

# Ultrametric Component Analysis with Application to Analysis of Text and of Emotion

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## Abstract

We review the theory and practice of determining what parts of a data set are ultrametric. It is assumed that the data set, to begin with, is endowed with a metric, and we include discussion of how this can be brought about if a dissimilarity, only, holds. The basis for part of the metric-endowed data set being ultrametric is to consider triplets of the observables (vectors). We develop a novel consensus of hierarchical clusterings. We do this in order to have a framework (including visualization and supporting interpretation) for the parts of the data that are determined to be ultrametric. Furthermore a major objective is to determine locally ultrametric relationships as opposed to non-local ultrametric relationships. As part of this work, we also study a particular property of our ultrametricity coefficient, namely, it being a function of the difference of angles of the base angles of the isosceles triangle. This work is completed by a review of related work, on consensus hierarchies, and of a major new application, namely quantifying and interpreting the emotional content of narrative.

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# 1 Introduction

## 1.1 Metric and Ultrmetric

A metric space  $(X, d)$  consists of a set  $X$  on which is defined a distance function  $d$  which assigns to each pair of points of  $X$  a distance between them, and satisfies the following four axioms for any triplet of points  $x, y, z$ :

$$\text{A1: } \forall x, y \in X, d(x, y) \geq 0 \text{ (positiveness)}$$

$$\text{A2: } \forall x, y \in X, d(x, y) = 0 \text{ iff } x = y \text{ (reflexivity)}$$

$$\text{A3: } \forall x, y \in X, d(x, y) = d(y, x) \text{ (symmetry)}$$

$$\text{A4: } \forall x, y, z \in X, d(x, z) \leq d(x, y) + d(y, z) \text{ (triangular inequality)}$$

When considering an ultrmetric space we need to consider the strong triangular inequality or ultrmetric inequality defined as:

$$\text{A5: } d(x, z) \leq \max \{d(x, y), d(y, z)\} \text{ (ultrmetric inequality)}$$

and this in addition to the positivity, reflexivity and symmetry properties (properties A1, A2, A3) for any triple of point  $x, y, z \in X$ .

## 1.2 Transforming to a Metric

Very briefly, we summarize the various mappings that are commonly and widely applied.

1. Starting with observations with readings on attributes, arranged as rows and columns of a matrix,  $X$ , then the spectral reduction of the variance-covariance matrix,  $X'X$ , where  $X$  may in addition have preprocessing involving centering of attributes, or centering and reduction to unit variance of attributes, yields the principal components, i.e. an orthonormal coordinate system endowed with the Euclidean distance. The input is assumed to be endowed with the Euclidean distance also. The three cases considered (centering, reduction, or not) provide for PCA of (i) the sums of squares and cross-products matrix, (ii) variances-covariances, and (iii) correlations.

2. When qualitative data are at issue, then Correspondence Analysis is the more appropriate analysis method, through projection into a Euclidean factor space. See [30, 32]. For contingency table data, a natural model for the data as quantified by the  $\chi^2$  statistic is the degree of fit between observed data and the expected data given by the outer product of the marginal, and also the row and column mass, densities. It is this discrepancy from the model provided by this product of densities that is interpreted in the Correspondence Analysis.
3. Principal Coordinates Analysis takes distances as input and reconstructs a coordinate space. See section 2.1.

### 1.3 Transforming to an Ultrametric

Here we consider the transforming of data endowed with a metric to the same set endowed with an ultrametric. We do not require a metric, and instead of a distance the algorithm performing the transformation to an ultrametric can work on input consisting of dissimilarities. (Recall that a dissimilarity is not required to satisfy the triangular inequality.)

#### 1.3.1 Agglomerative Hierarchical Clustering Algorithms

If  $X$  is endowed with a metric, then this metric can be mapped onto an ultrametric. In practice, endowing  $X$  with a metric can be relaxed to a dissimilarity. An often used mapping from metric to ultrametric is by means of an agglomerative hierarchical clustering algorithm. A succession of  $n - 1$  pairwise merge steps takes place by making use of the closest pair of singletons and/or clusters at each step. Here  $n$  is the number of observations, i.e. the cardinality of set  $X$ . Closeness between singletons is furnished by whatever distance or dissimilarity is in use. For closeness between singleton or non-singleton clusters, we need to define an inter-cluster distance or dissimilarity. This can be defined with reference to the cluster compactness or other property that we wish to optimize at each step of the algorithm.

#### 1.3.2 Hierarchical Clustering and Associated Ultrametric

A hierarchy,  $H$ , is defined as a binary, rooted, node-ranked tree, also termed a dendrogram [1, 19, 26, 29]. A hierarchy defines a set of embedded subsets of a given set of objects  $X$ , indexed by the set  $I$ . These subsets are totally ordered by an index function  $\nu$ , which is a stronger condition than the partial order required by the subset relation. A bijection exists between a hierarchy and an ultrametric space.

Let us show these equivalences between embedded subsets, hierarchy, and binary tree, through the constructive approach of inducing  $H$  on a set  $I$ .

Hierarchical agglomeration on  $n$  observation vectors with indices  $i \in I$  involves a series of  $1, 2, \dots, n - 1$  pairwise agglomerations of observations or clusters, with properties that follow.

In order to simplify notation, let us use the index  $i$  to represent also the observation, and also the observation vector. Hence for  $i = 3$  and the third – in some sequence – observation vector,  $x_i = x_3$ , we will use  $i$  to also represent  $x_i$  in such a case.

A hierarchy  $H = \{q|q \in 2^I\}$  such that (i)  $I \in H$ , (ii)  $i \in H \forall i$ , and (iii) for each  $q \in H, q' \in H : q \cap q' \neq \emptyset \implies q \subset q'$  or  $q' \subset q$ . Here we have denoted the power set of set  $I$  by  $2^I$ . An indexed hierarchy is the pair  $(H, \nu)$  where the positive function defined on  $H$ , i.e.,  $\nu : H \rightarrow \mathbb{R}^+$ , satisfies:  $\nu(i) = 0$  if  $i \in H$  is a singleton; and (ii)  $q \subset q' \implies \nu(q) < \nu(q')$ . Here we have denoted the positive reals, including 0, by  $\mathbb{R}^+$ . Function  $\nu$  is the agglomeration level. Take  $q \subset q''$  and  $q' \subset q''$ , and let  $q''$  be the lowest level cluster for which this is true. Then if we define  $D(q, q') = \nu(q'')$ ,  $D$  is an ultrametric.

A hierarchical clustering tree is referred to as a dendrogram.

### 1.3.3 Agglomerative Criterion and Its Associated Ultrametric

In practice, we start with a Euclidean or alternative dissimilarity, use some criterion such as minimizing the change in variance resulting from the agglomerations, or the minimal linkage (in terms of the initial input dissimilarities, and the redefined dissimilarities between clusters as they are formed) and then define  $\nu(q)$  as the dissimilarity associated with the agglomeration carried out.

For observations  $i, i' \in I$ , let  $q$  be the least, with respect to set inclusion, cluster containing both  $i$  and  $i'$ :  $i \neq i'; i \in q$  and  $i' \in q$ . Then the ultrametric distance between  $i$  and  $i'$  is  $D(i, i') = \nu(q)$ .

With the initial dissimilarity or distance,  $d(i, i')$ , the hierarchical agglomerative clustering is specified as follows: given a set of clusters comprising a partition  $P$  of  $I$ , and defining a level of the hierarchical clustering, then agglomerate  $q, q' \in P$  such that  $d(q, q') \leq d(q'', q'''), \forall q'', q''' \in P$ . Finally the agglomerative clustering criterion allows us to define  $d$ .

For the single link agglomerative clustering method,  $d(q, q') = \min_{i, i'} d(i, i'), i \in q, i \in q'$ . Single link is also referred to as the minimum link or nearest neighbour method. The ultrametric distance, read off the dendrogram,  $d_u(i, i')$  is such that  $d_u(i, i') \leq d(i, i')$  and it is the maximum such value over all ultrametric distance matches (i.e., mappings) of the input dissimilarity or distance data. For this reason the single link agglomerative clustering algorithm is referred to as the subdominant ultrametric, and also the maximum inferior ultrametric. See [1, 18].

Parenthetically we note the close link between the single link method and the minimal spanning tree. Kruskal's algorithm [21] takes the edges, i.e. pairs  $(i, i')$  in increasing order of their distance (or dissimilarity),  $d(i, i')$ , and connects them if a graph cycle does not result. In this way we end up with a tree (because cycles are excluded), that is spanning (all points  $i, i'$  are included), and that is minimal (the sum of edge weights or dissimilarities  $d(i, i')$  over all connected pairs,  $(i, i')$ , is minimal). Note that the link formed by a pair  $(i, i')$  is preserved in the minimal spanning tree, whereas the merger of two clusters  $(q, q')$  in the single link hierarchical clustering does not preserve this informa-

tion; it follows that transforming the minimal spanning tree into the single link hierarchical clustering only requires the processing of the  $n - 1$  edges; whereas the transforming of the single link hierarchy into the minimal spanning tree requires more processing, determining the minimal distance between members of clusters for every pair of clusters (immediate if the clusters are singletons). The latter task is easily seen to be quadratic if we approach it as the building from the start of the single link hierarchical clustering, because in doing so we have the required (closest link between clusters) information.

In the case of the complete link method, we are dealing with a maximum link method. The ultrametric distance, read off the dendrogram,  $d_u(i, i')$  is such that  $d_u(i, i') \geq d(i, i')$  and it is a non-unique minimal such value over all ultrametric distance matches of the input dissimilarity or distance data. For this reason the complete link agglomerative clustering algorithm is referred to as a minimal superior ultrametric. See [1]. Another name used is furthest neighbour algorithm.

## 2 Quantifying How Metric or How Ultrametric A Data Set Is

### 2.1 Quantifying How Metric a Data Set Is

We will take a practical and applicable framework in order to show how the positive (or non-negative) eigenvalues of a spectral decomposition provides a way to measure how metric a data set is.

Consider Principal Coordinates Analysis, also referred to as Classical Multidimensional Scaling and metric scaling, and associated with the names of Torgerson [48] and Gower. It takes distances as input and produces coordinate values as output.

Consider the initial data matrix,  $X$ , of dimensions  $n \times m$ , and the “sums of squares and cross products” matrix of the rows:

$$A = XX'$$

$$a_{ik} = \sum_j x_{ij}x_{kj}.$$

If  $d_{ik}$  is the Euclidean distance between objects  $i$  and  $k$  (using row vectors  $i$  and  $k$  of matrix  $X$ ) we have that:

$$\begin{aligned} d_{ik}^2 &= \sum_j (x_{ij} - x_{kj})^2 \\ &= \sum_j x_{ij}^2 + \sum_j x_{kj}^2 - 2 \sum_j x_{ij}x_{kj} \\ &= a_{ii} + a_{kk} - 2a_{ik}. \end{aligned} \tag{1}$$

In Principal Coordinates Analysis, we are given the distances and we want to obtain  $X$ . We will assume that the columns of this matrix are centered, i.e.

$$\sum_i x_{ij} = 0.$$

It will now be shown that matrix  $A$  can be constructed from the distances using the following formula:

$$a_{ik} = -\frac{1}{2}(d_{ik}^2 - d_i^2 - d_k^2 + d^2) \quad (2)$$

where

$$\begin{aligned} d_i^2 &= \frac{1}{n} \sum_k d_{ik}^2 \\ d_k^2 &= \frac{1}{n} \sum_i d_{ik}^2 \\ d^2 &= \frac{1}{n^2} \sum_i \sum_k d_{ik}^2. \end{aligned}$$

This result may be proved by substituting for the distance terms (using equation 1), and knowing that by virtue of the centering of the row vectors of matrix  $X$ , we have

$$\sum_i a_{ik} = 0$$

(since  $a_{ik} = \sum_j x_{ij}x_{kj}$ ; and consequently in the term  $\sum_i \sum_j x_{ij}x_{kj}$  we can separate out  $\sum_i x_{ij}$  which equals zero). Similarly (by virtue of the symmetry of  $A$ ) we use the fact that

$$\sum_k a_{ik} = 0.$$

Having thus been given distances, we have constructed matrix  $A = XX'$ . We now wish to reconstruct matrix  $X$ ; or, since this matrix has in fact never existed, we require some matrix  $X$  which satisfies  $XX' = A$ .

If matrix  $A$  is positive, symmetric and semidefinite, it will have rank  $p \leq n$ . We may derive  $p$  non-zero eigenvalues,  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p > 0$ , with corresponding eigenvectors  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_p$ . Consider the scaled eigenvectors, defined as  $\mathbf{f}_i = \sqrt{\lambda_i} \mathbf{u}_i$ . Then the matrix  $X = (\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_p)$  is a possible coordinate matrix. This is shown as follows. We have, in performing the eigen-decomposition of  $A$ :

$$A\mathbf{u}_i = \lambda_i \mathbf{u}_i$$

and by requirement

$$XX'\mathbf{u}_i = \lambda_i \mathbf{u}_i.$$

In the left hand side,  $X \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_p \end{pmatrix} \mathbf{u}_i = X \begin{pmatrix} \sqrt{\lambda_1} \mathbf{u}_1 \\ \sqrt{\lambda_2} \mathbf{u}_2 \\ \vdots \\ \sqrt{\lambda_p} \mathbf{u}_p \end{pmatrix} \mathbf{u}_i = X \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \sqrt{\lambda_i} \\ \vdots \\ 0 \end{pmatrix}$  since

eigenvectors are mutually orthogonal. Continuing:

$$(\sqrt{\lambda_1} \mathbf{u}_1, \sqrt{\lambda_2} \mathbf{u}_2, \dots, \sqrt{\lambda_p} \mathbf{u}_p) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \sqrt{\lambda_i} \\ \vdots \\ 0 \end{pmatrix} = \lambda_i \mathbf{u}_i.$$

Thus we have succeeded in constructing a matrix  $X$ , in an orthonormal basis, having been initially given a set of distances.

