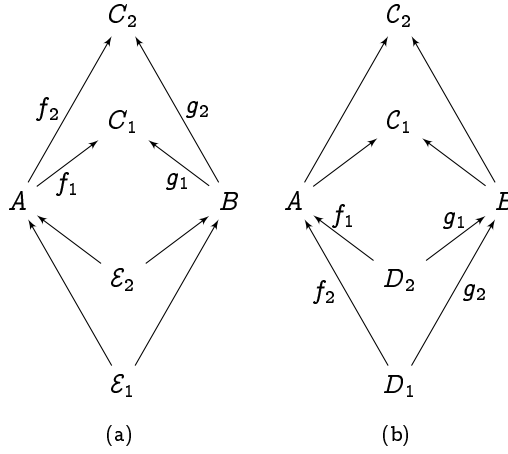


FIGURE 10. Diagram representation of the meet and join of vector spaces A and B of a given diagram \mathcal{D} when A and B have more than one common target (case (a) with targets C_1 and C_2) or more than one common source (case (b) with sources D_1 and D_2), respectively.

937 Dually, observe that, for each $k \in J$, the quotient set of the coequalizer $\text{coeq}((F_k)_{k \in I})$ is given
 938 by: the factor $A \oplus B/\theta$, where θ is the equivalence generated by the set

$$\bigcap_{k \in J} \{ (f_{ik}(x), f_{jk}(x)) \mid x \in Y_k \text{ and } f_{ik}, f_{jk} \in \text{Hom}(D_k, A \oplus B) \}, \text{ that is,}$$

$$\{ (f_{ik}(x), f_{jk}(x)) \mid x \in \bigcup_{k \in J} Y_k \text{ and } f_{ik}, f_{jk} \in \bigcup_{k \in J} \text{Hom}(D_k, A \oplus B) \}.$$

939 The corresponding linear map h is the canonical projection map $A \oplus B \twoheadrightarrow A \oplus B/\theta$. Furthermore,
 940 the universal property again derives from the conjugation of the universal properties valid to each
 941 coequalizer $\text{coeq}(\mathcal{F}_k)$. □

942 **Corollary B.2.6.** *Let $C_1 < C_2 < \dots < C_n$, $D_m < D_{m-1} < \dots < C_1$ and A, B be vector spaces in*
 943 *a diagram such that $D_1 < A, B < C_n$. Then, the equalizer of $\bigcup_k \text{Hom}(A \oplus B, C_k)$ is just the*
 944 *equalizer of $\text{Hom}(A \oplus B, C_n)$ while the coequalizer of $\bigcup_k \text{Hom}(D_k, A \oplus B)$ is just the coequalizer*
 945 *of $\text{Hom}(D_m, A \oplus B)$.*

946 *Proof.* This result is due to the assumption of the commutativity of all diagrams together with
 947 proposition B.2.5. □

948 **Remark B.2.7.** *Both Proposition B.2.5 and Corollary B.2.6 now link to Theorem 3.2.3*
 949 *establishing the completeness of persistence lattices. Indeed, both of the lattice operations*
 950 *extend to arbitrary joins $\bigvee_i D_i$ given by*

$$\bigwedge S = \{ x \in X : f_i(x) = f_j(x), \text{ for all } f_i, f_j \in \bigcup_k \text{Hom}(X, C_k) \}.$$

951 and meets $\bigwedge_i D_i$ given by

$$\bigvee_{\ell} A_{\ell} = (\oplus_{\ell} A_{\ell}) / \bigcap_k \langle (f_i(x), f_j(x)) \mid x \in \oplus_k D_k \rangle$$

952 that are a great deal dependent from the biggest element of the correspondent total orders
953 determined by $\bigcup_k \text{Hom}(X, C_k)$ and $\oplus_k D_k$, respectively.

954

APPENDIX C. GLOSSARY OF DEFINITIONS

955 In the following we present a list of basic concepts of lattice theory and category theory that will
956 help the reader, that is unfamiliar with such, through this paper. These concepts are presented by
957 order of appearance. For more details please read [5], [18] or [26].

- 958 ☆ Preorder \equiv a binary relation R that satisfies *reflexivity* (i.e., for all $x \in A$, xRx) and
959 *transitivity* (i.e., for all $x, y, z \in A$, xRy and yRz implies xRz).
- 960 ☆ Partial order \equiv a preorder \leq such that, for all $x, y \in A$, $x \leq y$ and $y \leq x$ implies $x = y$
961 (antisymmetry).
- 962 ☆ Poset \equiv an order structure (P, \leq) consisting of a set P and a partial order \leq .
- 963 ☆ Total order \equiv a poset such that every pair of elements is related, that is, for all $x, y \in A$,
964 $x \leq y$ or $y \leq x$.
- 965 ☆ Antitotal order \equiv a partial order for which no two distinct elements are related.
- 966 ☆ Lattice \equiv a poset for which all pairs of elements have an infimum and a supremum.
- 967 ☆ Complete lattice \equiv a poset for which every subset has a supremum and an infimum.
- 968 ☆ Associativity \equiv for all x, y, z , $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$.
- 969 ☆ Idempotency \equiv for all x , $x \wedge x = x = x \vee x$.
- 970 ☆ Comutativity \equiv for all x, y , $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.
- 971 ☆ Absorption \equiv for all x, y , $x \wedge (x \vee y) = x = x \vee (x \wedge y)$.
- 972 ☆ Modularity \equiv for all x, y, z , $y \leq x$ implies $x \wedge (y \vee z) = y \vee (x \wedge z)$.
- 973 ☆ Distributivity \equiv for all x, y, z , $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ or $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.
- 974 ☆ Heyting algebra \equiv a bounded distributive lattice such that for all a and b there is a greatest
975 element x such that $a \wedge x \leq b$.
- 976 ☆ Implication operation, $a \Rightarrow b \equiv$ the greatest element x in a Heyting algebra such that
977 $a \wedge x \leq b$.
- 978 ☆ Join-irreducible element \equiv an element x for which $x = y \vee z$ implies $x = y$ or $x = z$, for
979 all y, z .
- 980 ☆ Meet-irreducible element \equiv an element x for which $x = y \wedge z$ implies $x = y$ or $x = z$, for
981 all y, z .
- 982 ☆ Boolean algebra \equiv a distributive lattice with a unary operation \neg and nullary operations
983 0 and 1 such that $a \vee 0 = a$ and $a \wedge 1 = a$, as well as $a \vee \neg a = 1$ and $a \wedge \neg a = 0$.
- 984 ☆ Category \equiv a class of objects and morphisms between them such that their composition is
985 a well defined associative operation and that an identity morphism exists.
- 986 ☆ Functor \equiv a map between two categories A and B that associates to each object of A
987 an object of B and to each morphism in A a morphism in B so that the image of an
988 identity morphism in A is an identity morphism in B , and the image of the composition of
989 morphisms in A is the composition of their images in B .

- 990 ★ Pullback \equiv the limit of a diagram constituted by two morphisms with a common codomain.
991 ★ Pushout \equiv the colimit of a diagram constituted by two morphisms with a common domain.
992 ★ Equalizer \equiv the limit of the diagram consisting of two objects X and Y and two parallel
993 morphisms $f, g : X \rightarrow Y$.
994 ★ Coequalizer \equiv the colimit of the diagram consisting of two objects X and Y and two parallel
995 morphisms $f, g : X \rightarrow Y$ (dual concept of equalizer).

996

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1003

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