Improved Oracle Complexity of Variance Reduced Methods for Nonsmooth Convex Stochastic Composition Optimization

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Abstract

We consider the nonsmooth convex composition optimization problem where the objective is a composition of two finite-sum functions and analyze stochastic compositional variance reduced gradient (SCVRG) methods for them. SCVRG and its variants have recently drawn much attention given their edge over stochastic compositional gradient descent (SCGD); but the theoretical analysis exclusively assumes strong convexity of the objective, which excludes several important examples such as Lasso, logistic regression, principle component analysis and deep neural nets. In contrast, we prove non-asymptotic incremental first-order oracle (IFO) complexity of SCVRG or its novel variants for nonsmooth convex composition optimization and show that they are provably faster than SCGD and gradient descent. More specifically, our method achieves the total IFO complexity of $O\left((m+n)\log\left(1/\epsilon\right)+1/\epsilon^3\right)$ which improves that of $O\left(1/\epsilon^{3.5}\right)$ and $O\left((m+n)/\sqrt{\epsilon}\right)$ obtained by SCGD and accelerated gradient descent (AGD) respectively. Experimental results confirm that our methods outperform several existing methods, e.g., SCGD and AGD, on sparse mean-variance optimization problem.

Keywords: Large-scale nonsmooth convex optimization, composition optimization, stochastic gradient, variance reduction, incremental first-order oracle complexity.

1 Introduction

1.1 Motivations

The popular stochastic variance reduced gradient (SVRG) methods [14] are well studied and proven well suited for minimizing the sum of a large number of loss functions with the nonsmooth regularization penalty. Despite their popularity, SVRG and its variants can not address the problem of a minimizing

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nonlinear function involving a composition between two finite-sum functions, which covers a wide range of applications including reinforcement learning [32], nonparametric statistics [11], risk management [25], multi-stage stochastic programming [31], system control [15], model-based stochastic search methods [10] and deep learning [9].

In this paper, we consider the *nonsmooth convex composition problem* where the objective is a composition of two finite-sum functions, given by

$$\min_{\mathbf{x}\in\mathbb{R}^d} \ \Phi(\mathbf{x}) = f(\mathbf{x}) + r(\mathbf{x}), \tag{1}$$

where $r : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is an extended real-valued closed convex function and $f : \mathbb{R}^d \to \mathbb{R}$ is a continuously differentiable convex function, given by

$$f(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} f_i\left(\frac{1}{m}\sum_{j=1}^{m} g_j(\mathbf{x})\right).$$

where $f_i : \mathbb{R}^l \to \mathbb{R}$ ($i \in [n]$) and $g_j : \mathbb{R}^d \to \mathbb{R}^l$ ($j \in [m]$) are continuously differentiable functions. We assume throughout that there exists at least one optimal solution $\mathbf{x}^* \in \mathbb{R}^d$ to problem (1). We assume the smoothness of f_i and g_j but do not require either of them to be convex or monotone. We also allow r to be a nonsmooth penalty function, e.g., ℓ_1-norm.

Example 1.1 (Risk-Averse Learning)**.** *Consider the mean-variance minimization problem*

$$\min_{\mathbf{x}} \ \frac{1}{n}\sum_{i=1}^{n} h(\mathbf{x}, a_i, b_i) + \frac{\lambda}{n}\sum_{i=1}^{n}\left[h(\mathbf{x}, a_i, b_i) - \frac{1}{n}\sum_{i=1}^{n} h(\mathbf{x}, a_i, b_i)\right]^2,$$

where $h(\mathbf{x}, \mathbf{a}_i, b_i)$ is some loss function on a sample data $(\mathbf{a}_i, b_i)$ and $\lambda > 0$ is a regularization parameter. Here the variance term is the composition of the mean square function and an expected loss.

Example 1.2 (Reinforcement Learning [32])**.** *Given a controllable Markov chain with states $1, 2, \ldots, S$, we are aiming at estimating the value-per-state of a fixed control policy π, i.e., the Bellman equations:*

$$\gamma P^\pi V^\pi + \mathbf{r}^\pi = V^\pi.$$

where $\gamma \in (0, 1)$ is a discount factor, $P^\pi_{s\tilde{s}}$ is the transition probability from state s to state $\tilde{s}$ and r^π_s is the expected state transition reward at state s. The solution V^π to the Bellman equation is the value vector with V^π_s being the total expected reward starting at state s. In the black box simulation environment, solving the Bellman equation becomes a special case of the stochastic composition optimization problem:

$$\min_{\mathbf{x}\in X} \left\| \frac{1}{m}\left[\sum_{j=1}^{m}\left(I - \gamma P^\pi_j\right)\mathbf{x} - \mathbf{r}^\pi_j\right]\right\|^2$$

where P^π_j and $\mathbf{r}^\pi_j$ are sampled from a simulator.

Generally speaking, the composition problem formulated as Eq(1) is substantially more challenging than its non-composition counterpart. The main reason is the lack of linearity [33] in the sampling probabilities of the objective in (1), making us inaccessible to an *unbiased* samples of the gradient. Therefore, the computational cost of SVRG on problem (1) is very expensive since the per-iteration cost is proportional to n. Despite of this, there has been some efforts on developing new compositional variance reduced methods for solving problem (1) with a theoretical guarantee. The existing works include the IFO complexity of $O\left((m+n+\tilde{\kappa_2}^3)\log(1/\epsilon)\right)$ for smooth and strongly convex objective [18] and that of $O\left(m+n+(m+n)^{0.8}/\epsilon\right)$ for smooth and possibly nonconvex objective [19].

However, it is still demanding to ask how to analyze stochastic compositional variance reduced (SCVRG) methods for nonsmooth convex composition optimization problems. Firstly, the assumption of the smooth and strongly convex objective is too restricted and excludes several interesting applications [21]. Secondly, the IFO complexity of SCVRG for nonsmooth convex composition optimization should be better than $O\left(m+n+(m+n)^{0.8}/\epsilon\right)$, which is obtained for smooth nonconvex composition optimization. More specifically, [4] presented an improved IFO complexity that $m+n$ is **independent** of $1/\epsilon^p$ and $p \geq 1$, which is significantly better than $O\left(m+n+(m+n)^{0.8}/\epsilon\right)$ for large-scale problems. However, the analysis in [19] for smooth convex composition optimization can not obtain such improved IFO complexity.

Given the promising property of SCVRG for smooth and strongly convex/nonconvex composition optimization problems, there is an urgent need for investigating an improved IFO complexity for nonsmooth convex composition optimization problems. This raises the central question of this paper:

> Can we prove an improved IFO complexity of SCVRG and its variants for convex composition optimization problem (1), where $m+n$ is **independent** of $1/\epsilon^p$ and $p \geq 1$ and also show that SCVRG outperforms stochastic compositional gradient descent (SCGD) and accelerated gradient descent (AGD)?

1.2 Related Works

Johnson and Zhang's seminal work [14] inspired a burst of following research on variance reduction for stochastic optimization. This technique has been successfully extended from the strongly convex objective to general objective, e.g., convex objective [4] and nonconvex objective [30, 3, 23, 2, 17]. Reddi et al. [23] and Allen-Zhu and Hazan [3] proved the IFO complexity of $O(n+n^{2/3}/\epsilon))$ of variance reduced gradient method for smooth nonconvex stochastic optimization, where n is the number of component functions and ϵ is the tolerance. This complexity bound improves the IFO complexity of gradient descent, i.e., $O(n/\epsilon))$, when n is very large. In addition to its theoretical guarantee under mild assumption, the variance reduced gradient method has been also extended and analyzed in asynchronous parallel setting [24]. There were also some efforts on understanding other techniques to accelerate stochastic gradient methods, such as stochastic average gradient [26, 6, 7] and stochastic dual coordinate ascent [29, 28].

However, there are concerns on applying variance reduction for stochastic optimization. First of all, the existing theoretical results in the literature of variance reduction are based on the stationary gap, which can not translate to optimality gap or low training loss and test error in general. Furthermore, several recent works [8, 16, 12, 13] have suggested that variance in the stochastic algorithms can actually help avoid local minimum and saddle points, which doubts against the necessity of variance reduction for smooth nonconvex stochastic optimization, as well as smooth nonconvex stochastic composition

Table 1: The IFO complexity of all stochastic composition optimization methods. ϵ is the tolerance. m is the number of inner functions. n is the number of outer functions. κ is the condition number of the objective while κ_1 and κ_2 are defined in [18]. GradientDescent refers to gradient-type method containing standard and accelerated gradient descent methods. Accelerated SCGD refers to the method in [33] only applied when the objective is smooth. SCVRG refers to stochastic compositional gradient method with variance reduction for the inner function. Accelerated SCVRG refers to stochastic compositional gradient method with variance reduction for the inner and outer functions. We hide the dependence of the IFO complexity on some other parameters, such as the Lipschitz constant of f, f_i and g_j, the upper bound of the norm of ∇f_i and ∂g_j and the distances between the initial point and the optimal set for a clean comparison.

	Convex	Strongly Convex
GradientDescent	$O\left((m+n)/\sqrt{\epsilon}\right)$ [20]	$O\left((m+n)\sqrt{\kappa}\log(1/\epsilon)\right)$ [20]
SCGD	$O\left(1/\epsilon^4\right)$ [33]	$O\left(1/\epsilon^{1.5}\right)$ [33]
Accelerated SCGD	$O\left(1/\epsilon^{3.5}\right)$ [33]	$O\left(1/\epsilon^{1.25}\right)$ [33]
Extrap-Smoothing SCGD	$O\left(1/\epsilon^{3.5}\right)$ [34]	$O\left(1/\epsilon^{1.25}\right)$ [34]
SCVRG	$O\left((m+n)\log\left(1/\epsilon\right)+1/\epsilon^4\right)$	$O\left((m+n+\kappa_1^4)\log(1/\epsilon)\right)$ [18]
Accelerated SCVRG	$O\left((m+n)\log\left(1/\epsilon\right)+1/\epsilon^3\right)$	$O\left((m+n+\kappa_2^3)\log(1/\epsilon)\right)$ [18]

optimization. However, one can reap the benefit of variance reduction by envisioning a two-stage algorithm which uses SCGD as an initialization and turns to SCVRG as an efficient tool to approach a good local minimum.

Another related stream of research is the algorithmic design for stochastic composition optimization. Wang et al. [33] proposed and analyzed a class of stochastic compositional gradient/subgradient methods (SCGD), with two iterates of different time scales instead of one iterate in stochastic gradient descent. Wang et al. [34] proposed an extrapolation-smoothing SCGD, which improved the IFO complexity over [33]. Yang et al. [36] further proposed a class of multi-level stochastic gradient methods for the multi-level composition optimization problem with a solid theoretical guarantee. On the other hand, variance reduction was firstly proposed to accelerate stochastic compositional gradient method in [18], where the IFO complexity is shown linear for the strongly convex objective. Yu and Huang [37] proposed a variance reduced alternating direction method of multipliers for linearly constrained stochastic composition optimization. Very recently, Liu et al. [19] have obtained an $O((m+n)^{0.2})$ improvement of similar algorithms over gradient descent method for smooth nonconvex composition optimization. *To the best of our knowledge, no existing works provide the improved IFO complexity analysis of SCVRG and its variants for nonsmooth convex composition optimization.*

1.3 Contributions

In this paper, we present a unified framework to analyze stochastic compositional variance reduced gradient method that applies to composition optimization problem with nonsmooth convex objective. This is more general than the smooth strongly convex objective considered in [18]. We use an *iterative stochastic analysis based on different potential functions* to show that our method achieves better IFO complexity. We refer the reader to Table 1 for the detail of our results.

Our major contributions are summarized as follows:

1. We provide an IFO complexity analysis of SCVRG and its variants for nonsmooth convex composition optimization. This is also the first stochastic compositional variance reduced gradient method that is able to address the nonsmooth regularization penalty $r(\cdot)$ without deteriorating the IFO complexity.

2. We obtain an IFO complexity of $O\left((m+n)\log\left(1/\epsilon\right)+1/\epsilon^3\right)$ for composition optimization problem with nonsmooth convex objective. This improves the best known IFO complexity, i.e., $O\left(1/\epsilon^{3.5}\right)$ obtained by extrapolation-smoothing SCGD and $O\left((m+n)/\sqrt{\epsilon}\right)$ obtained by accelerated gradient descent method and provides a new benchmark for the nonsmooth convex stochastic composition problem.

3. We develop a new iterative stochastic analysis approach and obtain an improved IFO complexity bound which is better than $O\left((m+n)+(m+n)^{0.8}/\epsilon\right)$ in [19]. Our bound on the objective gap is also more reasonable and general for nonsmooth convex composition problem than the bound on the norm of the gradient obtained in [19].

4. We describe the application of our method to sparse mean-variance optimization problem and conduct extensive experiments to show that our methods outperform other competing methods.

1.4 Notations and Organization

Throughout the paper, we denote vectors by bold lower case letters, e.g., $\mathbf{x}$, and matrices by regular upper case letters, e.g., X. The transpose of a real vector $\mathbf{x}$ is denoted as $\mathbf{x}^\top$. $\|\mathbf{x}\|$ and $\|X\|$ denote the vector ℓ_2 norm and the matrix spectral norm for a vector $\mathbf{x}$ and a matrix X. For a scalar $x\in\mathbb{R}$, $\lfloor x \rfloor$ is the largest integer which is smaller than x. For two nonnegative sequences $\{a_t\}$ and $\{b_t\}$, we write $a_t = O(b_t)$ if there exists a constant $C>0$ such that $a_t \le Cb_t$ for each $t\ge 0$, and $a_t = o(b_t)$ if there exists a nonnegative sequence $\{c_t\}$ such that $a_t \le c_t b_t$ for each $t\ge 0$ and $c_t \to 0$ as $t\to\infty$. We denote the gradient[1] of $f:\mathbb{R}^d\to\mathbb{R}$ at $\mathbf{x}$ as $[\partial g(\mathbf{x})]^\top \nabla f(g(\mathbf{x}))\in\mathbb{R}^d$, where $\partial g(\mathbf{x})\in\mathbb{R}^{l\times d}$ is the Jacobian of $g:\mathbb{R}^d\to\mathbb{R}^l$ at $\mathbf{x}$. The subscript, e.g., $\mathbf{x}_t^s$, denotes the iterate at the t-th iteration in the s-th epoch. The sets $\mathcal{A}_t$ and $\mathcal{B}_t$ are denoted as the set of randomly selected index at the t-th iteration, where their batch sizes are A and B. We also denote $\mathbb{E}\left[\cdot \mid \zeta\right]$ as taking conditional expectation given the variable ζ and $\mathbb{E}$ as taking expectation over all random variables.

The rest of the paper is organized as follows. Section 2 states the incremental first-order oracle (IFO) and our method. Section 3 states the IFO complexity bound of our method in convex setting, with the proofs and technical details deferred to the appendices. Section 4 demonstrates an application of our method to sparse mean-variance optimization problem and numerical results. Conclusions and future works come in Section 5.

[1]The gradient operator always calculates the gradient with respect to the first level variable. More specifically, $\nabla f(g(\mathbf{x}))$ refers to the gradient of $f(\mathbf{y})$ at $\mathbf{y}=g(\mathbf{x})$, not the gradient of $f(g(\mathbf{x}))$ at $\mathbf{x}$.

Algorithm 1 Stochastic Compositional Variance Reduced Gradient Method (SCVRG)

Input: $\tilde{\mathbf{x}}^0 = \mathbf{x}_{k_0}^0 = \mathbf{x}^0 \in \mathbb{R}^d$, first epoch length k_0, stepsize $\eta > 0$ and the number of epochs S.
Initialization: $l = 0$ and $T = k_0 \cdot 2^S - k_0$.
for $s = 0, 1, \ldots, S$ **do**
  $\mathbf{x}_0^{s+1} = \mathbf{x}_{k_s}^s$, $\tilde{\mathbf{g}}^{s+1} = g(\tilde{\mathbf{x}}^s)$,
  $\tilde{\mathbf{f}}^{s+1} = [\partial g(\tilde{\mathbf{x}}^s)]^\top \nabla f(\tilde{\mathbf{g}}^{s+1})$ and $k_{s+1} = 2^{s+1} \cdot k_0$.
  for $t = 0, 1, \ldots, k_{s+1} - 1$ **do**
    Query the IFO and obtain a mini-batch of function samples $g_j(\mathbf{x}_t^{s+1})$ and $g_j(\tilde{\mathbf{x}}^s)$ where $j \in \mathcal{A}_t \subset [m]$ and the cardinality of $\mathcal{A}_t$ is A.
    Update the auxiliary iterate $\mathbf{g}_t^{s+1}$ by a *variance reduction* scheme, i.e., (2).
    Query the IFO and obtain gradient samples $\nabla f_{i_t}(\mathbf{g}_t^{s+1})$ and $\nabla f_{i_t}(\tilde{\mathbf{g}}^{s+1})$ and Jacobian samples $\partial g_{j_t}(\mathbf{x}_t^{s+1})$ and $\partial g_{j_t}(\tilde{\mathbf{x}}^s)$.
    Update the auxiliary iterate $\mathbf{f}_t^{s+1}$ by a *variance reduction* scheme, i.e., (3).
    Update $l = l + 1$ and $\eta_{t+1}^{s+1} = \frac{\eta\sqrt{T}}{\sqrt{2T-l}}$.
    Update the main $\mathbf{x}_{t+1}^{s+1}$ by a *proximal* step, i.e., (4).
  end for
  $\tilde{\mathbf{x}}^{s+1} = \frac{1}{k_{s+1}} \sum_{t=0}^{k_{s+1}-1} \mathbf{x}_t^{s+1}$.
end for
Output: $\tilde{\mathbf{x}}^S$.