In practice, we might be given dissimilarities rather than distances. Then, matrix  $A$  will be symmetric and have zero values on the diagonal but will not be positive semidefinite. In this case negative eigenvalues are obtained. These are inconvenient but may often be ignored if the approximate Euclidean representation (given by the eigenvectors corresponding to positive eigenvalues) is satisfactory. Thus the Euclidean content of the data is quantified using the non-negative eigenvalues in the spectral decomposition. Their proportion of the sum of absolute values of eigenvalues is a single measure of how metric a data set is.

## 2.2 Quantifying How Ultrametric a Data Set Is

### 2.2.1 Ultrametricity Coefficient of Lerman

The principle adopted in any constructive assessment of ultrametricity is to construct an ultrametric on data and see what discrepancy there is between input data and induced ultrametric data structure. Quantifying ultrametricity using a constructive approach is less than perfect as a solution, given the potential complications arising from known problems, e.g. chaining in single link, and non-uniqueness, or even inversions, with other methods. The conclusion here is that the ‘‘measurement tool’’ used for quantifying ultrametricity itself occupies an overly prominent role relative to that which we seek to measure. For such reasons, we need an independent way to quantify ultrametricity.

Lerman’s [26] H-classifiability index is as follows. From the isosceles triangle principle, given a distance  $d$  where  $d(x, y) \neq d(y, z)$  we have  $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ , it follows that the largest and second largest of the numbers  $d(x, y), d(y, z), d(x, z)$  are equal. Lerman’s H-classifiability measure essentially looks at how close these two numbers (largest, second largest) are. So as to avoid influence of distribution of the distance values, Lerman’s measure is based on ranks (of these distances) only. For further discussion of it, see [31].

There are two drawbacks with Lerman’s index. Firstly, ultrametricity is



associated with  $H = 0$  but non-ultrametricity is not bounded. In extensive experimentation, we found maximum values for  $H$  in the region of 0.24. The second problem with Lerman’s index is that for floating point coordinate values, especially in high dimensions, the strict equality necessitated for an equilateral triangle is nearly impossible to achieve. However our belief is that approximate equilateral triangles are very likely to arise in important cases of high-dimensional spaces with data points at hypercube vertex locations. We would prefer therefore that the quantifying of ultrametricity should “gracefully” take account of triplets which are “close to” equilateral. Note that for some authors, the equilateral case is considered to be “trivial” or a “trivial limit” [49]. For us, however, it is an important case, together with the other important case of ultrametricity (i.e., isosceles-with-small-base, which we will write in that way for clarity).

### 2.2.2 Ultrametricity Coefficient of Rammal, Toulouse and Virasoro

The quantifying of how ultrametric a data set is by Rammal et al. [39, 40] was influential for us in this work. The Rammal ultrametricity index is given by  $\sum_{x,y} (d(x,y) - d_c(x,y)) / \sum_{x,y} d(x,y)$  where  $d$  is the metric distance being assessed, and  $d_c$  is the subdominant ultrametric. The latter is also the ultrametric associated with the single link hierarchical clustering method. The Rammal et al. index is bounded by 0 (= ultrametric) and 1. As pointed out in [39, 40] this index suffers from “the chaining effect and from sensitivity to fluctuations”. The single link hierarchical clustering method, yielding the subdominant ultrametric, is, as is well known, subject to such difficulties.

### 2.2.3 Ultrametricity Coefficients of Treves and of Hartmann

Treves [49] considers triplets of points giving rise to minimal, median and maximal distances. In the plot of  $d_{\min}/d_{\max}$  against  $d_{\text{med}}/d_{\max}$ , the triangular inequality, the ultrametric inequality, and the “trivial limit” of equilateral triangles, occupy definable regions.

Hartmann [16] considers  $d_{\max} - d_{\text{med}}$ . Now, Lerman [26] uses ranks in order to give (translation, scale, etc.) invariance to the sensitivity (i.e., instability, lack of robustness) of distances. Hartmann instead fixes the remaining distance  $d_{\min}$ .

We seek to avoid, as far as possible, lack of invariance due to use of distances. We seek to quantify both isosceles-with-small-base configurations, as well as equilateral configurations. Finally, we seek a measure of ultrametricity bounded by 0 and 1.

### 2.2.4 Bayesian Network Modeling

Latent ultrametric distances were estimated by Schweinberger and Snijders [44] using a Bayesian and maximum likelihood approach in order to represent transitive structures among pairwise relationships. As they state, “The observed

network is generated by hierarchically nested latent transitive structures, expressed by ultrametrics”. Multiple, nested transitive structures are at issue. “Ultrametric structures imply transitive structures” and as an informal way to characterize ultrametric structures (arising from embedded clusters, comprising “friends” and “close friends”): “Friends are likely to agree, and unlikely to disagree; close friends are very likely to agree, and very unlikely to disagree.”

Issues however in the statistical model-based approach to determining ultrametricity include that convergence to an optimal fit is not guaranteed and there can be an appreciable computational requirement. Our approach (to be described in the next subsection) in contrast is fast and can be achieved through sampling which supposes that there is a homogeneous ultrametricity pertaining to the data used. If sampling is used (for computational reasons) then we assume that the text is “textured” in the same way throughout, or that it is sufficiently “unified”. For one theme in regard to content, or one origin, or one author, such an assumption seems a reasonable one.

### 2.2.5 Our $\alpha_\epsilon$ -Ultrametricity Coefficient

We define a coefficient of ultrametricity termed  $\alpha_\epsilon$ , parametrized by an approximation factor,  $\epsilon$ , which is specified algorithmically as follows.

1. All triplets of points are considered, with a distance (by default, Euclidean) defined on these points. Since for a large number of points,  $n$ , the number of triplets,  $n(n-1)(n-2)/6$  would be computationally prohibitive, we may wish to randomly (uniformly) sample coordinates ( $i \sim \{1..n\}, j \sim \{1..n\}, k \sim \{1..n\}$ ).
2. We check for possible alignments (implying degenerate triangles) and exclude such cases.
3. Next we select the smallest angle as less than or equal to 60 degrees. (We use the well-known definition of the cosine of the angle facing side of length  $x$  as:  $(y^2 + z^2 - x^2)/2yz$ .) This is our first necessary property for being a strictly isosceles ( $< 60$  degrees) or equilateral ( $= 60$  degrees) ultrametric triangle.
4. For the two other angles subtended at the triangle base, we seek an angular difference of strictly less than  $\epsilon$ , defined in our past work as 2 degrees or 0.03490656 radians. This condition is an approximation to the ultrametric configuration, based on an arbitrary choice of small angle. This condition is targeting a configuration that may not be exactly ultrametric but nonetheless is very close to ultrametric.
5. Among all triplets (1) satisfying our exact properties (2, 3) and close approximation property (4), we define our ultrametricity coefficient as the relative proportion of these triplets. Approximately ultrametric data will yield a value of 1. On the other hand, data that is non-ultrametric in the

sense of not respecting conditions 3 and 4 will yield a low value, potentially reaching 0.

In summary, the  $\alpha_\epsilon$  index is defined in this way:

Consider a triplet of points, that defines a triangle. If the smallest internal angle,  $a$ , in this triangle is  $\leq 60$  degrees, and, for the two other internal angles,  $b$  and  $c$ , if  $|b - c| < \epsilon$  radians, then this triangle is an ultrametric one. We look for the overall proportion of such ultrametric triangles in our data.

### 2.3 Application to Partial Ultrametric Embedding

In [34], we discuss permutation representations of a data stream. Since hierarchies can also be represented as permutations, there is a ready way to associate data streams with hierarchies. In fact, early computational work on hierarchical clustering used permutation representation to great effect (cf. [45]).

In [33], in analysis of time series, chronological or other ordered 1-dimensional signals, data was fingerprinted in the domains of chaotic (found to be less ultrametric), biomedical, meteorological, financial data. To analyze data streams in this way, in [33] we developed an approach to ultrametric embedding of time-varying signals, including biomedical, meteorological, financial and other. As opposed to the classical way of inducing a hierarchy, through use of an agglomerative hierarchical clustering algorithm, we looked for the ultrametric relationship – the strong triangular inequality – and, when found, counted such particular cases of adherence to inherent hierarchical properties in the data. The most non-ultrametric time series are found to be chaotic ones. Eyegaze trace data was found to be remarkably high in ultrametricity, which is likely to be due to extreme saccade movements. Some initial questions were raised in that work ([33]) in regard to the EEG data used, for sleeping, petit mal and irregular epilepsy cases.

This work was pursued by Khrennikov and his colleagues in modelling multi-agent systems. See [12]. Furthermore this work used Bose-Einstein and Fermi-Dirac statistical distributions (derived from quantum statistics of energy states of bosons and fermions, i.e. elementary particles with integer, and half odd integer, spin). In [11, 12] multi-agent behaviours are modelled using such energy distributions. The framework was an urn model, where balls can move, with loss of energy over time, and with possibilities to receive input energy, but potentially shared with other balls. See the cited works for a full description of the Monte Carlo system set up. Sequences of actions (and moves), viz. their histories, are coded such that triangle properties can be investigated (cf. also [33]). That leads to a characterization of how ultrametrically-embeddable the data is, *ab initio* (and not through imposing any hierarchical or other structure on the data with retrospective goodness of fit assessment). In [12], the case is presented for such analysis of behavioural histories being important for study of social and economic complexity.

Quantum statistical distributions have been noted in the foregoing work ([11, 12]). Van Rijsbergen [50] has set out various ways in which a quantum

physics formalism makes clearer what is being done in information retrieval and in data analysis generally.

The quantifying of the inherent ultrametric content of text, and finding that some are much more inherently hierarchical than others, was pursued in [37]. In this way, a large number of texts were fingerprinted. As data, the following were used: tales from the Brothers Grimm, Jane Austen novels, dream reports, air accident reports, and James Joyce’s *Ulysses*.

### 3 Transforming Data to Become More Metric or More Ultrametric

#### 3.1 Transforming Non-Metric to Metric Data

We recall that a metric satisfies symmetry and positive definiteness, together with the triangular inequality:  $\forall i, j, k, d(i, j) = d(j, i); d(i, j) \geq 0, d(i, j) = 0 \iff i = j; d(i, j) \leq d(i, k) + d(k, j)$ . A dissimilarity does not satisfy the triangular inequality. Hence a dissimilarity is not directly embedable in a metric space, such as a Euclidean 2D (planar) or 3D space typically used for visualization. Next we consider (i) mapping metric data to a best low dimensional projection, (ii) mapping non-metric, hence dissimilarity, data to a best low dimensional metric space, and (iii) transforming non-metric data into close if not best-fitting metric values.

Consider a set  $I$  endowed with a dissimilarity,  $d_I$ . Multidimensional scaling is an approach to take  $d_I$  into a distance,  $\delta_I$ , which therefore as a distance respects the triangular inequality. In multidimensional scaling it is just the ranks of  $d_I$  that are used (so  $d_{ij} < d_{kl}$  for all couples  $(i, j)$  and  $(k, l)$ ).

In Torgerson’s [48] metric multidimensional scaling, a low dimensional representation of Euclidean data is obtained. The least-squares best fit is provided by the orthonormal axes that define the principal axes of inertia. In non-metric multidimensional scaling [22], dissimilarities of a general sort, that are rank ordered, are mapped non-linearly into a low dimensional Euclidean representation. A goodness of fit measure between input dissimilarities and output distances in the low dimensional space, called “stress”, is used as an optimand. Iterative, steepest descent is used for optimization. Additional short description is to be found in [38, 30].

There are other ways of taking a dissimilarity into a distance. Consider the following from Cailliez and Pagès ([4] p. 257). The most common are from the following family, where  $r$  is a positive value:

$$\delta_{ij} = 0 \text{ if } i = j \ ; \ d_{ij} = (d_{ij}^r + c)^{1/r} \text{ if } i \neq j$$

where  $c$  is a positive constant chosen so that the triangular inequality is respected by the function  $\delta_I$ .

For  $r = 1$ , taking:

$$c = \max\{d_{ij} - d_{ik} - d_{jk} | i, j, k \in I^3\}$$

which is the minimum possible value for  $c$ . This provides for a simple transformation approach.

Many other transformations are possible, such as for example:

$$\delta_{ij} = d_{ij}^r \quad \text{for all } (i, j) \in I \times I$$

where  $r$  is the largest value between 0 and 1 such that  $\delta_I$  satisfies the triangular inequality.

