Communication Enhancement through Quantum Coherent Control of N Channels in an Indefinite Causal-Order Scenario

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Abstract: In quantum Shannon theory, transmission of information is enhanced by quantum features. Up to very recently, the trajectories of transmission remained fully classical. Recently, a new paradigm was proposed by playing quantum tricks on two completely depolarizing quantum channels i.e., using coherent control in space or time of the two quantum channels. We extend here this control to the transmission of information through a network of an arbitrary number N of channels with arbitrary individual capacity i.e., information preservation characteristics in the case of indefinite causal order. We propose a formalism to assess information transmission in the most general case of N channels in an indefinite causal order scenario yielding the output of such transmission. Then, we explicitly derive the quantum switch output and the associated Holevo limit of the information transmission for $N = 2$, $N = 3$ as a function of all involved parameters. We find in the case $N = 3$ that the transmission of information for three channels is twice that of transmission of the two-channel case when a full superposition of all possible causal orders is used.

Keywords: quantum control; indefinite causal order; quantum switch; Holevo information

1. Introduction

In information theory, the main tasks to perform are the transmission, codification, and compression of information [1]. Incorporating quantum phenomena, such as quantum superposition and quantum entanglement, into classical information theory gives rise to a new paradigm known as quantum Shannon theory [2]. In this paradigm, each figure of merit can be enhanced: the capacity to transmit information in a channel is increased [3], the security to share a message is improved [4], and the storing and compressing of information is optimized [5]. In all these enhancements, only the carriers and the channels of information are considered as quantum entities. On the other hand, connections between channels are still classical, that is, quantum channels are connected setting a definite causal order in space or time. However, principles of quantum mechanics and specifically the quantum superposition principle can be applied to the connections of channels [6], i.e., the trajectories either in space [7] or time [8].

Recently, it has been theoretically [9] and experimentally [10,11] shown that two completely depolarizing channels can surprisingly transmit classical information when combined under an indefinite causal order (i.e., when the order of application of the two channels is not one after another instead of a quantum superposition of the two possibilities). In this paper, we tackle the general

situation of an arbitrary number N of channels with arbitrary parameters associated with the control and depolarizing strength. As N is greater than two, the number of different causal orders increases as $N!$

The indefiniteness of causal order has been recently theoretically proposed as a novel resource for applications to quantum information theory [12,13] and quantum communication [14,15]. Initially, indefinite causal orders have been studied and implemented using two parties with the proposal of a quantum switch by Chiribella et al. [6] followed by experimental demonstrations [11,16–19]. The quantum switch is an example of quantum control where a switch can, like its classical counterpart, routes a target system to undergo through two operators in series following one causal order (1 then 2) or the other (2 then 1). However, this quantum switch can also trigger a whole new quantum trajectory where the ordering of the two operators is indefinite. Efforts to describe the quantum switch in a multipartite scenario of more than two quantum operations have recently started [20,21] with an application to reduce the number of queries for quantum computation [13].

Specifically, in a quantum N-switch used in a second-quantized Shannon theory context [22], the order of application of N channels $\mathcal{N}_j$ on a target system ρ is coherently controlled by a control system ρ_c. The state of ρ_c encodes for the temporal combination of the N channels applied to ρ. There are $N!$ different possibilities of definite causal orders using each channel once and only once, as sketched in Figures 1 and 2 for $N = 2$ and $N = 3$, respectively. In those figures, when the wiring passes through the channel, there is a single channel use, i.e., the target system passes once through one physical channel [16]. We discard all wirings with multiple uses of the same channel and missing channels [7]. For each causal order of channels, the overall operator is

$$\mathcal{N}_\pi := \pi(\mathcal{N}_1 \circ \cdots \circ \mathcal{N}_N), \tag{1}$$

where π is a permutation element of the symmetric group $S_N = \{\pi_k | k \in \{1, 2, \ldots, N!\}\}$, and k is associated with a specific definite causal order (equivalent to a single element of S_N) to combine the N channels where each channel is used once and only once.
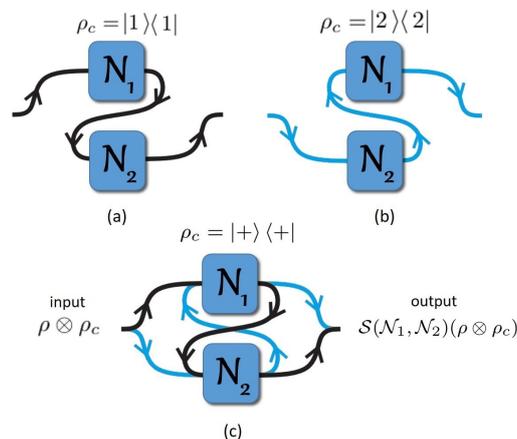

Figure 1. Concept of the quantum 2-switch. $\mathcal{N}_i = \mathcal{N}_{q_i}^D$ is a depolarizing channel applied to the quantum state ρ, where $1 - q_i$ is the strength of the depolarization. For two channels, depending on the control system ρ_c, there are 2! possibilities to combine the channels with definite causal order: (**a**) if ρ_c is in the state $|1\rangle\langle 1|$, the causal order will be $\mathcal{N}_2 \circ \mathcal{N}_1$, i.e., $\mathcal{N}_1$ is before $\mathcal{N}_2$; (**b**) on the other hand, if ρ_c is on the state $|2\rangle\langle 2|$, the causal order will be $\mathcal{N}_1 \circ \mathcal{N}_2$; (**c**) however, placing ρ_c in a superposition of its states (i.e., $\rho_c = |+\rangle\langle+|$, where $|+\rangle_c = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$) results in the indefinite causal order of $\mathcal{N}_1$ and $\mathcal{N}_2$ to become indefinite. In this situation, we said that the quantum channels are in a superposition of causal orders. This device is called a quantum 2-switch [6] whose input and output are $\rho \otimes \rho_c$ and $\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2)(\rho \otimes \rho_c)$, respectively.

In a quantum N-switch, the control state ρ_c in the state $|1\rangle\langle 1|$ for instance fixes the order of application of the channels to be $\mathcal{N}_{\mathrm{Id}} = \mathcal{N}_1 \circ \mathcal{N}_2 \circ \cdots \circ \mathcal{N}_N$, whereas choosing $\rho_c = |k\rangle\langle k|$, $k \leq N!$ would assign another ordering $\mathcal{N}_{\pi_k} = \mathcal{N}_{\pi_k(1)} \circ \mathcal{N}_{\pi_k(2)} \circ \cdots \circ \mathcal{N}_{\pi_k(N)}$ (defined by the effect of the permutation element $\pi_k \in S_N$ on the order of channels). The key to accessing indefinite causal order of the channels is thus to put ρ_c in a superposition of the $|k\rangle\langle k|$ states (e.g., $\rho_c = |+\rangle\langle+|$ where $|+\rangle \equiv \frac{1}{\sqrt{N!}} \sum |k\rangle$).
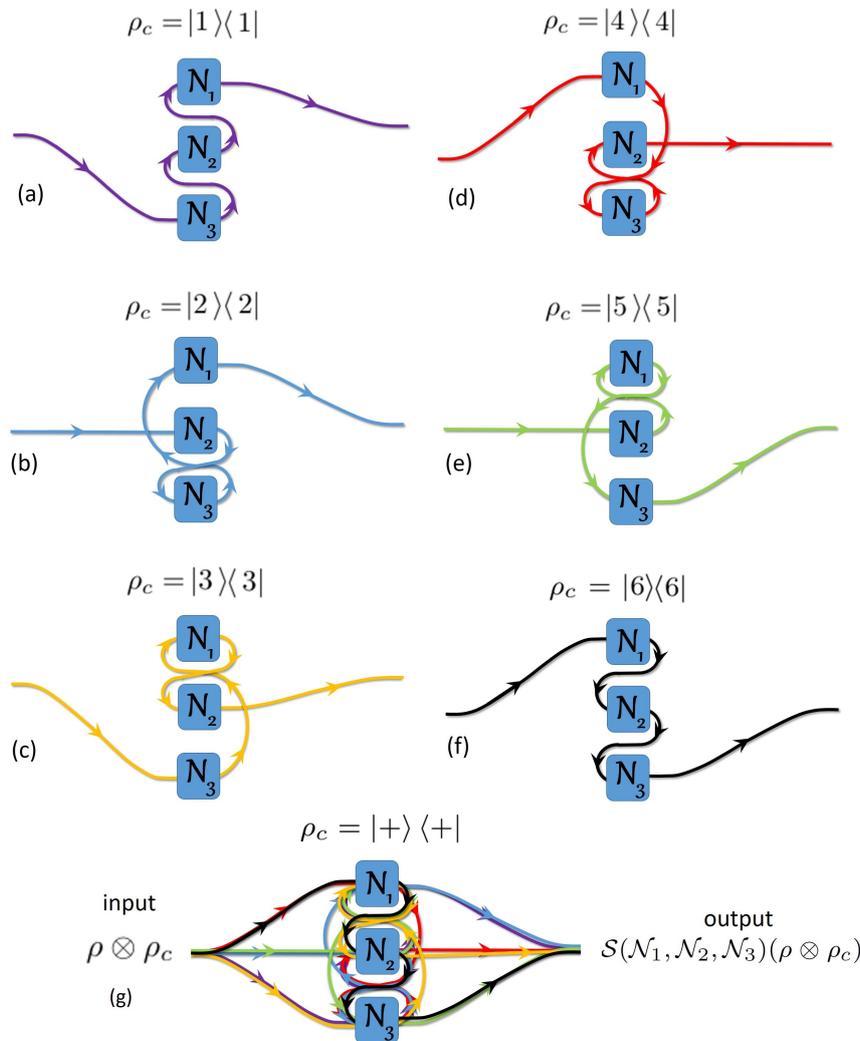

Figure 2. Concept of the quantum 3-switch. For three channels, depending on ρ_c, we have 3! possibilities to combine the channels in a definite causal order: (**a**) $\rho_c = |1\rangle\langle 1|$ encodes a causal order $\mathcal{N}_1 \circ \mathcal{N}_2 \circ \mathcal{N}_3$, i.e., $\mathcal{N}_3$ is applied first to ρ; (**b**) $\rho_c = |2\rangle\langle 2|$ encodes $\mathcal{N}_1 \circ \mathcal{N}_3 \circ \mathcal{N}_2$; (**c**) $\rho_c = |3\rangle\langle 3|$ encodes $\mathcal{N}_2 \circ \mathcal{N}_1 \circ \mathcal{N}_3$; (**d**) $\rho_c = |4\rangle\langle 4|$ encodes $\mathcal{N}_2 \circ \mathcal{N}_3 \circ \mathcal{N}_1$; (**e**) $\rho_c = |5\rangle\langle 5|$ encodes $\mathcal{N}_3 \circ \mathcal{N}_1 \circ \mathcal{N}_2$; (**f**) $\rho_c = |6\rangle\langle 6|$ encodes $\mathcal{N}_3 \circ \mathcal{N}_2 \circ \mathcal{N}_1$; (**g**) finally, if $\rho_c = |+\rangle\langle+|$, where $|+\rangle = \frac{1}{\sqrt{6}} \sum_{k=1}^{6} |k\rangle$ we shall have a superposition of six different causal orders. This is an indefinite causal order called quantum 3-switch whose input and output are $\rho \otimes \rho_c$ and $\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3)(\rho \otimes \rho_c)$, respectively. Notice that, for each superposition with m different causal orders, there are $\binom{N!}{m}$ (with $m = 1, 2, \ldots, 6$) possible combinations of causal orders to build such superposition with $N = 3$ channels, where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ is the binomial coefficient. The input and output of each channel are fixed. The arrows along the wire just indicate that the target system enters in or exits from the channel.

