From Dissipativity Theory to Compositional Construction of Finite Markov Decision Processes

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ABSTRACT

This paper is concerned with a compositional approach for constructing finite Markov decision processes of interconnected discrete-time stochastic control systems. The proposed approach leverages the interconnection topology and a notion of so-called *stochastic storage functions* describing joint dissipativity-type properties of subsystems and their abstractions. In the first part of the paper, we derive dissipativity-type compositional conditions for quantifying the error between the interconnection of stochastic control subsystems and that of their abstractions. In the second part of the paper, we propose an approach to construct finite Markov decision processes together with their corresponding stochastic storage functions for classes of discrete-time control systems satisfying some *incremental passivablity* property. Under this property, one can construct finite Markov decision processes by a suitable discretization of the input and state sets. Moreover, we show that for linear stochastic control systems, the aforementioned property can be readily checked by some matrix inequality. We apply our proposed results to the temperature regulation in a circular building by constructing compositionally a finite Markov decision process of a network containing 200 rooms in which the compositionality condition does not require any constraint on the number or gains of the subsystems. We employ the constructed finite Markov decision process as a substitute to synthesize policies regulating the temperature in each room for a bounded time horizon. We also illustrate the effectiveness of our results on an example of fully connected network.

KEYWORDS

Finite Markov Decision Processes, Interconnected Stochastic Control Systems, Compositionality, Dissipativity Theory

1 INTRODUCTION

Large-scale interconnected systems have received significant attentions in the last few years due to their presence in real life systems including power networks, air traffic control, and so on. Each complex real-world system can be regarded as an interconnected system composed of several subsystems. Since these large-scale networks of systems are inherently difficult to analyze and control, one can develop compositional schemes to employ the abstractions of the given subsystems as a replacement in the controller design process. Those abstractions allow us to design controllers for them, and then refine the controllers to the ones for the concrete subsystems, while provide us with the quantified errors for the overall interconnected system in this controller synthesis detour.

Construction of finite abstractions was introduced in recent years as a method to reduce the complexity of controller synthesis problems in particular for enforcing complex logical properties. Finite abstractions are abstract descriptions of the continuous-space control systems in which each discrete state corresponds to a collection of continuous states of the original system. Since the abstractions are finite, algorithmic approaches from computer science are applicable to synthesize controllers enforcing complex logic properties including those expressed as linear temporal logic formulae.

In the past few years, there have been several results on the construction of (in)finite abstractions for stochastic systems. Existing results for *continuous-time* systems include infinite approximation techniques for jump-diffusion systems [1], finite bisimilar abstractions for incrementally stable stochastic switched systems [2] and randomly switched stochastic systems [3], and finite bisimilar abstractions for incrementally stable stochastic control systems without discrete dynamics [4]. Compositional modelling and analysis for the safety verification of stochastic hybrid systems are investigated in [5] in which random behaviour occurs only over the discrete components – this limits their applicability to systems with continuous probabilistic evolutions.

Recently, compositional construction of infinite abstractions is discussed in [6] using small-gain type conditions and of finite bisimilar abstractions in [7] based on a new notion of disturbance bisimilarity relation.

For *discrete-time* stochastic models with continuous state spaces, finite approximations are initially proposed in [8] for formal verification and synthesis of this class of systems. The algorithms are improved in terms of scalability in [9, 10]. Those techniques have been implemented in the tool FAUST^2 [11]. Extension of the techniques to infinite horizon properties is proposed in [12] and formal abstraction-based policy synthesis is discussed in [13]. Recently, compositional construction of finite abstractions is discussed in [14] using dynamic Bayesian networks, and infinite abstractions (reduced order models) in [15] and [16] using small-gain type conditions and dissipativity-type properties of subsystems and their abstractions, respectively, all for discrete-time stochastic control systems. Our proposed approach extends the abstraction techniques in [14] from verification to synthesis, by proposing a different quantification of the abstraction error, and leveraging the dissipativity properties of subsystems and structure of interconnection topology to show the compositonal results for the finite Markov decision processes. Although the results in [15] deal only with infinite abstractions (reduced order models), our proposed approach considers finite Markov decision processes as abstractions which are the main tools for automated synthesis of controllers for complex logical properties. To the best of our knowledge, this is the first time a closed form dynamical representation of the abstract finite Markov decision processes is used to facilitate the use of dissipativity properties of subsystems in the error quantification.