### 3.2 Transforming Metric, or Non-Metric, to Minimal Superior Ultrametric, and to Maximal Inferior Ultrametric

In section 1.3.1, agglomerative hierarchical clustering was described.

Rather than the triangular inequality, ultrametric data are characterized by the properties of a metric together with the ultrametric, or strong triangular, inequality,  $d_u(i, j) \leq \max\{d_u(i, k), d_u(j, k)\}$ . Hierarchical agglomerative clustering algorithms are a general and widely-used class of algorithm for inducing an ultrametric on dissimilarity or distance input, or coordinate data on which a metric or a dissimilarity has been defined.

Johnson [19] makes every triangle isosceles by making its smallest side the base. This leads to the maximal inferior ultrametric, also known as the subdominant ultrametric, or ultrametric produced by the single linkage hierarchical clustering agglomerative algorithm. See Benzécri [1] (p. 144). The maximal inferior ultrametric is unique.

The minimal superior ultrametric is not though. In this case, Benzécri [1] (p. 144) discusses the adaptive modification of triangle edges to enforce the ultrametric inequality, i.e. isosceles triangles with small base.

### 3.3 Approximating an Ultrametric for Similarity Metric Space Searching

In [31] we show that, in much work over the years, nearest neighbour searching has been made more efficient through the use of more easily determined feasibility bounds. An early example is Fukunaga and Narendra [14], a chapter review is in [29], and a survey is to be found in [6]. Rendering given distances as ultrametric is a powerful way to facilitate nearest neighbour searching. Furthermore “stretching the triangular inequality” (a phrase used by [5]) so that it becomes the strong triangular inequality, or ultrametric inequality, gives a unifying view of some algorithms of this type.

Fast nearest neighbour finding often makes use of pivots to establish bounds on points to be searched, and points to be bypassed as unfeasible [3, 6].

A full discussion can be found in [31]. Fast nearest neighbour searching in metric spaces often appeals to heuristics. The link with ultrametric spaces gives rise instead to a unifying view.

Hjaltason and Samet [17] discuss heuristic nearest neighbour searching in terms of embedding the given metric space points in lower dimensional spaces. From our discussion in this section, we see that there is evidently another alternative direction for facilitating fast nearest neighbour searching: viz., taking the metric space as an ultrametric one, and if it does not quite fit this perspective then “stretch” it (transform it locally) so that it does so.

## 4 Emotion as the Doorway into the Subconscious

### 4.1 Emotion as an Essential Component of Perception

In [36, 37] we have described how Matte Blanco’s psycho-analytical theories lend themselves very well to the following viewpoint, a viewpoint that was noted first in [23]. If a Euclidean, or more generally metric, space is a good framework for time, causality, semantics (e.g. through all pairwise distance relationships, cf. [35]), and Matte Blanco’s *asymmetric* thought processes (see e.g. [36]), then metric space collaterally is useful for mapping conscious activities. By implication, metric space is a good model for consciousness. On the other hand, ultrametric spaces are a good way of envisioning Matte Blanco’s *symmetric* thought processes, atemporal thoughts, and episodic (as opposed to semantic memory, for example in studies of Alzheimer’s disease, see [47]).

In the PhD thesis of Marco Tonti, [47], Matte Blanco’s work is taken in the domain of human emotion, which is closely associated with the human subconscious. Emotion is “the component of thought that can be attributed to [...] unconscious rules” ([47], p. 78). His aim in his thesis is to demarcate that component of thought. Since tangible objects are imbued with emotion (“an object in a bag of symmetry which is deeply irradiated with emotion”, and we can go a lot further in considering “a highly emotionated object”, pp. 87, 88), the problem addressed by Tonti is that “...some mechanism of connecting the experiential/situational dimension of an individual with the deep/unconscious dimension, in terms of emotionated objects, is required.” (p. 87).

Furthermore, “The fundamental theoretical hypothesis of this dissertation is that emotions are an essential component in the process of perception (and therefore evaluation) of the world. Consequently, the amount of emotions involved in such a task is expected to be reflected in the way of categorizing/expressing evaluations of a subject.” (p. 119).

An important point made is that “...emotions are a fundamental and building force for cognition, and not just an attribution of affective aspects or a physical response.” (p. 164). Emotions for Tonti are a fundamental constituent of thought and intelligence. They are not juxtaposed but rather are an essential part of the latter domains, viz. thought and intelligence. The common perception is to take emotions as “the opposite” (in some sense) of thought. However, for Tonti, intelligence would not be possible without emotion. Emotions are not just the product of an appraisal by us of some thing, but the very reason for which some elements of the world are relevant. Emotions are also regulatory

functions in our (human) process of segmenting reality. (p. 9). An emotion is not just a datum but is constitutive of our internal reality.

In his objective of having “emotion measuring techniques”, Tonti notes that emotion is usually taken as a *reaction* to events, and this can be assessed in some physical manner, such as via a report by the subject, biomarkers and bodily signals, etc.

Counterposed to this reflection of emotion, Tonti sets out a more ambitious aim (p. 122): “In the rest of the chapter are presented two ways of measuring emotions conceptually based on the unconscious functioning of the mind, which seeks for the emotional activity not in the presumed emotional responses to an arbitrary stimulus, but in the influence of emotionality in the way of responding. If emotions are a component of thought, then it would be possible to observe the trails of their presence and influence, in the diverse degrees, in the overall process of mind functioning.”

The big advantage of posing the problem of demarcating emotion in this way is that it opens up better vantage points on tracking emotion, and on emotion in context (of other emotions, or emotionated objects, of the subject’s mind processes, and so on).

Furthermore Tonti considers metric, as well as ultrametric, mathematical frameworks.

“The metric content index is therefore a measure of the amount of structure embedded in the neural representations that inform subject choice: It is high when individual memory items are classified using semantic cues, which leads to a more concentrated distribution of errors. It is low either when performance is random (in which case performance measures are also low), or when episodic access to the identity of each famous face is prevalent, semantic relationships remain largely unused, and errors, when made, tend to be more randomly distributed.” (p.146). The foregoing is with reference to the balance or strength of semantic memory versus episodic memory, useful in observing Alzheimer’s patients (where the latter increases in dominance; see [47], p. 14 and elsewhere).

Then the ultrametric understanding of the subconscious is described in these terms: “...an important idea is the one that sees the unconscious in terms of a topological semantic structure with specific features. If we imagine the experience of a life as encoded in a network of representations of facts, ideas, relations and so on, it can hypothesized that some specific distance between the elements can be defined. In this novel interpretation [...] brought forward by Lauro-Grotto (called “ultrametric”), the fabric of this network is modified in a way that, in certain circumstances, the distance between a group  $G$  of otherwise distinct objects and a third object  $X$  is considered to be the same for each of them. If we consider the distance as the probability of going from  $X$  to each of the object in  $G$ , we should conclude that the probability of going from  $X$  to any of the objects in  $G$  is the same, i.e. they are structurally considered to be equivalent.” (p. 129)

This work of [47] is motivation for the new work that follows. We will seek to directly point out emotionated objects, and rank them in accordance with how well they present a characteristic related to the subconscious. This characteristic

is that of being inherently ultrametric.

## 4.2 Data With Relatively High Ultrametricity

From the Dreambank repository [7, 8, 10, 43] we selected various collections. See Murtagh [36, 37] for further description and analyses. One set of 139 dream reports, from one individual, Barbara Sanders, was particularly reliable (according to [10]).

In order to have a text that ought to contain vestiges of ultrametricity because of subconscious thinking, admittedly subconscious thinking that was afterwards reported on in a fully conscious way, we took these Barbara Sanders dream reports. In discussion of this data provided in Domhoff [7] he notes that there is “astonishing consistency” shown in dreams such as these over long periods of time. Taking the set of 139 of the Barbara Sanders dream reports, we used the 2000 most frequently occurring words used in these dream reports including function words. Then we took 30 words to carry out some experimentation with their ultrametric properties. These are listed as follows.

Note that our processing converts all upper case to lower case. The thirty words selected were as follows:

“tyler”, “jared”, “car”, “road”, “derek”, “john”, “jamie”, “peter”, “arrow”, “dragon”, “football”, “lance”, “room”, “bedroom”, “family”, “game”, “mabel”, “crew”, “director”, “assistant”, “balloon”, “ship”, “balloons”, “pudgy”, “valerie”, “dolly”, “cat”, “gun”, “howard”, “horse”

We selected these words to have some personal names, some words that could be metaphors for the commonplace or the fearful, and some words that could be commonplace and hence banal. Names of people are: Tyler, Jared, Derek, John, Jamie, Peter, Lance, Mabel, Valerie, and Howard.

We carried out, firstly, a Correspondence Analysis of all 139 dream reports crossed by the 2000-word set. Then we use for the subsequent analysis the Euclidean, factor space, with full dimensionality, for the 30 selected terms. Figure 1 shows the principal factor plane projection.

The full Euclidean, factor, space dimensionality is the minimum of  $2000 - 1$  and  $139 - 1$ , hence: 138. By selecting 30 terms of interest, our data to be further explored is a table that crosses 30 chosen terms by 138 Euclidean space coordinates, viz. the factor projections. The projections on factors 1 and 2 are to be seen in Figure 1.

## 4.3 A New Principle for Metric and Ultrametric Component Analysis

In seeking ultrametric relationships in data, our approach in section 2 has been to single out triplets of the observables that respect the necessary ultrametric properties.

In section 1.3, the well established data analytic approach imposes an ultrametric topology on data, and then interprets this retrospectively.



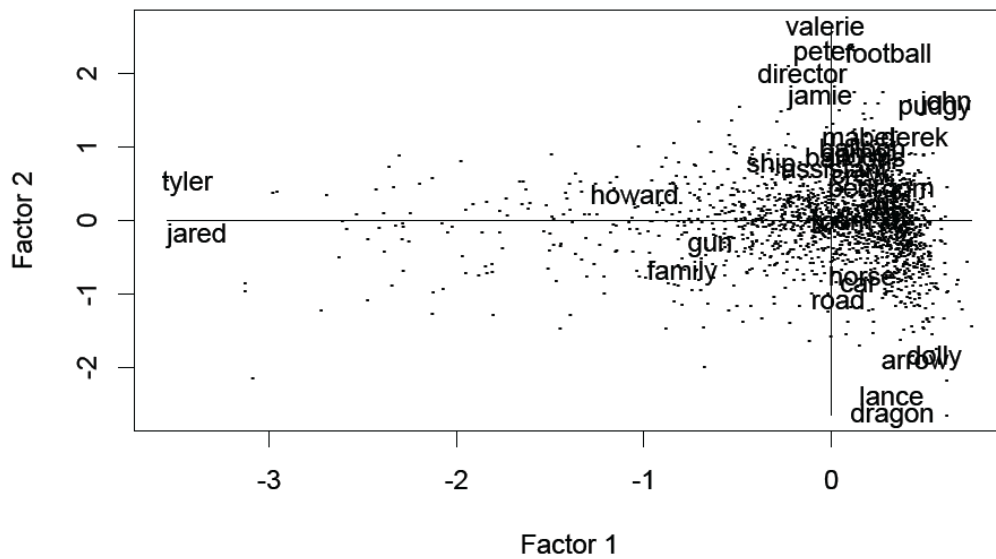


Figure 1: 2000 terms, 30 of them indicated - see how tyler and jared; lance and dragon; valerie, football, peter, director, jamie; etc. come out close. Note this is a planar approximation – 2.2% and 1.37% of the inertia explained by these factors, respectively.

The former, quantifying, approach does not give us insight into what could become part of the ultrametric component in the data, were it possible to bend (or “stretch”, the term used in subsection 3.3) the data in order for the data to become more ultrametric.

The latter approach suffers from having any number of different hierarchies that can be induced on the data, in accordance with criteria (or definitions) for doing so.

We will introduce a new approach to determining the ultrametric component of a data set, by combining both quantifying and transforming approaches. We do so as follows.

1. Determine an ultrametric consensus space of given ultrametric spaces. Operationally, we fit hierarchies, using different agglomerative criteria, to our data, and we determine a consensus hierarchy.
2. From the consensus hierarchy, implying a mapping of our data into this ultrametric topology, we determine with respect to our original, input measurements, just how close the mapped ultrametric relationships are to our input relationships. We retain the better mapped relationships.

In sections 5 and 6 we discuss these two stages of our new principle of ultrametric component analysis.

## 5 Consensus Ultrametric Sets from Two Hierarchies

### 5.1 Ultrametric Consensus: Definition and Algorithm

In Appendix A1, there is a short review of consensus approaches for hierarchical clustering. Here we pose the following consensus problem. Firstly we note how we define ultrametric relations on the basis of any given triangle of observations or points. Therefore, in seeking a consensus between hierarchies, we start by checking each triangle defined from the terminals, that are associated with our observations.

In an agglomerative hierarchical clustering with pairwise agglomerations, and assuming that no input distances are identical (which will not always be respected in practice; however our assumption of unique input distances is aided by the Euclidean, hence continuous, values taken to start with), then it follows that ultrametric cases associated with equilateral triangles will not arise. The isosceles-with-small-base case is the only one that is applicable. Hence, in our considerations here we exclude tied distances that would lead to no smallest base side: we require that there be a “base” side in any triplet.