2 Algorithm

In this section, we focus on the algorithmic design of stochastic composition optimization under the black-box sampling environment with the access to an increment first-order oracle (IFO). This is a typical simulation oracle available in both online and batch learning [27] and a standard tool in the complexity analysis [1, 23, 35].

Definition 2.1. *Given some $\mathbf{x} \in \mathbb{R}^d$ and $j \in [m]$, the IFO returns a vector $g_j(\mathbf{x})$ or a matrix $\partial g_j(\mathbf{x})$. Alternatively, given some $\mathbf{y} \in \mathbb{R}^l$ and $i \in [n]$, the IFO returns a value $f_i(\mathbf{y})$ or a vector $\nabla f_i(\mathbf{y})$.*

We proceed to propose new standard and accelerated stochastic variance reduced gradient methods for nonsmooth convex stochastic composition optimization, denoted as SCVRG and Accelerated SCVRG for short, see Algorithm 1 and Algorithm 2.

In Algorithm 1, the *variance reduction* scheme is applied for estimating the value of the function vector $g(\cdot)$ and the gradient vector $[\partial g(\cdot)]^\top \cdot \nabla f(g(\cdot))$ at the iterate $\mathbf{x}_t^{s+1}$. More specifically, given a reference point $\tilde{\mathbf{x}}^s$ and the function vector $\tilde{\mathbf{g}}^{s+1} = g(\tilde{\mathbf{x}}^s)$, we can estimate the function vector $g(\mathbf{x}_t^{s+1})$ by

$$\mathbf{g}_t^{s+1} = \frac{1}{A} \sum_{j \in \mathcal{A}_t} g_j(\mathbf{x}_t^{s+1}) - \frac{1}{A} \sum_{j \in \mathcal{A}_t} g_j(\tilde{\mathbf{x}}^s) + \tilde{\mathbf{g}}^{s+1}. \tag{2}$$

where $\mathcal{A}_t \subset [m]$ is a subset with the cardinality A. Further given a reference gradient vector $\tilde{\mathbf{f}}^{s+1} = [\partial g(\tilde{\mathbf{x}}^s)]^\top \nabla f(\tilde{\mathbf{g}}^{s+1})$, we can estimate the gradient vector $\left[\partial g(\mathbf{x}_t^{s+1})\right]^\top \cdot \nabla f(g(\mathbf{x}_t^{s+1}))$ by

$$\mathbf{f}_t^{s+1} = \left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) - [\partial g_{j_t}(\tilde{\mathbf{x}}^s)]^\top \nabla f_{i_t}(\tilde{\mathbf{g}}^{s+1}) + \tilde{\mathbf{f}}^{s+1}. \tag{3}$$

Algorithm 2 Accelerated Stochastic Compositional Variance Reduced Gradient Method (Accelerated SCVRG)

Input: $\tilde{\mathbf{x}}^0 = \mathbf{x}_{k_0}^0 = \mathbf{x}^0 \in \mathbb{R}^d$, first epoch length k_0, stepsize $\eta > 0$ and the number of epochs S.
Initialization: $l = 0$ and $T = k_0 \cdot 2^S - k_0$.
for $s = 0, 1, \ldots, S$ **do**
  $\mathbf{x}_0^{s+1} = \mathbf{x}_{k_s}^s$, $\tilde{\mathbf{g}}^{s+1} = g(\tilde{\mathbf{x}}^s)$, $\tilde{\mathbf{G}}^{s+1} = \partial g(\tilde{\mathbf{x}}^s)$,
  $\tilde{\mathbf{f}}^{s+1} = [\partial g(\tilde{\mathbf{x}}^s)]^\top \nabla f(\tilde{\mathbf{g}}^{s+1})$ and $k_{s+1} = 2^{s+1} \cdot k_0$.
  for $t = 0, 1, \ldots, k_{s+1} - 1$ **do**
    Query the IFO and obtain a mini-batch of function samples $g_j(\mathbf{x}_t^{s+1})$ and $g_j(\tilde{\mathbf{x}}^s)$ where $j \in \mathcal{A}_t \subset [m]$ and the cardinality of $\mathcal{A}_t$ is A.
    Update the auxiliary iterate $\mathbf{g}_t^{s+1}$ by a *variance reduction* scheme, i.e., (2).
    Query the IFO and obtain a mini-batch of function samples $\partial g_j(\mathbf{x}_t^{s+1})$ and $\partial g_j(\tilde{\mathbf{x}}^s)$ where $j \in \mathcal{B}_t \subset [m]$ and the cardinality of $\mathcal{B}_t$ is B.
    Update the auxiliary iterate $\mathbf{G}_t^{s+1}$ by a *variance reduction* scheme, i.e., (5).
    Query the IFO and obtain gradient samples $\nabla f_{i_t}(\mathbf{g}_t^{s+1})$ and $\nabla f_{i_t}(\tilde{\mathbf{g}}^{s+1})$.
    Update the auxiliary iterate $\mathbf{f}_t^{s+1}$ by a *variance reduction* scheme, i.e., (6).
    Update $l = l + 1$ and $\eta_{t+1}^{s+1} = \frac{\eta\sqrt{T}}{\sqrt{2T-l}}$.
    Update the main $\mathbf{x}_{t+1}^{s+1}$ by a *proximal* step, i.e., (4).
  end for
  $\tilde{\mathbf{x}}^{s+1} = \frac{1}{k_{s+1}} \sum_{t=0}^{k_{s+1}-1} \mathbf{x}_t^{s+1}$.
end for
Output: $\tilde{\mathbf{x}}^S$.

We compute a new stepsize η_{t+1}^{s+1} and update the main iterate $\mathbf{x}_{t+1}^{s+1}$ by

$$\mathbf{x}_{t+1}^{s+1} = \underset{\mathbf{x} \in \mathbb{R}^d}{\operatorname{argmin}} \left\{ \left\langle \mathbf{f}_t^{s+1}, \mathbf{x} \right\rangle + \frac{1}{2\eta_{t+1}^{s+1}} \left\| \mathbf{x} - \mathbf{x}_t^{s+1} \right\|^2 + r(\mathbf{x}) \right\}. \tag{4}$$

Finally, we use the average of all iterates $\mathbf{x}_t^{s+1}$ for $0 \leq t \leq k_{s+1} - 1$ as the reference point for the next epoch. The final output is the reference point of the last iteration, i.e., $\tilde{\mathbf{x}}^S$.

In Algorithm 2, the *variance reduction* scheme is further applied to estimate the Jacobian matrix $\partial g(\cdot)$ at the iterate $\mathbf{x}_t^{s+1}$, serving as the critical role in the acceleration of convergence. Indeed, the acceleration stems from the refined estimation of the unknown quantity $\partial g(\mathbf{x}_t^{s+1})$. Given a reference Jacobian matrix $\tilde{\mathbf{G}}^{s+1} = \partial g(\tilde{\mathbf{x}}^s)$, we can estimate the Jacobian matrix $\partial g(\mathbf{x}_t^{s+1})$ by

$$\mathbf{G}_t^{s+1} = \frac{1}{B} \sum_{j \in \mathcal{B}_t} \partial g_j(\mathbf{x}_t^{s+1}) - \frac{1}{B} \sum_{j \in \mathcal{B}_t} \partial g_j(\tilde{\mathbf{x}}^s) + \tilde{\mathbf{G}}^{s+1}, \tag{5}$$

where $\mathcal{B}_t \subset [m]$ is a subset with the cardinality B. Then we can obtain a refined estimation of the gradient vector $\left[\partial g(\mathbf{x}_t^{s+1})\right]^\top \cdot \nabla f(g(\mathbf{x}_t^{s+1}))$ by using $\mathbf{G}_t^{s+1}$ and $\tilde{\mathbf{G}}^{s+1}$, given by

$$\mathbf{f}_t^{s+1} = \left[\mathbf{G}_t^{s+1}\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \left[\tilde{\mathbf{G}}^{s+1}\right]^\top \nabla f_{i_t}(\tilde{\mathbf{g}}^{s+1}) + \tilde{\mathbf{f}}^{s+1}. \tag{6}$$

We compute a new stepsize η_{t+1}^{s+1} and update the main iterate $\mathbf{x}_{t+1}^{s+1}$ by (4). Finally, we use the average of all iterates $\mathbf{x}_t^{s+1}$ for $0 \leq t \leq k_{s+1}-1$ as the reference point for the next epoch. The final output is the reference point of the last iteration, i.e., $\tilde{\mathbf{x}}^S$.

Discussion: In terms of IFO complexity per-epoch, a full gradient vector and a full Jacobian matrix are computed at the point $\tilde{\mathbf{x}}^s$, requiring $m+n$ IFO queries. Therefore, the IFO complexity of SCVRG for the s-th epoch is $m+n+k_s \cdot A$ while that of Accelerated SCVRG for the s-th epoch is $m+n+k_s \cdot (A+B)$ since Accelerated SCVRG further carries out a variance reduction scheme for the Jacobian matrix. In the next section, we prove that the total IFO complexity, where $m+n$ is **independent** of $1/\epsilon^p$ and $p \geq 1$, can be attained through selecting appropriate sample sizes A and B.

3 Main Results

We present our main theoretical results in this section. For the ease of presentation, we defer the proofs for the theorems and technical lemmas to the appendices. We start by adopting a standard definition of ϵ-optimal solution for nonsmooth convex stochastic composition optimization, given by

Definition 3.1. *Given $\epsilon \in (0,1)$, we say $\mathbf{x} \in \mathbb{R}^d$ is an ϵ-optimal solution to problem* (1) *if*

$$\mathbb{E}\left[\Phi(\mathbf{x})\right] - \Phi(\mathbf{x}^*) \leq \epsilon.$$

where $\mathbf{x}^* \in \mathbb{R}^d$ is an optimal solution to problem (1).

Throughout this paper, we measure the efficiency of different algorithms by comparing the number of IFO queries to achieve an ϵ-optimal solution. To conduct our analysis, we make the following standard assumption on f, f_i and g_j, where $i \in [n]$ and $j \in [m]$.

Assumption 3.1. *The objective f and r are both convex, i.e.,*

$$\begin{aligned} f(\mathbf{x}) - f(\mathbf{y}) - \langle \nabla f(\mathbf{y}), \mathbf{x}-\mathbf{y}\rangle &\geq 0, \quad \mathbf{x},\mathbf{y} \in \mathbb{R}^d, \\ r(\mathbf{x}) - r(\mathbf{y}) - \langle \xi, \mathbf{x}-\mathbf{y}\rangle &\geq 0, \quad \mathbf{x},\mathbf{y} \in \mathbb{R}^d. \end{aligned}$$

where $\xi \in \partial r(\mathbf{y})$ is a subgradient of r.

Assumption 3.2. *The proximal mapping of the objective r, given by the following problem,*

$$\operatorname*{argmin}_{\mathbf{x}\in\mathbb{R}^d} \left\{ \langle \mathbf{g}, \mathbf{x}\rangle + \frac{1}{2\eta}\|\mathbf{x}-\mathbf{y}\|^2 + r(\mathbf{x}) \right\}.$$

is easily computed for any $\mathbf{g}, \mathbf{y} \in \mathbb{R}^d$ and $\eta > 0$.

Assumption 3.3. *For $i \in [n]$ and $j \in [m]$, there exist some constants $0 < L_f, L_g, L_\phi < \infty$ such that*

$$\begin{aligned} \|\nabla f_i(\mathbf{x}) - \nabla f_i(\mathbf{y})\| &\leq L_f \|\mathbf{x}-\mathbf{y}\|, \quad \mathbf{x},\mathbf{y} \in \mathbb{R}^l, \\ \|\partial g_j(\mathbf{x}) - \partial g_j(\mathbf{y})\| &\leq L_g \|\mathbf{x}-\mathbf{y}\|, \quad \mathbf{x},\mathbf{y} \in \mathbb{R}^d, \end{aligned}$$

and

$$\left\| [\partial g_j(\mathbf{x})]^\top \nabla f_i(g(\mathbf{x})) - [\partial g_j(\mathbf{y})]^\top \nabla f_i(g(\mathbf{y})) \right\| \leq L_\phi \|\mathbf{x}-\mathbf{y}\|, \quad \mathbf{x},\mathbf{y} \in \mathbb{R}^d.$$

The above assumption implies that ∇f is Lipschitz continuous with a constant $L_\phi > 0$, i.e.,

$$\begin{aligned}
& \|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\| \\
= & \left\| \left[\frac{1}{m}\sum_{j=1}^{m} g_j(\mathbf{x})\right]^\top \left[\frac{1}{n}\sum_{i=1}^{n} \nabla f_i(g(\mathbf{x}))\right] - \left[\frac{1}{m}\sum_{j=1}^{m} g_j(\mathbf{y})\right]^\top \left[\frac{1}{n}\sum_{i=1}^{n} \nabla f_i(g(\mathbf{y}))\right] \right\| \\
\leq & \frac{1}{mn}\sum_{j=1}^{m}\left[\sum_{i=1}^{n}\left[\left\|[\partial g_j(\mathbf{x})]^\top \nabla f_i(g(\mathbf{x})) - [\partial g_j(\mathbf{y})]^\top \nabla f_i(g(\mathbf{y}))\right\|\right]\right] \\
\leq & L_\phi \|\mathbf{x} - \mathbf{y}\|.
\end{aligned}$$

Intuitively, the constants L_f, L_g and L_ϕ jointly characterize the smoothness and complexity of stochastic composition optimization. They do not admit any straightforward dependence relation.

Assumption 3.4. *For $j \in [m]$, there exists a constant $0 < B_g < \infty$ such that*

$$\|\partial g_j(\mathbf{x})\| \leq B_g, \quad \mathbf{x} \in \mathbb{R}^d.$$

The following two assumptions are only required for analyzing Accelerated SCVRG and attaining the improved IFO complexity of $O\left((m+n)\log(1/\epsilon) + 1/\epsilon^3\right)$.

Assumption 3.5. *For $i \in [n]$, the component objective $f_i(g)$ is convex, i.e.,*

$$f_i(g(\mathbf{x})) - f_i(g(\mathbf{y})) - \left\langle [\partial g(\mathbf{y})]^\top \cdot \nabla f_i(g(\mathbf{y})), \mathbf{x} - \mathbf{y} \right\rangle \geq 0, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d,$$

Assumption 3.6. *For $i \in [n]$, there exists a constant $0 < B_f < \infty$ such that*

$$\|\nabla f_i(\mathbf{x})\| \leq B_f, \quad \mathbf{x} \in \mathbb{R}^l.$$

Remark 3.7. *We easily see that Example 1.2 satisfy Assumption 3.1-3.6.*

3.1 Incremental First-order Oracle Complexity

Our first main result provides an IFO complexity of Algorithm 1.

Theorem 3.8. *Given the initial vector $\mathbf{x}^0 \in \mathbb{R}^d$ satisfies that*

$$\left\|\mathbf{x}^0 - \mathbf{x}^*\right\|^2 \leq D_x, \quad \Phi(\mathbf{x}^0) - \Phi(\mathbf{x}^*) \leq D_\phi,$$

and the first epoch length $k_0 > 0$ and the number of epochs $S > 0$ satisfy that

$$k_0 = \lfloor \frac{D_x}{2\eta D_\phi} \rfloor + 1, \qquad S = \log_2\left(\frac{6D_\phi}{\epsilon}\right),$$

and the sample size $A > 0$ satisfies that

$$A = \frac{2B_g^4 L_f^2}{\eta^2 L_\phi^2},$$

where $\eta > 0$ satisfies that

$$\eta = \min\left\{1,\ \frac{1}{10L_\phi},\ \frac{2D_\phi}{D_x},\ \frac{D_\phi}{6B_\phi D_x},\ \frac{\epsilon}{552\sqrt{2}B_\phi D_x}\right\},$$

and $\epsilon \in (0,1)$ is a tolerance and $B_\phi = \max\{L_\phi, L_\phi^2\}$, then the total IFO complexity, i.e., the number of IFO queries to achieve an ϵ-optimal solution that satisfies

$$\mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^{s+1}) - \Phi(\mathbf{x}^*)\right] \leq \epsilon,$$

is

$$O\left((m+n)\cdot\log\left(\frac{1}{\epsilon}\right) + \frac{1}{\epsilon^4}\right).$$

where we omit the dependence of the IFO complexity on the Lipschitz constant L_ϕ, L_f and L_g, the upper bound of the norm of B_g and the distances between the initial point and the optimal set D_x and D_ϕ.

Our second main result provides an improved IFO complexity of Algorithm 2.