The paper is organized as follows: Section 2 is devoted to the general theoretical framework for the investigation of the transmission of classical information over N noisy channels with an arbitrary degree of depolarization, i.e., arbitrary level of noise. Section 2 also gives the channels representation in terms of Kraus operators performed from those operators for a single depolarizing channel. In Section 3, following the previous formalism, we explicitly analyze the case $N = 2$, generalizing the outcomes

in the literature [9] to any degree of depolarization and level of coherent control. Similarly, the case $N = 3$ is developed in the same section. Finally, conclusions and perspectives are given in Section 5.

2. Transmission over Multiple Channels in Quantum Superposition of Causal Order

In the current development, the sender prepares the target system in the state ρ, where the information to transmit is encoded. A control system ρ_c is associated with the target system to coherently control the causal order for the application of N quantum communication channels. We relate the basis for the quantum state ρ_c mapping their elements on those of the symmetric group of permutations $S_N : \frac{1}{N!} \sum_{k,k'} |k\rangle \langle k'|$. Then, the sender introduces as input $\rho \otimes \rho_c$ to a network of N partially depolarizing channels $\mathcal{N}_i = \mathcal{N}_{q_i}^D, 1 \leq i \leq N$ applied in series (i.e., the output of one channel becomes the input of the next channel). Throughout this work, the N depolarizing channels $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_N$ can have different depolarization strengths $1 - q_j$, (thus, $\mathcal{N}_{q_j}^D$ is sometimes used for $\mathcal{N}_j$ to improve the readability).

After the network, the receiver gets the output state $\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_N)(\rho \otimes \rho_c)$, where $\mathcal{S}$ is the quantum N-switch channel. No information is encoded by the sender into the control system controlling the way information is transmitted. Eventually, the receiver retrieves the information decoded in ρ.

Communication quantum channels in a network are mathematically described with completely positive trace preserving maps (CPTP). Here, we adopt the Kraus decomposition [2] $\mathcal{N}_{(\rho)} = \sum_i K_i \rho K_i^\dagger$ to describe the action of a total depolarizing channel $\mathcal{N}$ on the quantum state ρ ($i \in \{1, 2, \ldots, d\}$): $\mathcal{N}_\rho = \text{Tr}[\rho] \frac{\mathbb{I}}{d}$. The set of d^2 non-unique and generally non-unitary Kraus operators $\{K_i\}$ satisfies the completeness condition $\sum_{i=1}^{d^2} K_i K_i^\dagger = \mathbb{I}$. Thus, to describe the action of the j-th partially depolarizing channel $\mathcal{N}_j$ on a d-dimensional quantum system ρ, we write as in [9]

$$\mathcal{N}_{q_j}^D(\rho) = q_j \rho + (1 - q_j) \text{Tr}[\rho] \frac{\mathbb{I}_t}{d} = q_j \rho + \frac{1 - q_j}{d^2} \sum_{i_j=1}^{d^2} U_{i_j}^j \rho U_{i_j}^{j\dagger}$$

$$= \frac{1 - q_j}{d^2} \sum_{i_j=0}^{d^2} U_{i_j}^j \rho U_{i_j}^{j\dagger}, \tag{2}$$

where each $\mathcal{N}_j = \mathcal{N}_{q_j}^D$ is thus decomposed on an orthonormal basis $\{U_{i_j}^j\}|_{i_j=1}^{d^2}$. Then, we define $K_{i_j}^j = \frac{\sqrt{1-q_j}}{d} U_{i_j}^j$ for $i_j \neq 0$, where the added non-unitary operator $U_0^j = \frac{d\sqrt{q_j}}{\sqrt{1-q_j}} \mathbb{I}_t$, for $i_j = 0$. In addition, $\mathcal{N}_j$ has no noise when $q_j = 1$. On the other hand, $\mathcal{N}_j$ is completely depolarizing when $q_j = 0$. The results reported in [7,9] are mainly related to two completely depolarizing ($q_1 = q_2 = 0$) channels $\mathcal{N}_1$ and $\mathcal{N}_2$, despite the fact that generalization is outlined. Below, we extend the results from [9] to the case of a quantum switch with N channels $\mathcal{N}_j$ with arbitrary individual depolarization strengths q_j.

The Formalism for a Quantum N-Switch Channel $S(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_N)$

We define the control state ρ_c as $\rho_c = |\psi_c\rangle \langle \psi_c| = \sum_{k,k'=1}^{N!} \sqrt{P_k P_{k'}} |k\rangle \langle k'|$, where P_k is the probability to apply the causal order k (corresponding to the permutation π_k as it was previously stated) to the channels such that $\sum_{k=1}^{N!} P_k = 1$.

The action of the quantum N-switch channel $\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_N)$ can be expressed through generalized Kraus operators $W_{i_1 i_2 \ldots i_N}$ for the full quantum channel resulting from the switching of N channels as

$$\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_N)(\rho \otimes \rho_c) = \sum_{\{i_j\}|_{j=1}^N} W_{\mathbf{i}}(\rho \otimes \rho_c) W_{\mathbf{i}}^\dagger, \tag{3}$$

where $W_{\mathbf{i}} := W_{i_1 i_2 \ldots i_N} = \sum_{k=1}^{N!} K_{\pi_k} \otimes |k\rangle \langle k|$ and K_{π_k} has been defined similarly to Equation (1): $K_{\pi_k} := \pi_k(K_{i_1}^1 \cdots K_{i_N}^N)$, where π_k acts on the index j, and the sum over $\{i_j\}|_{j=1}^N$ means all i_j associated with

each channel $\mathcal{N}_j$ vary from 0 to d^2. We verify (see Appendix A) that these generalized Kraus operators satisfy the completeness property $\sum_{\{i_j\}|_{j=1}^N} W_{\mathbf{i}} W_{\mathbf{i}}^\dagger = \mathbb{I}_t \otimes \mathbb{I}_c$, where identity operators in the target and control systems spaces are denoted $\mathbb{I}_t$ and $\mathbb{I}_c$, respectively. This check of completeness suggests how the i_j indices allow the systematic reordering of the sums by isolating and grouping the $i_j = 0$ cases. To distinguish those terms, we introduce the number z of indices i_j equal to zero. The sums over the indices i_j can then be rearranged as

$$\sum_{\{i_j\}|_{j=1}^N} \rightarrow \sum_{z=0}^N \sum_{A_z \in \mathbf{A}_z^N} \sum_{b \in B_z}, \tag{4}$$

where A_z is the set of z indices equal to zero ($i_a = 0, \forall a \in A_z$) and B_z is the complementary set of indices in $\{1, 2 \ldots, N\} : i_b \neq 0, i_b \in \{1, 2, \ldots, d^2\}$ for all $b \in B_z$. Then, $U_{\pi_k} U_{\pi_k}^\dagger = d^{2z} h_{A_z} \mathbb{I}_t$, where $h_{A_z} = \prod_{a \in A_z} \frac{q_a}{1 - q_a}$ and $h_{A_0} = 1$.

Introducing the Kraus operators $W_{\mathbf{i}}$ into $\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_N)$, Equation (3) can be written as a sum of $N + 1$ matrices $\mathcal{S}_z$ whose $N! \times N!$ elements are matrices of dimension $d \times d$ involving exactly z factors U_{i_j} equal to the identity operator. The overall dimension $\mathcal{S}_z$ is thus $dN! \times dN!$

$$\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_N)(\rho \otimes \rho_c) = \sum_{z=0}^N \mathcal{S}_z, \tag{5}$$

and (see Appendices A and B)

$$\mathcal{S}_z = \sum_{k,k'=1}^{N!} \sqrt{P_k P_{k'}} \sum_{A_z \in \mathbf{A}_z^N} f_{A_z} \cdot Q_{A_z}^{k,k'} \otimes |k\rangle \langle k'| \tag{6}$$

with

$$f_{A_z} = d^{2(z-N)} \prod_{j=1}^N (1 - q_j) \prod_{a \in A_z} \frac{q_a}{1 - q_a}, $$

where $\mathbf{A}_z^N$ is the collection of all possible subsets A_z of z subscripts in $\{1, 2, \ldots, N\}$ corresponding to the z indices equal to zero (i.e., $i_a = 0 \; \forall a \in A_z$). The following subsections provide details of examples with $N = 2$ and $N = 3$. The coefficients $Q_{A_z}^{k,k'}$ are given by

$$Q_{A_z}^{k,k'} = \sum_{\{i_b | b \in B_z\}} \pi_k \left(U_{i_1} \cdots U_{i_N} \right) \rho \left[\pi_{k'} \left(U_{i_1} \cdots U_{i_N} \right) \right]^\dagger. \tag{7}$$

The $U_{i_j}^j$ of Equation (2) have been simplified in U_{i_j}. We can see from Equation (7) that the elements of the matrix $\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_N)(\rho \otimes \rho_c)$ will always be a linear combination of ρ and $\mathbb{I}_t$, whatever N channels. Note also the operators U_{i_j} for $j \in A_z$ are identity operators $\mathbb{I}_t$ by construction. Thus, arguments $U_{i_1} \cdots U_{i_N}$ under π_k in Equation (7) involves N elements, z of them in A_z and $N - z$ in B_z. The matrix $\mathcal{S}$ and the pivotal Equations (5)–(7) contain all information about the correlations between precise causal orders coherently controlled by ρ_c and the output of the quantum switch. $\mathcal{S}$ is a function of several parameters: the involved causal orders π_k via the probabilities P_k, the depolarization strengths q_i's of each individual channel $\mathcal{N}_i$, the dimension d of the target system undergoing the operations of those channels, and the number of channels N. Notably, the sum over k and k' in Equation (6) can be restricted to a subset of definite causal orders via the probabilities P_k, i.e., a subset of superposition of m causal orders among the $N!$ existing ones for advanced quantum control. This handle had remained unexplored up to now. It was not accessible to former explorations limited to two channels. In the current work, we consider only superpositions of all causal orders. The control of causal orders will be presented elsewhere.