Our ultrametric consensus algorithm is as follows:

1. Consider all triplets of observations (i.e., singletons associated with terminal nodes in the dendrogram),  $i, j, k$ .

2. Consider also the cophenetic matrix, i.e. the matrix of ultrametric distances,  $d_u$ . Read off:  $d_u(i, j), d_u(j, k), d_u(i, k)$ . Determine:  $d_u^{(1)} < d_u^{(2)} = d_u^{(3)}$ .

3. Next consider that two dendrograms produced by agglomerative hierarchical cluster are at issue. Determine for each:

$$d_{u1}^{(1)} < d_{u1}^{(2)} = d_{u1}^{(3)}$$

$$d_{u2}^{(1)} < d_{u2}^{(2)} = d_{u2}^{(3)}$$

4. Recapitulate: we have selected the same triplet,  $i, j, k$ , in the two hierarchies. We look at the ultrametric relationship between the three points,  $i, j, k$ . These three points must have the property of forming an isosceles triangle with small base. Without loss of generality we ignore the case of an equilateral triangle. (Why is this justified? Firstly, due to each agglomeration in the hierarchical agglomerative algorithm taking two nodes at a time; secondly, given the real values of distance or dissimilarity.)

Let  $i$  be the triangle apex in the first hierarchy, and  $j$  and  $k$  then are the points at the small triangle base. In that case, the distance between  $j$  and  $k$  is the small base side of the triangle connecting the three points. We need to see if the same triangle apex point is present in the second hierarchy. That is all we need to do, in order to have a morphological equivalence between the two cases of ultrametricity, extracted from the hierarchies.

5. For the ultrametric relationships associated with the two hierarchical clusterings to be the same, we require that the base side of the triangles involve the same observations in the two cases:

Given the lowest ranked triangle side in the two cases:

$$d_{u1}^{(1)}(i_1, i_2), d_{u2}^{(1)}(i_3, i_4)$$

we require that the base side involve the same observations: either  $i_1 = i_3; i_2 = i_4$  or  $i_1 = i_4; i_2 = i_3$ .

6. Count these matches between the hierarchies. The count is that of isosceles-with-small-base that are consistent for the two hierarchies. The total number of triangles considered, for  $n$  observations (hence  $n$  terminal nodes in the hierarchical tree), is  $n \cdot (n-1) \cdot (n-2) / (3 \cdot 2 \cdot 1)$ . This furnishes a coefficient of ultrametric consensus between the two hierarchies, or ultrametric embedding of the same set of observations.

7. For all isosceles-with-small-base cases, these determine a consensus between the two hierarchies. The consensus is the set of triplets of points (i.e. singleton clusters, or terminals in the dendrogram, or observations) that (i) respect the ultrametric condition of having a triangle with small base – which is a necessity, given the hierarchical or tree structure of the data; and (ii) that have the same apex point in the two hierarchies considered.

## 5.2 Application: Ultrametric Consensus of Hierarchies

Figure 5 shows hierarchical clusterings with three agglomerative criteria. The ultrametric relationship consistency between these hierarchies was found to be the following. Firstly, there are 4060 triangles in total. For Ward and Single link, there were 2369 matching isosceles-with-small-base triangles. For Ward and Complete link, there were 3084 matching ultrametric cases. For Single link and Complete link, there were 3128 matching cases.

## 5.3 Inversions in Hierarchical Clustering

A requirement for our ultrametric consensus algorithm is that the hierarchy does not contain inversions. In section 5 in step 4 it is necessary that the lowest ranked triplet (or triangle) side is the distance associated with the first (in the sequence of agglomerations) agglomeration between observations. When there are inversions in the hierarchy, which we will now exemplify, this first agglomeration involving triplet vertices may be associated with a distance, hence triangle side, that is not the lowest ranked.

Figure 6 shows quite a few inversions in the sequence of agglomerations, meaning that there is non-monotonic increase in inter-cluster (cluster encompassing singleton cluster also) distance as the agglomerations are carried out. Inversions occur for particular agglomerative criteria. See [29]. These include the centroid and median criteria. These therefore must be avoided in such ultrametric consensus finding since the base of the isosceles triangle is not necessarily found with such hierarchies.

A condition for a hierarchy to be guaranteed not to give rise to inversions was provided in [2]. This is formulated in one form of Bruynooghe's *reducibility property* [2] as:

$$\text{Inversion impossible if: } d(i, j) < d(i, k) \text{ or } d(j, k) \Rightarrow d(i, j) < d(i \cup j, k)$$

## 5.4 Ultrametric Consensus

Taking the 30 selected terms, originally in a 2000-dimensional space that leads to the semantic space that jointly represents the projected terms and observations (the latter: dream reports), we find the following for the number of consistent ultrametric triplets from the total set of 4060 triplets.

To recapitulate the ultrametric consensus algorithm used for the two hierarchies, using their derived cophenetic, i.e. ultrametric, distances:

For each triplet, take the pairwise ultrametric (cophenetic) distances; rank them, for the triplet in both cases; the ranks are 1, 2.5, 2.5 for the case of isosceles-with-small-base; rarely but possibly the ranks are 2, 2, 2 for the case of an equilateral triangle. Knowing now the small side of the isosceles triangle, check if this is based on the same pair (of points in the triplet, hence of triangle

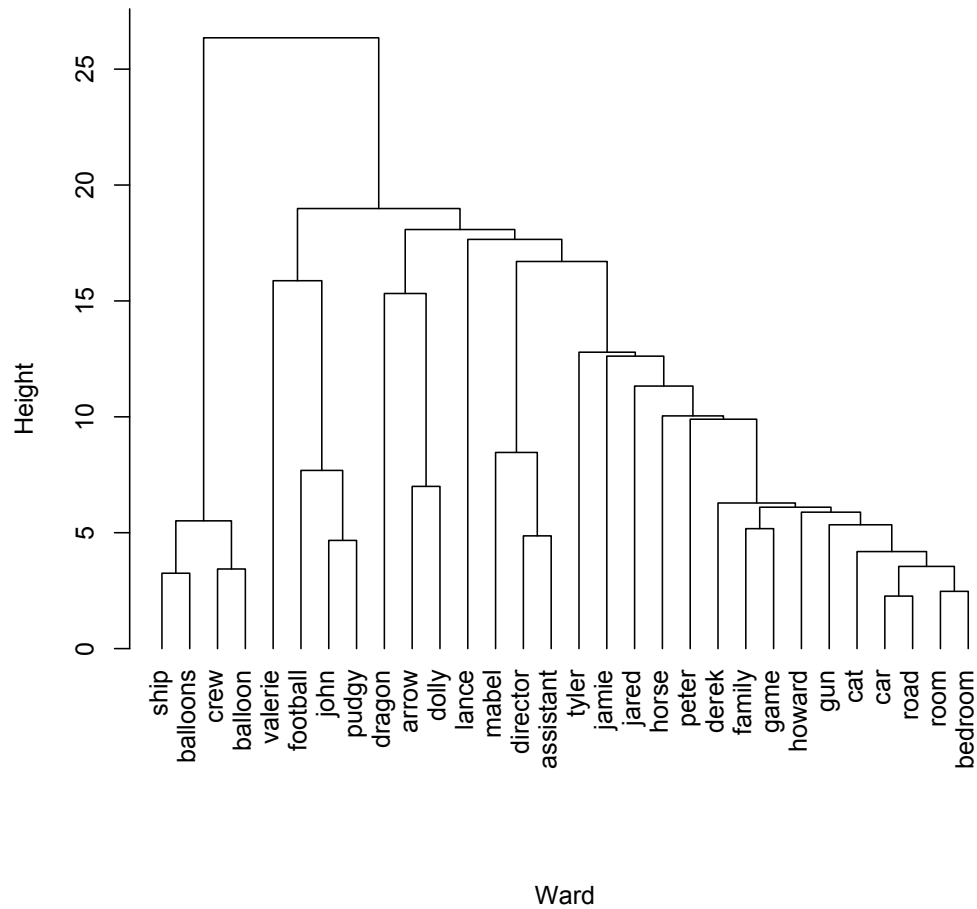


Figure 2: Hierarchical clustering using the Ward minimum variance agglomerative criterion. (Relative to Figures 3, 4, and just as one example, the similar clustering sequence of “ship”, “balloons”, “crew”, “balloon” is to be noted in all three cases.)

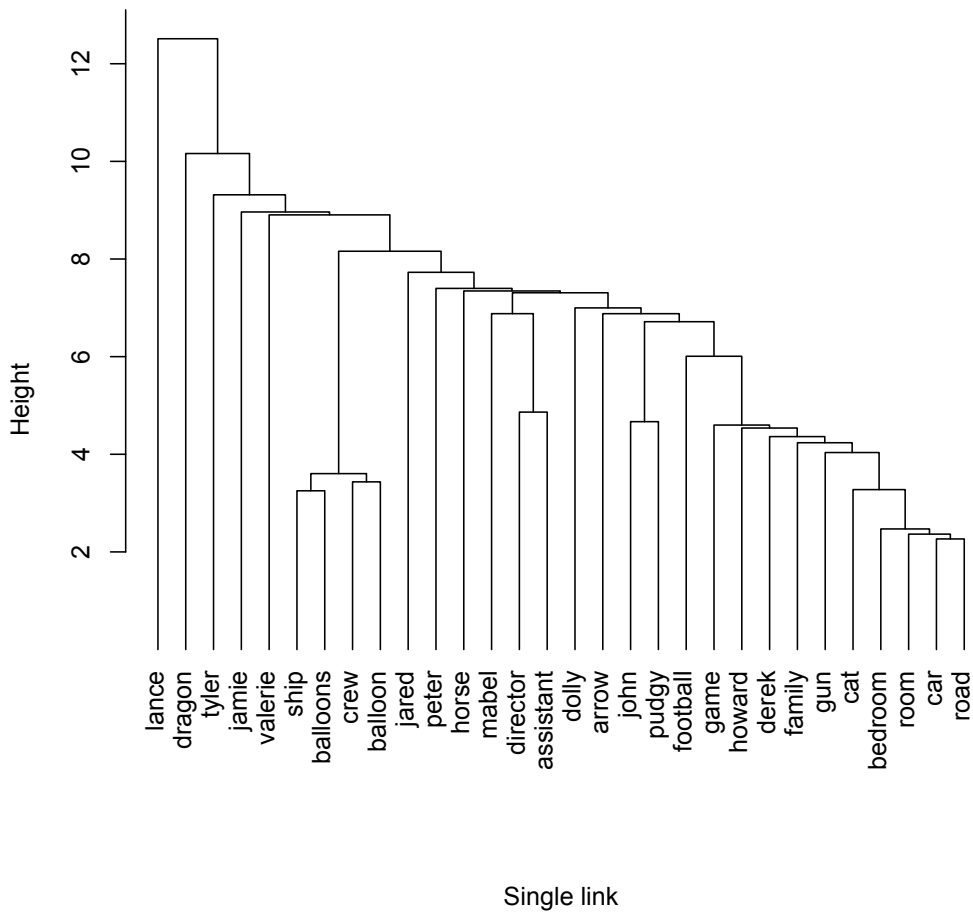


Figure 3: Hierarchical clustering using the single link agglomerative criterion. (Relative to Figures 2, 4, and just as one example, the similar clustering sequence of “ship”, “balloons”, “crew”, “balloon” is to be noted in all three cases.)

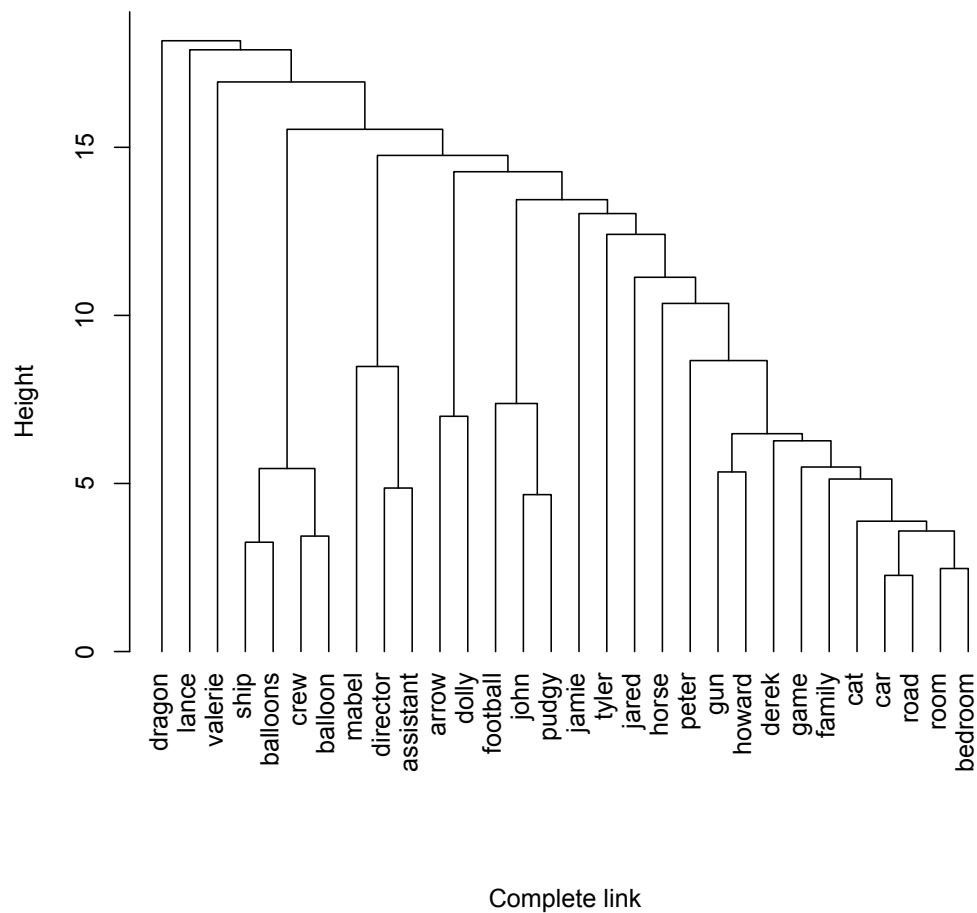


Figure 4: Hierarchical clustering using the complete link agglomerative criterion. (Relative to Figures 2, 3, and just as one example, the similar clustering sequence of “ship”, “balloons”, “crew”, “balloon” is to be noted in all three cases.)