In particular, we provide a compositional approach for the construction of finite Markov decision processes of interconnected discrete-time stochastic control systems. The proposed compositional technique leverages the interconnection structure and joint dissipativity-type properties of subsystems and their abstractions characterized via a notion of so-called *stochastic storage functions*. The provided compositionality conditions can enjoy the structure of interconnection topology and be potentially satisfied independently of the number or gains of the subsystems (cf. case study

section). The stochastic storage functions of subsystems are utilized to quantify the error in probability between the interconnection of concrete stochastic subsystems and that of their finite Markov decision processes. As a consequence, one can leverage the proposed results here to solve particularly safety/reachability problems over the finite interconnected systems and then carry the results over the concrete interconnected ones.

We also propose an approach to construct finite Markov decision processes together with their corresponding stochastic storage functions for classes of stochastic control subsystems satisfying some *incremental passivability* property. Under this property, one can construct a finite Markov decision process by a suitable discretization of the input and state sets. Moreover, we show that for linear stochastic control systems, the mentioned property can be readily verified by some matrix inequality. Finally, we illustrate the effectiveness of the results using the temperature regulation in a circular building by constructing compositionally a finite Markov decision process of a network containing 200 rooms in which the compositionality condition does not require any constraint on the number or gains of the subsystems. We leverage the constructed finite Markov decision process as a substitute to synthesize policies regulating the temperature in each room for a bounded time horizon. We benchmark our results against the compositional abstraction technique of [14] which is based on construction of finite dynamic Bayesian networks.

2 DISCRETE-TIME STOCHASTIC CONTROL SYSTEMS

2.1 Notation

The following notation is used throughout the paper. We denote the set of nonnegative integers by $\mathbb{N} := \{0, 1, 2, \dots\}$ and the set of positive integers by $\mathbb{N}_{\geq 1} := \{1, 2, 3, \dots\}$. The symbols $\mathbb{R}$, $\mathbb{R}_{>0}$, and $\mathbb{R}_{\geq 0}$ denote the set of real, positive and nonnegative real numbers, respectively. For any set X we denote by 2^X the power set of X that is the set of all subsets of X. Given N vectors $x_i \in \mathbb{R}^{n_i}$, $n_i \in \mathbb{N}_{\geq 1}$, and $i \in \{1, \dots, N\}$, we use $x = [x_1; \dots; x_N]$ to denote the corresponding vector of dimension $\sum_i n_i$. Given a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm of x. Symbols I_n and $\mathbf{1}_n$ denote, respectively, the identity matrix in $\mathbb{R}^{n\times n}$ and the column vector in $\mathbb{R}^{n\times 1}$ with all its elements equal to one. We denote by $\mathrm{diag}(a_1, \dots, a_N)$ a diagonal matrix in $\mathbb{R}^{N\times N}$ with diagonal matrix entries $a_1, \dots, a_N$ starting from the upper left corner. Given functions $f_i : X_i \to Y_i$, for any $i \in \{1, \dots, N\}$, their Cartesian product $\prod_{i=1}^N f_i : \prod_{i=1}^N X_i \to \prod_{i=1}^N Y_i$ is defined as $(\prod_{i=1}^N f_i)(x_1, \dots, x_N) = [f_1(x_1); \dots; f_N(x_N)]$. For any set A we denote by $A^{\mathbb{N}}$ the Cartesian product of a countable number of copies of A, i.e., $A^{\mathbb{N}} = \prod_{k=0}^{\infty} A$. Given a measurable function $f : \mathbb{N} \to \mathbb{R}^n$, the (essential) supremum of f is denoted by $\|f\|_\infty := (\mathrm{ess})\sup\{\|f(k)\|, k \geq 0\}$. A function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is said to be a class $\mathcal{K}$ function if it is continuous, strictly increasing, and $\gamma(0) = 0$. A class $\mathcal{K}$ function $\gamma(\cdot)$ is said to be a class $\mathcal{K}_\infty$ if $\lim_{r\to\infty} \gamma(r) = \infty$.