In the following subsections, we will give the explicit expressions of the quantum switch matrices for the quantum N-switch channel for $N = 2$ and $N = 3$. We access these matrices of the quantum N-switch channel via the systematic ordering of the terms in Equations (3) as settled in Equations (5)–(7).

The explicit calculation of the quantum N-switch channel gives important insights on the transmission of information coherently controlled by ρ_c in a fascinating multi-parameter space. We briefly review below some of the intriguing behaviors associated with the parameters exploration in the $N = 2$ and the $N = 3$ cases. We show indeed in those cases how the nature and number of the causal orders in the control state superposition, the dimension of the target system, and the level of noise all play a role. We underline that the $N = 3$ case is still untouched experimentally.

To derive Equation (6) for particular cases of N, we first introduce the definitions of $W_{\mathbf{i}}$ and ρ_c into Equation (3). Introducing the definitions of the Kraus operators in terms of $U_{i_j}^j$ operators and applying the same reordering on the sums as in Equation (4) leads to Equation (6). In the following subsection, specific developments for $N = 2$ and $N = 3$ to evaluate the $Q_{A_z}^{k,k'}$ are given simplifying $Q_{A_z}^{k,k'}$ in Equation (7) by following the relations presented in Appendix B (Equations (A4)–(A6)).

3. The Quantum Switch Matrices for $N = 2$ and $N = 3$

To show the usefulness of Equations (6), we derive general expressions to investigate the transmission of information through two and three channels in an indefinite causal order. Our method can be easily applied to any number of depolarizing channels provided that $\{U_i\}_{i=1}^{d^2}$ are unitary operators setting an orthonormal basis for the space of $d \times d$ matrices.

3.1. Evaluation of $\mathcal{S}$ for $N = 2$

To explicitly evaluate Equation (5) with two channels, we identify the two permutations in S_2: $\pi_1 = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right)$ and $\pi_2 = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$. Equation (5) for the quantum 2-switch channel matrix acting on the input state $\rho \otimes \rho_c$ writes

$$\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2)(\rho \otimes \rho_c) = \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2. \tag{8}$$

The collection of all subsets of subscripts in $\{1, 2\}$ are $\mathbf{A}_0^2 = \{\varnothing\}$, $\mathbf{A}_1^2 = \{\{1\}, \{2\}\}$ and $\mathbf{A}_2^2 = \{\{1, 2\}\}$. Then, the corresponding complementary collections are $\mathbf{B}_0^2 = \{\{1, 2\}\}$, $\mathbf{B}_1^2 = \{\{2\}, \{1\}\}$ and $\mathbf{B}_2^2 = \{\varnothing\}$.

Coefficients for $\mathcal{S}_0$. In this case, we use $\mathbf{A}_0^2 = \{\varnothing\}$ to calculate the coefficients $Q_\varnothing^{k,k'}$, $k, k' \in \{1, 2\}$. The $Q_\varnothing^{k,k'}$ then reads

$$
\begin{aligned}
Q_\varnothing^{1,1} &= \sum_{i_1, i_2} \pi_1(U_{i_1} U_{i_2}) \rho \pi_1(U_{i_1} U_{i_2})^\dagger \\
&= \sum_{i_1, i_2} (U_{i_1} U_{i_2}) \rho (U_{i_2}^\dagger U_{i_1}^\dagger) \\
&= d \sum_{i_1, i_2} U_{i_1} U_{i_1}^\dagger = d^3 \mathbb{I}. \\
Q_\varnothing^{1,2} &= \sum_{i_1, i_2} \pi_1(U_{i_1} U_{i_2}) \rho \pi_2(U_{i_1} U_{i_2})^\dagger \\
&= \sum_{i_1, i_2} (U_{i_1} U_{i_2}) \rho (U_{i_1}^\dagger U_{i_2}^\dagger) \\
&= d \sum_{i_1,} U_{i_1} \mathrm{tr}(\rho U_{i_1}^\dagger) = d^2 \rho,
\end{aligned}
\tag{9}
$$

where we have used Equations (A4) and (A6) for $Q_\varnothing^{1,1}$, Equation (A4) with $X = U_{i_2} \rho$ and Equation (A5) for $Q_\varnothing^{1,2}$. Likewise, we have $Q_\varnothing^{\alpha, \alpha'} = d^3 \mathbb{I}$, for $(\alpha, \alpha') \in \mathfrak{A} \equiv \{(1, 1), (2, 2)\}$ and $Q_\varnothing^{\beta, \beta'} = d^2 \rho$, for $(\beta, \beta') \in \mathfrak{B} \equiv \{(1, 2), (2, 1)\}$. Then, we may write

$$\mathcal{S}_0 = \sum_{(\alpha, \alpha') \in \mathfrak{A}} \frac{r_0 \mathbb{I}}{d} \sqrt{P_\alpha P_{\alpha'}} \otimes |\alpha\rangle \langle \alpha'| + \sum_{(\beta, \beta') \in \mathfrak{B}} \frac{r_0 \rho}{d^2} \sqrt{P_\beta P_{\beta'}} \otimes |\beta\rangle \langle \beta'|, \tag{10}$$

where $r_0 = p_1 p_2$ with $p_i = 1 - q_i$.

Coefficients for $\mathcal{S}_1$. In this case, $\mathbf{A}_1^2 = \{\{1\}, \{2\}\}$ and $\mathbf{B}_1^2 = \{\{2\}, \{1\}\}$. Let us first consider the coefficient $Q_{\{1\}}^{\gamma, \gamma'} = \sum_{i_2} \pi_\gamma (\mathbb{I} \cdot U_{i_2}) \rho \pi_{\gamma'} (\mathbb{I} \cdot U_{i_2})^\dagger = d\mathbb{I}$, using the general relations (A4)–(A6), for $(\gamma, \gamma') \in \mathfrak{G} \equiv \{(1,1), (1,2), (2,1), (2,2)\}$. Since indices are dumb, it can be shown that $Q_{\{2\}}^{\gamma, \gamma'} = Q_{\{1\}}^{\gamma, \gamma'}$ for all (γ, γ'). Then, the term $\mathcal{S}_1$ can be written as

$$\mathcal{S}_1 = \sum_{k,k'} \frac{r_1}{d} \sqrt{P_k P_{k'}} \mathbb{I} \otimes |k\rangle\langle k'| = \frac{r_1}{d} \mathbb{I} \otimes \rho_c, \tag{11}$$

where $r_1 = q_1 p_2 + q_2 p_1$.

Coefficients for $\mathcal{S}_2$. Finally, let us consider the term $\mathcal{S}_2$. In this case $\mathbf{A}_2^2 = \{\{1,2\}\}$ and hence $\mathbf{B}_2^2 = \{\varnothing\}$. Note that $Q_{\{1,2\}}^{k,k'} = \rho$ for all k and k'. Thus, the term with $z = 2$ reads

$$\mathcal{S}_2 = \sum_{k,k'} r_2 \rho \sqrt{P_k P_{k'}} \otimes |k\rangle\langle k'| = r_2 \rho \otimes \rho_c, \tag{12}$$

with $r_2 = q_1 q_2$. By expanding the matrices $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{S}_2$ in the control qubit basis, $\{|1\rangle, |2\rangle\}$, we are able to write

$$\mathcal{S}_0 = \begin{pmatrix} \frac{r_0}{d} \mathbb{I} P_1 & \frac{r_0 \rho}{d^2} \sqrt{P_1 P_2} \\ \frac{r_0 \rho}{d^2} \sqrt{P_2 P_1} & \frac{r_0}{d} \mathbb{I} P_2 \end{pmatrix},$$

$$\mathcal{S}_1 = \begin{pmatrix} \frac{r_1}{d} \mathbb{I} P_1 & \frac{r_1}{d} \mathbb{I} \sqrt{P_1 P_2} \\ \frac{r_1}{d} \mathbb{I} \sqrt{P_2 P_1} & \frac{r_1}{d} \mathbb{I} P_2 \end{pmatrix}, \tag{13}$$

$$\mathcal{S}_2 = \begin{pmatrix} r_2 \rho P_1 & r_2 \rho \sqrt{P_1 P_2} \\ r_2 \rho \sqrt{P_2 P_1} & r_2 \rho P_2 \end{pmatrix},$$

where $\mathbb{I} = \mathbb{I}_t$. Summing those matrices according to Equation (5), we find that the quantum 2-switch channel matrix $\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2)$ has diagonal elements $a_k = P_k[(r_0 + r_1)\mathbb{I}/d + r_2\rho]$, for $k = 1,2$ and off-diagonal elements $b = \sqrt{P_1 P_2}[(r_0 + d^2 r_2)\rho/d^2 + \frac{r_1}{d}\mathbb{I}]$, with $r_0 = p_1 p_2$, $r_1 = q_1 p_2 + q_2 p_1$ and $r_2 = q_1 q_2$. Thus,

$$\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2)(\rho \otimes \rho_c) = \begin{pmatrix} a_1 & b \\ b & a_2 \end{pmatrix}. \tag{14}$$

Note that the diagonal and off-diagonal elements a_1, a_2 and b are matrices and are linear combinations of matrices ρ and $\mathbb{I}_t$. This property is non-unique for case $N = 2$; instead, it is general for N channels, an advisable aspect from Equation (7) and Equations (A4) and (A5). Indeed, Equation (14) gives as particular outputs the predicted Holevo capacity of Figure 3 in [10] and expressions of Holevo information in [9].

We end up this subsection stressing that Figure 1 sketches different ways to connect channels $\mathcal{N}_1$ and $\mathcal{N}_2$ in either (a) and (b) a definite causal order and (c) for an indefinite causal order combining the 2! possible orders.