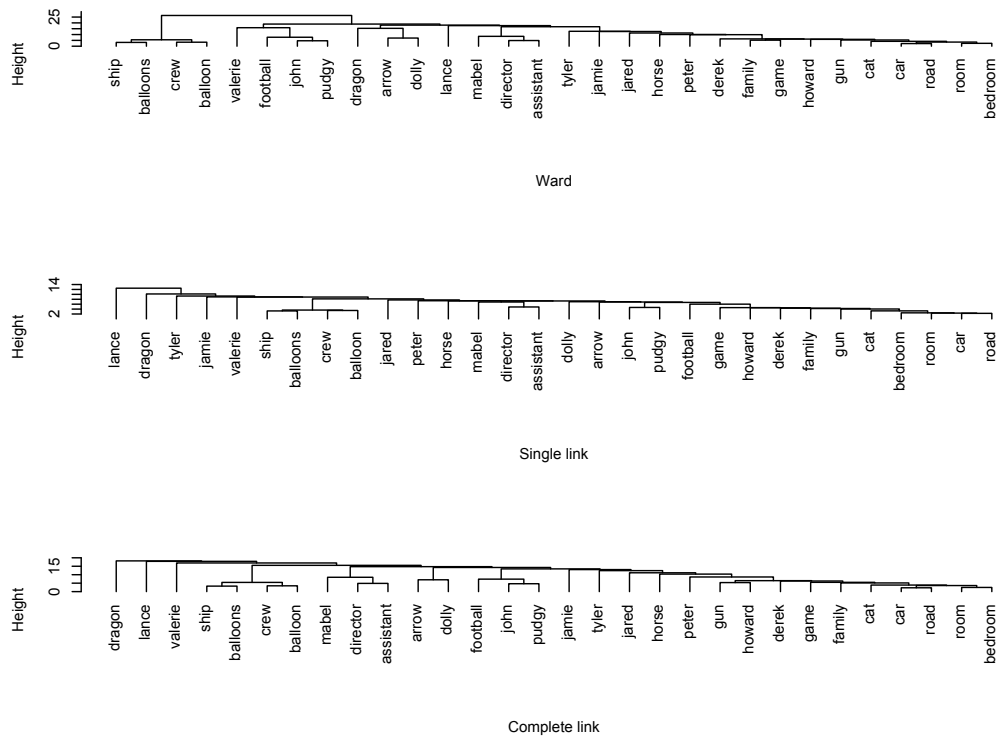


Figure 5: For ease of comparison, this shows together Figures 2, 3, 4. Hierarchical clusterings using (top) the Ward minimum variance agglomerative criterion, (middle) the single link method, and (bottom) the complete link method. (Just as one example, the clustering of “ship”, “balloons”, “crew”, “balloon” is to be noted in all three cases.)



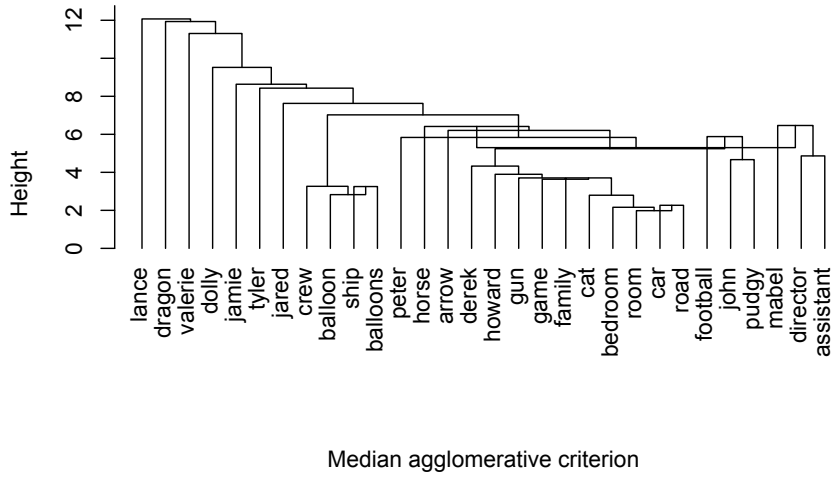


Figure 6: Inversions in the sequence of agglomerations. That is,  $i$  and  $j$  merge, and the distance of this new cluster to another cluster is *smaller* than the defining distance of the  $i, j$  merger. Hence, there is non-monotonic change in the level index, given by the distance defining the merger agglomeration.

	Ward	Average	Single	Complete	McQuitty
Ward	4060	3050	2369	3084	3014
Average	3050	4060	3311	3480	3830
Single	2369	3311	4060	3128	3336
Complete	3084	3480	3128	4060	3327
McQuitty	3014	3830	3336	3327	4060

Table 1: Hierarchies – see Figures 2, 3, 4 – are compared by looking at each triplet. Given that each hierarchy uses the same set of  $n = 30$  observations, there are  $n \cdot (n - 1) \cdot (n - 2) / (3 \cdot 2 \cdot 1) = 4060$  triplets. The values in this table give the numbers of isocles-with-small-base triplets found, that are morphologically consistent, i.e. with the same triangle base pair, across the two hierarchies compared.

	Ward	Average	Single	Complete	McQuitty
Ward	4060	2103	1714	2478	1978
Average	2103	4060	3608	1762	2955
Single	1714	3608	4060	1500	2770
Complete	2478	1762	1500	4060	2134
McQuitty	1978	2955	2770	2134	4060

Table 2: Using randomly generated data (see text for details), the table shows the numbers of morphologically consistent, vertex-labelled, ultrametric triangles between pairs of hierarchies.

vertices) in the case of the two triplets. If it is, we have ultrametric consistency between the two triplets.

Table 1 shows that a great deal of consensus can be obtained from the hierarchies. Comparing a hierarchy against itself, as done for the main diagonal elements of the table, is just done to find what the expected full consistency.

In Table 2 we benchmark the Table 1 results by taking some random data.

We “mirrored” the real data used throughout this work in the following way. First, we generated a matrix with uniformly distributed values on  $[0,1]$ , of dimensions  $139 \times 2000$ . We mapped it into a Euclidean representation, for full consistency of treatment with the data used throughout this work. Then using the full dimensionality, 138 (i.e. one less than the minimum of the row and column dimensions of the input data matrix) of the Euclidean, Correspondence Analysis mapping into the Euclidean space, we selected 30 vectors. (How we did this was as follows. We used exactly the same indices as used in our real data, selecting from the 2000 vectors. Hence, we replicated what was done on the real data, in order to furnish a comparable benchmark in all respects.)

We then induced hierarchies, with the agglomerative criteria noted – Ward, Average, Single, Complete, McQuitty. Figures 7, 8 and 9 show three of these hierarchies.

Table 1 has, throughout, higher values of ultrametric consistency between hierarchies, compared to Table 2. While this is fully consistent in turn with greater ultrametricity in the initial data, which we have demonstrated to be so, nonetheless we issue the caveat here that what we are dealing with are hierarchies that are induced on the data. Hence we need to be careful in drawing too many conclusions from this, insofar as the inducing of hierarchical relationship are forcing an increase in ultrametricity. See also section 6 below.

## 5.5 From the Ultrametric Consensus Set to a Consensus Hierarchy

We form a consensus hierarchy as follows. For the two hierarchies, with ultrametric (cophenetic) distances that are denoted  $d_{u1}$  and  $d_{u2}$ , we form a new set of ultrametric distances, denoted  $d_u^{\text{Cons}}$ , as follows. As above, a morphologically consistent isosceles-with-small-base triplet means that the apex vertex has the

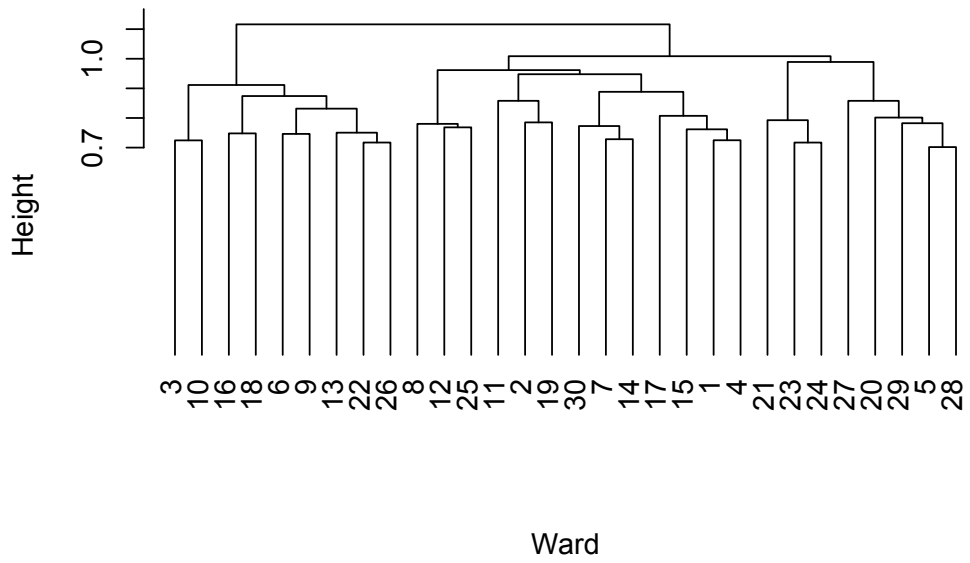


Figure 7: Hierarchical clustering using the Ward minimum variance agglomerative criterion. Input data are randomly generated. (See text for details.)

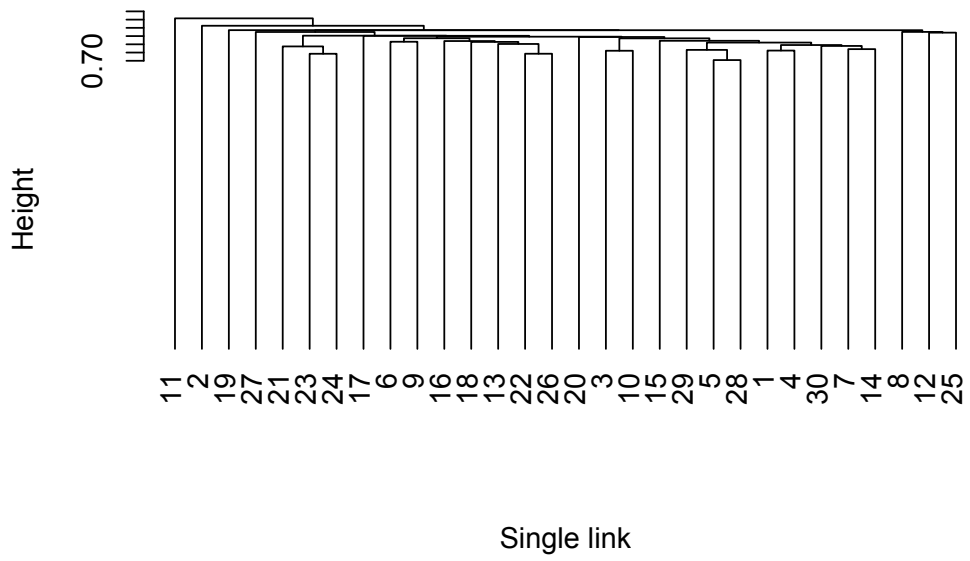


Figure 8: Hierarchical clustering using the single link agglomerative criterion. Input data are randomly generated. (See text for details.)