2.2 Discrete-Time Stochastic Control Systems

We consider stochastic control systems in discrete time (dt-SCS) defined over a general state space and characterized by the tuple

$$\Sigma = (X, U, W, \varsigma, f, Y_1, Y_2, h_1, h_2), \tag{1}$$

where X is a Borel space as the state space of the system. We denote by $(X, \mathcal{B}(X))$ the measurable space with $\mathcal{B}(X)$ being the Borel sigma-algebra on the state space. Sets U and W are Borel spaces as the *external* and *internal* input spaces of the system. Notation ς denotes a sequence of independent and identically distributed (i.i.d.) random variables on a set V_ς

$$\varsigma := \{\varsigma(k) : \Omega \to V_\varsigma,\ k \in \mathbb{N}\}.$$

The map $f : X \times U \times W \times V_\varsigma \to X$ is a measurable function characterizing the state evolution of the system. Finally, sets Y_1 and Y_2 are Borel spaces as the external and internal output spaces of the system, respectively. Maps $h_1 : X \to Y_1$ and $h_2 : X \to Y_2$ are measurable functions that map a state $x \in X$ to its external and internal outputs $y_1 = h_1(x)$ and $y_2 = h_2(x)$, respectively.

For given initial state $x(0) \in X$ and input sequences $\nu(\cdot) : \mathbb{N} \to U$ and $w(\cdot) : \mathbb{N} \to W$, evolution of the state of dt-SCS Σ can be written as

$$\Sigma : \begin{cases} x(k+1) = f(x(k), \nu(k), w(k), \varsigma(k)) \\ y_1(k) = h_1(x(k)) \\ y_2(k) = h_2(x(k)), \end{cases} \qquad k \in \mathbb{N}. \tag{2}$$

Given the dt-SCS in (1), we are interested in *Markov policies* to control the system.

Definition 2.1. A Markov policy for the dt-SCS Σ in (1) is a sequence $\rho = (\rho_0, \rho_1, \rho_2, \dots)$ of universally measurable stochastic kernels ρ_n [17], each defined on the input space U given $X \times W$ and such that for all $(x_n, w_n) \in X \times W$, $\rho_n(U(x_n, w_n)|(x_n, w_n)) = 1$. The class of all such Markov policies is denoted by Π_M.

We associate respectively to U and W the sets $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{W}}$ to be collections of sequences $\{\nu(k) : \Omega \to U,\ k \in \mathbb{N}\}$ and $\{w(k) : \Omega \to W,\ k \in \mathbb{N}\}$, in which $\nu(k)$ and $w(k)$ are independent of $\varsigma(t)$ for any $k, t \in \mathbb{N}$ and $t \geq k$. For any initial state $a \in X$, $\nu(\cdot) \in \boldsymbol{\mathcal{U}}$, and $w(\cdot) \in \boldsymbol{\mathcal{W}}$, the random sequences $x_{a\nu w} : \Omega \times \mathbb{N} \to X$, $y^1_{a\nu w} : \Omega \times \mathbb{N} \to Y_1$ and $y^2_{a\nu w} : \Omega \times \mathbb{N} \to Y_2$ that satisfy (2) are called respectively the *solution process* and external and internal *output trajectory* of Σ under external input ν, internal input w and initial state a.

In this sequel we assume that the state space X of Σ is a subset of $\mathbb{R}^n$. System Σ is called finite if X, U, W are finite sets and infinite otherwise. In this paper we are interested in studying interconnected discrete-time stochastic control systems without internal inputs and outputs that result from the interconnection of dt-SCS having both internal and external inputs and outputs. In this case, the interconnected dt-SCS without internal input and output in indicated by the simplified tuple $(X, U, \varsigma, f, Y, h)$ with $f : X \times U \times V_\varsigma \to X$.

2.3 General Markov Decision Processes

A dt-SCS Σ in (1) can be *equivalently* represented as a general Markov decision process (gMDP) [18]

$$\Sigma = (X, W, U, T_{\mathsf{x}}, Y_1, Y_2, h_1, h_2),$$

where the map $T_{\mathsf{x}} : \mathcal{B}(X) \times X \times U \times W \to [0, 1]$, is a conditional stochastic kernel that assigns to any $x \in X$, $w \in W$ and $\nu \in U$ a probability measure $T_{\mathsf{x}}(\cdot|x, \nu, w)$ on the measurable space $(X, \mathcal{B}(X))$ so that for any set $A \in \mathcal{B}(X)$,

$$\mathbb{P}(x(k+1) \in A \,|\, x(k), \nu(k), w(k)) = \int_A T_{\mathsf{x}}(d\bar{x}|x(k), \nu(k), w(k)).$$

For given inputs $\nu(\cdot)$, $w(\cdot)$, the stochastic kernel T_{x} captures the evolution of the state of Σ and can be uniquely determined by the pair (ς, f) from (1).