3.2. Evaluation of $\mathcal{S}$ for $N = 3$

In this section, we explicitly evaluate expression (5) considering three channels. Let us label the six elements of S_3 according to the following set of permutations $\pi_1 = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}\right)$, $\pi_2 = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix}\right)$, $\pi_3 = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix}\right)$, $\pi_4 = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right)$, $\pi_5 = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right)$ and $\pi_6 = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{smallmatrix}\right)$. Equation (5) for the quantum 3-switch channel matrix acting on input state $\rho \otimes \rho_c$ reads

$$\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3)(\rho \otimes \rho_c) = \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3. \tag{15}$$

Coefficients for $\mathcal{S}_0$. In this case, note that $\mathbf{A}_0^3 = \{\varnothing\}$, hence $\mathbf{B}_0^3 = \{\{1, 2, 3\}\}$. In addition, the sum in $Q_\varnothing^{1,k'}$ is over the indices $\{i_1, i_2, i_3\}$. These can be computed explicitly

$$Q_\varnothing^{1,1} = \sum_{i_1, i_2, i_3} \pi_1(U_{i_1} U_{i_2} U_{i_3}) \rho \pi_1(U_{i_1} U_{i_2} U_{i_3})^\dagger = d^5 \mathbb{I}. \tag{16}$$

Likewise,

$$Q_\varnothing^{1,4} = \sum_{i_1, i_2, i_3} \pi_1(U_{i_1} U_{i_2} U_{i_3}) \rho \pi_4(U_{i_1} U_{i_2} U_{i_3})^\dagger = d^4 \rho. \tag{17}$$

The remaining coefficients for $\mathcal{S}_0$ are

$$\begin{aligned}
Q_\varnothing^{1,2} &= \sum_{i_1, i_2, i_3} \pi_1(U_{i_1} U_{i_2} U_{i_3}) \rho \pi_2(U_{i_1} U_{i_2} U_{i_3})^\dagger \\
&= \sum_{i_1, i_2, i_3} (U_{i_1} U_{i_2} U_{i_3}) \rho (U_{i_2}^\dagger U_{i_3}^\dagger U_{i_1}^\dagger) \\
&= d \sum_{i_1, i_3} U_{i_1} \operatorname{Tr}(U_{i_3} \rho) U_{i_3}^\dagger U_{i_1}^\dagger = d^2 \sum_{i_1} U_{i_1} \rho U_{i_1}^\dagger \\
&= d^3 \mathbb{I}, \\
Q_\varnothing^{1,3} &= \sum_{i_1, i_2, i_3} \pi_1(U_{i_1} U_{i_2} U_{i_3}) \rho \pi_3(U_{i_1} U_{i_2} U_{i_3})^\dagger \\
&= \sum_{i_1, i_2, i_3} (U_{i_1} U_{i_2} U_{i_3}) \rho (U_{i_3}^\dagger U_{i_1}^\dagger U_{i_2}^\dagger) \\
&= d \sum_{i_1, i_2} U_{i_1} U_{i_2} \mathbb{I} U_{i_1}^\dagger U_{i_2}^\dagger = d^2 \sum_{i_1} \operatorname{Tr}(U_{i_2} \mathbb{I}) U_{i_2}^\dagger \\
&= d^3 \mathbb{I}, \\
Q_\varnothing^{1,5} &= \sum_{i_1, i_2, i_3} \pi_1(U_{i_1} U_{i_2} U_{i_3}) \rho \pi_5(U_{i_1} U_{i_2} U_{i_3})^\dagger \\
&= \sum_{i_1, i_2, i_3} (U_{i_1} U_{i_2} U_{i_3}) \rho (U_{i_2}^\dagger U_{i_1}^\dagger U_{i_3}^\dagger) \\
&= d \sum_{i_1, i_3} U_{i_1} \operatorname{Tr}(U_{i_3} \rho) U_{i_1}^\dagger U_{i_3}^\dagger = d^3 \sum_{i_3} \operatorname{Tr}(U_{i_3} \rho) U_{i_3}^\dagger \\
&= d^4 \rho, \\
Q_\varnothing^{1,6} &= \sum_{i_1, i_2, i_3} \pi_1(U_{i_1} U_{i_2} U_{i_3}) \rho \pi_6(U_{i_1} U_{i_2} U_{i_3})^\dagger \\
&= \sum_{i_1, i_2, i_3} (U_{i_1} U_{i_2} U_{i_3}) \rho (U_{i_1}^\dagger U_{i_2}^\dagger U_{i_3}^\dagger) \\
&= d \sum_{i_1, i_3} U_{i_1} \operatorname{Tr}(U_{i_3} \rho U_{i_1}^\dagger) U_{i_3}^\dagger = d^2 \sum_{i_1} U_{i_1} \rho U_{i_1}^\dagger \\
&= d^3 \mathbb{I}.
\end{aligned} \tag{18}$$

The coefficients $Q_\varnothing^{k,k'}$ with $k \geq 2$ can be computed using these expressions from Equations (18). For instance, consider the following

$$Q_\varnothing^{2,6} = \sum_{i_1, i_2, i_3} \pi_2(U_{i_1} U_{i_2} U_{i_3}) \rho \pi_6(U_{i_1} U_{i_2} U_{i_3})^\dagger = \sum_{i_1, i_2, i_3} (U_{i_1} U_{i_3} U_{i_2}) \rho (U_{i_1}^\dagger U_{i_2}^\dagger U_{i_3}^\dagger), \tag{19}$$

which is equivalent to expression $Q_\varnothing^{1,4}$ because the indices i's are dumb. Thus, one can calculate explicitly the remaining coefficients. Results are thus summarized in the following list:

$$\begin{aligned}
Q_\varnothing^{i,i'} &= d^3 \mathbb{I}, \forall \, (i, i') \in \mathfrak{I} \equiv \{(1, 6), (2, 4), (3, 5), (4, 2), \\
&\qquad (1, 2), (2, 1), (3, 4), (4, 3), (5, 6), \\
&\qquad (6, 5), (5, 3), (6, 1), (1, 3), (2, 5), \\
&\qquad (3, 1), (4, 6), (5, 2), (6, 4)\}, \\
Q_\varnothing^{j,j'} &= d^4 \rho, \forall \, (j, j') \in \mathfrak{I} \equiv \{(1, 4), (2, 6), (3, 2), (4, 5), \\
&\qquad (5, 1), (6, 3), (1, 5), (2, 3), (3, 6), \\
&\qquad (4, 1), (5, 4), (6, 2)\}, \\
Q_\varnothing^{k,k'} &= d^5 \mathbb{I}, \forall \, (k, k') \in \mathfrak{K} \equiv \{(1, 1), (2, 2), (3, 3), \\
&\qquad (4, 4), (5, 5), (6, 6)\}.
\end{aligned} \tag{20}$$

After calculating all these coefficients, we obtain

$$S_0 = \sum_{(i,i')\in\mathfrak{I}} \frac{s_0}{d^3}\mathbb{I}\sqrt{P_iP_{i'}}\otimes|i\rangle\langle i'| + \sum_{(j,j')\in\mathfrak{J}} \frac{s_0\rho}{d^2}\sqrt{P_jP_{j'}}\otimes|j\rangle\langle j'|$$
$$+ \sum_{(k,k')\in\mathfrak{K}} \frac{s_0}{d}\mathbb{I}\sqrt{P_kP_{k'}}\otimes|k\rangle\langle k'|, \quad (21)$$

where $s_0 = p_1p_2p_3$.

Coefficients for S_1. In this case $\mathbf{A}_1^3 = \{\{1\},\{2\},\{3\}\}$ and $\mathbf{B}_1^3 = \{\{2,3\},\{1,3\},\{1,2\}\}$. Let us first consider the coefficient $Q_{\{1\}}^{k,k'}$, so that sum must be accomplished over the indices $\{i_2,i_3\}$, hence $Q_{\{1\}}^{k,k'} = \sum_{i_2,i_3}\pi_k(\mathbb{I}\cdot U_{i_2}\cdot U_{i_3})\rho\pi_{k'}(\mathbb{I}\cdot U_{i_2}\cdot U_{i_3})^\dagger$. Using the relations (A4)–(A6), we obtain

$$\begin{aligned}
Q_{\{1\}}^{\ell,\ell'} &= d^2\rho, \forall\,(\ell,\ell')\in\mathfrak{L}_1 \equiv \{(2,3),(3,2),(2,4),(4,2),\\
&\quad (3,5),(5,3),(3,6),(6,3),(4,5),\\
&\quad (5,4),(4,6),(6,4),(5,1),(1,5),\\
&\quad (1,2),(2,1),(1,6),(6,1)\},\\
Q_{\{1\}}^{m,m'} &= d^3\mathbb{I}, \forall\,(m,m')\in\mathfrak{M}_1 \equiv \{(1,1),(2,2),(3,3),(4,4),\\
&\quad (5,5),(6,6),(1,3),(1,4),(4,1)\\
&\quad (3,1),(2,5),(5,2),(2,6),(6,2),\\
&\quad (3,4),(4,3),(5,6),(6,5)\},
\end{aligned} \quad (22)$$

$$\begin{aligned}
Q_{\{2\}}^{\ell,\ell'} &= d^2\rho, \forall\,(\ell,\ell')\in\mathfrak{L}_2 \equiv \{(1,4),(1,5),(1,6),(2,4),\\
&\quad (2,5),(2,6),(3,4),(3,5),(3,6)\\
&\quad (4,1),(4,2),(4,3),(5,1),(5,2),\\
&\quad (5,3),(6,1),(6,2),(6,3)\},\\
Q_{\{2\}}^{m,m'} &= d^3\mathbb{I}, \forall\,(m,m')\in\mathfrak{M}_2 \equiv \{(1,1),(1,2),(1,3),(2,1),\\
&\quad (2,2),(2,3),(3,1),(3,2),(3,3)\\
&\quad (4,4),(4,5),(4,6),(5,4),(5,5),\\
&\quad (5,6),(6,4),(6,5),(6,6)\},
\end{aligned} \quad (23)$$

$$\begin{aligned}
Q_{\{3\}}^{\ell,\ell'} &= d^2\rho, \forall\,(\ell,\ell')\in\mathfrak{L}_3 \equiv \{(1,3),(1,4),(1,6),(2,3),\\
&\quad (2,4),(2,6),(3,1),(3,2),(3,5)\\
&\quad (4,1),(4,2),(4,5),(5,3),(5,4),\\
&\quad (5,6),(6,1),(6,2),(6,5)\},\\
Q_{\{3\}}^{m,m'} &= d^3\mathbb{I}, \forall\,(m,m')\in\mathfrak{M}_3 \equiv \{(1,1),(1,2),(1,5),(2,1),\\
&\quad (2,2),(2,5),(3,3),(3,4),(3,6)\\
&\quad (4,3),(4,4),(4,6),(5,1),(5,2),\\
&\quad (5,5),(6,3),(6,4),(6,6)\}.
\end{aligned} \quad (24)$$

Hence, the matrix S_1 can be computed

$$S_1 = \frac{1}{d^2}\sum_{s=1}^{3}\left(t_s d\sum_{(\ell,\ell')\in\mathfrak{M}_s}\sqrt{P_\ell P_{\ell'}}\,\mathbb{I}\otimes|\ell\rangle\langle\ell'| + t_s\sum_{(m,m')\in\mathfrak{L}_s}\sqrt{P_m P_{m'}}\,\rho\otimes|m\rangle\langle m'|\right), \quad (25)$$

where $t_1 = p_2p_3q_1$, $t_2 = p_1p_3q_2$ and $t_3 = p_1p_2q_3$.