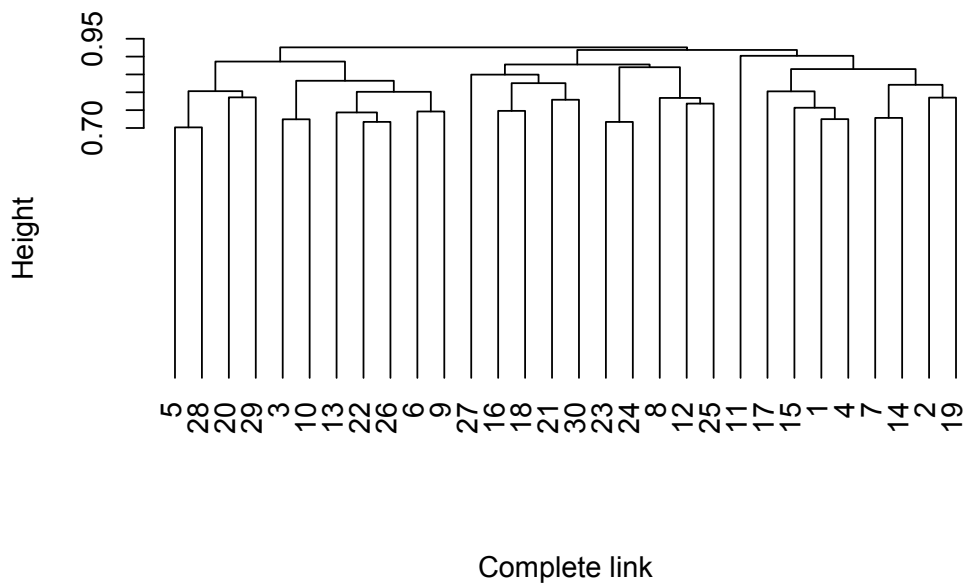


Figure 9: Hierarchical clustering using the complete link agglomerative criterion. Input data are randomly generated. (See text for details.)

same label.

1. For all isosceles-with-small-base triples that are morphologically consistent, use the minimum, between the two hierarchies considered, pairwise distances:

$$d_u^{\text{Cons}}(i, j) \leftarrow \min\{d_{u1}(i, j), d_{u2}(i, j)\}$$

2. For morphologically inconsistent triplets, we put all

$$d_u^{\text{Cons}}(i, j), d_u^{\text{Cons}}(i, k), d_u^{\text{Cons}}(j, k)$$

in the triplet considered to be equal to the minimum of all triplet pair distances, i.e. the single minimum value,

$$\min\{d_{u1}(i, j), d_{u1}(i, k), d_{u1}(j, k), d_{u2}(i, j), d_{u2}(i, k), d_{u2}(j, k)\}$$

We use the minimum in view of the maximal inferior ultrametric properties that ensue, i.e. the optimal fit from below (cf. section 1.3.3).

The consensus hierarchy is such that the new hierarchy determined from the hierarchies associated with  $u1$  and  $u2$  is the same as the new hierarchy determined from the hierarchies associated with  $u2$  and  $u1$ . It is commutative and unique for a given input pair of hierarchies.

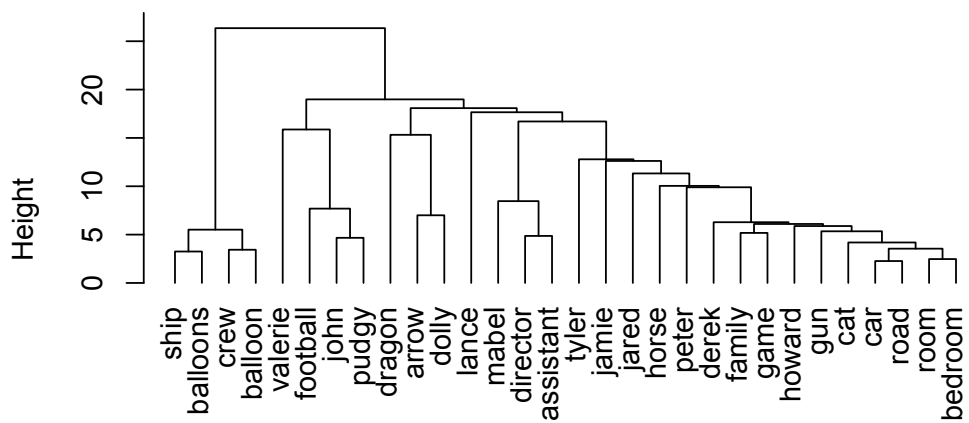
A less good property of this consensus hierarchy is that it is a function of the pair of hierarchies used as input. A different pair of input hierarchies will yield a somewhat different outcome.

## 6 Taking the Ultrametric Consensus Set Beyond Model Fitting: Inherent Ultrametric Properties

### 6.1 Coming from a Space Endowed at least with a Dissimilarity, and Analyzing it when Mapped into an Ultrametric Space

Thus we have a way to quantify ultrametric consensus, and the consensus set in terms of triplets of observations. Our next concern though is to recognize that the hierarchical clustering may be a good ultrametric embedding of our data. But it is induced on our data. More informally we can say that it is forced on our data.

Now we will relate ultrametric consensus back to our original, empirical and known data.



Consensus hierachy

Figure 10: Consensus hierarchical clustering using the Ward minimum variance and single link agglomerative criteria, with the Barbara Sanders data. Hence consensus of Figures 2 and 3.

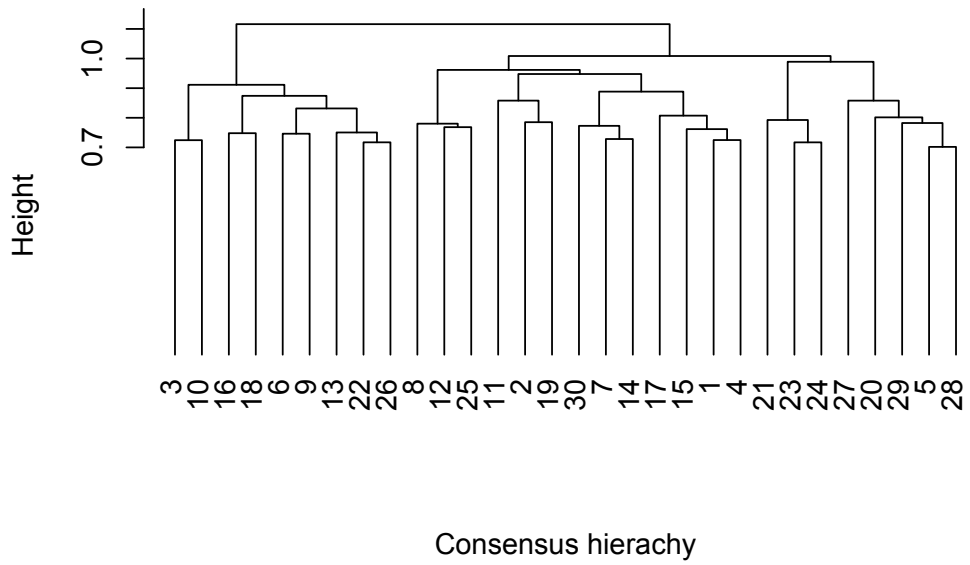


Figure 11: Consensus hierarchical clustering using the Ward minimum variance and single link agglomerative criteria. Random data used as input. Hence consensus of Figures 7 and 8.



## 6.2 Ultrametric Triplet Sets Through Other Pairs of Hierarchies

A further reason to relate ultrametric consensus back to our original, empirical and known data is as follows.

In the foregoing we looked at Ward and single link as the pair of hierarchies from which we looked for consensus ultrametric triplets, and then proceeded to find a subset of those triplets, 163 of them, that were  $\alpha_\epsilon$ -ultrametric in the original metric data space.

Looking at the single and complete link hierarchies gave 190  $\alpha_\epsilon$ -ultrametric triplets. Looking at Ward and complete link gave 193  $\alpha_\epsilon$ -ultrametric triplets.

We recall that any such consensus hierarchy provides us with the candidate set of  $\alpha_\epsilon$ -ultrametric parts of our data, for a given value of  $\epsilon$ . We can easily determine the most ultrametric parts of our data. This we do from a consensus hierarchy pair in Appendix 3.

## 6.3 Just How Good Are the Ultrametric-Respecting Triplets Relative to the Input Data?

Figure 12 shows the values of  $\alpha_\epsilon$ . In previous work [31, 33, 37], we took  $\epsilon = 2$  degrees, or 0.034906585 radians.

Arising out of this look at a continuum of  $\epsilon$  values, we conclude that neither the very large  $\alpha_\epsilon$  values, nor any other consideration, make us alter our view that  $\epsilon = 2$  degrees, or 0.034906585 radians is a satisfactory compromise between a small difference from identity ( $\epsilon = 0$ ), while allowing a certain number of triangles to have this property of sufficiently nearly identical base angles.

For the real data, we find 163 triplets, that are, as we know, consistent in isosceles-with-small-base morphology across two hierarchies (Ward, single link used), and that have an  $\alpha_\epsilon$  value (i.e., difference in angles at the triplet triangle base)  $\leq 0.034906585$  radians. For the random data, we find 549 such triplets. These numbers correspond to a threshold in Figure 12 of (ordinate) 0.034906585 radians.

Such triplets therefore have the following properties.

1. Each such triplet is embedded in an ultrametric space. This implies that the associated triangle, in metric space terms, has a small base, and is isosceles.
2. An amendment needs to be made to the foregoing point (as we have noted before): we can have equilateral triangles. The likelihood is small if (as in this work) we embed our data in a Euclidean space. Our assertion here is that the number of identical distances on input is zero.
3. By taking two hierarchical clusterings, each of which impose an ultrametric embedding on the data, we determine a candidate set of consistent ultrametric triplet properties between the two hierarchies.

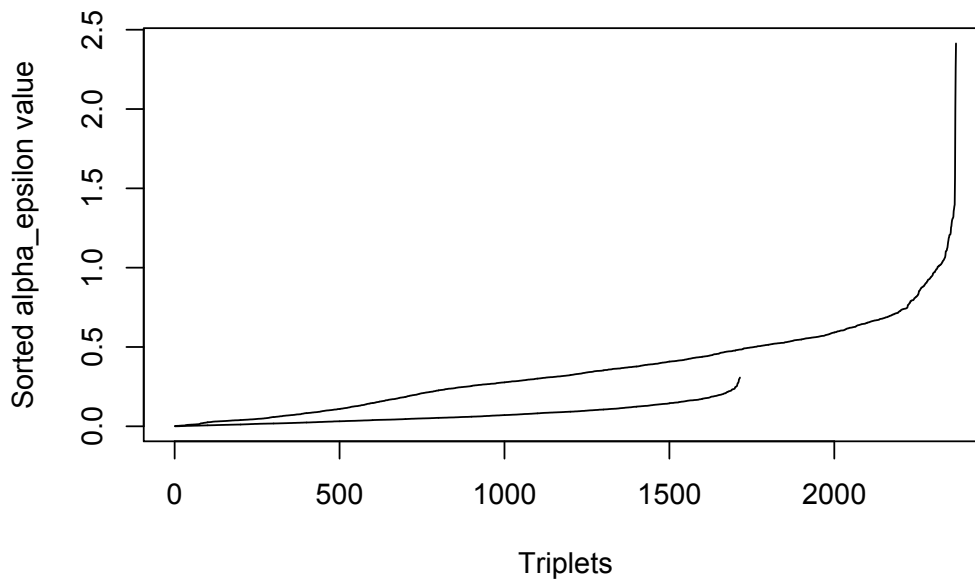


Figure 12: [CORRECTION: ORDINATE IS EPSILON VALUES, IN RADIANS. SEE NEXT FIGURE.] The upper curve relates to the consensus of the Ward and single link hierarchies, see Figures 2 and 3. The lower curve relates to the consensus of the Ward and single link hierarchies from random data, see Figures 7 and 8. From Tables 1 and 2, we note that there were, respectively, 2369 and 1714 ultrametric-respecting triplets. The figure shows the sorted values of the  $\alpha_\epsilon$  measure, determined from the input (i.e. to the hierarchical clustering) data for the triplets.

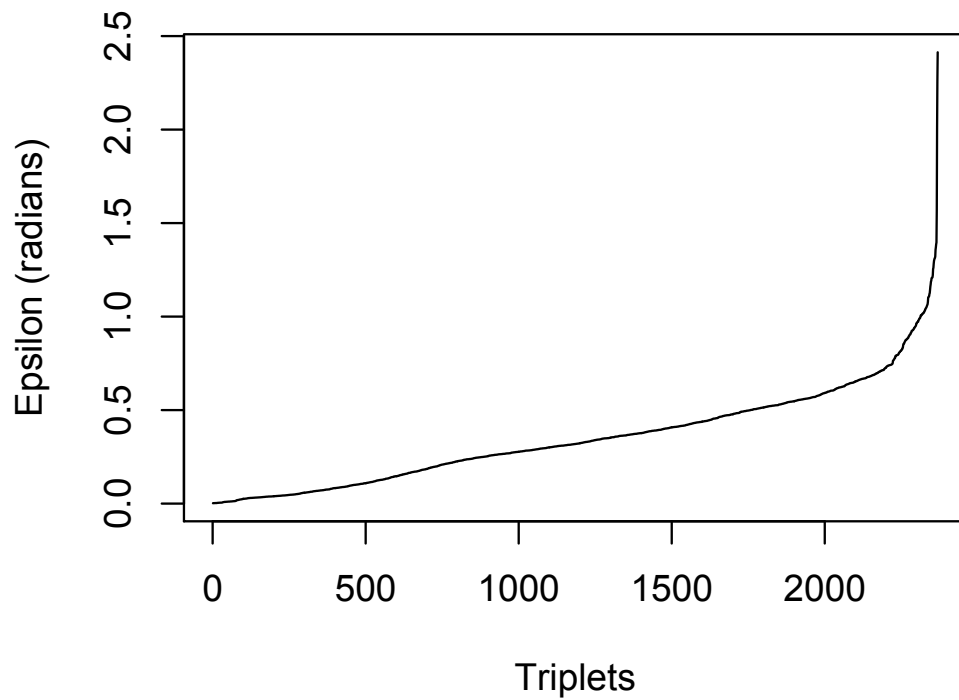


Figure 13: CORRECTED VERSION OF PREVIOUS FIGURE – LABEL ON ORDINATE. From the consensus of two hierarchies (Ward and single link hierarchies, see Figures 2 and 3) all triplets are, by construction, ultrametric. However, with reference to their origin, as points in a Euclidean space (30 points in a 139-dimensional space), how close were they to being inherently ultrametric? I.e. with very small  $\epsilon$ . We see in the figure that very few are genuinely ultrametric, in this sense. Typically we use  $\epsilon = 0.034906585$ .