Theorem 3.9. *Given the initial vector $\mathbf{x}^0 \in \mathbb{R}^d$ satisfies that*

$$\left\|\mathbf{x}^0 - \mathbf{x}^*\right\|^2 \leq D_x, \quad \Phi(\mathbf{x}^0) - \Phi(\mathbf{x}^*) \leq D_\phi,$$

and the first epoch length $k_0 > 0$ and the number of epochs $S > 0$ satisfy that

$$k_0 = \lfloor \frac{D_x}{2\eta D_\phi} \rfloor + 1, \qquad S = \log_2\left(\frac{6D_\phi}{\epsilon}\right),$$

and the sample sizes $A > 0$ and $B > 0$ satisfy that

$$A = \frac{1}{\eta^{2\alpha}}\cdot\max\left\{\frac{2B_g^4 L_f^2}{L_\phi^2},\ \frac{8B_g^4 L_f^2}{15L_\phi^2}\right\}, \qquad B = \frac{8B_f^2 L_g^2}{15\eta^{2\alpha}L_\phi^2},$$

where $\eta > 0$ satisfies

$$\eta = \min\left\{1,\ \frac{1}{46L_\phi},\ \frac{2D_\phi}{D_x},\ \frac{2D_\phi}{3L_\phi D_x},\ \left(\frac{\epsilon}{120\sqrt{2}L_\phi D_x}\right)^{\frac{1}{\alpha}}\right\},$$

and $\epsilon \in (0,1)$ is a tolerance, then the total IFO complexity, i.e., the number of IFO queries to achieve an ϵ-optimal solution that satisfies

$$\mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^{s+1}) - \Phi(\mathbf{x}^*)\right] \leq \epsilon,$$

is

$$O\left((m+n)\cdot\log\left(\frac{1}{\epsilon}\right) + \frac{1}{\epsilon^{3+\frac{1}{\alpha}}}\right).$$

where we omit the dependence of the IFO complexity on the Lipschitz constant L_ϕ, L_f and L_g, the upper bound of the norm of B_g and the distances between the initial point and the optimal set D_x and D_ϕ.

Remark 3.10. *This holds true for any $\alpha > 1$ and implies that the total IFO complexity is*

$$O\left((m+n) \cdot \log\left(\frac{1}{\epsilon}\right) + \frac{1}{\epsilon^3}\right).$$

Remark 3.11. *We are focusing on the large-scale finite-sum optimization problems, i.e., m and n are extremely large. In fact, our methods achieve superior performance than batch methods since A and B are independent of m and n, as confirmed by our experimental results. When m and n are relatively small, the batch methods will attain better performance than our methods.*

3.2 Discussion

We provide a comprehensive comparison between our methods and the existing methods. In specific, we compare the key aspects of the complexity for AGD, Accelerated SCGD and Accelerated SCVRG for nonsmooth convex composition optimization. The comparison is based on the IFO complexity to achieve an ϵ-optimal solution.

1. **Dependence on** $m + n$: The number of IFO queries of AGD and Accelerated SCVRG depend explicitly on $m + n$. In contrast, the IFO complexity of Accelerated SCGD is independent of $m + n$. However, this comes at the expense of worse dependence on ϵ. The IFO complexity of AGD is proportional to $m + n$ while $m + n$ is **independent** of $1/\epsilon^p$ and $p \geq 1$ for Accelerated SCVRG. In fact, $m+n$ is nearly independent of ϵ since $\log(1/\epsilon) = o(1/\epsilon^p)$ and $p \geq 1$. This makes Accelerated SCVRG superior over other competing methods in practical performance.

2. **Dependence on** ϵ: The dependence on ϵ follows from the complexity bound of the stochastic algorithms. More specifically, Accelerated SCGD depend as $O(1/\epsilon^{3.5})$ on ϵ while Accelerated SCVRG converges as $O(1/\epsilon^3)$ and AGD converge as $O(1/\sqrt{\epsilon})$. This speedup in convergence over Accelerated SCGD is especially significant when medium to high accuracy solutions are required (i.e., ϵ is small).

3. **Dependence on shrinking stepsize**: It is beneficial to compare the stepsizes used by different algorithms. There is an undesirable property of Accelerated SCGD is that its stepsizes shrink as the number of iterations T increase. In contrast, the stepsizes of Accelerated SCVRG fall into the interval of $\left[\frac{\eta}{\sqrt{2}}, \eta\right]$ and that of AGD is independent of T, which implies no dependence on the shrinking stepsizes. This is especially crucial to the effectiveness and robustness of the algorithms when a huge number of iterations are required – a case which is now very common in learning complex models such as deep neural networks.

4 Experiments

In this section, we show the application of SCVRG and Accelerated SCVRG to sparse mean-variance optimization problem. From experiment results we conclude that our methods outperform other competing methods.

4.1 Sparse Mean-variance Optimization

Sparse mean-variance optimization (SpMO) problem [22] is a class of risk-averse learning models for high-dimensional sparse regression portfolio management. Given a group of d assets and the reward vectors observed at n time points, i.e., $\{\mathbf{r}_i\}_{i=1}^n \subset \mathbb{R}^d$, the goal of SpMO is to maximize the return of the investment as well as controlling the investment risk. In specific, let $\mathbf{x} \in \mathbb{R}^d$ be the quantities invested to each portfolio, SpMO aims the following optimization problem,

$$\min_{\mathbf{x}\in\mathbb{R}^d} \quad \left[\frac{1}{n}\sum_{i=1}^{n}\left(\langle \mathbf{r}_i, \mathbf{x}\rangle - \frac{1}{n}\sum_{j=1}^{n}\langle \mathbf{r}_j, \mathbf{x}\rangle\right)^2 - \frac{1}{n}\sum_{i=1}^{n}\langle \mathbf{r}_i, \mathbf{x}\rangle + \lambda\left\|\mathbf{x}\right\|_1\right]. \tag{7}$$

We claim that problem (7) is in the form of problem (1). Indeed, assume that $m = n$, then the problem can be represented as

$$\min_{\mathbf{x}\in\mathbb{R}^{d\times p}} \ \Phi(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} f_i\left(\frac{1}{m}\sum_{j=1}^{m} g_j(\mathbf{x})\right) + r(\mathbf{x})$$

where $f_i : \mathbb{R}^{d+1} \to \mathbb{R}$ being a quadratic function and $g_j : \mathbb{R}^d \to \mathbb{R}^{d+1}$ being a linear function, i.e.,

$$\begin{aligned} f_i(\mathbf{x}, y) = &\quad \left(\langle \mathbf{r}_i, \mathbf{x}\rangle + y\right)^2 - \langle \mathbf{r}_i, \mathbf{x}\rangle, && \mathbf{x} \in \mathbb{R}^d,\ y \in \mathbb{R}, \\ g_j(\mathbf{x}) = &\quad \begin{pmatrix} \mathbf{x} \\ -\langle \mathbf{r}_j, \mathbf{x}\rangle \end{pmatrix}, && \mathbf{x} \in \mathbb{R}^d, \end{aligned}$$

and $r : \mathbb{R}^d \to \mathbb{R}$ is a nonsmooth function, i.e., $r(\mathbf{x}) = \lambda\left\|\mathbf{x}\right\|_1$. We observe that problem (7) satisfies Assumption 3.1-Assumption 3.5. We also say that problem (7) satisfies Assumption 3.6 since $\|x\|$ is bounded. Otherwise, if $\|x\| \to +\infty$, then $r(\mathbf{x}) \to +\infty$ as $\|x\| \to +\infty$ which results in a contradiction.

In the experiment, we set $n \in \{500, 1000, 2000, 5000\}$ and $d = 100$ together with different regularization parameter $\lambda \in \{10^{-4}, 10^{-5}, 10^{-6}\}$. The reward vectors are generated by the following procedure,

1. Set a Gaussian distribution on $\mathbb{R}^d$ with zero mean and **positive semi-definite** co-variance matrix. More specifically, we let $\Sigma = L^\top L$ where $L \in \mathbb{R}^{d\times r}$ and $r = 30$. Each element of L is drawn from the normal distribution.

2. Draw a random vector from this Gaussian distribution and set $\mathbf{r}_i$ as its component-wise absolute value to make sure problem (7) has a solution.

Remark 4.1. *We highlight that problem* (7) *is* ***neither smooth nor strongly convex*** *since the co-variance matrix is only assumed to be semi-definite positive. Therefore, the method in [18] is excluded from our experiment. We solve problem* (7) *by adopting our SCVRG and Accelerated SCVRG, together with the baseline methods, e.g., AGD, SCGD and Accelerated SCGD. We use the implementation of SCGD and Accelerated SCGD provided by the authors.*

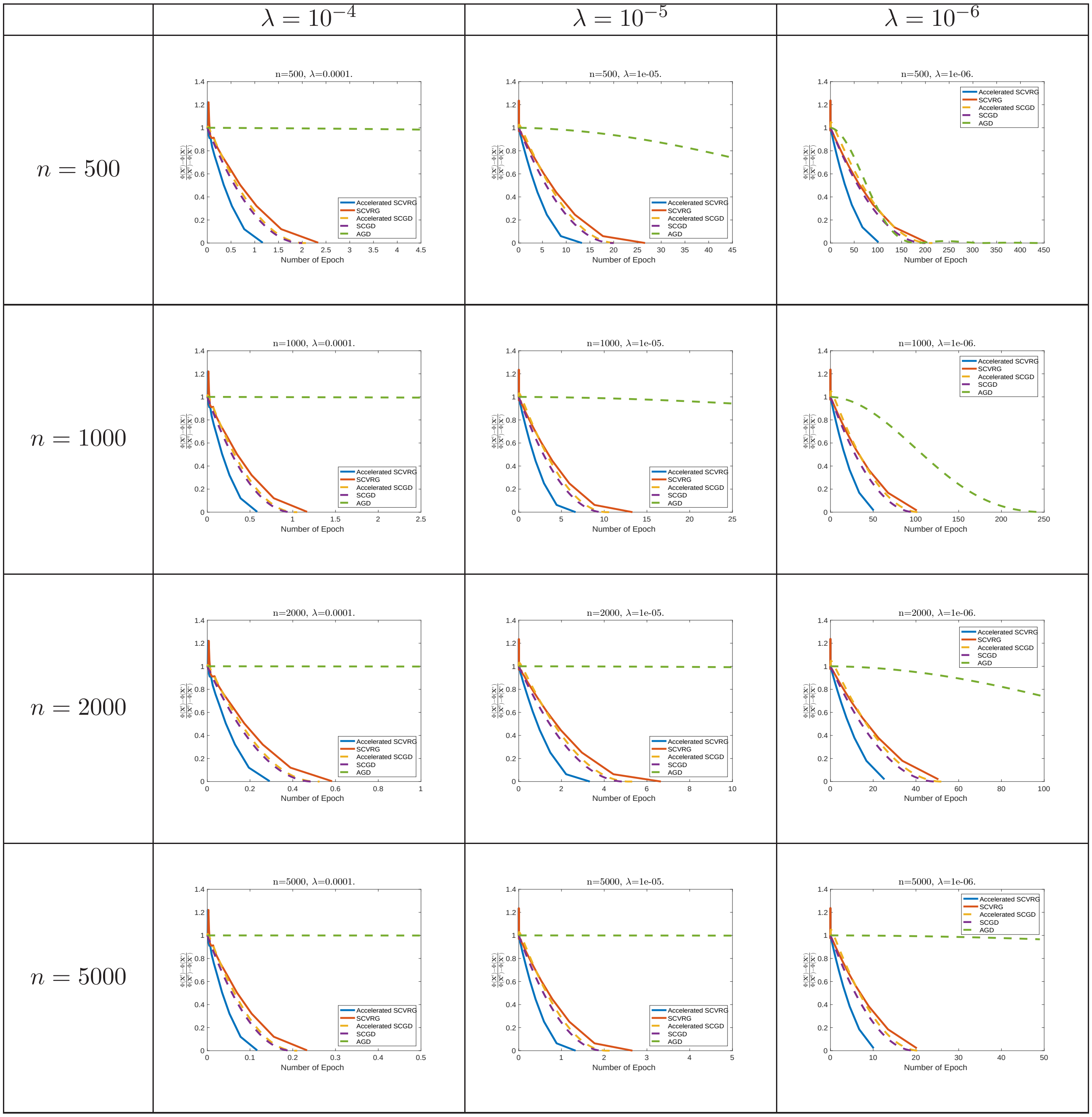

Figure 1: The empirical rate of estimating sparse mean-variance problem on synthetic data ($d = 100$), where $\mathbf{x}^t$ is the output at the t-th epoch, and $\mathbf{x}^*$ is the optimal solution. SCVRG is Algorithm 1 and Accelerated SCVRG is Algorithm 2. SCGD is Algorithm 1 in [33] and Accelerated SCGD is Algorithm 1 in [34]. Accelerated SCVRG performs consistently the best, followed by Accelerated SCGD and SCVRG, outperforms SCGD and AGD.

4.2 Experimental Results

In Figure 1, we reported our results where x-axis shows the number of epochs and y-axis shows the residue. Because the underlying true value $\mathbf{x}^*$ is unknown, we run $K = 5000$ iterations of AGD and take the output as an optimal $\mathbf{x}^*$ instead.

Firstly, we observe that Accelerated SCVRG performs consistently the best, outperforming SCVRG and all other competing methods including Accelerated SCGD, SCGD and AGD. This agrees with our theoretical results presented in Table 1 that Accelerated SCVRG is the best in terms of the IFO complexity. Accelerated SCVRG also performs very robust thanks to the use of variance reduced Jacobian matrix. Secondly, we observe that SCVRG is competitive with Accelerated SCGD in terms of convergence rate, implying that the IFO complexity of SCVRG can be further improved. Finally, we find that AGD is competitive with Accelerated SCVRG in terms of convergence rate when N is relatively small. This makes sense since AGD turns out to be the best method for small/medium-scale problems, as illustrated in Table 1.

5 Conclusion

We develop a unified framework to analyze SCVRG for convex composition optimization and establish the IFO complexity under reasonable assumptions. Our framework provides the new IFO complexity benchmarks that improve the best-known results prior to this paper. The extensive experiments conducted on sparse mean-variance optimization problem demonstrate that our method outperforms other competing methods. For future direction, it still remains open if the IFO complexity can be further improved to match that of stochastic variance reduced gradient method for stochastic optimization.

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A Proof Outline

In this section, we list the assumptions in our paper and some major steps to give a whole picture of the proof.

Proof Outlines:

1. We provide two basic lemmas, which concerns with convex objective and common variance; see Lemmas B.1 and B.3.
2. We bound the term of $\mathbb{E}\left[\left\|\mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]$ using the term $\|\tilde{\mathbf{x}}^s - \mathbf{x}^*\|^2$, $\left\|\mathbf{x}_t^{s+1} - \mathbf{x}^*\right\|^2$ and the size of $\mathcal{A}_t$ and $\mathcal{B}_t$, i.e., A and B; see Lemmas B.3 and B.4.
3. We bound the term of $\mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^{s+1})\right] - \Phi(\mathbf{x}^*)$ using the term $\Phi(\mathbf{x}^0) - \Phi(\mathbf{x}^*)$, $\left\|\mathbf{x}^0 - \mathbf{x}^*\right\|^2$ and the parameters η and k_0; see Lemmas C.1, C.2, D.1 and D.2.
4. We provide the explicit IFO complexity of Algorithm 1 and Algorithm 2 in terms of $m+n$, L_ϕ, L_f and L_g; see Theorem C.3 and D.3.

Assumption A.1. *The objective f and r are both convex, i.e.,*

$$\begin{aligned} f(\mathbf{x}) - f(\mathbf{y}) - \langle \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle &\geq 0, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d, \\ r(\mathbf{x}) - r(\mathbf{y}) - \langle \xi, \mathbf{x} - \mathbf{y} \rangle &\geq 0, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d. \end{aligned}$$

where $\xi \in \partial r(\mathbf{y})$ is a subgradient of r.

Assumption A.2. *The proximal mapping of the objective r, given by the following problem,*

$$\operatorname*{argmin}_{\mathbf{x} \in \mathbb{R}^d} \left\{ \langle \mathbf{g}, \mathbf{x} \rangle + \frac{1}{2\eta} \|\mathbf{x} - \mathbf{y}\|^2 + r(\mathbf{x}) \right\}.$$

is easily computed for any $\mathbf{g}, \mathbf{y} \in \mathbb{R}^d$ and $\eta > 0$.

