

Characterizing SW-Efficiency in the Social Choice Domain

Haris Aziz

NICTA and UNSW, 2033 Sydney, Australia

Abstract

Recently, Dogan, Dogan and Yildiz (2015) presented a new efficiency notion for the random assignment setting called SW (social welfare)-efficiency and characterized it. In this note, we generalize the characterization for the more general domain of randomized social choice.

Keywords: Social decision schemes, Social choice theory, Pareto optimal, Social Welfare, Stochastic Dominance

JEL: C63, C70, C71, and C78

1. Introduction

The *random assignment setting* captures the scenario in which n agents express preferences over n objects and the outcome is a probabilistic assignment. For the the setting, two interesting efficiency notions are ex post efficiency and SD (stochastic dominance)-efficiency [1, 3, 5, 6, 8, 10]. The assignment setting can be considered as a special case of voting where each discrete assignment can be viewed as a voting alternative [2, 4, 7].

Recently, Dogan et al. [9] presented a new notion of efficiency called SW (social welfare)-efficiency for the random assignment setting. They characterize SW-efficiency. In this note, we generalize the characterization to the more general voting setting.

2. Preliminaries

Consider the social choice setting in which there is a set of agents $N = \{1, \dots, n\}$, a set of alternatives $A = \{a_1, \dots, a_m\}$ and a preference profile $\succsim = (\succsim_1, \dots, \succsim_n)$ such that each \succsim_i is a complete and transitive relation over A . We write $a \succsim_i b$ to denote that agent i values alternative a at least as much as alternative b and use \succ_i for the strict part of \succsim_i , i.e., $a \succ_i b$ iff $a \succsim_i b$ but not $b \succsim_i a$. Finally, \sim_i denotes i 's indifference relation, i.e., $a \sim_i b$ iff both $a \succsim_i b$ and $b \succsim_i a$.

Email address: haris.aziz@nicta.com.au (Haris Aziz)

a. The alternatives in A could be any discrete structures: voting outcomes, house allocation, many-to-many two-sided matching, or coalition structures. A utility profile $u = (u_1, \dots, u_n)$ specified for each agent $i \in N$ his utility for $u_i(a)$ for each alternative $a \in A$. A utility profile is *consistent* with the preference profile \succsim , if for each $i \in N$ and $a, b \in A$, $u_i(a) \geq u_i(b)$ if $a \succsim_i b$. Two alternatives $a, b \in A$ are *Pareto indifferent* if $a \sim_i b$ for all $i \in N$. For any alternative $a \in A$, we will denote by $D(a)$ the set $\{b \in A : \exists i \in N, a \succ_i b\}$.

We will also consider randomized outcomes that are lotteries over A . A lottery is a probability distribution over A . We denote the set of lotteries by $\Delta(A)$. For a lottery $p \in \Delta(A)$, we denote by $p(a)$ the probability of alternative $a \in A$ in lottery p . We denote by $\text{supp}(p)$ the set $\{a \in A : p(a) > 0\}$. A lottery p is *interesting* if there exist $a, b \in \text{supp}(p)$ such that there exist $i, j \in N$ such that $a \succ_i b$ and $b \succ_j a$. A lottery is *degenerate* if it puts probability one on a single alternative.

Under *stochastic dominance (SD)*, an agent prefers a lottery that, for each alternative $x \in A$, has a higher probability of selecting an alternative that is at least as good as x . Formally, $p \succsim_i^{SD} q$ iff $\forall y \in A: \sum_{x \in A: x \succsim_i y} p(x) \geq \sum_{x \in A: x \succsim_i y} q(x)$. It is well-known that $p \succsim^{SD} q$ iff p yields at least as much expected utility as q for any von-Neumann-Morgenstern utility function consistent with the ordinal preferences [4, 8]. A lottery is *SD-efficient* if it is Pareto optimal with respect to the SD relation.

3. SW-efficiency

We now consider SW-efficiency as introduced by Dogan et al. [9]. Although Dogan et al. [9] defined SW-efficiency in the context of random assignment, the definition extends in a straightforward manner to the case of voting.

Definition 1 (SW-efficiency). *A lottery p is SW-efficient if there exists no other lottery q that SW dominates it. Lottery q SW dominates p if for any utility profile for which p maximizes welfare, q maximises welfare, and there exists at least one utility profile for which q maximised welfare but p does not.*

Lemma 1. *Consider a Pareto optimal alternative $a \in A$ and a non-empty set $D(a) = \{b \in A : \exists i \in N, a \succ_i b\}$. Then, there exists a utility profile u such that $\sum_{i \in N} u_i(a) > \sum_{i \in N} u_i(b)$ for all $b \in D(a)$.*

Proof. We can construct the require utility function profile u as follows. Whenever $a \succ_i b$, make the difference $u_i(a) - u_i(b)$ huge. Whenever $b \succ_j a$, make the difference $u_j(b) - u_j(a)$ arbitrarily small. Hence the value $u_i(a) - u_i(b)$ is large enough that it makes up for all j for which $u_j(b) - u_j(a) > 0$. Hence $\sum_{i \in N} u_i(a) - u_i(n) > 0$. \square

Lemma 2. *SW-efficiency implies SD-efficiency implies ex post efficiency.*

Proof. It is well-known that SD-efficiency implies ex post efficiency [4].