4. A hierarchy, as noted, imposes an ultrametric structure on a set of data. Furthermore each such hierarchy, using as it does an agglomerative clustering criterion, imposes a somewhat different hierarchical or ultrametric structure on the data. This is so because each hierarchy can be said to be approximating the data, and the criteria are different (albeit based on relative compactness or density or linkage).
5. We chose to work with the Ward minimum variance, and single link, agglomerative clustering criteria because they were found to be more distinct and contrasting. The minimum variance criterion seeks good compactness for the clusters formed (embedded in the hierarchy), while the single link criterion is based on minimal connecting edges between clusters.
6. We have a consistent set of ultrametric triplets coming from the two hierarchies. In this candidate set of ultrametric triplets, and given that the hierarchies were “imposed” on the data (even if in a particular best fit way), we ask how good the ultrametric properties are on the original input data,
7. Taking all candidate sets of ultrametric triplets, we determine their  $\alpha_\epsilon$  values for  $\epsilon = 2$  degrees.
8.  $\alpha_\epsilon$  values are determined from the input data, i.e. the Euclidean-embedded, Correspondence Analysis output, that is full dimensional.

## 6.4 Interpretation of Ultrametric Triplets

The figures to follow use the same planar projection as was used in Figure 1. The inherent dimensionality is 138, and this was the basis for the  $\alpha_\epsilon$  property testing, and before that the metric context for inducing the hierarchical clustering.

In Figure 14, the 163 triplets are depicted by a line segment connecting the three sides of each triangle. The small base side is in red. Recall that this is a depiction of isosceles with small base triangles that satisfy the  $\alpha_\epsilon$  property. They are depicted here projected from their 138-dimensional space into the principal factor plane.

We can look at detail in Figure 14 by taking terms separately.

Figure 15 depicts three ultrametric triplets respecting the  $\alpha_\epsilon$  property for  $\epsilon = 2$  degrees. (To recap: the angles at the small base of the isosceles must be different by less than or equal to 2 degrees). In terms of the 30 selected terms on which we are working, these triplets are as follows: 5, 29, 1; 1, 7, 10; and 1, 7, 12. The small base is formed by the first two of the terms.

The terms of the three ultrametric triplets are as follows: 1 = “tyler”; 5 = “derek”; 7 = “jamie”; 10 = “dragon”; 12 = “lance”; and 29 = “howard”. In Figure 15, the small base in the isosceles triangles is in red. For example, therefore, “howard” and “derek” form a small base, with triangle apex “tyler”. The names “howard” and “derek” are clustered closely relative to the more different

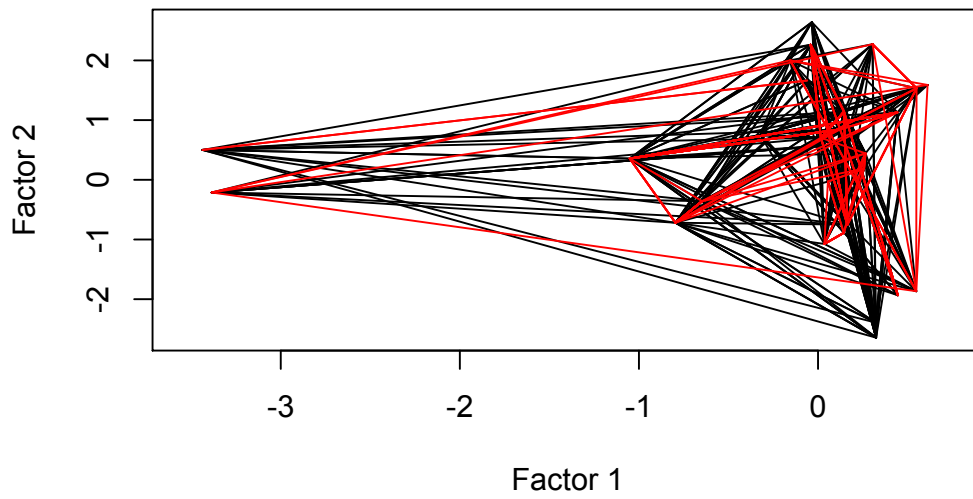


Figure 14: The set of 163 ultrametric triplets, coming from the consensus Ward and single link hierarchies, and respecting the  $\alpha_\epsilon$  property for  $\epsilon = 2$  degrees. Edges drawn, the small base in the isocetes triangle shown in red. Planar projection of the 138-dimensional space. Cf. the same planar projection in Figure 1.

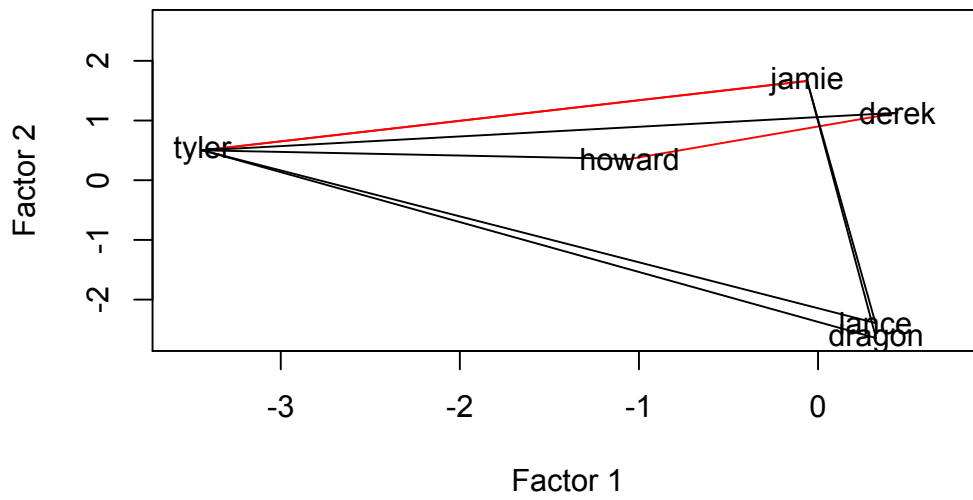


Figure 15: The first three of the set of 163 ultrametric triplets, coming from the consensus Ward and single link hierarchies, and respecting the  $\alpha_\epsilon$  property for  $\epsilon = 2$  degrees. Edges drawn, the small base in the isocles triangle shown in red. Planar projection of the 138-dimensional space. Cf. the same planar projection in Figure 1.

name, “tyler”. Similarly the close pair, “jamie” and “tyler” are contrasted both with the name “lance” and the term “dragon”.

The dates of the 139 Barbara Sanders dreams used were from 09/29/80 (29 September 1980) to 01/25/97 (25 January 1997).

According to [9], personages in the Barbara Sanders dreams are:

- Howard: her ex-husband, divorced, died suddenly in 1997.
- Derek: a man she had an affair with from 1994, and broke with in 1996.
- Darryl: ex-boyfriend.
- Dwight: favourite brother.

According to a look at the dream reports, the following are other people and things:

- Tyler (1980), “co-worker”
- Jamie (1981), “old friend”, “homosexual”
- Mabel (1981), community college friend. 1991, “co-worker”.
- Peter (1980), “a para, a student at the community college”, married.
- horse (1981 to 1991, but mostly in a 1980 and a 1981 dream).
- dragon (1993) (no other names mentioned in this dream about a dragon)
- Lance (1992) “is black and used to be the city manager assistant to the disability rights city group”; “married”.
- game, gun, family: words used (in differing contexts) throughout the years.

The full 163 triplet set of ultrametric cases is in Appendix A3. The following is a small extracted set of these. Note that labels are alphabetically ordered for the two base vertices of the triangle, columns 1 and 2, and the triangle apex is in column 3. Then comes the  $\alpha_\epsilon$  difference of base angles, in radians.

Note, in the following, the later period (i.e. 1992, 1993) dragon, Lance, counterposed to the earlier period (i.e. 1980, 1981) presence of Tyler, Jamie, Mabel, Peter, horse.

[9,]	"tyler"	"jamie"	"dragon"	"0.002900765"
[10,]	"tyler"	"jamie"	"lance"	"0.002900765"
[47,]	"family"	"gun"	"jamie"	"0.01004061"
[48,]	"family"	"gun"	"tyler"	"0.01007054"
[68,]	"peter"	"horse"	"jamie"	"0.01281991"
[69,]	"peter"	"horse"	"tyler"	"0.012843"

[40,]	"peter"	"mabel"	"dragon"	"0.009091976"
[41,]	"peter"	"mabel"	"lance"	"0.009091976"
[52,]	"peter"	"horse"	"dragon"	"0.01065118"
[53,]	"peter"	"horse"	"lance"	"0.01065118"
[81,]	"mabel"	"horse"	"dragon"	"0.01887733"
[82,]	"mabel"	"horse"	"lance"	"0.01887733"
[26,]	"game"	"howard"	"jamie"	"0.005031676"
[27,]	"game"	"howard"	"tyler"	"0.005045841"
[16,]	"game"	"howard"	"dragon"	"0.003874804"
[17,]	"game"	"howard"	"lance"	"0.003874804"
[102,]	"derek"	"howard"	"dragon"	"0.02580133"
[103,]	"derek"	"howard"	"lance"	"0.02580133"
[134,]	"derek"	"game"	"dragon"	"0.03133084"
[135,]	"derek"	"game"	"lance"	"0.03133084"

At least as regards this selection, we observe association of:

- Tyler, Jamie;
- family, gun;
- Peter, horse; Mabel, horse; Peter, Mabel;
- Howard, game; Derek, game; Howard, Derek.

We also see close association, in this selection, between:

- Tyler, Jamie;
- Howard, Derek.

In this short exploration, we see how we can focus in on particular associations of names or of terms, and also on contrasting names and terms. Through ultrametricity we have a ranking from near perfect ( $\alpha_\epsilon = 0$ ) up to our imposed limit of a 2 degree difference between base angles in triangle configurations.

## 7 Conclusion

Our analysis of triangle properties can be related to triads, or triadic patterns, in other areas of speech or writing. Veale [51] notes the following: “Recent work by cognitive scientists Jeffrey Loewenstein and Chip Heath shows that the AAB pattern in stories – which they call the repetition-break plot structure – is considered more enjoyable by readers than the equivalent AAA (unbroken repetition) or ABC (no repetition) patterns. Many narrative jokes use explicit



repetition to enforce an AAA pattern in the minds of an audience, so that AAB repetition-break comes as an incongruous and potentially humorous surprise.”

He cites [27] and [42]. He gives various examples, and adds: “There are whole genres of jokes involving a priest, a rabbi and an imam; or an Irishman, and Englishman and a Scotsman; or a trio of nuns, hookers, husbands or some other stock characters, in which two of the three act somewhat predictably while the zany actions of the third provide the humorous departure.”

The above is noted as another example of where an ultrametric triangle provides a visualization for the two base vertices of a triangle, followed by a quite distinct, and possibly distant, triangle apex.

Future work will aim at addressing humour and jokes, through use of an encoding of successive assertions (rather than the word-based approach that was developed in this work described in this paper).

## Appendix

### A1. Comparing and Combining Hierarchical Clusterings

The ultrametric distance associated with a dendrogram (see section 1.3.2) is commonly termed cophenetic distance [46]. This distance is the smallest one for which two observations are in the same cluster. Informally expressed it is the distance to the closest common ancestor. Comparing this distance with the input distances used leads to a correlation measure, cophenetic correlation [41]. The latter is a goodness of fit measure between input and the hierarchy that is fit to the data.

By comparing the cophenetic distance matrices we also have a means of assessing how close two dendrograms are to one another. Hence this is the most immediate and most widely used approach to comparison of hierarchical clusterings. See [25] for discussion and applications.

Given two or more hierarchical clusterings, Zheng et al. [52] combine ultrametric distances in a comparative study involving (i) various ultrametric distances derived from a dendrogram, including cophenetic distance; rank distance; and cluster, partition and subtree cardinality measures; (ii) adding ultrametric distance matrices over the set of dendrograms, and then using the transitive dissimilarity (to be explained next) as a means to force this aggregated set of ultrametries to be, itself, ultrametric.

On a path connecting vertices  $i$  and  $j$ , consider the transitive dissimilarity as the maximum edge length on that path. Then over all paths between the two vertices, the minimal transitive dissimilarity is determined. This is an ultrametric. (A modified Floyd-Warshall transitive closure algorithm is used to determine these new distances between all  $i$  and  $j$ , in [52]).