Assumption A.3. *For $i \in [n]$ and $j \in [m]$, there exist some constants $0 < L_f, L_g, L_\phi < \infty$ such that*

$$\begin{aligned} \|\nabla f_i(\mathbf{x}) - \nabla f_i(\mathbf{y})\| &\leq L_f \|\mathbf{x} - \mathbf{y}\|, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^l, \\ \|\partial g_j(\mathbf{x}) - \partial g_j(\mathbf{y})\| &\leq L_g \|\mathbf{x} - \mathbf{y}\|, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d, \\ \left\|[\partial g_j(\mathbf{x})]^\top \nabla f_i(g(\mathbf{x})) - [\partial g_j(\mathbf{y})]^\top \nabla f_i(g(\mathbf{y}))\right\| &\leq L_\phi \|\mathbf{x} - \mathbf{y}\|, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d. \end{aligned}$$

The above assumption implies that ∇f is Lipschitz continuous with a constant $L_\phi > 0$, i.e.,

$$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\| \leq L_\phi \|\mathbf{x} - \mathbf{y}\|.$$

Assumption A.4. *For $j \in [m]$, there exists a constant $0 < B_g < \infty$ such that*

$$\|\partial g_j(\mathbf{x})\| \leq B_g, \quad \mathbf{x} \in \mathbb{R}^d.$$

Assumption A.5. *For $i \in [n]$, the component objective $f_i(g)$ is convex, i.e.,*

$$f_i(g(\mathbf{x})) - f_i(g(\mathbf{y})) - \left\langle [\partial g(\mathbf{y})]^\top \cdot \nabla f_i(g(\mathbf{y})), \mathbf{x} - \mathbf{y} \right\rangle \geq 0, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d,$$

Assumption A.6. *For $i \in [n]$, there exists a constant $0 < B_f < \infty$ such that*

$$\|\nabla f_i(\mathbf{x})\| \leq B_f, \quad \mathbf{x} \in \mathbb{R}^l.$$

B Proof of Technical Lemmas

Lemma B.1. *For any* $\mathbf{x} \in \mathbb{R}^d$*, we have*

$$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{x}^*)\|^2 \leq 2L_\phi \left[\Phi(\mathbf{x}) - \Phi(\mathbf{x}^*)\right],$$

where $\mathbf{x}^*$ *is one optimal solution.*

Proof. Given an optimal solution $\mathbf{x}^*$, we define $h(\mathbf{x})$ as

$$h(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{x}^*) - \langle \nabla f(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^* \rangle,$$

and obtain that ∇h s Lipschitz continuous with a constant $L_\phi > 0$ since $\nabla h(\mathbf{x}) = \nabla f(\mathbf{x}) - \nabla f(\mathbf{x}^*)$. Furthermore, $\mathbf{x}^*$ is an optimal solution of h since $\nabla h(\mathbf{x}^*) = 0$ and h is convex. Therefore, we have

$$h(\mathbf{x}^*) \leq h\left(\mathbf{x} - \frac{1}{L_\phi}\nabla h(\mathbf{x})\right) \leq h(\mathbf{x}) - \left\langle \nabla h(\mathbf{x}), \frac{1}{L_\phi}\nabla h(\mathbf{x})\right\rangle + \frac{L_\phi}{2}\left\|\frac{1}{L_\phi}\nabla h(\mathbf{x})\right\|^2 = h(\mathbf{x}) - \frac{1}{2L_\phi}\|\nabla h(\mathbf{x})\|^2.$$

This implies that

$$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{x}^*)\|^2 = \|\nabla h(\mathbf{x})\|^2 \leq 2L_\phi\left[h(\mathbf{x}) - h(\mathbf{x}^*)\right] = 2L_\phi\left[f(\mathbf{x}) - f(\mathbf{x}^*) - \langle \nabla f(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^*\rangle\right].$$

In addition, we have

$$-\langle \nabla f(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^*\rangle = \langle \xi, \mathbf{x} - \mathbf{x}^*\rangle \leq r(\mathbf{x}) - r(\mathbf{x}^*),$$

where the first equality since $\mathbf{x}^*$ is one optimal solution and the first inequality comes from Assumption 3.1. Therefore, we conclude that

$$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{x}^*)\|^2 \leq 2L_\phi\left[f(\mathbf{x}) - f(\mathbf{x}^*) + r(\mathbf{x}) - r(\mathbf{x}^*)\right] = 2L_\phi\left[\Phi(\mathbf{x}) - \Phi(\mathbf{x}^*)\right].$$

This completes the proof. □

Lemma B.2. *In both Algorithm 1 and 2, the following statement holds true,*

$$\left\|\mathbb{E}\left[\left(\left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \nabla f(\mathbf{x}_t^{s+1})\right) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right\|^2 \leq \frac{2B_g^4 L_f^2}{A}\left[\|\tilde{\mathbf{x}}^s - \mathbf{x}^*\|^2 + \left\|\mathbf{x}_t^{s+1} - \mathbf{x}^*\right\|^2\right]. \quad (8)$$

where $\mathbf{x}^*$ *is one optimal solution.*

Proof. We have

$$\begin{aligned}
& \left\|\mathbb{E}\left[\left(\left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \nabla f(\mathbf{x}_t^{s+1})\right) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right\|^2 \\
= & \left\|\mathbb{E}\left[\left(\left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(g(\mathbf{x}_t^{s+1}))\right) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right\|^2 \\
\leq & \ \mathbb{E}\left[\left\|\left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(g(\mathbf{x}_t^{s+1}))\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
\leq & \ \mathbb{E}\left[\left\|\partial g_{j_t}(\mathbf{x}_t^{s+1})\right\|^2 \cdot \left\|\nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \nabla f_{i_t}(g(\mathbf{x}_t^{s+1}))\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
\leq & \ B_g^2 L_f^2 \cdot \mathbb{E}\left[\left\|\mathbf{g}_t^{s+1} - g(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right],
\end{aligned}$$

where the first inequality holds due to Jensen's inequality and the last inequality comes from Assumption 3.3 and Assumption 3.4. Then it suffices to show that

$$\mathbb{E}\left[\left\|\mathbf{g}_t^{s+1} - g(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \leq \frac{2B_g^2}{A}\left[\|\tilde{\mathbf{x}}^s - \mathbf{x}^*\|^2 + \left\|\mathbf{x}_t^{s+1} - \mathbf{x}^*\right\|^2\right].$$

Indeed, we have

$$\begin{aligned}
& \mathbb{E}\left[\left\|\mathbf{g}_t^{s+1} - g(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
= \; & \mathbb{E}\left[\left\|\frac{1}{A}\sum_{j\in\mathcal{A}_t} g_j(\mathbf{x}_t^{s+1}) - \frac{1}{A}\sum_{j\in\mathcal{A}_t} g_j(\tilde{\mathbf{x}}^s) + \tilde{\mathbf{g}}^{s+1} - g(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
= \; & \frac{1}{A^2}\cdot\mathbb{E}\left[\left\|\sum_{j\in\mathcal{A}_t}\left(g_j(\mathbf{x}_t^{s+1}) - g_j(\tilde{\mathbf{x}}^s) - g(\mathbf{x}_t^{s+1}) + \tilde{\mathbf{g}}^{s+1}\right)\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
= \; & \frac{1}{A^2}\cdot\sum_{j\in\mathcal{A}_t}\mathbb{E}\left[\left\|g_j(\mathbf{x}_t^{s+1}) - g_j(\tilde{\mathbf{x}}^s) - g(\mathbf{x}_t^{s+1}) + \tilde{\mathbf{g}}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
= \; & \frac{1}{A^2}\cdot\sum_{j\in\mathcal{A}_t}\mathbb{E}\left[\left\|g_j(\mathbf{x}_t^{s+1}) - g_j(\tilde{\mathbf{x}}^s) - \mathbb{E}\left[g_j(\mathbf{x}_t^{s+1}) - g_j(\tilde{\mathbf{x}}^s)\right]\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
\leq \; & \frac{1}{A^2}\cdot\sum_{j\in\mathcal{A}_t}\mathbb{E}\left[\left\|g_j(\mathbf{x}_t^{s+1}) - g_j(\tilde{\mathbf{x}}^s)\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
\leq \; & \frac{B_g^2}{A^2}\cdot\sum_{j\in\mathcal{A}_t}\left\|\mathbf{x}_t^{s+1} - \tilde{\mathbf{x}}^s\right\|^2 \\
= \; & \frac{B_g^2}{A}\cdot\left\|\mathbf{x}_t^{s+1} - \tilde{\mathbf{x}}^s\right\|^2 \\
\leq \; & \frac{2B_g^2}{A}\left[\left\|\mathbf{x}_t^{s+1} - \mathbf{x}^*\right\|^2 + \|\tilde{\mathbf{x}}^s - \mathbf{x}^*\|^2\right],
\end{aligned} \tag{9}$$

where the third equality holds true since the indices in $\mathcal{A}_t$ are drawn independently, the first inequality holds true since $\mathbb{E}\|\xi - \mathbb{E}[\xi]\|^2 \leq \mathbb{E}\|\xi\|^2$, and the second inequality comes from Assumption 3.4. This completes the proof. □

Lemma B.3. *In Algorithm 1, the gap between $\nabla f(\mathbf{x}_t^{s+1})$ and its approximation $\mathbf{f}_t^{s+1}$ is upper bounded by $\|\tilde{\mathbf{x}}^s - \mathbf{x}^*\|^2$ and $\left\|\mathbf{x}_t^{s+1} - \mathbf{x}^*\right\|^2$ in terms of conditional expectation, given by*

$$\begin{aligned}
& \mathbb{E}\left[\left\|\mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
\leq \; & 6L_\phi\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*)\right] + 6L_\phi\left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right] + \left(\frac{12B_g^4L_f^2}{A} + 12L_\phi^2\right)\cdot\left[\left\|\mathbf{x}_t^{s+1} - \mathbf{x}^*\right\|^2 + \|\tilde{\mathbf{x}}^s - \mathbf{x}^*\|^2\right].
\end{aligned}$$

Proof. We have

$$
\begin{aligned}
& \mathbb{E}\left[\left\|\mathbf{f}_t^{s+1}-\nabla f(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1},\tilde{\mathbf{x}}^s\right] \\
= & \mathbb{E}\left[\left\|\left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \left[\partial g_{j_t}(\tilde{\mathbf{x}}^s)\right]^\top \nabla f_{i_t}(\tilde{\mathbf{g}}^{s+1}) + \tilde{\mathbf{f}}^{s+1} - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1},\tilde{\mathbf{x}}^s\right] \\
\leq & 3\left\|\nabla f(\mathbf{x}_t^{s+1}) - \nabla f(\mathbf{x}^*)\right\|^2 + 3\left\|\nabla f(\tilde{\mathbf{x}}^s) - \nabla f(\mathbf{x}^*)\right\|^2 \\
& +3\mathbb{E}\left[\left\|\left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \left[\partial g_{j_t}(\tilde{\mathbf{x}}^s)\right]^\top \nabla f_{i_t}(\tilde{\mathbf{g}}^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1},\tilde{\mathbf{x}}^s\right].
\end{aligned}
$$

Applying Lemma B.1 yields that

$$
\left\|\nabla f(\mathbf{x}_t^{s+1}) - \nabla f(\mathbf{x}^*)\right\|^2 + \left\|\nabla f(\tilde{\mathbf{x}}^s) - \nabla f(\mathbf{x}^*)\right\|^2 \leq 2L_\phi\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*)\right] + 2L_\phi\left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right]. \quad (10)
$$

Furthermore, we have

$$
\begin{aligned}
& \mathbb{E}\left[\left\|\left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \left[\partial g_{j_t}(\tilde{\mathbf{x}}^s)\right]^\top \nabla f_{i_t}(\tilde{\mathbf{g}}^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1},\tilde{\mathbf{x}}^s\right] \\
\leq & 2\mathbb{E}\left[\left\|\left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(g(\mathbf{x}_t^{s+1}))\right\|^2 \mid \mathbf{x}_t^{s+1},\tilde{\mathbf{x}}^s\right] \\
& +2\mathbb{E}\left[\left\|\left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(g(\mathbf{x}_t^{s+1})) - \left[\partial g_{j_t}(\tilde{\mathbf{x}}^s)\right]^\top \nabla f_{i_t}(\tilde{\mathbf{g}}^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1},\tilde{\mathbf{x}}^s\right] \\
\leq & 2\mathbb{E}\left[\left(\left\|\partial g_{j_t}(\mathbf{x}_t^{s+1})\right\|^2 \cdot \left\|\nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \nabla f_{i_t}(g(\mathbf{x}_t^{s+1}))\right\|^2\right) \mid \mathbf{x}_t^{s+1},\tilde{\mathbf{x}}^s\right] \\
& +2\mathbb{E}\left[\left\|\left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(g(\mathbf{x}_t^{s+1})) - \left[\partial g_{j_t}(\tilde{\mathbf{x}}^s)\right]^\top \nabla f_{i_t}(\tilde{\mathbf{g}}^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1},\tilde{\mathbf{x}}^s\right] \\
\leq & 2B_g^2 L_f^2 \cdot \mathbb{E}\left[\left\|\mathbf{g}_t^{s+1} - g(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1},\tilde{\mathbf{x}}^s\right] + 2L_\phi^2\left\|\mathbf{x}_t^{s+1} - \tilde{\mathbf{x}}^s\right\|^2 \\
\leq & \frac{4B_g^4 L_f^2}{A} \cdot \left[\left\|\mathbf{x}_t^{s+1} - \mathbf{x}^*\right\|^2 + \left\|\tilde{\mathbf{x}}^s - \mathbf{x}^*\right\|^2\right] + 4L_\phi^2\left[\left\|\mathbf{x}_t^{s+1} - \mathbf{x}^*\right\|^2 + \left\|\tilde{\mathbf{x}}^s - \mathbf{x}^*\right\|^2\right], \quad (11)
\end{aligned}
$$

where the first inequality comes from the triangle inequality, the third inequality comes from Assumption 3.3 and Assumption 3.4 and the last inequality comes from (9) and the triangle inequality.

Combining (10) and (11) yields that

$$
\begin{aligned}
& \mathbb{E}\left[\left\|\mathbf{f}_t^{s+1}-\nabla f(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1},\tilde{\mathbf{x}}^s\right] \\
\leq & 6L_\phi\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*)\right] + 6L_\phi\left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right] + \left(\frac{12B_g^4 L_f^2}{A} + 12L_\phi^2\right) \cdot \left[\left\|\mathbf{x}_t^{s+1} - \mathbf{x}^*\right\|^2 + \left\|\tilde{\mathbf{x}}^s - \mathbf{x}^*\right\|^2\right]
\end{aligned}
$$

This completes the proof. □

Lemma B.4. *In Algorithm 2, the gap between* $\nabla f(\mathbf{x}_t^{s+1})$ *and its approximation* $\mathbf{f}_t^{s+1}$ *is upper bounded by* $\|\tilde{\mathbf{x}}^s - \mathbf{x}^*\|^2$ *and* $\left\|\mathbf{x}_t^{s+1} - \mathbf{x}^*\right\|^2$ *in terms of conditional expectation, given by*

$$
\begin{aligned}
& \mathbb{E}\left[\left\|\mathbf{f}_t^{s+1}-\nabla f(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1},\tilde{\mathbf{x}}^s\right] \\
\leq & 30L_\phi\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*)\right] + 30L_\phi\left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right] + \left(\frac{24B_g^4 L_f^2}{A} + \frac{24B_f^2 L_g^2}{B}\right) \cdot \left[\left\|\mathbf{x}_t^{s+1} - \mathbf{x}^*\right\|^2 + \left\|\tilde{\mathbf{x}}^s - \mathbf{x}^*\right\|^2\right].
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
& \mathbb{E}\left[\left\|\mathbf{f}_t^{s+1}-\nabla f(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
= \ & \mathbb{E}\left[\left\|\left[\mathbf{G}_t^{s+1}\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \left[\tilde{\mathbf{G}}^{s+1}\right]^\top \nabla f_{i_t}(\tilde{\mathbf{g}}^{s+1}) + \tilde{\mathbf{f}}^{s+1} - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
\leq \ & 3\left\|\nabla f(\mathbf{x}_t^{s+1}) - \nabla f(\mathbf{x}^*)\right\|^2 + 3\left\|\nabla f(\tilde{\mathbf{x}}^s) - \nabla f(\mathbf{x}^*)\right\|^2 \\
& + 3\mathbb{E}\left[\left\|\left[\mathbf{G}_t^{s+1}\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \left[\tilde{\mathbf{G}}^{s+1}\right]^\top \nabla f_{i_t}(\tilde{\mathbf{g}}^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right].
\end{aligned}
$$

Similarly, we obtain that

$$
\left\|\nabla f(\mathbf{x}_t^{s+1}) - \nabla f(\mathbf{x}^*)\right\|^2 + \left\|\nabla f(\tilde{\mathbf{x}}^s) - \nabla f(\mathbf{x}^*)\right\|^2 \leq 2L_\phi\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*)\right] + 2L_\phi\left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right]. \quad (12)
$$

Furthermore, we have

$$
\begin{aligned}
& \mathbb{E}\left[\left\|\left[\mathbf{G}_t^{s+1}\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \left[\tilde{\mathbf{G}}^{s+1}\right]^\top \nabla f_{i_t}(\tilde{\mathbf{g}}^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
\leq \ & 2\mathbb{E}\left[\left\|\left[\mathbf{G}_t^{s+1}\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \left[\partial g(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(g(\mathbf{x}_t^{s+1}))\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
& + 2\mathbb{E}\left[\left\|\left[\partial g(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(g(\mathbf{x}_t^{s+1})) - \left[\tilde{\mathbf{G}}^{s+1}\right]^\top \nabla f_{i_t}(\tilde{\mathbf{g}}^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
\leq \ & 2\mathbb{E}\left[\left\|\left[\mathbf{G}_t^{s+1}\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \left[\partial g(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(g(\mathbf{x}_t^{s+1}))\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
& + 4\mathbb{E}\left[\left\|\left[\partial g(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(g(\mathbf{x}_t^{s+1})) - \left[\partial g(\mathbf{x}^*)\right]^\top \nabla f_{i_t}(g(\mathbf{x}^*))\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
& + 4\mathbb{E}\left[\left\|\left[\tilde{\mathbf{G}}^{s+1}\right]^\top \nabla f_{i_t}(\tilde{\mathbf{g}}^{s+1}) - \left[\partial g(\mathbf{x}^*)\right]^\top \nabla f_{i_t}(g(\mathbf{x}^*))\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right], \qquad (13)
\end{aligned}
$$

where the first and second inequalities comes from the triangle inequality. In addition, it follows from Assumption 3.3, Assumption 3.5 and the similar derivation in Lemma B.3 that

$$
\mathbb{E}\left[\left\|\left[\partial g(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(g(\mathbf{x}_t^{s+1})) - \left[\partial g(\mathbf{x}^*)\right]^\top \nabla f_{i_t}(g(\mathbf{x}^*))\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \leq 2L_\phi\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*)\right], \quad (14)
$$

$$
\mathbb{E}\left[\left\|\left[\tilde{\mathbf{G}}^{s+1}\right]^\top \nabla f_{i_t}(\tilde{\mathbf{g}}^{s+1}) - \left[\partial g(\mathbf{x}^*)\right]^\top \nabla f_{i_t}(g(\mathbf{x}^*))\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \leq 2L_\phi\left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right]. \qquad (15)
$$