Coefficients for S_2. In this case, $\mathbf{A}_2^3 = \{\{1,2\},\{1,3\},\{2,3\}\}$ and hence $\mathbf{B}_2^3 = \{\{3\},\{2\},\{1\}\}$. Let us consider

$$Q_{\{2,3\}}^{k,k'} = \sum_{i_1}\pi_k(U_{i_1}\cdot\mathbb{I}\cdot\mathbb{I})\rho\pi_{k'}(U_{i_1}\cdot\mathbb{I}\cdot\mathbb{I})^\dagger = d\mathbb{I},$$

where the operators $\mathbb{I}$ have been written for the sake of clarity as the permutations π_k act on sets of three elements. In a similar way, $Q^{k,k'}_{\{1,3\}} = Q^{k,k'}_{\{1,2\}} = d\mathbb{I}$. Thus, we obtain

$$
\begin{aligned}
\mathcal{S}_2 = \tfrac{p_1 p_2 p_3}{d^2} \sum_{k,k'} \sqrt{P_k P_{k'}} \left(\tfrac{q_2 q_3}{p_2 p_3} Q^{k,k'}_{\{2,3\}} \right.\\
\left. + \tfrac{q_1 q_3}{p_1 p_3} Q^{k,k'}_{\{1,3\}} + \tfrac{q_1 q_2}{p_1 p_2} Q^{k,k'}_{\{1,2\}} \right) \otimes |k\rangle\langle k'| = \tfrac{s_2}{d} \mathbb{I} \otimes \rho_c
\end{aligned}
\tag{26}
$$

where $s_2 = q_1 q_2 p_3 + q_1 q_3 p_2 + q_2 q_3 p_1$.

Coefficients for $\mathcal{S}_3$. Finally, note that $Q^{k,k'}_{\{1,2,3\}} = \rho$ for all k and k'. Thus, the term with $z = 3$ reads

$$
\mathcal{S}_3 = s_3 \sum_{k,k'} \sqrt{P_k P_{k'}} \rho \otimes |k\rangle\langle k'| = s_3 \rho \otimes \rho_c,
\tag{27}
$$

where $s_3 = q_1 q_2 q_3$ and uses the definition of the control qudit.

For three channels, Figure 2 shows different ways to connect channels $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_3$ in either (a)–(f) a definite causal order, or (g) in an indefinite causal order taking into account all 3! causal orders. The quantum 3-switch matrix is again calculated with Equation (5) (see Appendix C)

$$
\mathcal{S}(\mathcal{N}_1,\mathcal{N}_2,\mathcal{N}_3)(\rho \otimes \rho_c) =
\begin{pmatrix}
\mathcal{A}_1 & \mathcal{B} & \mathcal{C} & \mathcal{D} & \mathcal{E} & \mathcal{F} \\
\mathcal{B} & \mathcal{A}_2 & \mathcal{G} & \mathcal{H} & \mathcal{I} & \mathcal{J} \\
\mathcal{C} & \mathcal{G} & \mathcal{A}_3 & \mathcal{K} & \mathcal{L} & \mathcal{M} \\
\mathcal{D} & \mathcal{H} & \mathcal{K} & \mathcal{A}_4 & \mathcal{N} & \mathcal{P} \\
\mathcal{E} & \mathcal{I} & \mathcal{L} & \mathcal{N} & \mathcal{A}_5 & \mathcal{Q} \\
\mathcal{F} & \mathcal{J} & \mathcal{M} & \mathcal{P} & \mathcal{Q} & \mathcal{A}_6
\end{pmatrix},
\tag{28}
$$

where the diagonal and the off-diagonal elements whose expressions are given in Appendix C are also linear combinations of matrices ρ and $\mathbb{I}_t$. From the definition of symmetric matrices [23], we can see that the quantum switch matrices (14) and (28) are block-symmetric matrices with respect to the main diagonal. This could be seen as general from the fact $Q^{k,k'}_{A_z} = Q^{k',k}_{A_z}$ due to Equations (6) and (7) because indices in the sums are dumb. Thus, as the number of channels increases, the number of different $d \times d$ matrices involved in the quantum N-switch matrix $\mathcal{S}$ scales as $N!(N!+1)/2$. Notice that those matrices also characterize information transmission of any definite causal ordering π_k of channels $\mathcal{N}_{\pi_k}$ when setting $P_k = 1$ and $P_s = 0$ for all $s \neq k$.

Matrices in Equation (14) or (28) are written in the basis of the control system ρ_c which maps and weights the chosen causal orders. To know the best rate to communicate classical information with two and three channels, in the following section, we diagonalize matrices (14) and (28) to compute the Holevo information limit χ, which quantifies how much classical information can be transmitted through a channel in a single use. χ gives a lower bound on the classical capacity [3,7,24].

4. Holevo Information Limit for Two and Three Channels

We compute the Holevo information limit (Holevo information for shortness in the following) $\chi(\mathcal{S})$ for $N = 2$ and $N = 3$ channels through a generalization of the mutual information (see for example [25]) and supplementary information of [9]. The Holevo information $\chi(\mathcal{S})$ is found by maximizing mutual information, and it can be shown that maximization over the ρ pure states is sufficient [25]. The Holevo information is then given by

$$
\chi_{\text{QNS}}(\mathcal{S}) = \log d + H(\tilde{\rho}_c^{(N)}) - H^{\min}(\mathcal{S}),
\tag{29}
$$

where d is the dimension of the target system ρ, $H(\tilde{\rho}_c^{(N)})$ is the von-Neumann entropy of the output control system $\tilde{\rho}_c^{(N)}$ for N channels, and $H^{\min}(\mathcal{S})$ is the minimum of the entropy at the output of the

channel $\mathcal{S}$. The minimization of $H^{\min}(\mathcal{S}) \equiv \min\limits_{\rho} H^{\min}(\mathcal{S}(\rho))$ is over all input states ρ going on the channel $\mathcal{S}$ [25]. To evaluate Equation (29):

1. The diagonalization and minimization of $H^{\min}(\mathcal{S})$ is performed on all possible states given by ρ. It is done analytically for $N = 2$ channels and arbitrary q_i. For $N = 3$ channels, we compute the eigenvalues of the full quantum 3-switch matrix $\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3) (\rho \otimes \rho_c)$ numerically.
2. $\tilde{\rho}_c^{(N)}$ was analytically calculated following [9].
3. We deduce $H(\tilde{\rho}_c^{(N)})$ from the analytical expressions of $\tilde{\rho}_c^{(N)}$.

4.1. Holevo Information Limit for $N = 2$ Channels

4.1.1. Calculation of $H^{\min}$

We calculate the minimum output entropy $H^{\min}(\mathcal{S})$ of the channel $\mathcal{S} \equiv \mathcal{S}(\mathcal{N}_1, \ldots, \mathcal{N}_N)$

$$H^{\min}(\mathcal{S}) \equiv \min_{\rho} H^{\min}(\mathcal{S}(\rho)) = \min_{\rho} \sum_i -\lambda_{\mathcal{S}(\rho),i} \log[\lambda_{\mathcal{S}(\rho),i}],$$

where the minimization is a priori over all input states ρ and $\{\lambda_{\mathcal{S}(\rho),i}\}_{i=1}^{d}$ are the eigenvalues of $\mathcal{S}(\rho)$. In fact, it is sufficient to minimize over the states [25] and the eigenvalues $\{\lambda_{\rho,i}\}_{i=1}^{d}$ sum up to 1. As $H^{\min}(\mathcal{S}(\rho))$ is concave, the minimization is done as in Reference [9] : the eigenvalues $\{\lambda_{\mathcal{S}(\rho),i}\}_{i=1}^{d}$ are taken at the border of the interval $[0, 1]^{\times d}$ and, as they sum up to one, the minimization is simplified to the cases where all λ but one are set to zero, and the last one is equal to 1.

In this situation, $\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2) (\rho \otimes \rho_c)$ has only four non-zero matrix elements, (see Equation (14)), which can be rewritten as 2×2 matrices

$$\begin{pmatrix} a_0 p & b \\ b & a_0 q \end{pmatrix}, \tag{30}$$

where a_0 and b are $d \times d$ matrices and linear combinations of ρ and $\mathbb{I}_t$:

$$a_0 = (r_0 + r_1)\mathbb{I}/d + r_2 \rho, \quad b = \sqrt{P_1 P_2}[(r_0 + d^2 r_2)\rho/2 + r_1 \mathbb{I}/d], \tag{31}$$

with $p \equiv P_1$, $q \equiv P_2$ are the control probabilities with $p + q = 1$.

Using the commutativity of ρ and $\mathbb{I}_t$ (so they have the same eigenvectors), we then retrieve analytically $a_{\pm}$, the matrix-eigenvalues of $\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_N) (\rho \otimes \rho_c)$

$$a_{\pm} = \frac{a_0}{2} \pm \sqrt{b^2 + a_0^2 (p - \frac{1}{2})^2}. \tag{32}$$

The existence of this last expression is warranted by the positivity of the discriminant [26], considering the positivity of ρ and the structure of a_0 and b, which are linear combinations of $\mathbb{I}$ and ρ.