Consider a lottery p that is not SD-efficient. Then there exists another lottery q that SD-dominates it. Hence p does not maximize welfare for any consistent utility profile because q yields more utility for each utility profile. \square

Lemma 3. *An interesting lottery is not SW-efficient.*

Proof. If an interesting lottery p is not SD-efficient, we are already done because by Lemma 2, p is not SW-efficient. So let us assume p is SD-efficient and hence ex post efficient. Since p is interesting, there exists at least one $a \in \text{supp}(p)$ such that $a \succ_i b$ for some $b \in \text{supp}(p)$ and $i \in N$. Note that a is Pareto optimal. By Lemma 1, there exists a utility profile u such that $\sum_{i \in N} u_i(a) > \sum_{i \in N} u_i(b)$ for all $b \in D(a)$ where $D(a) \cap \text{supp}(p) \neq \emptyset$. Hence, there exists a utility profile u such that $\sum_{i \in N} u_i(a) > \sum_{i \in N} u_i(b)$ for all $b \in \text{supp}(p) \cap D(a)$. This means that $\sum_{i \in N} u_i(a) > \sum_{i \in N} u_i(p)$. \square

Lemma 4. *An uninteresting lottery over Pareto optimal alternatives is SW-efficient.*

Proof. An uninteresting lottery p over Pareto optimal alternatives is SD-efficient. Assume that there is another lottery q that SW-dominates p . Then $\text{supp}(q)$ contains one alternative b that is not Pareto indifferent to alternatives $\text{supp}(p)$. This means that there exists a utility profile u such that welfare is maximized by p but not by b . Hence q does not SW-dominate p . \square

Theorem 1. *A lottery is SW-efficient iff it is ex post efficient and uninteresting.*

Proof. By Lemma 4, an ex post efficiency and uninteresting lottery is SW-efficient.

We now prove that if lottery is not ex post efficient or uninteresting, it is not SW-efficient. Due to Lemma 2, if a lottery is not ex post efficient, it is not SW-efficient. Similarly, by Lemma 3, if a lottery is interesting, it is not SW-efficient. \square

Lemma 5. *If A contains no Pareto indifferent alternatives, then if a lottery is uninteresting and not degenerate, then it is not ex post efficient.*

Proof. Assume that a lottery p is uninteresting and not degenerate. Since p is not degenerate, $|\text{supp}(p)| \geq 2$. Since p is uninteresting, there do not exist $a, b \in \text{supp}(p)$ such that there exist $i, j \in N$ such that $a \succ_i b$ and $b \succ_j a$. Thus either a Pareto dominates b , or b Pareto dominates a or a and b are Pareto indifferent. The third case is not possible because we assumed that A does not contain Pareto indifferent alternatives. Since a Pareto dominates b or b Pareto dominates a , $\text{supp}(p)$ contains a Pareto dominated alternative. Hence p is not ex post efficient. \square

Theorem 2. *If A contains no Pareto indifferent alternatives, then a lottery is SW-efficient iff it is ex post efficient and degenerate.*

Proof. Assume that A contains no Pareto indifferent alternatives. If a lottery p is SW-efficient, then by Theorem 1, it is ex post efficient and uninteresting. By Lemma 5, since p is ex post efficient, it is degenerate.

Now assume that a lottery p is ex post efficient and degenerate. Since p is degenerate, it is uninteresting by definition. Since it is both ex post efficient and uninteresting, then by Theorem 1, it is SW-efficient. \square

Lemma 6. *An assignment problem with strict preferences does not admit Pareto indifferent discrete assignments.*

Proof. Consider two discrete assignments M and M' such that all agents are indifferent among them. Then this means that each agent gets the same item in both M' and M . But this implies that $M' = M$. \square

Corollary 1 (Dogan et al. [9]). *If preferences are strict, the only undominated probabilistic assignments are the Pareto efficient deterministic assignments.*

Proof. No two discrete assignments are completely indifferent for all agents. Hence only random assignment that is SW-efficient is a discrete Pareto optimal assignment. \square

Acknowledgments

NICTA is funded by the Australian Government through the Department of Communications and the Australian Research Council through the ICT Centre of Excellence Program.

References

- [1] Abdulkadiroğlu, A., Sönmez, T., 2003. Ordinal efficiency and dominated sets of assignments. *Journal of Economic Theory* 112 (1), 157–172.
- [2] Aziz, H., 2014. A characterization of stochastic dominance efficiency. *Economic Theory Bulletin*, 205–212.
- [3] Aziz, H., Brandl, F., Brandt, F., 2015. Universal dominance and welfare for plausible utility functions. *Journal of Mathematical Economics* 60, 123–133.
- [4] Aziz, H., Brandt, F., Brill, M., 2013. On the tradeoff between economic efficiency and strategyproofness in randomized social choice. In: *Proceedings of the 12th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*. IFAAMAS, pp. 455–462.
- [5] Aziz, H., Mackenzie, S., Xia, L., Ye, C., 2015. Ex post efficiency of random assignments. In: *Proceedings of the 14th International Conference on Autonomous Agents and Multi-Agent Systems (AAMAS)*. pp. 1639–1640.
- [6] Bogomolnaia, A., Moulin, H., 2001. A new solution to the random assignment problem. *Journal of Economic Theory* 100 (2), 295–328.

- [7] Carroll, G., 2010. An efficiency theorem for incompletely known preferences. *Journal of Economic Theory* 145 (6), 2463–2470.
- [8] Cho, W. J., 2012. Probabilistic assignment: A two-fold axiomatic approach, Mimeo.
- [9] Dogan, B., Dogan, S., Yildiz, K. ., 2015. A new efficiency criterion for probabilistic assignments. In: *The proceedings of the 3rd International Workshop on Matching Under Preferences (MATCHUP)*.
- [10] McLennan, A., 2002. Ordinal efficiency and the polyhedral separating hyperplane theorem. *Journal of Economic Theory* 105 (2), 435–449.