Combining (12)-(15) yields that

$$
\begin{aligned}
\mathbb{E}\left[\left\|\mathbf{f}_t^{s+1}-\nabla f(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \leq \ & 30L_\phi\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*)\right] + 30L_\phi\left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right] \\
& + 6\mathbb{E}\left[\left\|\left[\mathbf{G}_t^{s+1}\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \left[\partial g(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(g(\mathbf{x}_t^{s+1}))\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right].
\end{aligned}
$$

The remaining step is to bound the term $\mathbb{E}\left[\left\|\left[\mathbf{G}_t^{s+1}\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \left[\partial g(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(g(\mathbf{x}_t^{s+1}))\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]$. Indeed, we have

$$
\begin{aligned}
& \mathbb{E}\left[\left\|\left[\mathbf{G}_t^{s+1}\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \left[\partial g(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(g(\mathbf{x}_t^{s+1}))\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
\leq\ & 2\mathbb{E}\left[\left\|\left[\mathbf{G}_t^{s+1}\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \left[\partial g(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
& +2\mathbb{E}\left[\left\|\left[\partial g(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \left[\partial g(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(g(\mathbf{x}_t^{s+1}))\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
\leq\ & 2\mathbb{E}\left[\left\|\mathbf{G}_t^{s+1} - \partial g(\mathbf{x}_t^{s+1})\right\|^2 \cdot \left\|\nabla f_{i_t}(\mathbf{g}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
& +2\mathbb{E}\left[\left\|\partial g(\mathbf{x}_t^{s+1})\right\|^2 \cdot \left\|\nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \nabla f_{i_t}(g(\mathbf{x}_t^{s+1}))\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
\leq\ & 2B_f^2 \cdot \mathbb{E}\left[\left\|\mathbf{G}_t^{s+1} - \partial g(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + 2B_g^2 \cdot \mathbb{E}\left[\left\|\nabla f_{i_t}(\mathbf{g}_t^{s+1}) - \nabla f_{i_t}(g(\mathbf{x}_t^{s+1}))\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
\leq\ & 2B_f^2 \cdot \mathbb{E}\left[\left\|\mathbf{G}_t^{s+1} - \partial g(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + 2B_g^2 L_f^2 \cdot \mathbb{E}\left[\left\|\mathbf{g}_t^{s+1} - g(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
\leq\ & 2B_f^2 \cdot \mathbb{E}\left[\left\|\mathbf{G}_t^{s+1} - \partial g(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{4B_g^4 L_f^2}{A}\left[\left\|\mathbf{x}_t^{s+1} - \mathbf{x}^*\right\|^2 + \left\|\tilde{\mathbf{x}}^s - \mathbf{x}^*\right\|^2\right],
\end{aligned}
$$

where the first inequality comes from the triangle inequality, the third inequality comes from Assumption 3.4 and Assumption 3.6, the fourth inequality comes from Assumption 3.3 and the last inequality comes from (9). Then we try to bound the term $\mathbb{E}\left[\left\|\mathbf{G}_t^{s+1} - \partial g(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]$ as follows,

$$
\begin{aligned}
& \mathbb{E}\left[\left\|\mathbf{G}_t^{s+1} - \partial g(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
= \ & \mathbb{E}\left[\left\|\frac{1}{B}\sum_{j\in\mathcal{B}_t} \partial g_j(\mathbf{x}_t^{s+1}) - \frac{1}{B}\sum_{j\in\mathcal{B}_t} \partial g_j(\tilde{\mathbf{x}}^s) + \tilde{\mathbf{G}}^{s+1} - \partial g(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
= \ & \frac{1}{B^2}\cdot \mathbb{E}\left[\left\|\sum_{j\in\mathcal{B}_t}\left(\partial g_j(\mathbf{x}_t^{s+1}) - \partial g_j(\tilde{\mathbf{x}}^s) - \partial g(\mathbf{x}_t^{s+1}) + \tilde{\mathbf{G}}^{s+1}\right)\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
= \ & \frac{1}{B^2}\cdot \sum_{j\in\mathcal{B}_t} \mathbb{E}\left[\left\|\partial g_j(\mathbf{x}_t^{s+1}) - \partial g_j(\tilde{\mathbf{x}}^s) - \partial g(\mathbf{x}_t^{s+1}) + \tilde{\mathbf{G}}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
= \ & \frac{1}{B^2}\cdot \sum_{j\in\mathcal{B}_t} \mathbb{E}\left[\left\|\partial g_j(\mathbf{x}_t^{s+1}) - \partial g_j(\tilde{\mathbf{x}}^s) - \mathbb{E}\left[\partial g_j(\mathbf{x}_t^{s+1}) - \partial g_j(\tilde{\mathbf{x}}^s)\right]\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
\leq \ & \frac{1}{B^2}\cdot \sum_{j\in\mathcal{B}_t} \mathbb{E}\left[\left\|\partial g_j(\mathbf{x}_t^{s+1}) - \partial g_j(\tilde{\mathbf{x}}^s)\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
\leq \ & \frac{L_g^2}{B^2}\cdot \sum_{j\in\mathcal{B}_t} \left\|\mathbf{x}_t^{s+1} - \tilde{\mathbf{x}}^s\right\|^2 \\
= \ & \frac{L_g^2}{B}\cdot \left\|\mathbf{x}_t^{s+1} - \tilde{\mathbf{x}}^s\right\|^2 \\
\leq \ & \frac{2L_g^2}{B}\left[\left\|\mathbf{x}_t^{s+1} - \mathbf{x}^*\right\|^2 + \left\|\tilde{\mathbf{x}}^s - \mathbf{x}^*\right\|^2\right],
\end{aligned}
$$

where the third equality holds true since the indices in $\mathcal{B}_t$ are drawn independently, the first inequality holds true since $\mathbb{E}\left\|\xi - \mathbb{E}\left[\xi\right]\right\|^2 \leq \mathbb{E}\left\|\xi\right\|^2$, and the second inequality comes from Assumption 3.3.

Therefore, we conclude that

$$
\begin{aligned}
& \mathbb{E}\left[\left\|\mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
\leq \ & 30L_\phi\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*)\right] + 30L_\phi\left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right] + \left(\frac{24B_g^4L_f^2}{A} + \frac{24B_f^2L_g^2}{B}\right)\cdot\left[\left\|\mathbf{x}_t^{s+1} - \mathbf{x}^*\right\|^2 + \left\|\tilde{\mathbf{x}}^s - \mathbf{x}^*\right\|^2\right].
\end{aligned}
$$

This completes the proof. □

C Proof of Theorem 3.7

Lemma C.1. *In Algorithm 1, for any $\mathbf{x} \in \mathbb{R}^d$, we have*

$$
\begin{aligned}
0 \quad \leq \quad & \Phi(\mathbf{x}) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{\eta L_\phi}{2}\left\|\mathbf{x} - \mathbf{x}_t^{s+1}\right\|^2 + \frac{\eta}{2(1-\eta L_\phi)}\mathbb{E}\left[\left\|\mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
& + \frac{1}{2\eta L_\phi}\left\|\mathbb{E}\left[\left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \\
& + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x} - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x} - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right].
\end{aligned}
$$

Proof. For any $\mathbf{x} \in \mathbb{R}^d$, it follows from the update of the main iterate $\mathbf{x}_{t+1}^{s+1}$ that

$$
r(\mathbf{x}) - r(\mathbf{x}_{t+1}^{s+1}) + \left\langle \mathbf{x} - \mathbf{x}_{t+1}^{s+1}, \mathbf{f}_t^{s+1} + \frac{1}{\eta_{t+1}^{s+1}}\left(\mathbf{x}_{t+1}^{s+1} - \mathbf{x}_t^{s+1}\right)\right\rangle \geq 0.
$$

Equivalently, we have

$$
\begin{aligned}
0 \quad \leq \quad & r(\mathbf{x}) - r(\mathbf{x}_{t+1}^{s+1}) + \left\langle \mathbf{x} - \mathbf{x}_{t+1}^{s+1}, \mathbf{f}_t^{s+1}\right\rangle + \frac{1}{\eta_{t+1}^{s+1}} \cdot \left\langle \mathbf{x} - \mathbf{x}_{t+1}^{s+1}, \mathbf{x}_{t+1}^{s+1} - \mathbf{x}_t^{s+1}\right\rangle \\
= \quad & r(\mathbf{x}) - r(\mathbf{x}_{t+1}^{s+1}) + \left\langle \mathbf{x} - \mathbf{x}_t^{s+1}, \mathbf{f}_t^{s+1}\right\rangle + \left\langle \mathbf{x}_t^{s+1} - \mathbf{x}_{t+1}^{s+1}, \mathbf{f}_t^{s+1}\right\rangle \\
& + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x} - \mathbf{x}_t^{s+1}\right\|^2 - \left\|\mathbf{x} - \mathbf{x}_{t+1}^{s+1}\right\|^2 - \left\|\mathbf{x}_{t+1}^{s+1} - \mathbf{x}_t^{s+1}\right\|^2\right] \\
= \quad & r(\mathbf{x}) - r(\mathbf{x}_{t+1}^{s+1}) + \left\langle \mathbf{x} - \mathbf{x}_t^{s+1}, \mathbf{f}_t^{s+1}\right\rangle + \left\langle \mathbf{x}_t^{s+1} - \mathbf{x}_{t+1}^{s+1}, \nabla f(\mathbf{x}_t^{s+1})\right\rangle + \left\langle \mathbf{x}_t^{s+1} - \mathbf{x}_{t+1}^{s+1}, \mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1})\right\rangle \\
& + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x} - \mathbf{x}_t^{s+1}\right\|^2 - \left\|\mathbf{x} - \mathbf{x}_{t+1}^{s+1}\right\|^2 - \left\|\mathbf{x}_{t+1}^{s+1} - \mathbf{x}_t^{s+1}\right\|^2\right] \\
\leq \quad & r(\mathbf{x}) - r(\mathbf{x}_{t+1}^{s+1}) + \left\langle \mathbf{x} - \mathbf{x}_t^{s+1}, \mathbf{f}_t^{s+1}\right\rangle + f(\mathbf{x}_t^{s+1}) - f(\mathbf{x}_{t+1}^{s+1}) + \frac{L_\phi}{2}\left\|\mathbf{x}_t^{s+1} - \mathbf{x}_{t+1}^{s+1}\right\|^2 \\
& + \left\langle \mathbf{x}_t^{s+1} - \mathbf{x}_{t+1}^{s+1}, \mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1})\right\rangle + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x} - \mathbf{x}_t^{s+1}\right\|^2 - \left\|\mathbf{x} - \mathbf{x}_{t+1}^{s+1}\right\|^2 - \left\|\mathbf{x}_{t+1}^{s+1} - \mathbf{x}_t^{s+1}\right\|^2\right] \\
\leq \quad & r(\mathbf{x}) - r(\mathbf{x}_{t+1}^{s+1}) + \left\langle \mathbf{x} - \mathbf{x}_t^{s+1}, \mathbf{f}_t^{s+1}\right\rangle + f(\mathbf{x}_t^{s+1}) - f(\mathbf{x}_{t+1}^{s+1}) + \left(\frac{L_\phi}{2} - \frac{1}{2\eta_{t+1}^{s+1}}\right)\left\|\mathbf{x}_t^{s+1} - \mathbf{x}_{t+1}^{s+1}\right\|^2 \\
& + \left(\frac{1}{2\eta_{t+1}^{s+1}} - \frac{L_\phi}{2}\right)\left\|\mathbf{x}_t^{s+1} - \mathbf{x}_{t+1}^{s+1}\right\|^2 + \frac{\eta_{t+1}^{s+1}}{2(1-\eta_{t+1}^{s+1}L_\phi)}\left\|\mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \\
& + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x} - \mathbf{x}_t^{s+1}\right\|^2 - \left\|\mathbf{x} - \mathbf{x}_{t+1}^{s+1}\right\|^2\right] \\
= \quad & r(\mathbf{x}) - r(\mathbf{x}_{t+1}^{s+1}) + \left\langle \mathbf{x} - \mathbf{x}_t^{s+1}, \mathbf{f}_t^{s+1}\right\rangle + f(\mathbf{x}_t^{s+1}) - f(\mathbf{x}_{t+1}^{s+1}) + \frac{\eta_{t+1}^{s+1}}{2(1-\eta_{t+1}^{s+1}L_\phi)}\left\|\mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \\
& + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x} - \mathbf{x}_t^{s+1}\right\|^2 - \left\|\mathbf{x} - \mathbf{x}_{t+1}^{s+1}\right\|^2\right],
\end{aligned}
$$

where the first equality holds true since $\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{2}\left[\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2\right]$, the second inequality comes from Assumption 3.3, and the third inequality comes from the Young inequality.

Taking the conditional expectation of both side on $\mathbf{x}_t^{s+1}$ and $\tilde{\mathbf{x}}^s$ yields that

$$\begin{aligned}
0 \;\leq\;& r(\mathbf{x}) - \mathbb{E}\left[r(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \left\langle \mathbf{x} - \mathbf{x}_t^{s+1}, \mathbb{E}\left[\left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right\rangle + f(\mathbf{x}_t^{s+1}) \\
& - \mathbb{E}\left[f(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{\eta_{t+1}^{s+1}}{2(1-\eta_{t+1}^{s+1}L_\phi)}\mathbb{E}\left[\left\|\mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
& + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x} - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x} - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right] \\
=\;& r(\mathbf{x}) - \mathbb{E}\left[r(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \left\langle \mathbf{x} - \mathbf{x}_t^{s+1}, \mathbb{E}\left[\left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] - \nabla f(\mathbf{x}_t^{s+1})\right\rangle \\
& + \left\langle \mathbf{x} - \mathbf{x}_t^{s+1}, \nabla f(\mathbf{x}_t^{s+1})\right\rangle + f(\mathbf{x}_t^{s+1}) - \mathbb{E}\left[f(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
& + \frac{\eta_{t+1}^{s+1}}{2(1-\eta_{t+1}^{s+1}L_\phi)}\mathbb{E}\left[\left\|\mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x} - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x} - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right] \\
\leq\;& \Phi(\mathbf{x}) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \left\langle \mathbf{x} - \mathbf{x}_t^{s+1}, \mathbb{E}\left[\left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] - \nabla f(\mathbf{x}_t^{s+1})\right\rangle \\
& + \frac{\eta_{t+1}^{s+1}}{2(1-\eta_{t+1}^{s+1}L_\phi)}\mathbb{E}\left[\left\|\mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x} - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x} - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right] \\
\leq\;& \Phi(\mathbf{x}) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{\eta L_\phi}{2}\left\|\mathbf{x} - \mathbf{x}_t^{s+1}\right\|^2 + \frac{\eta_{t+1}^{s+1}}{2(1-\eta_{t+1}^{s+1}L_\phi)}\mathbb{E}\left[\left\|\mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
& + \frac{1}{2\eta L_\phi}\left\|\mathbb{E}\left[\left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \\
& + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x} - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x} - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right] \\
\leq\;& \Phi(\mathbf{x}) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{\eta L_\phi}{2}\left\|\mathbf{x} - \mathbf{x}_t^{s+1}\right\|^2 + \frac{\eta}{2(1-\eta L_\phi)}\mathbb{E}\left[\left\|\mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
& + \frac{1}{2\eta L_\phi}\left\|\mathbb{E}\left[\left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \\
& + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x} - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x} - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right],
\end{aligned}$$

where the second inequality comes from Assumption 3.1, the third inequality comes from the Young inequality with $\eta L_\phi > 0$ and the last inequality holds true since

$$\frac{\eta_{t+1}^{s+1}}{2(1-\eta_{t+1}^{s+1}L_\phi)} = \frac{1}{2(\frac{1}{\eta_{t+1}^{s+1}} - L_\phi)} \leq \frac{1}{2(\frac{1}{\eta} - L_\phi)} = \frac{\eta}{2(1-\eta L_\phi)}.$$

where the inequality comes from the fact that $\eta_{t+1}^{s+1} \in \left[\frac{\eta}{\sqrt{2}}, \eta\right]$. This completes the proof. □