The commutativity properties of ρ and $\mathbb{I}_t$ are inherited to $a_{\pm}$. Then, the eigenvalues of $\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2) (\rho \otimes \rho_c)$ for two causal orders are the eigenvalues of $a_{\pm}$. Thus, to diagonalize $a_{\pm}$, we just replace ρ by its eigenvalues, labeled as $\lambda_{\rho,i}$, in Equation (32), which generalizes the procedure obtained in [9]. Our procedure gives access to the transmission of information in a more general situation, where the depolarization strengths q_i can be different for each channel, and it can take any value between 0 and 1. Equation (32) gives the eigenvalues of the matrix $\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2)(\rho \otimes \rho_c)$ ($s = \pm 1$)

$$\lambda_{s,i} = \frac{\alpha_0}{2} + s\sqrt{pq\beta^2 + \alpha_0^2(p - \frac{1}{2})^2}$$

$$\text{with: } \alpha_0 \equiv \frac{1 - q_1 q_2}{d} + q_1 q_2 \lambda_{\rho,i} \tag{33}$$

$$\beta \equiv \frac{p_1 q_2 + q_1 p_2}{d} + (\frac{p_1 p_2}{d^2} + q_1 q_2)\lambda_{\rho,i}.$$

The eigenvalues of $\lambda_{s,i}$ are well defined because of the positivity of discriminant [26]. Finally, using the concavity of the entropy, the minimum of the entropy $H^{\min}$ for a state is reached by setting just one $\lambda_{\rho,i}$ to one and all the others to zero; with this, we obtain

$$- H^{\min}(\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2)) = \sum_{\substack{s\in\{\pm 1\} \\ k\in\{0,1\}}} (d-1)^{1-k} \lambda_{s,k} \log(\lambda_{s,k}), \tag{34}$$

$$\lambda_{s,k} = \frac{\alpha_{0,k}}{2} + s\sqrt{pq\beta_k^2 + \alpha_{0,k}^2 (p - \frac{1}{2})^2}, \tag{35}$$

$$\alpha_{0,k} = \frac{1 - q_1 q_2}{d} + k q_1 q_2, \tag{36}$$

$$\beta_k = \frac{p_1 q_2 + q_1 p_2}{d} + k\left(\frac{p_1 p_2}{d^2} + q_1 q_2\right). \tag{37}$$

It is easy to show $\beta_k \leq \alpha_{0,k}$, then $\lambda_{\pm,i} \geq 0$ ($\lambda_{s,k} \geq 0$) as expected. In addition, $0 \leq \lambda_{s,k} \leq 1$ and then $-H^{\min}(\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2)) \leq 0$.

If one of $q_1 = 1$, i.e., channel 1 is free of depolarization, then $\alpha_{0,k} = \frac{p_2}{d} + k q_2 = \beta_k$ and

$$- H^{\min}(\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2)) = (d-1)\frac{p_2}{d} \log(\frac{p_2}{d}) + (\frac{p_2}{d} + q_2) \log(\frac{p_2}{d} + q_2) \tag{38}$$

depends only on the probability of depolarization for channel 2. Thus, $-H^{\min}(\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2))$ reaches its maximum value of zero only if $q_1 = q_2 = 1$. Alternatively, it is direct to show that the discriminant reaches its maximum value when $p = \frac{1}{2}$, which is the case studied by Ebler et al. [9]. In addition, if $q_1 = q_2 = 0$, i.e., both channels are fully depolarizing, then $\alpha_{0,k} = \frac{1}{d}$, $\beta_k = \frac{k}{d^2}$, so $-H^{\min}(\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2))$ reaches the minimum value

$$- H^{\min}(\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2)) = -\log(2d) + \frac{1}{2d^2} \log(\frac{d+1}{d-1}) + \frac{1}{2d} \log(1 - \frac{1}{d^2}). \tag{39}$$

For the sake of shortness, the entropy $H^{\min}(\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2))$ will be denoted simply as $H^{\min}(\mathcal{S}_N)$. To illustrate the range of parameters of Equation (38), we plot the entropy $H^{\min}(\mathcal{S}_2)$ map for two noisy channels. Figure 3 shows the entropy $H^{\min}(\mathcal{S}_2)$. The plots are contour surfaces of $H^{\min}(\mathcal{S}_2)$ when q_1, q_2 vary from 0 to 1. Each plot contains thirty surfaces distributed in their complete range shown in the color-chart. We plot several cases of $H^{\min}(\mathcal{S}_2)$ when the dimension of the target is $d = 2, 3, 10$ and 100.

4.1.2. Derivation of $\tilde{\rho}_c^{(2)}$

To obtain the output state of the control system $\tilde{\rho}_c^{(N)}$ after N channels, we calculate

$$\mathrm{Tr}_{XIJ}\left[(\mathcal{S}(\mathcal{N}_1, \ldots, \mathcal{N}_N)(\rho \otimes \rho_c)) \otimes \mathbb{I})(\omega_{XIJAC})\right],$$

where ω_{XIJAC} is an extended input state with a pure conditional state as described in [9]. A direct calculation shows:

$$\mathrm{Tr}_{XIJ}\left[(\mathcal{S}(\mathcal{N}_1, \ldots, \mathcal{N}_N)(\rho \otimes \rho_c)) \otimes \mathbb{I})(\omega_{XIJAC})\right] =$$

$$= \mathrm{Tr}_{XIJ}\left[\frac{1}{d^2} \sum_{x,i,j} p_x |x\rangle\langle x| |i\rangle\langle i| |j\rangle\langle j| \otimes \mathcal{S}(\mathcal{N}_1, \ldots, \mathcal{N}_N)(\rho' \otimes \rho_c)\right]$$

$$= \frac{\mathbb{I}}{d} \otimes \tilde{\rho}_c^{(N)}, \tag{40}$$

where $\rho' = X(i)Z(j)\rho Z(j)^\dagger X(i)^\dagger$ and $X(i)\,|l\rangle = |i \oplus l\rangle$, $Z(j)\,|l\rangle = e^{2\pi ijl}\,|l\rangle$ are the known Heisenberg$-$Weyl operators [25]. To isolate the term $\tilde{\rho}_c^{(N)}$, we apply the following relations:

$$\text{Tr}_{XIJ}\left[\sum_{XIJ} p_x\,|x\rangle\langle x|\,|i\rangle\langle i|\,|j\rangle\langle j|\,\rho'\right] = d\mathbb{I}, \tag{41}$$

$$\text{Tr}_{XIJ}\left[\sum_{XIJ} p_x\,|x\rangle\langle x|\,|i\rangle\langle i|\,|j\rangle\langle j|\,\mathbb{I}\right] = d^2\mathbb{I}, \tag{42}$$

which are valid for $N \geq 2$, and they are obtained by direct calculation following the former definitions. Then, for $N = 2$, we find that the output control state is

$$\tilde{\rho}_c^{(2)} = p_1 p_2 [P_1\,|1\rangle\langle 1| + P_2\,|2\rangle\langle 2| + \frac{\sqrt{P_1 P_2}}{d^2}\,(|0\rangle\langle 1| + |1\rangle\langle 0|)] + \rho_c\,(1 - p_1 p_2), \tag{43}$$

where $p_i = 1 - q_i$.

Using the two previous results for χ_{QNS}, Figure 4 shows the transmission map of information for two noisy channels. The plots are contour surfaces of χ_{Q2S} when q_1, q_2 and $P_1 = P_2 = p$ vary from 0 to 1. The maximum capacity is trivially reached when $q_1 = q_2 = 1$ simultaneously reaching the value $\chi_{Q2S} = \log d$. The minimum capacity is zero, reached in the boundary of the front sides with $(q_1 = 0, p = 0, 1)$, $(q_2 = 0, p = 0, 1)$, $(q_1 = 0, q_2 = 1)$, and $(q_1 = 1, q_2 = 0)$. Notably, for $q_1 = q_2 = 0$, there are values higher than the minimum. This phenomenon is observed in the protuberance of plots near $\chi_{Q2S} = 0$. For larger values of d, the protuberance occurs sharply near $q_1 = 0$ and $q_2 = 0$ faces. Note the nearest surface to those faces are for $\chi_{Q2S} = 10^{-3}, 10^{-3}, 10^{-4}, 10^{-7}$ respectively for each plot $d = 2, 3, 10, 100$.
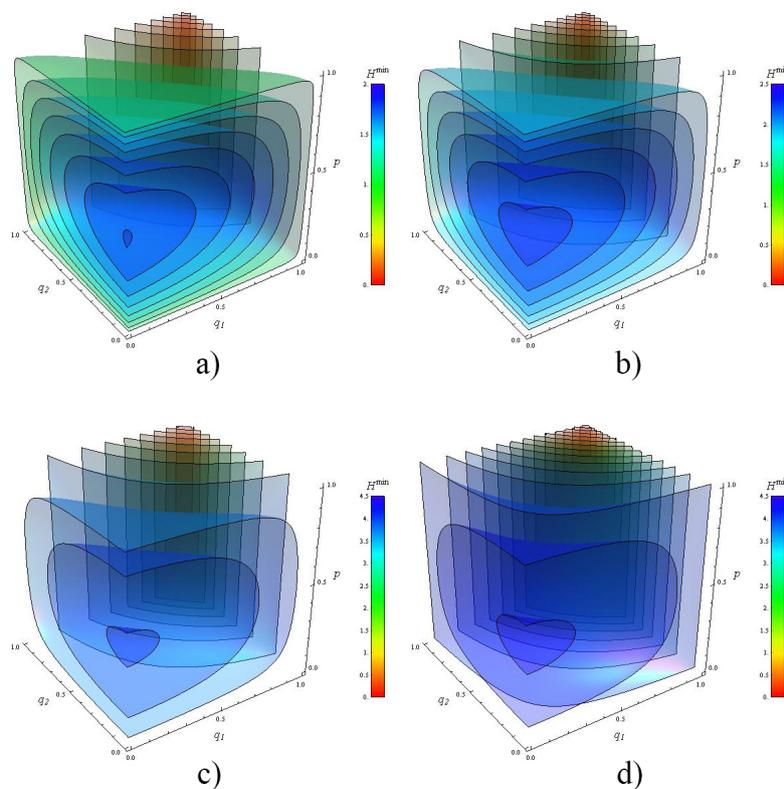