Path and subtree combinatorial optimization represents a commonly used approach for tree difference or modification, e.g. [13] or [24].

Morlini and Zani [28] define and study in depth a new dissimilarity measure between dendrograms that takes into account the partitions and clusters, as well

as the embeddedness relationships.

## A2. Metrics and Ultrametrics based on 3-Way Distances

Our  $\alpha_\epsilon$ -ultrametricity measure involves looking at, and testing for ultrametricity, in triangles. Each triangle must be isosceles-with-small-base, or equilateral, to be a case of an ultrametric relation in the triplet considered. The measure  $\alpha_\epsilon$  considers a triplet  $i, j, k$  and looks at distances  $d_{ij}, d_{ik}, d_{jk}$  or angles between them.

An idea is to take the consideration of ultrametric or not, using the triplet  $i, j, k$ , to the different context of a 3-way distance,  $d_{ijk}$ . In [20], such 3-way distances are defined, and termed: semi-perimeter distance, star distance, inertial distance, and restriction of 3-way distances to 2-way. Three-way incidence tables, i.e. tensors, are considered, and the minimal spanning tree is generalized to triplets where “connecting edges by an extremal point” becomes instead “connecting triangles by a common side”.

The matching of triplet index matrices (tensors) and fitting a tree under the  $\ell_\infty$  or max norm are considered in [15].

## A3. Ultrametric Alpha Epsilon-Respecting Triplets Using Ward and Single Link

There are 163  $\alpha_\epsilon$  property triplets in the following. The 2 degrees capped difference between the base angles (of the isosceles-with-small-base triangles) equals 0.034906585 radians. Column 4 in the following gives the actual difference of angles. The other columns are such that column 3 is the triangle apex.

	[,1]	[,2]	[,3]	[,4]
[1,]	"family"	"gun"	"arrow"	"0.001851383"
[2,]	"car"	"road"	"cat"	"0.002350057"
[3,]	"road"	"bedroom"	"balloon"	"0.002403404"
[4,]	"assistant"	"horse"	"dragon"	"0.002445828"
[5,]	"assistant"	"horse"	"valerie"	"0.002493735"
[6,]	"assistant"	"horse"	"ship"	"0.002621979"
[7,]	"assistant"	"horse"	"balloons"	"0.002705753"
[8,]	"assistant"	"horse"	"balloon"	"0.002866144"
[9,]	"tyler"	"jamie"	"dragon"	"0.002900765"
[10,]	"tyler"	"jamie"	"lance"	"0.002900765"
[11,]	"john"	"director"	"balloon"	"0.002964512"
[12,]	"john"	"dolly"	"balloon"	"0.002971291"
[13,]	"bedroom"	"cat"	"pudgy"	"0.003025777"
[14,]	"assistant"	"horse"	"crew"	"0.003123132"
[15,]	"car"	"bedroom"	"cat"	"0.003820139"
[16,]	"game"	"howard"	"dragon"	"0.003874804"
[17,]	"game"	"howard"	"lance"	"0.003874804"
[18,]	"family"	"assistant"	"lance"	"0.003954736"
[19,]	"game"	"howard"	"ship"	"0.004253321"
[20,]	"game"	"howard"	"balloons"	"0.004444716"

[21,]	"game"	"howard"	"director"	"0.004576373"
[22,]	"game"	"howard"	"dolly"	"0.004598292"
[23,]	"road"	"bedroom"	"cat"	"0.004736657"
[24,]	"game"	"howard"	"balloon"	"0.004836543"
[25,]	"director"	"dolly"	"valerie"	"0.004974686"
[26,]	"game"	"howard"	"jamie"	"0.005031676"
[27,]	"game"	"howard"	"tyler"	"0.005045841"
[28,]	"director"	"dolly"	"ship"	"0.00514599"
[29,]	"car"	"road"	"game"	"0.005168709"
[30,]	"director"	"dolly"	"balloons"	"0.005252192"
[31,]	"game"	"howard"	"crew"	"0.005558694"
[32,]	"game"	"howard"	"arrow"	"0.006075735"
[33,]	"mabel"	"assistant"	"game"	"0.006910084"
[34,]	"family"	"gun"	"dragon"	"0.007645213"
[35,]	"mabel"	"assistant"	"road"	"0.007757971"
[36,]	"family"	"gun"	"valerie"	"0.007849058"
[37,]	"family"	"gun"	"ship"	"0.008419404"
[38,]	"family"	"gun"	"balloons"	"0.008814132"
[39,]	"family"	"gun"	"director"	"0.009087052"
[40,]	"peter"	"mabel"	"dragon"	"0.009091976"
[41,]	"peter"	"mabel"	"lance"	"0.009091976"
[42,]	"family"	"gun"	"dolly"	"0.009132605"
[43,]	"family"	"gun"	"john"	"0.009603296"
[44,]	"family"	"gun"	"balloon"	"0.009629988"
[45,]	"peter"	"assistant"	"dragon"	"0.009635763"
[46,]	"peter"	"mabel"	"ship"	"0.009780957"
[47,]	"family"	"gun"	"jamie"	"0.01004061"
[48,]	"family"	"gun"	"tyler"	"0.01007054"
[49,]	"peter"	"mabel"	"balloons"	"0.01011235"
[50,]	"peter"	"assistant"	"ship"	"0.0103711"
[51,]	"bedroom"	"assistant"	"lance"	"0.01064924"
[52,]	"peter"	"horse"	"dragon"	"0.01065118"
[53,]	"peter"	"horse"	"lance"	"0.01065118"
[54,]	"peter"	"assistant"	"balloons"	"0.01072554"
[55,]	"peter"	"mabel"	"balloon"	"0.01075504"
[56,]	"family"	"gun"	"jared"	"0.0108418"
[57,]	"family"	"gun"	"football"	"0.01129312"
[58,]	"football"	"pudgy"	"director"	"0.01135783"
[59,]	"peter"	"assistant"	"balloon"	"0.0114148"
[60,]	"peter"	"horse"	"ship"	"0.01142248"
[61,]	"family"	"gun"	"pudgy"	"0.01150559"
[62,]	"peter"	"horse"	"balloons"	"0.01178966"
[63,]	"peter"	"mabel"	"crew"	"0.01181512"
[64,]	"family"	"gun"	"peter"	"0.01193978"
[65,]	"football"	"dolly"	"crew"	"0.01230778"
[66,]	"peter"	"horse"	"balloon"	"0.0124934"
[67,]	"peter"	"assistant"	"crew"	"0.0125593"
[68,]	"peter"	"horse"	"jamie"	"0.01281991"
[69,]	"peter"	"horse"	"tyler"	"0.012843"
[70,]	"mabel"	"assistant"	"family"	"0.0129161"

[71,]	"mabel"	"assistant"	"car"	"0.01301065"
[72,]	"peter"	"horse"	"jared"	"0.01340517"
[73,]	"jared"	"football"	"crew"	"0.01352421"
[74,]	"peter"	"horse"	"crew"	"0.01362332"
[75,]	"game"	"gun"	"assistant"	"0.01414658"
[76,]	"family"	"howard"	"peter"	"0.01492026"
[77,]	"road"	"room"	"family"	"0.01493067"
[78,]	"mabel"	"assistant"	"cat"	"0.01692686"
[79,]	"family"	"howard"	"mabel"	"0.01835927"
[80,]	"car"	"room"	"crew"	"0.01850425"
[81,]	"mabel"	"horse"	"dragon"	"0.01887733"
[82,]	"mabel"	"horse"	"lance"	"0.01887733"
[83,]	"mabel"	"horse"	"valerie"	"0.01925078"
[84,]	"derek"	"gun"	"balloon"	"0.01943399"
[85,]	"car"	"cat"	"mabel"	"0.019929"
[86,]	"mabel"	"horse"	"ship"	"0.02025176"
[87,]	"mabel"	"horse"	"balloons"	"0.02090672"
[88,]	"arrow"	"mabel"	"valerie"	"0.021144"
[89,]	"mabel"	"assistant"	"bedroom"	"0.02164229"
[90,]	"mabel"	"horse"	"balloon"	"0.02216332"
[91,]	"room"	"cat"	"assistant"	"0.0221949"
[92,]	"arrow"	"mabel"	"ship"	"0.02227254"
[93,]	"car"	"road"	"director"	"0.02242459"
[94,]	"arrow"	"mabel"	"balloons"	"0.02301388"
[95,]	"family"	"howard"	"horse"	"0.02331366"
[96,]	"bedroom"	"cat"	"balloons"	"0.02342833"
[97,]	"mabel"	"horse"	"crew"	"0.02418521"
[98,]	"bedroom"	"family"	"lance"	"0.024372"
[99,]	"arrow"	"mabel"	"balloon"	"0.0244434"
[100,]	"mabel"	"assistant"	"dragon"	"0.02544971"
[101,]	"bedroom"	"cat"	"peter"	"0.02573532"
[102,]	"derek"	"howard"	"dragon"	"0.02580133"
[103,]	"derek"	"howard"	"lance"	"0.02580133"
[104,]	"mabel"	"assistant"	"valerie"	"0.02600296"
[105,]	"car"	"bedroom"	"balloon"	"0.02604236"
[106,]	"derek"	"family"	"balloon"	"0.02616928"
[107,]	"derek"	"howard"	"valerie"	"0.02645994"
[108,]	"car"	"road"	"valerie"	"0.02659476"
[109,]	"arrow"	"mabel"	"crew"	"0.02676692"
[110,]	"car"	"road"	"ship"	"0.02690026"
[111,]	"pudgy"	"horse"	"ship"	"0.02716962"
[112,]	"mabel"	"assistant"	"ship"	"0.02750808"
[113,]	"pudgy"	"horse"	"balloons"	"0.02801975"
[114,]	"derek"	"howard"	"ship"	"0.02828986"
[115,]	"mabel"	"assistant"	"balloons"	"0.02851275"
[116,]	"mabel"	"assistant"	"dolly"	"0.02930135"
[117,]	"gun"	"howard"	"peter"	"0.0293042"
[118,]	"road"	"bedroom"	"dolly"	"0.02939051"
[119,]	"derek"	"howard"	"balloons"	"0.02954464"
[120,]	"pudgy"	"horse"	"balloon"	"0.02964163"

[121,]	"assistant"	"pudgy"	"ship"	"0.02978986"
[122,]	"room"	"cat"	"horse"	"0.03020312"
[123,]	"road"	"bedroom"	"peter"	"0.03026897"
[124,]	"car"	"road"	"balloons"	"0.03030422"
[125,]	"derek"	"howard"	"director"	"0.03040631"
[126,]	"mabel"	"assistant"	"john"	"0.03043118"
[127,]	"car"	"road"	"football"	"0.03048117"
[128,]	"mabel"	"assistant"	"balloon"	"0.03049399"
[129,]	"derek"	"howard"	"dolly"	"0.03054964"
[130,]	"assistant"	"pudgy"	"balloons"	"0.03072369"
[131,]	"football"	"pudgy"	"dragon"	"0.03104475"
[132,]	"football"	"pudgy"	"lance"	"0.03104475"
[133,]	"peter"	"arrow"	"ship"	"0.03109828"
[134,]	"derek"	"game"	"dragon"	"0.03133084"
[135,]	"derek"	"game"	"lance"	"0.03133084"
[136,]	"mabel"	"assistant"	"jamie"	"0.03144317"
[137,]	"mabel"	"assistant"	"tyler"	"0.03151111"
[138,]	"family"	"howard"	"dragon"	"0.03191839"
[139,]	"jared"	"director"	"crew"	"0.03197288"
[140,]	"derek"	"howard"	"john"	"0.03202227"
[141,]	"jared"	"dolly"	"crew"	"0.03206637"
[142,]	"peter"	"arrow"	"balloons"	"0.03212724"
[143,]	"pudgy"	"horse"	"crew"	"0.03222202"
[144,]	"car"	"room"	"assistant"	"0.03243193"
[145,]	"assistant"	"pudgy"	"balloon"	"0.03250582"
[146,]	"director"	"pudgy"	"crew"	"0.03258847"
[147,]	"pudgy"	"dolly"	"crew"	"0.03268863"
[148,]	"family"	"howard"	"valerie"	"0.03275107"
[149,]	"jared"	"john"	"crew"	"0.03299317"
[150,]	"family"	"game"	"dragon"	"0.03302905"
[151,]	"cat"	"gun"	"game"	"0.03313567"
[152,]	"mabel"	"assistant"	"jared"	"0.03320492"
[153,]	"derek"	"howard"	"jamie"	"0.03337614"
[154,]	"derek"	"howard"	"tyler"	"0.03346827"
[155,]	"football"	"pudgy"	"ship"	"0.03347461"
[156,]	"car"	"road"	"balloon"	"0.03358371"
[157,]	"mabel"	"assistant"	"crew"	"0.0338856"
[158,]	"bedroom"	"gun"	"howard"	"0.03408125"
[159,]	"peter"	"arrow"	"balloon"	"0.03410934"
[160,]	"mabel"	"assistant"	"football"	"0.03414623"
[161,]	"derek"	"game"	"ship"	"0.03438099"
[162,]	"mabel"	"assistant"	"pudgy"	"0.03457698"
[163,]	"football"	"pudgy"	"balloons"	"0.03465437"

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