Lemma C.2. *In Algorithm 1, we assume that $k_0 \geq 1$ and the sample size $A > 0$ satisfies*

$$A = \frac{2B_g^4 L_f^2}{\eta^2 L_\phi^2},$$

where $\eta > 0$ satisfies

$$\begin{aligned} \eta &\leq \min\left\{1,\ \frac{1}{10L_\phi}\right\}, \\ \eta &\leq \frac{1}{92\sqrt{2}B_\phi}\cdot\frac{1}{\eta T}, \end{aligned}$$

then we have

$$\mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^{s+1}) - \Phi(\mathbf{x}^*)\right] \leq \frac{2\left[\Phi(\mathbf{x}^0) - \Phi(\mathbf{x}^*)\right]}{2^S} + \frac{12\eta B_\phi\left\|\mathbf{x}^* - \mathbf{x}^0\right\|^2}{2^S} + \frac{\left\|\mathbf{x}^* - \mathbf{x}^0\right\|^2}{2^S\eta\cdot k_0}.$$

where $B_\phi = \max\{L_\phi, L_\phi^2\} > 0$ and $\mathbf{x}^*$ is an optimal solution, i.e.,

$$\mathbf{x}^* = \operatorname*{argmin}_{\mathbf{x}\in\mathbb{R}^d}\ \Phi(\mathbf{x}).$$

Proof. Combining Lemma B.3, Lemma B.3 and Lemma C.1 yields that

$$\begin{aligned} 0 \ \leq\ & \Phi(\mathbf{x}^*) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{\eta L_\phi}{2}\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 + \frac{\eta}{2(1-\eta L_\phi)}\mathbb{E}\left[\left\|\mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\ & + \frac{1}{2\eta L_\phi}\left\|\mathbb{E}\left[\left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \\ & + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x}^* - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right] \\ \leq\ & \Phi(\mathbf{x}^*) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{\eta L_\phi}{2}\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 + \frac{3\eta L_\phi}{1-\eta L_\phi}\left[\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*)\right] + \left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right]\right] \\ & + \frac{6\eta}{1-\eta L_\phi}\cdot\left(\frac{B_g^4 L_f^2}{A} + L_\phi^2\right)\cdot\left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 + \left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2\right] \\ & + \frac{1}{\eta L_\phi}\cdot\frac{B_g^4 L_f^2}{A}\left[\left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2 + \left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2\right] \\ & + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x}^* - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right]. \end{aligned}$$

Plugging the sample size of $\mathcal{A}_t$,

$$A = \frac{2B_g^4 L_f^2}{\eta^2 L_\phi^2}$$

into the above inequality yields that

$$\begin{aligned} 0 \ \leq\ & \Phi(\mathbf{x}^*) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{\eta L_\phi}{2}\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 + \frac{3\eta L_\phi}{1-\eta L_\phi}\left[\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*)\right] + \left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right]\right] \\ & + \frac{6\eta}{1-\eta L_\phi}\cdot\left(\frac{\eta^2 L_\phi^2}{2} + L_\phi^2\right)\cdot\left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 + \left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2\right] + \frac{\eta L_\phi}{2}\left[\left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2 + \left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2\right] \\ & + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x}^* - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right]. \end{aligned}$$

We observe that

$$\frac{3\eta L_\phi}{1-\eta L_\phi} = \frac{3L_\phi}{\frac{1}{\eta} - L_\phi} \leq \frac{3L_\phi}{10L_\phi - L_\phi} = \frac{1}{3}. \tag{16}$$

where the inequality holds true since $0 < \eta \leq \frac{1}{10L_\phi}$ and

$$\frac{6\eta}{1-\eta L_\phi} \cdot \left(\frac{\eta^2 L_\phi^2}{2} + L_\phi^2\right) = \frac{3\eta^3 L_\phi^2}{1-\eta L_\phi} + \frac{6\eta L_\phi^2}{1-\eta L_\phi} \leq \frac{\eta L_\phi}{3} + \frac{20\eta L_\phi^2}{3} \leq 7\eta B_\phi,$$

where $B_\phi = \max\{L_\phi, L_\phi^2\} > 0$ and the inequality comes from (16) and the fact that $0 < \eta \leq 1$ and $1 - \eta L_\phi > \frac{9}{10}$. Therefore, we conclude that

$$\begin{aligned}
0 \quad \leq \quad & \Phi(\mathbf{x}^*) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{\eta L_\phi}{2}\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 + \frac{1}{3}\cdot\left[\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*)\right] + \left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right]\right] \\
& + 7\eta B_\phi \cdot \left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 + \left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2\right] + \frac{\eta L_\phi}{2}\left[\left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2 + \left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2\right] \\
& + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x}^* - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right] \\
\leq \quad & \Phi(\mathbf{x}^*) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{\eta B_\phi}{2}\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 + \frac{1}{3}\cdot\left[\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*)\right] + \left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right]\right] \\
& + \frac{15\eta B_\phi}{2} \cdot \left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 + \left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2\right] + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x}^* - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right] \\
= \quad & \Phi(\mathbf{x}^*) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{1}{3}\cdot\left[\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*)\right] + \left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right]\right] \\
& + \frac{\eta B_\phi}{2} \cdot \left[16\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 + 15\left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2\right] + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x}^* - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right] \\
= \quad & \Phi(\mathbf{x}^*) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{1}{3}\cdot\left[\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*)\right] + \left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right]\right] \\
& + \frac{\eta B_\phi}{2} \cdot \left[15\left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2 - 30\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2\right] + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x}^* - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right] \\
& + 23\eta B_\phi \cdot \left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2,
\end{aligned}$$

where the second inequality since $L_\phi \leq B_\phi$. Furthermore, we observe that

$$\frac{1}{\eta_t^{s+1}} - \frac{1}{\eta_{t+1}^{s+1}} \geq \frac{1}{2\eta\sqrt{T}\sqrt{2T}} = \frac{1}{2\sqrt{2}\eta T} \geq 46\eta B_\phi,$$

where the last inequality comes from the fact that

$$0 < \eta \leq \frac{1}{92\sqrt{2}B_\phi} \cdot \frac{1}{\eta T}.$$

This implies that

$$
\begin{aligned}
& \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x}^* - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right] + 23\eta B_\phi \cdot \left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 \\
\leq & \left(\frac{1}{2\eta_{t+1}^{s+1}} + 23\eta B_\phi\right) \cdot \left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 - \frac{1}{2\eta_{t+1}^{s+1}} \cdot \mathbb{E}\left[\left\|\mathbf{x}^* - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
\leq & \frac{1}{2\eta_t^{s+1}} \cdot \left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 - \frac{1}{2\eta_{t+1}^{s+1}} \cdot \mathbb{E}\left[\left\|\mathbf{x}^* - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right].
\end{aligned}
$$

Therefore, we conclude that

$$
\begin{aligned}
& \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] - \Phi(\mathbf{x}^*) \\
\leq & \frac{1}{3} \cdot \left[\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*)\right] + \left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right]\right] + \frac{\eta B_\phi}{2} \cdot \left[15\left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2 - 30\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2\right] \\
& + \frac{1}{2\eta_t^{s+1}} \cdot \left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 - \frac{1}{2\eta_{t+1}^{s+1}} \cdot \mathbb{E}\left[\left\|\mathbf{x}^* - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right].
\end{aligned}
$$

Taking the expectation of both sides, summing it up over $t = 0, 1, 2, \ldots, k_{s+1} - 1$ and dividing both sides by k_{s+1}, we arrive at

$$
\begin{aligned}
& \mathbb{E}\left[\sum_{t=0}^{k_{s+1}-1} \frac{\Phi(\mathbf{x}_{t+1}^{s+1}) + 15\eta B_\phi \cdot \left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2}{k_{s+1}} - \Phi(\mathbf{x}^*)\right] \\
\leq & \mathbb{E}\left[\frac{1}{3}\left(\sum_{t=0}^{k_{s+1}-1} \frac{\Phi(\mathbf{x}_t^{s+1})}{k_{s+1}} - \Phi(\mathbf{x}^*) + \Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right) + \frac{15\eta B_\phi}{2} \cdot \left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2 + \frac{1}{2\eta_0^{s+1} \cdot k_{s+1}} \cdot \left\|\mathbf{x}^* - \mathbf{x}_0^{s+1}\right\|^2\right. \\
& \left. - \frac{1}{2\eta_{k_{s+1}}^{s+1} \cdot k_{s+1}} \cdot \left\|\mathbf{x}^* - \mathbf{x}_{k_{s+1}}^{s+1}\right\|^2\right].
\end{aligned}
$$

Multiplying both sides of the above inequality by 3 and rearranging yields that

$$
\begin{aligned}
& 2 \cdot \mathbb{E}\left[\sum_{t=0}^{k_{s+1}-1} \frac{\Phi(\mathbf{x}_t^{s+1}) + \frac{45\eta B_\phi}{2}\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2}{k_{s+1}} - \Phi(\mathbf{x}^*)\right] \\
\leq & \mathbb{E}\left[\frac{3\left[\Phi(\mathbf{x}_0^{s+1}) - \Phi(\mathbf{x}^*)\right] - 3\left[\Phi(\mathbf{x}_{k_{s+1}}^{s+1}) - \Phi(\mathbf{x}^*)\right]}{k_{s+1}} + \Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*) + \frac{45\eta B_\phi}{2}\left\|\tilde{\mathbf{x}}^s - \mathbf{x}^*\right\|^2\right. \\
& \left. + \frac{3}{2\eta_0^{s+1} \cdot k_{s+1}} \cdot \left\|\mathbf{x}^* - \mathbf{x}_0^{s+1}\right\|^2 - \frac{3}{2\eta_{k_{s+1}}^{s+1} \cdot k_{s+1}} \cdot \left\|\mathbf{x}^* - \mathbf{x}_{k_{s+1}}^{s+1}\right\|^2\right].
\end{aligned}
$$

According to Assumption 3.1 and the definition of $\tilde{\mathbf{x}}^{s+1}$, we obtain that

$$
\begin{aligned}
\Phi(\tilde{\mathbf{x}}^{s+1}) & \leq \sum_{t=0}^{k_{s+1}-1} \frac{\Phi(\mathbf{x}_t^{s+1})}{k_{s+1}}, \\
\left\|\tilde{\mathbf{x}}^{s+1} - \mathbf{x}^*\right\|^2 & \leq \sum_{t=0}^{k_{s+1}-1} \frac{\left\|\mathbf{x}_t^{s+1} - \mathbf{x}^*\right\|^2}{k_{s+1}},
\end{aligned}
$$

which implies that

$$\mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^{s+1}) - \Phi(\mathbf{x}^*) + \frac{45\eta B_\phi}{2}\left\|\tilde{\mathbf{x}}^{s+1} - \mathbf{x}^*\right\|^2\right] \leq \mathbb{E}\left[\sum_{t=0}^{k_{s+1}-1} \frac{\Phi(\mathbf{x}_t^{s+1}) + \frac{45\eta B_\phi}{2}\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2}{k_{s+1}} - \Phi(\mathbf{x}^*)\right].$$

Then we arrive at

$$\begin{aligned}
& 2\cdot\mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^{s+1}) - \Phi(\mathbf{x}^*) + \frac{45\eta B_\phi}{2}\left\|\tilde{\mathbf{x}}^{s+1} - \mathbf{x}^*\right\|^2\right] \\
\leq\ & \mathbb{E}\left[\frac{3\left[\Phi(\mathbf{x}_0^{s+1}) - \Phi(\mathbf{x}^*)\right] - 3\left[\Phi(\mathbf{x}_{k_{s+1}}^{s+1}) - \Phi(\mathbf{x}^*)\right]}{k_{s+1}} + \Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*) + \frac{45\eta B_\phi}{2}\left\|\tilde{\mathbf{x}}^s - \mathbf{x}^*\right\|^2\right. \\
& \left. + \frac{3}{2\eta_0^{s+1}\cdot k_{s+1}}\cdot\left\|\mathbf{x}^* - \mathbf{x}_0^{s+1}\right\|^2 - \frac{3}{2\eta_{k_{s+1}}^{s+1}\cdot k_{s+1}}\cdot\left\|\mathbf{x}^* - \mathbf{x}_{k_{s+1}}^{s+1}\right\|^2\right].
\end{aligned}$$

Combining the fact that $\mathbf{x}_{k_{s+1}}^{s+1} = \mathbf{x}_0^{s+2}$, $\eta_{k_{s+1}}^{s+1} = \eta_0^{s+2}$ and $k_{s+2} = 2k_{s+1}$, we have

$$\begin{aligned}
& 2\cdot\mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^{s+1}) - \Phi(\mathbf{x}^*) + \frac{45\eta B_\phi}{2}\left\|\mathbf{x}^* - \tilde{\mathbf{x}}^{s+1}\right\|^2 + \frac{3\left\|\mathbf{x}^* - \mathbf{x}_0^{s+2}\right\|^2}{4\eta_0^{s+2}\cdot k_{s+1}} + \frac{3\left[\Phi(\mathbf{x}_0^{s+2}) - \Phi(\mathbf{x}^*)\right]}{2k_{s+1}}\right] \\
\leq\ & \mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*) + \frac{45\eta B_\phi}{2}\left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2 + \frac{3\left\|\mathbf{x}^* - \mathbf{x}_0^{s+1}\right\|^2}{4\eta_0^{s+1}\cdot k_s} + \frac{3\left[\Phi(\mathbf{x}_0^{s+1}) - \Phi(\mathbf{x}^*)\right]}{2k_s}\right].
\end{aligned}$$

Finally, we telescope the above inequality for $s = 0, 1, 2, \ldots, S$ and obtain that

$$\begin{aligned}
& \mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^{s+1}) - \Phi(\mathbf{x}^*)\right] \\
\leq\ & \frac{1}{2^{S+1}}\cdot\mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^0) - \Phi(\mathbf{x}^*) + \frac{45\eta B_\phi}{2}\left\|\mathbf{x}^* - \tilde{\mathbf{x}}^0\right\|^2 + \frac{3\left\|\mathbf{x}^* - \mathbf{x}_0^1\right\|^2}{4\eta_0^1\cdot k_0} + \frac{3\left[\Phi(\mathbf{x}_0^1) - \Phi(\mathbf{x}^*)\right]}{2k_0}\right] \\
\leq\ & \frac{2\left[\Phi(\mathbf{x}^0) - \Phi(\mathbf{x}^*)\right]}{2^S} + \frac{12\eta B_\phi\left\|\mathbf{x}^* - \mathbf{x}^0\right\|^2}{2^S} + \frac{\left\|\mathbf{x}^* - \mathbf{x}^0\right\|^2}{2^S\eta\cdot k_0}.
\end{aligned}$$

This completes the proof. □

Theorem C.3. *Given the initial vector $\mathbf{x}^0 \in \mathbb{R}^d$ satisfies that*

$$\left\|\mathbf{x}^0 - \mathbf{x}^*\right\|^2 \leq D_x, \quad \Phi(\mathbf{x}^0) - \Phi(\mathbf{x}^*) \leq D_\phi,$$

and the first epoch length $k_0 > 0$ and the number of epochs $S > 0$ satisfy that

$$\begin{aligned}
k_0 &= \lfloor\frac{D_x}{2\eta D_\phi}\rfloor + 1, \\
S &= \log_2\left(\frac{6D_\phi}{\epsilon}\right),
\end{aligned}$$

and the sample size $A > 0$ satisfies that

$$A = \frac{2B_g^4 L_f^2}{\eta^2 L_\phi^2},$$

where $\eta > 0$ satisfies that

$$\eta = \min\left\{1,\ \frac{1}{10L_\phi},\ \frac{2D_\phi}{D_x},\ \frac{D_\phi}{6B_\phi D_x},\ \frac{\epsilon}{552\sqrt{2}B_\phi D_x}\right\},$$

and $\epsilon \in (0,1)$ is a tolerance and $B_\phi = \max\{L_\phi, L_\phi^2\}$, then the total IFO complexity, i.e., the number of IFO queries to achieve an ϵ-optimal solution that satisfies

$$\mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^{s+1}) - \Phi(\mathbf{x}^*)\right] \leq \epsilon,$$

is

$$O\left((m+n)\cdot \log\left(\frac{1}{\epsilon}\right) + \frac{1}{\epsilon^4}\right).$$

where we omit the dependence of the IFO complexity on some other parameters, such as the Lipschitz constant L_ϕ, L_f and L_g, the upper bound of the norm of B_g and the distances between the initial point and the optimal set D_x and D_ϕ.