Figure 3. Entropy map for two noisy channels. The 3D graphs represent contour surfaces of the Von-Neumann entropy $H^{\min}(\mathcal{S}_2)$ when the depolarizing parameters q_1, q_2, and the probabilities $P_1 = P_2 = p$ are varied from 0 to 1. We plot several cases when the dimension d of the target ρ is: (**a**) $d = 2$; (**b**) $d = 3$; (**c**) $d = 10$; and (**d**) $d = 100$. The value of $H^{\min}(\mathcal{S}_2)$ is also depicted by the color in the bar.
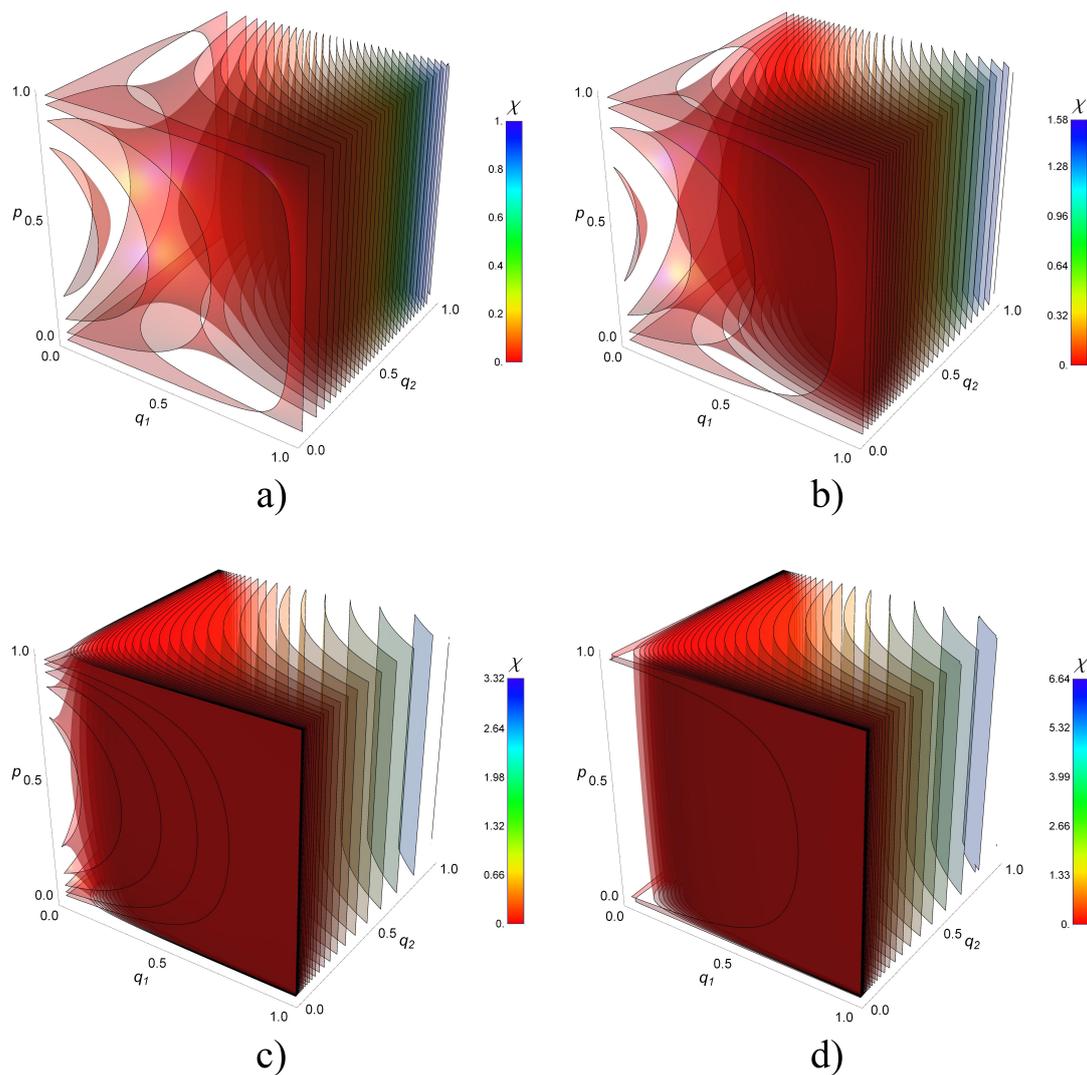
Figure 4. Transmission map of information for two noisy channels. The 3D graphs represent contour surfaces of the Holevo information χ_{Q2S} when the depolarising parameters q_1, q_2 and the probabilities $P_1 = P_2 = p$ varied from 0 to 1. We plot several cases for the dimension d of the target system: (**a**) $d = 2$, (**b**) $d = 3$, (**c**) $d = 10$ and (**d**) $d = 100$. In all these cases there are thirty contour surfaces of χ_{Q2S}. The values of χ_{Q2S} are shown in the color bars.

4.2. Holevo Information for $N = 3$ Channels

We numerically calculate the eigenvalues of the entropy $H^{\min}$ for $N = 3$ channels from Equation (28). Then, using relations (41) from (21), (25), (26) and (27), we find that the output state is

$$\widetilde{\rho}_c^{(3)} = (s_2 + s_3)\rho_c + \frac{s_0}{d^2} \left(\sum_{(k,k') \in \mathfrak{I},\mathfrak{J}} \sqrt{P_k P_{k'}} |k\rangle\langle k'| + d^2 \sum_{(k,k') \in \mathfrak{K}} \sqrt{P_k P_{k'}} |k\rangle\langle k'| \right) \tag{44}$$
$$+ \frac{1}{d^2} \left(\sum_{s=1}^{3} \sum_{(\ell,\ell') \in \mathfrak{L}_s} \sqrt{P_\ell P_{\ell'}} r_s |\ell\rangle\langle \ell'| + d^2 \sum_{s=1}^{3} \sum_{(m,m') \in \mathfrak{M}_s} \sqrt{P_m P_{m'}} r_s |m\rangle\langle m'| \right).$$

As before, putting those outcomes together in (29), Figure 5a,b give the Holevo information χ_{Q2S} and χ_{Q3S} for two and three channels, respectively, as a function of the depolarization strengths q_i and the dimension d of the target system. Our model enables us to exhibit a wealth of different behaviors

as a function of d, q_i, and N from the fully noisy situation to the identity channel transmission. For the sake of simplicity, we restrict our graphical analysis to equal depolarization strengths, i.e., $q_1 = q_2 = q_3$, with a balanced superposition of $N!$ causal orders, that is, with equally weighted probabilities $P_k = 1/N!$ for each case $N = 2, 3$.
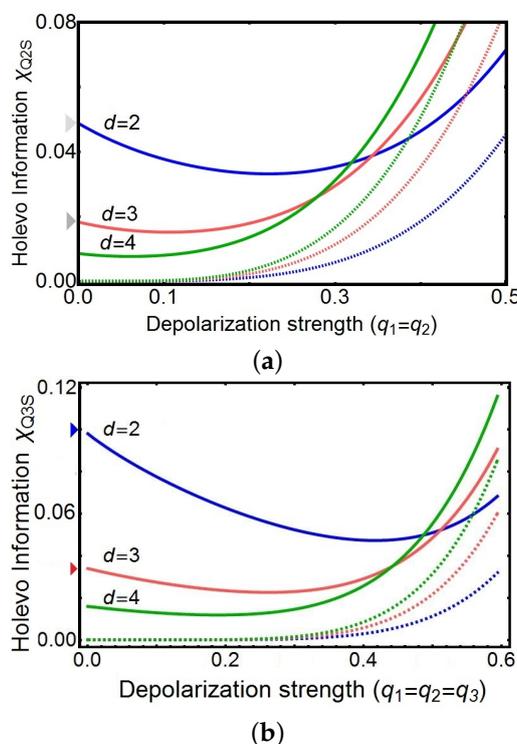

Figure 5. Transmission of information for $N = 2$ and $N = 3$ channels. Holevo information as a function of the depolarization strengths q_i of the channels. We plot the subcases of equal depolarization strengths, i.e., $q_1 = q_2 = q_3 = q$, with equally weighted probabilities P_k for indefinite causal orders (solid line) with (**a**) $N = 2$ and (**b**) $N = 3$ channels. The transmission of information first decreases to a minimal value for Holevo information and then the transmission of information increases with q. For completely depolarizing channels, i.e., $q = 0$, the transmission of information is nonzero and decreases as d increases. A comparison is shown between the Holevo information when the channels are in a definite causal order (dashed line). A full superposition of $N!$ causal orders is used.

The analysis of these results allows us to draw the following conclusions for those particular cases:

- For a fixed dimension d, the Holevo information for indefinite causal order is always higher than the one obtained using one of the definite causal order shown in Figure 2. This is especially the case for totally depolarized channels i.e., $q_i = 0, \forall i$. For completely clean channels ($q = 1$), the Holevo information for indefinite and definite causal order converges to the same value depending on d (not shown).
- Two regions can be distinguished. In the strongly depolarized region (roughly $q < 0.3$ for $N = 2$ and $q < 0.5$ for $N = 3$), the increase of the dimension d of the target system is detrimental to the Holevo information transmitted by the quantum switch. In contrast, in the moderately depolarized region ($q > 0.3$ for $N = 2$ and $q > 0.5$ for $N = 3$), the Holevo information increases both with q and d, a maximum (not shown) as expected for completely clean channels.
- In the strongly depolarized region, increasing the number of channels to $N = 3$ is definitively advantageous for information extraction. For instance, in the case of totally depolarized channels ($q = 0$), the Holevo information is approximately doubled with $N = 3$ with respect to $N = 2$ for all values of the dimension d calculated up to $d = 10$

In fact, Table 1 gives the values of the ratio χ_{Q3S}/χ_{Q2S}, finding that the Holevo information is approximately doubled for $N = 3$ with respect to $N = 2$.

Table 1. Values of the Holevo information ratio χ_{Q3S}/χ_{Q2S}. The mean value of the ratio is 1.9328 ± 0.0617.

d	χ_{Q2S}	χ_{Q3S}	χ_{Q2S}/χ_{Q3S}
2	0.0487	0.0980	2.0123
3	0.0183	0.0339	1.8524
4	0.0085	0.0159	1.8705
5	0.0046	0.0087	1.8913
6	0.0027	0.0053	1.9629
7	0.0018	0.0034	1.8888
8	0.0012	0.0023	1.9166
9	0.0008	0.0016	2
10	0.0006	0.0012	2

5. Conclusions

Communication enhancement is a challenging task in quantum information processing due to imperfection of communication channels subjected to depolarization. Causal order has been proposed as a disruptive procedure to improve communication, compression of quantum information, bringing the quantum possibilities into a new frontier. We have analyzed the quantum control of N operators in the context of the second-quantized Shannon theory and in the specific case of superposition of causal orders, extending the results in the current literature. We obtained a general expression for $\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_N)$ for the quantum N-switch for an arbitrary number of channels with any depolarizing strength, thus providing an operational formula enabling the exploration of communication channels controlled by causal orders. This formula is useful to explore computationally the cases with an increasing N. A detailed analysis to assess the information transmission for the cases of $N = 2$ and $N = 3$ channels is presented: an increasing number of channels improves the transmission of information. In particular, we remarkably found that the Holevo information is doubled when the number of channels goes from $N = 2$ to $N = 3$.