Proof. Firstly, we check if the choices of parameter satisfy the requirement in Lemma C.2. Indeed, we observe that $k_0 \geq 1$ and the sample size $A > 0$ satisfies

$$A = \frac{2B_g^4 L_f^2}{\eta^2 L_\phi^2},$$

where $\eta > 0$ satisfies

$$\eta \leq \min\left\{1,\ \frac{1}{10L_\phi}\right\}.$$

Then it suffices to check if the following statement holds true,

$$\eta \leq \frac{1}{92\sqrt{2}B_\phi}\cdot\frac{1}{\eta T}.$$

We observe that

$$\begin{aligned}
\frac{1}{92\sqrt{2}B_\phi}\cdot\frac{1}{\eta T} &\geq \frac{1}{92\sqrt{2}B_\phi}\cdot\frac{1}{\eta k_0\cdot 2^S} \\
&= \frac{1}{92\sqrt{2}B_\phi}\cdot\frac{\epsilon}{\eta k_0\cdot 6D_\phi} \\
&\geq \frac{1}{92\sqrt{2}B_\phi}\cdot\frac{\epsilon}{\left(\frac{D_x}{2\eta D_\phi}+1\right)\cdot 6D_\phi} \\
&\geq \frac{\epsilon}{552\sqrt{2}B_\phi D_x} \\
&\geq \eta,
\end{aligned}$$

where the first inequality comes from the fact that $T = k_0 \cdot 2^S - k_0$, the equality comes from the fact that $S = \log_2\left(\frac{6D_\phi}{\epsilon}\right)$, the second inequality comes from the fact that $k_0 = \lfloor\frac{D_x}{2\eta D_\phi}\rfloor + 1$, the third inequality comes from the fact that $0 < \eta \leq \frac{2D_\phi}{D_x}$ and the last inequality comes from the fact that $0 < \eta < \frac{\epsilon}{552\sqrt{2}B_\phi D_x}$. Therefore, we arrive at

$$\mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^{s+1}) - \Phi(\mathbf{x}^*)\right] \leq \frac{2\left[\Phi(\mathbf{x}^0) - \Phi(\mathbf{x}^*)\right]}{2^S} + \frac{12\eta B_\phi \left\|\mathbf{x}^* - \mathbf{x}^0\right\|^2}{2^S} + \frac{\left\|\mathbf{x}^* - \mathbf{x}^0\right\|^2}{2^S \eta \cdot k_0}.$$

Furthermore, we have

$$\begin{aligned}
\frac{\Phi(\mathbf{x}^0) - \Phi(x^*)}{2^S} &\leq \frac{\epsilon}{3}, \\
\frac{12\eta B_\phi \left\|\mathbf{x}^0 - \mathbf{x}^*\right\|^2}{2^S} &\leq \frac{\epsilon}{6} \cdot \frac{12\eta B_\phi D_x}{D_\phi} \leq \frac{\epsilon}{3}, \\
\frac{\left\|x^0 - x^*\right\|^2}{2^S \eta k_0} &\leq \frac{\epsilon}{3}.
\end{aligned}$$

where the second inequality comes from the fact that $0 < \eta \leq \frac{D_\phi}{6B_\phi D_x}$ and the third inequality comes from the fact that $k_0 = \lfloor \frac{D_x}{2\eta D_\phi} \rfloor + 1$. Therefore, the total IFO complexity, i.e., the number of IFO queries to achieve an ϵ-optimal solution that satisfies

$$\mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^{s+1}) - \Phi(\mathbf{x}^*)\right] \leq \epsilon,$$

is

$$\begin{aligned}
& S \cdot (m+n) + 2^S \cdot k_0 \cdot A \\
= \; & (m+n) \cdot \log_2\left(\frac{6D_\phi}{\epsilon}\right) + \left(\frac{6D_\phi}{\epsilon}\right) \cdot \left(\lfloor \frac{D_x}{2\eta D_\phi} \rfloor + 1\right) \cdot \frac{2B_g^4 L_f^2}{\eta^2 L_\phi^2} \\
= \; & O\left((m+n) \cdot \log\left(\frac{1}{\epsilon}\right) + \frac{1}{\epsilon^4}\right).
\end{aligned}$$

where we hide the dependence of the IFO complexity on some other parameters, such as the Lipschitz constant L_ϕ, L_f and L_g, the upper bound of the norm of B_g and the distances between the initial point and the optimal set D_x and D_ϕ. □

D Proof of Theorem 3.8

Lemma D.1. *In Algorithm 2, for any $\mathbf{x} \in \mathbb{R}^d$, we have*

$$\begin{aligned}
0 \;\leq\; & \Phi(\mathbf{x}) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{\eta^\alpha L_\phi}{2}\left\|\mathbf{x} - \mathbf{x}_t^{s+1}\right\|^2 + \frac{\eta}{2(1-\eta L_\phi)}\mathbb{E}\left[\left\|\mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\
& + \frac{1}{2\eta^\alpha L_\phi}\left\|\mathbb{E}\left[\left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \\
& + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x} - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x} - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right],
\end{aligned}$$

where $\alpha > 1$ is a constant.

Proof. For any $\mathbf{x} \in \mathbb{R}^d$, it follows from the update of the main iterate $\mathbf{x}_{t+1}^{s+1}$ that

$$r(\mathbf{x}) - r(\mathbf{x}_{t+1}^{s+1}) + \left\langle \mathbf{x} - \mathbf{x}_{t+1}^{s+1}, \mathbf{f}_t^{s+1} + \frac{1}{\eta_{t+1}^{s+1}} \left(\mathbf{x}_{t+1}^{s+1} - \mathbf{x}_t^{s+1} \right) \right\rangle \geq 0.$$

Similar to Lemma C.1, we obtain that

$$\begin{aligned}
0 \quad \leq \quad & r(\mathbf{x}) - r(\mathbf{x}_{t+1}^{s+1}) + \langle \mathbf{x} - \mathbf{x}_t^{s+1}, \mathbf{f}_t^{s+1} \rangle + f(\mathbf{x}_t^{s+1}) - f(\mathbf{x}_{t+1}^{s+1}) + \frac{\eta_{t+1}^{s+1}}{2(1-\eta_{t+1}^{s+1} L_\phi)} \left\| \mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1}) \right\|^2 \\
& + \frac{1}{2\eta_{t+1}^{s+1}} \left[\left\| \mathbf{x} - \mathbf{x}_t^{s+1} \right\|^2 - \left\| \mathbf{x} - \mathbf{x}_{t+1}^{s+1} \right\|^2 \right],
\end{aligned}$$

Taking the conditional expectation of both side on $\mathbf{x}_t^{s+1}$ and $\tilde{\mathbf{x}}^s$ yields that

$$\begin{aligned}
0 \quad \leq \quad & r(\mathbf{x}) - \mathbb{E}\left[r(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] + \left\langle \mathbf{x} - \mathbf{x}_t^{s+1}, \mathbb{E}\left[\left[\partial g_{j_t}(\mathbf{x}_t^{s+1}) \right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] \right\rangle + f(\mathbf{x}_t^{s+1}) \\
& - \mathbb{E}\left[f(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] + \frac{\eta_{t+1}^{s+1}}{2(1-\eta_{t+1}^{s+1} L_\phi)} \mathbb{E}\left[\left\| \mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1}) \right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] \\
& + \frac{1}{2\eta_{t+1}^{s+1}} \left[\left\| \mathbf{x} - \mathbf{x}_t^{s+1} \right\|^2 - \mathbb{E}\left[\left\| \mathbf{x} - \mathbf{x}_{t+1}^{s+1} \right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] \right] \\
= \quad & r(\mathbf{x}) - \mathbb{E}\left[r(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] + \left\langle \mathbf{x} - \mathbf{x}_t^{s+1}, \mathbb{E}\left[\left[\partial g_{j_t}(\mathbf{x}_t^{s+1}) \right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] - \nabla f(\mathbf{x}_t^{s+1}) \right\rangle \\
& + \langle \mathbf{x} - \mathbf{x}_t^{s+1}, \nabla f(\mathbf{x}_t^{s+1}) \rangle + f(\mathbf{x}_t^{s+1}) - \mathbb{E}\left[f(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] \\
& + \frac{\eta_{t+1}^{s+1}}{2(1-\eta_{t+1}^{s+1} L_\phi)} \mathbb{E}\left[\left\| \mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1}) \right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] + \frac{1}{2\eta_{t+1}^{s+1}} \left[\left\| \mathbf{x} - \mathbf{x}_t^{s+1} \right\|^2 - \mathbb{E}\left[\left\| \mathbf{x} - \mathbf{x}_{t+1}^{s+1} \right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] \right] \\
\leq \quad & \Phi(\mathbf{x}) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] + \left\langle \mathbf{x} - \mathbf{x}_t^{s+1}, \mathbb{E}\left[\left[\partial g_{j_t}(\mathbf{x}_t^{s+1}) \right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] - \nabla f(\mathbf{x}_t^{s+1}) \right\rangle \\
& + \frac{\eta_{t+1}^{s+1}}{2(1-\eta_{t+1}^{s+1} L_\phi)} \mathbb{E}\left[\left\| \mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1}) \right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] + \frac{1}{2\eta_{t+1}^{s+1}} \left[\left\| \mathbf{x} - \mathbf{x}_t^{s+1} \right\|^2 - \mathbb{E}\left[\left\| \mathbf{x} - \mathbf{x}_{t+1}^{s+1} \right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] \right] \\
\leq \quad & \Phi(\mathbf{x}) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] + \frac{\eta^\alpha L_\phi}{2} \left\| \mathbf{x} - \mathbf{x}_t^{s+1} \right\|^2 + \frac{\eta_{t+1}^{s+1}}{2(1-\eta_{t+1}^{s+1} L_\phi)} \mathbb{E}\left[\left\| \mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1}) \right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] \\
& + \frac{1}{2\eta^\alpha L_\phi} \left\| \mathbb{E}\left[\left[\partial g_{j_t}(\mathbf{x}_t^{s+1}) \right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] - \nabla f(\mathbf{x}_t^{s+1}) \right\|^2 \\
& + \frac{1}{2\eta_{t+1}^{s+1}} \left[\left\| \mathbf{x} - \mathbf{x}_t^{s+1} \right\|^2 - \mathbb{E}\left[\left\| \mathbf{x} - \mathbf{x}_{t+1}^{s+1} \right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] \right] \\
\leq \quad & \Phi(\mathbf{x}) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] + \frac{\eta^\alpha L_\phi}{2} \left\| \mathbf{x} - \mathbf{x}_t^{s+1} \right\|^2 + \frac{\eta}{2(1-\eta L_\phi)} \mathbb{E}\left[\left\| \mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1}) \right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] \\
& + \frac{1}{2\eta^\alpha L_\phi} \left\| \mathbb{E}\left[\left[\partial g_{j_t}(\mathbf{x}_t^{s+1}) \right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] - \nabla f(\mathbf{x}_t^{s+1}) \right\|^2 \\
& + \frac{1}{2\eta_{t+1}^{s+1}} \left[\left\| \mathbf{x} - \mathbf{x}_t^{s+1} \right\|^2 - \mathbb{E}\left[\left\| \mathbf{x} - \mathbf{x}_{t+1}^{s+1} \right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] \right],
\end{aligned}$$

where the second inequality comes from Assumption 3.1, the third inequality comes from the Young inequality with $\eta^\alpha L_\phi > 0$ and $\alpha > 1$ and the last inequality holds true since

$$\frac{\eta_{t+1}^{s+1}}{2(1-\eta_{t+1}^{s+1}L_\phi)} = \frac{1}{2(\frac{1}{\eta_{t+1}^{s+1}} - L_\phi)} \leq \frac{1}{2(\frac{1}{\eta} - L_\phi)} = \frac{\eta}{2(1-\eta L_\phi)}.$$

where the inequality comes from the fact that $\eta_{t+1}^{s+1} \in \left[\frac{\eta}{\sqrt{2}}, \eta\right]$. This completes the proof. □

Lemma D.2. *In Algorithm 2, we assume that $k_0 \geq 1$ and the sample sizes $A > 0$ and $B > 0$ satisfy*

$$\begin{aligned} A &= \frac{1}{\eta^{2\alpha}} \cdot \max\left\{\frac{2B_g^4 L_f^2}{L_\phi^2}, \ \frac{8B_g^4 L_f^2}{15L_\phi^2}\right\}, \\ B &= \frac{8B_f^2 L_g^2}{15\eta^{2\alpha}L_\phi^2}, \end{aligned}$$

where $\alpha > 1$ and $\eta > 0$ satisfies

$$\begin{aligned} \eta &\leq \min\left\{1, \ \frac{1}{46L_\phi}\right\}, \\ \eta^\alpha &\leq \frac{1}{20\sqrt{2}L_\phi} \cdot \frac{1}{\eta T}, \end{aligned}$$

we have

$$\mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^{s+1}) - \Phi(\mathbf{x}^*)\right] \leq \frac{2\left[\Phi(\mathbf{x}^0) - \Phi(\mathbf{x}^*)\right]}{2^S} + \frac{3\eta^\alpha L_\phi \left\|\mathbf{x}^* - \mathbf{x}^0\right\|^2}{2^S} + \frac{\left\|\mathbf{x}^* - \mathbf{x}^0\right\|^2}{2^S \eta \cdot k_0}.$$

where $\mathbf{x}^*$ is an optimal solution, i.e.,

$$\mathbf{x}^* = \underset{\mathbf{x}\in\mathbb{R}^d}{\operatorname{argmin}} \ \Phi(\mathbf{x}).$$

Proof. Combining Lemma B.3, Lemma B.4, and Lemma D.1 yields that

$$\begin{aligned} 0 \leq\ & \Phi(\mathbf{x}^*) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{\eta^\alpha L_\phi}{2}\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 + \frac{\eta}{2(1-\eta L_\phi)}\mathbb{E}\left[\left\|\mathbf{f}_t^{s+1} - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] \\ & + \frac{1}{2\eta^\alpha L_\phi}\left\|\mathbb{E}\left[\left[\partial g_{j_t}(\mathbf{x}_t^{s+1})\right]^\top \nabla f_{i_t}(\mathbf{g}_t^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] - \nabla f(\mathbf{x}_t^{s+1})\right\|^2 \\ & + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x}^* - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right] \\ \leq\ & \Phi(\mathbf{x}^*) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{\eta^\alpha L_\phi}{2}\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 + \frac{15\eta L_\phi}{1-\eta L_\phi}\left[\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*)\right] + \left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right]\right] \\ & + \frac{12\eta}{1-\eta L_\phi} \cdot \left(\frac{B_g^4 L_f^2}{A} + \frac{B_f^2 L_g^2}{B}\right) \cdot \left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 + \left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2\right] \\ & + \frac{1}{\eta^\alpha L_\phi} \cdot \frac{B_g^4 L_f^2}{A}\left[\left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2 + \left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2\right] \\ & + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x}^* - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right]. \end{aligned}$$

Plugging the sample size of $\mathcal{A}_t$ and $\mathcal{B}_t$,

$$\begin{aligned}
A &= \frac{1}{\eta^{2\alpha}} \cdot \max\left\{\frac{2B_g^4 L_f^2}{L_\phi^2},\ \frac{8B_g^4 L_f^2}{15L_\phi^2}\right\}, \\
B &= \frac{8B_f^2 L_g^2}{15\eta^{2\alpha} L_\phi^2},
\end{aligned}$$

into the above inequality yields that

$$\begin{aligned}
0 \ \leq\ & \Phi(\mathbf{x}^*) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{\eta^\alpha L_\phi}{2}\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 + \frac{15\eta L_\phi}{1-\eta L_\phi}\left[\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*)\right] + \left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right]\right] \\
& + \frac{12\eta}{1-\eta L_\phi} \cdot \frac{15\eta^{2\alpha} L_\phi^2}{4} \cdot \left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 + \left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2\right] + \frac{\eta^\alpha L_\phi}{2}\left[\left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2 + \left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2\right] \\
& + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x}^* - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right].
\end{aligned}$$

We observe that

$$\frac{15\eta L_\phi}{1-\eta L_\phi} = \frac{15L_\phi}{\frac{1}{\eta} - L_\phi} \leq \frac{15L_\phi}{46L_\phi - L_\phi} = \frac{1}{3}. \tag{17}$$

where the inequality holds true since $0 < \eta \leq \frac{1}{46L_\phi}$ and

$$\frac{12\eta}{1-\eta L_\phi} \cdot \frac{15\eta^{2\alpha} L_\phi^2}{4} = \frac{15\eta L_\phi}{1-\eta L_\phi} \cdot 3\eta^{2\alpha} L_\phi \leq \eta^\alpha L_\phi,$$

where the inequality comes from (17) and the fact that $0 < \eta \leq 1$ and $\alpha > 1$. Therefore, we conclude that

$$\begin{aligned}
0 \ \leq\ & \Phi(\mathbf{x}^*) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{\eta^\alpha L_\phi}{2}\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 + \frac{1}{3} \cdot \left[\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*)\right] + \left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right]\right] \\
& + \eta^\alpha L_\phi \cdot \left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 + \left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2\right] + \frac{\eta^\alpha L_\phi}{2}\left[\left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2 + \left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2\right] \\
& + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x}^* - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right] \\
= \ & \Phi(\mathbf{x}^*) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{1}{3} \cdot \left[\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*)\right] + \left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right]\right] \\
& + \frac{\eta^\alpha L_\phi}{2} \cdot \left[4\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 + 3\left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2\right] + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x}^* - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right] \\
= \ & \Phi(\mathbf{x}^*) - \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right] + \frac{1}{3} \cdot \left[\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*)\right] + \left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*)\right]\right] \\
& + \frac{\eta^\alpha L_\phi}{2} \cdot \left[3\left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2 - 6\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2\right] + \frac{1}{2\eta_{t+1}^{s+1}}\left[\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2 - \mathbb{E}\left[\left\|\mathbf{x}^* - \mathbf{x}_{t+1}^{s+1}\right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s\right]\right] \\
& + 5\eta^\alpha L_\phi \cdot \left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2,
\end{aligned}$$