We give the matrices corresponding to quantum N-switches $\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_N)$ as a function of the number of channels, depolarization strengths, and dimension of the target system. We obtain other general properties for the general case of N channels such as the symmetric properties of matrices $Q_{A_z}^{k,k'}$ and thus of $\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_N)$. We also demonstrate that $\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_N)$ is always a linear combination of ρ and $\mathbb{I}_t$, whatever N channels. Expressions for the Holevo limit are equally accessible from our expressions and methodology. In addition, we showed that the depolarizing strengths can be used as control parameters to modify the information transmission on demand. We shall develop elsewhere the analysis of control via selected combinations of the $N!$ available causal order enabled by the present work.

Appendix A. Completeness Property for $W_{\mathbf{i}}$

We demonstrate here the completeness property

$$\sum_{\{i_s\}|_{s=1}^N} W_{\mathbf{i}} W_{\mathbf{i}}^{\dagger} = \mathbb{I}_t \otimes \mathbb{I}_c \tag{A1}$$

for the generalized Kraus operators $W_{\mathbf{i}}$ for the full quantum N-switch channel by relying on the reordering of the sums obtained by grouping terms with indices i_s equal to zero. $W_{\mathbf{i}} := W_{i_1 i_2 \ldots i_N} = \sum_{k=1}^{N!} K_{\pi_k} \otimes |k\rangle \langle k|$ and $K_{\pi_k} := \pi_k(K_{i_1}^1 \cdots K_{i_N}^N)$, where π_k acts on the subscripts j of the Kraus operators $K_{i_j}^j$. In the sum $\{i_j\}|_{j=1}^N$, each index in the set of indices $\{i_1, i_2, \ldots, i_N\}$ is associated with a channel $\mathcal{N}_j$ where $j \in \{1, 2, \ldots, N\}$ and varies from 0 to d^2. By introducing the definition of the Kraus operators $K_{i_j}^j = \frac{\sqrt{1-q_j}}{d} U_{i_j}^j$ into $W_{\mathbf{i}}$, the left side from Equation (A1) can be re-written as

$$h_N d^{-2N} \sum_{\{i_s\}|_{s=1}^N} \sum_{k=1}^{N!} U_{\pi_k} U_{\pi_k}^{\dagger} \otimes |k\rangle \langle k|, \tag{A2}$$

where $U_{\pi_k} := \pi_k(U_{i_1}^1 \cdots U_{i_N}^N)$ and $h_N = \prod_{j=1}^N 1 - q_j$. As $U_i U_i^{\dagger} = \mathbb{I}, \forall i > 0$, the product $U_{\pi_k} U_{\pi_k}^{\dagger}$ reduces almost to the identity $\mathbb{I}_t$ except for the factors $U_0 U_0^{\dagger} = \frac{d^2 q_j}{1-q_j} \mathbb{I}_t$. Using Equation (4) in the above expression, it can be rearranged into:

$$\sum_{\{i_j\}|_{j=1}^N} W_{i_1 i_2 \ldots i_N} W_{i_1 i_2 \ldots i_N}^{\dagger} = h_N d^{-2N} \sum_{k=1}^{N!} \sum_{z=0}^N d^{2z} \sum_{a \in \mathbf{A}_z^N} \sum_{b \in B_z} h_{A_z} \mathbb{I}_t \otimes |k\rangle \langle k|, \tag{A3}$$

where the sum over A_z is the sum of terms h_{A_z} over all the elements of $\mathbf{A}_z^N$, the set of all subset of z elements in $\{1, 2, \ldots, N\}$. This yields the factor $f_{A_z} = h_N h_{A_z} d^{2(z-N)}$ in Equation (6).

To prove Equation (A1), we then apply the total probability property $\sum_{z=0}^N \sum_{a \in A_z} h_{A_z} = \frac{1}{h_N}$ together with the property $\sum_{b \in B_z} d^{2(z-N)} = 1$. Thus, property (A1) is proved.

Appendix B. Relations to Evaluate Coefficients $Q_{A_z}^{k,k'}$

We recall below the relations needed to deduce explicitly matrices $\mathcal{S}_z$ and then $\mathcal{S}$ for the quantum N-switch from the sums and products of the $Q_{A_z}^{k,k'}$ factors

$$\sum_{i=1}^{d^2} U_i X \left[U_i\right]^{\dagger} = d \operatorname{Tr} X \, \mathbb{I}, \tag{A4}$$

$$\sum_{i=1}^{d^2} \operatorname{Tr}([U_i]^{\dagger} \rho) U_i = \sum_{i=1}^{d^2} \operatorname{Tr}(U_i \rho) \left[U_i\right]^{\dagger} = d \, \rho, \tag{A5}$$

where X is any $d \times d$ matrix and U_i an orthonormal basis for the $d \times d$ matrices. Applying Equation (A4) to $X = \mathbb{I}$,

$$\sum_{i=1}^{d^2} U_i \left[U_i\right]^{\dagger} = d^2 \mathbb{I}. \tag{A6}$$

Applying Equation (A4) to $X = \rho$, such that $\operatorname{Tr}(\rho) = 1$, we get a uniform randomization over the set of unitaries $U_{i \neq 0}$ that completely depolarizes the state ρ, thus giving the relation $\sum_i U_i \rho U_i^{\dagger} = d \mathbb{I}$.

Appendix C. Matrices $\mathcal{S}_z$ for the Quantum 3-Switch

Matrix $\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3)(\rho \otimes \rho_c)$ is a 6×6 block-matrix, block-symmetric matrix whose matrix elements are matrices of dimension $d \times d$. Since $Q_{A_z}^{k,k'} = Q_{A_z}^{k',k}$, for all A_z, then $\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3)(\rho \otimes \rho_c)$ is symmetric with respect to the main diagonal. Now, by expanding equations $\mathcal{S}_0$, $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ in the control qudit basis, $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle, |6\rangle\}$, we can find the quantum 3-switch matrix $\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3)(\rho \otimes \rho_c)$ having diagonal elements as

$$\mathcal{A}_k = \mathbb{I}P_k \frac{(s_0 + s_1 + s_2)}{d} + \rho P_k s_3, \qquad \text{for } k = 1, 2, \ldots, 6, \tag{A7}$$

where $s_1 = t_1 + t_2 + t_3$. The off-diagonal elements read

$$
\begin{aligned}
\mathcal{B} &= \mathbb{I}\sqrt{P_1 P_2}\frac{(d^2 s_2 + d^2 t_2 + d^2 t_3 + s_0)}{d^3} &+& \rho\sqrt{P_1 P_2}\frac{(d^2 s_3 + t_1)}{d^2} &,\\[4pt]
\mathcal{C} &= \mathbb{I}\sqrt{P_1 P_3}\frac{(d^2 s_2 + d^2 t_1 + d^2 t_2 + s_0)}{d^3} &+& \rho\sqrt{P_1 P_3}\frac{(d^2 s_3 + t_3)}{d^2} &,\\[4pt]
\mathcal{D} &= \mathbb{I}\sqrt{P_1 P_4}\frac{(t_1 + s_2)}{d} &+& \rho\sqrt{P_1 P_4}\frac{(d^2 s_3 + s_0 + t_2 + t_3)}{d^2} &,\\[4pt]
\mathcal{E} &= \mathbb{I}\sqrt{P_1 P_5}\frac{(s_2 + t_3)}{d} &+& \rho\sqrt{P_1 P_5}\frac{(d^2 s_3 + s_0 + t_1 + t_2)}{d^2} &,\\[4pt]
\mathcal{F} &= \mathbb{I}\sqrt{P_1 P_6}\frac{(d^2 s_2 + s_0)}{d^3} &+& \rho\sqrt{P_1 P_6}\frac{(d^2 s_3 + s_1)}{d^2} &,\\[4pt]
\mathcal{G} &= \mathbb{I}\sqrt{P_2 P_3}\frac{(s_2 + t_2)}{d} &+& \rho\sqrt{P_2 P_3}\frac{(d^2 s_3 + s_0 + t_1 + t_3)}{d^2} &,\\[4pt]
\mathcal{H} &= \mathbb{I}\sqrt{P_2 P_4}\frac{(d^2 s_2 + s_0)}{d^3} &+& \rho\sqrt{P_2 P_4}\frac{(d^2 s_3 + s_1)}{d^2} &,\\[4pt]
\mathcal{I} &= \mathbb{I}\sqrt{P_2 P_5}\frac{(d^2 s_2 + d^2 t_1 + d^2 t_3 + s_0)}{d^3} &+& \rho\sqrt{P_2 P_5}\frac{(d^2 s_3 + t_2)}{d^2} &,\\[4pt]
\mathcal{J} &= \mathbb{I}\sqrt{P_2 P_6}\frac{(s_2 + t_1)}{d} &+& \rho\sqrt{P_2 P_6}\frac{(d^2 s_3 + s_0 + t_2 + t_3)}{d^2} &,\\[4pt]
\mathcal{K} &= \mathbb{I}\sqrt{P_3 P_4}\frac{(d^2 s_2 + d^2 t_1 + d^2 t_3 + s_0)}{d^3} &+& \rho\sqrt{P_3 P_4}\frac{(d^2 s_3 + t_2)}{d^2} &,\\[4pt]
\mathcal{L} &= \mathbb{I}\sqrt{P_3 P_5}\frac{(d^2 s_2 + s_0)}{d^3} &+& \rho\sqrt{P_3 P_5}\frac{(d^2 s_3 + s_1)}{d^2} &,\\[4pt]
\mathcal{M} &= \mathbb{I}\sqrt{P_3 P_6}\frac{(s_2 + t_3)}{d} &+& \rho\sqrt{P_3 P_6}\frac{(d^2 s_3 + s_0 + t_1 + t_2)}{d^2} &,\\[4pt]
\mathcal{N} &= \mathbb{I}\sqrt{P_4 P_5}\frac{(s_2 + t_2)}{d} &+& \rho\sqrt{P_4 P_5}\frac{(d^2 s_3 + s_0 + t_1 + t_3)}{d^2} &,\\[4pt]
\mathcal{P} &= \mathbb{I}\sqrt{P_4 P_6}\frac{(d^2 s_2 + d^2 t_2 + d^2 t_3 + s_0)}{d^3} &+& \rho\sqrt{P_4 P_6}\frac{(d^2 s_3 + t_1)}{d^2} &,\\[4pt]
\mathcal{Q} &= \mathbb{I}\sqrt{P_5 P_6}\frac{(d^2 s_2 + d^2 t_1 + d^2 t_2 + s_0)}{d^3} &+& \rho\sqrt{P_5 P_6}\frac{(d^2 s_3 + t_3)}{d^2} &.
\end{aligned} \tag{A8}
$$

Those matrix elements are the entries of the matrix (28) in the main text.

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