Furthermore, we observe that

$$\frac{1}{\eta_t^{s+1}} - \frac{1}{\eta_{t+1}^{s+1}} \geq \frac{1}{2\eta\sqrt{T}\sqrt{2T}} = \frac{1}{2\sqrt{2}\eta T} \geq 10\eta^\alpha L_\phi,$$

where the last inequality comes from the fact that

$$0 < \eta^{\alpha} \leq \frac{1}{20\sqrt{2}L_{\phi}} \cdot \frac{1}{\eta T}.$$

This implies that

$$\begin{aligned}
& \frac{1}{2\eta_{t+1}^{s+1}} \left[\left\| \mathbf{x}^* - \mathbf{x}_t^{s+1} \right\|^2 - \mathbb{E}\left[\left\| \mathbf{x}^* - \mathbf{x}_{t+1}^{s+1} \right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] \right] + 5\eta^{\alpha} L_{\phi} \cdot \left\| \mathbf{x}^* - \mathbf{x}_t^{s+1} \right\|^2 \\
\leq \; & \left(\frac{1}{2\eta_{t+1}^{s+1}} + 5\eta^{\alpha} L_{\phi} \right) \cdot \left\| \mathbf{x}^* - \mathbf{x}_t^{s+1} \right\|^2 - \frac{1}{2\eta_{t+1}^{s+1}} \cdot \mathbb{E}\left[\left\| \mathbf{x}^* - \mathbf{x}_{t+1}^{s+1} \right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] \\
\leq \; & \frac{1}{2\eta_t^{s+1}} \cdot \left\| \mathbf{x}^* - \mathbf{x}_t^{s+1} \right\|^2 - \frac{1}{2\eta_{t+1}^{s+1}} \cdot \mathbb{E}\left[\left\| \mathbf{x}^* - \mathbf{x}_{t+1}^{s+1} \right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right].
\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}
& \mathbb{E}\left[\Phi(\mathbf{x}_{t+1}^{s+1}) \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right] - \Phi(\mathbf{x}^*) \\
\leq \; & \frac{1}{3} \cdot \left[\left[\Phi(\mathbf{x}_t^{s+1}) - \Phi(\mathbf{x}^*) \right] + \left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*) \right] \right] + \frac{\eta^{\alpha} L_{\phi}}{2} \cdot \left[3 \left\| \mathbf{x}^* - \tilde{\mathbf{x}}^s \right\|^2 - 6 \left\| \mathbf{x}^* - \mathbf{x}_t^{s+1} \right\|^2 \right] \\
& + \frac{1}{2\eta_t^{s+1}} \cdot \left\| \mathbf{x}^* - \mathbf{x}_t^{s+1} \right\|^2 - \frac{1}{2\eta_{t+1}^{s+1}} \cdot \mathbb{E}\left[\left\| \mathbf{x}^* - \mathbf{x}_{t+1}^{s+1} \right\|^2 \mid \mathbf{x}_t^{s+1}, \tilde{\mathbf{x}}^s \right].
\end{aligned}$$

Taking the expectation of both sides, summing it up over $t = 0, 1, 2, \ldots, k_{s+1} - 1$ and dividing both sides by k_{s+1}, we arrive at

$$\begin{aligned}
& \mathbb{E}\left[\sum_{t=0}^{k_{s+1}-1} \frac{\Phi(\mathbf{x}_{t+1}^{s+1}) + 3\eta^{\alpha} L_{\phi} \cdot \left\| \mathbf{x}^* - \mathbf{x}_t^{s+1} \right\|^2}{k_{s+1}} - \Phi(\mathbf{x}^*) \right] \\
\leq \; & \mathbb{E}\left[\frac{1}{3} \left(\sum_{t=0}^{k_{s+1}-1} \frac{\Phi(\mathbf{x}_t^{s+1})}{k_{s+1}} - \Phi(\mathbf{x}^*) + \Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*) \right) + \frac{3\eta^{\alpha} L_{\phi}}{2} \cdot \left\| \mathbf{x}^* - \tilde{\mathbf{x}}^s \right\|^2 + \frac{1}{2\eta_0^{s+1} \cdot k_{s+1}} \cdot \left\| \mathbf{x}^* - \mathbf{x}_0^{s+1} \right\|^2 \right. \\
& \left. - \frac{1}{2\eta_{k_{s+1}}^{s+1} \cdot k_{s+1}} \cdot \left\| \mathbf{x}^* - \mathbf{x}_{k_{s+1}}^{s+1} \right\|^2 \right].
\end{aligned}$$

Multiplying both sides of the above inequality by 3 and rearranging yields that

$$\begin{aligned}
& 2 \cdot \mathbb{E}\left[\sum_{t=0}^{k_{s+1}-1} \frac{\Phi(\mathbf{x}_t^{s+1}) + \frac{9\eta^{\alpha} L_{\phi}}{2} \left\| \mathbf{x}^* - \mathbf{x}_t^{s+1} \right\|^2}{k_{s+1}} - \Phi(\mathbf{x}^*) \right] \\
\leq \; & \mathbb{E}\left[\frac{3\left[\Phi(\mathbf{x}_0^{s+1}) - \Phi(\mathbf{x}^*) \right] - 3\left[\Phi(\mathbf{x}_{k_{s+1}}^{s+1}) - \Phi(\mathbf{x}^*) \right]}{k_{s+1}} + \Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*) + \frac{9\eta^{\alpha} L_{\phi}}{2} \left\| \tilde{\mathbf{x}}^s - \mathbf{x}^* \right\|^2 \right. \\
& \left. + \frac{3}{2\eta_0^{s+1} \cdot k_{s+1}} \cdot \left\| \mathbf{x}^* - \mathbf{x}_0^{s+1} \right\|^2 - \frac{3}{2\eta_{k_{s+1}}^{s+1} \cdot k_{s+1}} \cdot \left\| \mathbf{x}^* - \mathbf{x}_{k_{s+1}}^{s+1} \right\|^2 \right].
\end{aligned}$$

According to Assumption 3.1 and the definition of $\tilde{\mathbf{x}}^{s+1}$, we obtain that

$$\begin{aligned}\Phi(\tilde{\mathbf{x}}^{s+1}) &\leq \sum_{t=0}^{k_{s+1}-1} \frac{\Phi(\mathbf{x}_t^{s+1})}{k_{s+1}},\\ \left\|\tilde{\mathbf{x}}^{s+1} - \mathbf{x}^*\right\|^2 &\leq \sum_{t=0}^{k_{s+1}-1} \frac{\left\|\mathbf{x}_t^{s+1} - \mathbf{x}^*\right\|^2}{k_{s+1}},\end{aligned}$$

which implies that

$$\mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^{s+1}) - \Phi(\mathbf{x}^*) + \frac{9\eta^\alpha L_\phi}{2}\left\|\tilde{\mathbf{x}}^{s+1} - \mathbf{x}^*\right\|^2\right] \leq \mathbb{E}\left[\sum_{t=0}^{k_{s+1}-1} \frac{\Phi(\mathbf{x}_t^{s+1}) + \frac{9\eta^\alpha L_\phi}{2}\left\|\mathbf{x}^* - \mathbf{x}_t^{s+1}\right\|^2}{k_{s+1}} - \Phi(\mathbf{x}^*)\right].$$

Then we arrive at

$$\begin{aligned}&2\cdot\mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^{s+1}) - \Phi(\mathbf{x}^*) + \frac{9\eta^\alpha L_\phi}{2}\left\|\tilde{\mathbf{x}}^{s+1} - \mathbf{x}^*\right\|^2\right]\\ \leq\ & \mathbb{E}\left[\frac{3\left[\Phi(\mathbf{x}_0^{s+1}) - \Phi(\mathbf{x}^*)\right] - 3\left[\Phi(\mathbf{x}_{k_{s+1}}^{s+1}) - \Phi(\mathbf{x}^*)\right]}{k_{s+1}} + \Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*) + \frac{9\eta^\alpha L_\phi}{2}\left\|\tilde{\mathbf{x}}^s - \mathbf{x}^*\right\|^2\right.\\ &\left. + \frac{3}{2\eta_0^{s+1}\cdot k_{s+1}}\cdot\left\|\mathbf{x}^* - \mathbf{x}_0^{s+1}\right\|^2 - \frac{3}{2\eta_{k_{s+1}}^{s+1}\cdot k_{s+1}}\cdot\left\|\mathbf{x}^* - \mathbf{x}_{k_{s+1}}^{s+1}\right\|^2\right].\end{aligned}$$

Combining the fact that $\mathbf{x}_{k_{s+1}}^{s+1} = \mathbf{x}_0^{s+2}$, $\eta_{k_{s+1}}^{s+1} = \eta_0^{s+2}$ and $k_{s+2} = 2k_{s+1}$, we have

$$\begin{aligned}&2\cdot\mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^{s+1}) - \Phi(\mathbf{x}^*) + \frac{9\eta^\alpha L_\phi}{2}\left\|\mathbf{x}^* - \tilde{\mathbf{x}}^{s+1}\right\|^2 + \frac{3\left\|\mathbf{x}^* - \mathbf{x}_0^{s+2}\right\|^2}{4\eta_0^{s+2}\cdot k_{s+1}} + \frac{3\left[\Phi(\mathbf{x}_0^{s+2}) - \Phi(\mathbf{x}^*)\right]}{2k_{s+1}}\right]\\ \leq\ & \mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^s) - \Phi(\mathbf{x}^*) + \frac{9\eta^\alpha L_\phi}{2}\left\|\mathbf{x}^* - \tilde{\mathbf{x}}^s\right\|^2 + \frac{3\left\|\mathbf{x}^* - \mathbf{x}_0^{s+1}\right\|^2}{4\eta_0^{s+1}\cdot k_s} + \frac{3\left[\Phi(\mathbf{x}_0^{s+1}) - \Phi(\mathbf{x}^*)\right]}{2k_s}\right].\end{aligned}$$

Finally, we telescope the above inequality for $s = 0, 1, 2, \ldots, S$ and obtain that

$$\begin{aligned}&\mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^{s+1}) - \Phi(\mathbf{x}^*)\right]\\ \leq\ & \frac{1}{2^{S+1}}\cdot\mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^0) - \Phi(\mathbf{x}^*) + \frac{9\eta^\alpha L_\phi}{2}\left\|\mathbf{x}^* - \tilde{\mathbf{x}}^0\right\|^2 + \frac{3\left\|\mathbf{x}^* - \mathbf{x}_0^1\right\|^2}{4\eta_0^1\cdot k_0} + \frac{3\left[\Phi(\mathbf{x}_0^1) - \Phi(\mathbf{x}^*)\right]}{2k_0}\right]\\ \leq\ & \frac{2\left[\Phi(\mathbf{x}^0) - \Phi(\mathbf{x}^*)\right]}{2^S} + \frac{3\eta^\alpha L_\phi\left\|\mathbf{x}^* - \mathbf{x}^0\right\|^2}{2^S} + \frac{\left\|\mathbf{x}^* - \mathbf{x}^0\right\|^2}{2^S\eta\cdot k_0}.\end{aligned}$$

This completes the proof. □

Theorem D.3. *Given the initial vector $\mathbf{x}^0 \in \mathbb{R}^d$ satisfies that*

$$\left\|\mathbf{x}^0 - \mathbf{x}^*\right\|^2 \leq D_x, \quad \Phi(\mathbf{x}^0) - \Phi(\mathbf{x}^*) \leq D_\phi,$$

and the first epoch length $k_0 > 0$ *and the number of epochs* $S > 0$ *satisfy that*

$$\begin{aligned} k_0 &= \lfloor \frac{D_x}{2\eta D_\phi} \rfloor + 1, \\ S &= \log_2 \left(\frac{6D_\phi}{\epsilon} \right), \end{aligned}$$

and the sample sizes $A > 0$ *and* $B > 0$ *satisfy that*

$$\begin{aligned} A &= \frac{1}{\eta^{2\alpha}} \cdot \max \left\{ \frac{2B_g^4 L_f^2}{L_\phi^2}, \frac{8B_g^4 L_f^2}{15L_\phi^2} \right\}, \\ B &= \frac{8B_f^2 L_g^2}{15\eta^{2\alpha} L_\phi^2}, \end{aligned}$$

where $\eta > 0$ *satisfies*

$$\eta = \min \left\{ 1, \frac{1}{46L_\phi}, \frac{2D_\phi}{D_x}, \frac{2D_\phi}{3L_\phi D_x}, \left(\frac{\epsilon}{120\sqrt{2} L_\phi D_x} \right)^{\frac{1}{\alpha}} \right\},$$

and $\epsilon \in (0, 1)$ *is a tolerance, then the total IFO complexity, i.e., the number of IFO queries to achieve an* ϵ*-optimal solution that satisfies*

$$\mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^{s+1}) - \Phi(\mathbf{x}^*) \right] \leq \epsilon,$$

is

$$O\left((m+n) \cdot \log\left(\frac{1}{\epsilon} \right) + \frac{1}{\epsilon^{3+\frac{1}{\alpha}}} \right).$$

where we omit the dependence of the IFO complexity on some other parameters, such as the Lipschitz constant L_ϕ*,* L_f *and* L_g*, the upper bound of the norm of* B_g *and the distances between the initial point and the optimal set* D_x *and* D_ϕ*.*

Proof. Firstly, we check if the choices of parameter satisfy the requirement in Lemma D.2. Indeed, we observe that $k_0 \geq 1$ and the sample sizes $A > 0$ and $B > 0$ satisfy

$$\begin{aligned} A &= \frac{1}{\eta^{2\alpha}} \cdot \max \left\{ \frac{2B_g^4 L_f^2}{L_\phi^2}, \frac{8B_g^4 L_f^2}{15L_\phi^2} \right\}, \\ B &= \frac{8B_f^2 L_g^2}{15\eta^{2\alpha} L_\phi^2}, \end{aligned}$$

where $\eta > 0$ satisfies

$$\eta \leq \min \left\{ 1, \frac{1}{46L_\phi} \right\}.$$

Then it suffices to check if the following statement holds true,

$$\eta^\alpha \leq \frac{1}{20\sqrt{2} L_\phi} \cdot \frac{1}{\eta T}.$$

We observe that

$$\begin{aligned}
\frac{1}{20\sqrt{2}L_\phi}\cdot\frac{1}{\eta T} &\geq \frac{1}{20\sqrt{2}L_\phi}\cdot\frac{1}{\eta k_0\cdot 2^S} \\
&= \frac{1}{20\sqrt{2}L_\phi}\cdot\frac{\epsilon}{\eta k_0\cdot 6D_\phi} \\
&\geq \frac{1}{20\sqrt{2}L_\phi}\cdot\frac{\epsilon}{\left(\frac{D_x}{2\eta D_\phi}+1\right)\cdot 6D_\phi} \\
&\geq \frac{\epsilon}{120\sqrt{2}L_\phi D_x} \\
&\geq \eta^\alpha,
\end{aligned}$$

where the first inequality comes from the fact that $T = k_0\cdot 2^S - k_0$, the equality comes from the fact that $S = \log_2\left(\frac{6D_\phi}{\epsilon}\right)$, the second inequality comes from the fact that $k_0 = \lfloor\frac{D_x}{2\eta D_\phi}\rfloor + 1$, the third inequality comes from the fact that $0 < \eta \leq \frac{2D_\phi}{D_x}$ and the last inequality comes from the fact that $0 < \eta^\alpha < \frac{\epsilon}{120\sqrt{2}L_\phi D_x}$. Therefore, we arrive at

$$\mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^{s+1}) - \Phi(\mathbf{x}^*)\right] \leq \frac{2\left[\Phi(\mathbf{x}^0) - \Phi(\mathbf{x}^*)\right]}{2^S} + \frac{3\eta^\alpha L_\phi\left\|\mathbf{x}^* - \mathbf{x}^0\right\|^2}{2^S} + \frac{\left\|\mathbf{x}^* - \mathbf{x}^0\right\|^2}{2^S\eta\cdot k_0}.$$

Furthermore, we have

$$\begin{aligned}
\frac{\Phi(\mathbf{x}^0) - \Phi(x^*)}{2^S} &\leq \frac{\epsilon}{3}, \\
\frac{3\eta^\alpha L_\phi\left\|\mathbf{x}^0 - \mathbf{x}^*\right\|^2}{2^S} &\leq \frac{\epsilon}{6}\cdot\frac{3\eta^\alpha L_\phi D_x}{D_\phi} \leq \frac{\epsilon}{3}, \\
\frac{\left\|x^0 - x^*\right\|^2}{2^S\eta k_0} &\leq \frac{\epsilon}{3}.
\end{aligned}$$

where the second inequality comes from the fact that $0 < \eta \leq \frac{2D_\phi}{3L_\phi D_x}$ and the third inequality comes from the fact that $k_0 = \lfloor\frac{D_x}{2\eta D_\phi}\rfloor + 1$. Therefore, the total IFO complexity, i.e., the number of IFO queries to achieve an ϵ-optimal solution that satisfies

$$\mathbb{E}\left[\Phi(\tilde{\mathbf{x}}^{s+1}) - \Phi(\mathbf{x}^*)\right] \leq \epsilon,$$

is

$$\begin{aligned}
& S\cdot(m+n) + 2^S\cdot k_0\cdot(A+B) \\
= \; & (m+n)\cdot\log_2\left(\frac{6D_\phi}{\epsilon}\right) + \left(\frac{6D_\phi}{\epsilon}\right)\cdot\left(\lfloor\frac{D_x}{2\eta D_\phi}\rfloor + 1\right)\cdot\frac{1}{\eta^{2\alpha}}\cdot\left(\max\left\{\frac{2B_g^4L_f^2}{L_\phi^2}, \frac{8B_g^4L_f^2}{15L_\phi^2}\right\} + \frac{8B_f^2L_g^2}{15L_\phi^2}\right), \\
= \; & O\left((m+n)\cdot\log\left(\frac{1}{\epsilon}\right) + \frac{1}{\epsilon^{3+\frac{1}{\alpha}}}\right).
\end{aligned}$$

□