The alternative representation as gMDP is utilized in [14] to approximate a dt-SCS Σ with a *finite* $\widehat{\Sigma}$. Algorithm 1 adapted from [14] with some modifications presents this approximation. The algorithm first constructs a finite partition of state set X and input sets U, W. Then representative points $\bar{x}_i \in \mathsf{X}_i$, $\bar{\nu}_i \in \mathsf{U}_i$ and $\bar{w}_i \in \mathsf{W}_i$ are selected as abstract states and inputs. Transition probabilities in the finite gMDP $\widehat{\Sigma}$ are also computed according to (3). The output

maps $\hat{h}_1, \hat{h}_2$ are the same as h_1, h_2 with their domain restricted to finite state set $\hat{X}$ (cf. Step 7) and the output sets $\hat{Y}_1$, $\hat{Y}_2$ are just image of $\hat{X}$ under h_1, h_2, respectively (cf. Step 6).

Algorithm 1 Abstraction of dt-SCS Σ by a finite gMDP $\widehat{\Sigma}$

Require: input dt-SCS $\Sigma=(X, W, U, T_{\mathsf{x}}, Y_1, Y_2, h_1, h_2)$

1: Select finite partitions of sets X, U, W as $X = \cup_{i=1}^{n_x} \mathsf{X}_i$, $U = \cup_{i=1}^{n_v} \mathsf{U}_i$, $W = \cup_{i=1}^{n_w} \mathsf{W}_i$
2: For each X_i, U_i, and W_i, select single representative points $x_i \in \mathsf{X}_i$, $v_i \in \mathsf{U}_i$, $w_i \in \mathsf{W}_i$
3: Define $\hat{X} := \{x_i, i = 1, ..., n_x\}$ as the finite state set of gMDP $\widehat{\Sigma}$ with external and internal input sets $\hat{U} := \{v_i, i = 1, ..., n_v\}$ $\hat{W} := \{w_i, i = 1, ..., n_w\}$
4: Define the map $\Xi : X \to 2^X$ that assigns to any $x \in X$, the corresponding partition set it belongs to, i.e., $\Xi(x) = \mathsf{X}_i$ if $x \in \mathsf{X}_i$ for some $i = 1, 2, \ldots, n_x$
5: Compute the discrete transition probability matrix $\hat{T}_{\mathsf{x}}$ for $\widehat{\Sigma}$ as:

$$\hat{T}_{\mathsf{x}}(x'|x, v, w) = T_{\mathsf{x}}(\Xi(x')|x, v, w), \tag{3}$$

for all $x, x' \in \hat{X}$, $v \in \hat{U}$, $w \in \hat{W}$
6: Define output spaces $\hat{Y}_1 := h_1(\hat{X})$, $\hat{Y}_2 := h_2(\hat{X})$
7: Define output maps $\hat{h}_1 := h_1|_{\hat{X}}$ and $\hat{h}_2 := h_2|_{\hat{X}}$

Ensure: output finite gMDP $\widehat{\Sigma} = (\hat{X}, \hat{U}, \hat{W}, \hat{T}_{\mathsf{x}}, \hat{Y}_1, \hat{Y}_2, \hat{h}_1, \hat{h}_2)$

In the following theorem we give a dynamical representation of the finite gMDP, which is more suitable for the study of this paper. The proof of this theorem is provided in the Appendix.

Theorem 2.2. *Given a dt-SCS $\Sigma = (X, U, W, \varsigma, f, Y_1, Y_2, h_1, h_2)$, the finite gMDP $\widehat{\Sigma}$ constructed in Algorithm 1 can be represented as*

$$\hat{\Sigma} = (\hat{X}, \hat{U}, \hat{W}, \varsigma, \hat{f}, \hat{Y}_1, \hat{Y}_2, \hat{h}_1, \hat{h}_2), \tag{4}$$

where $\hat{f} : \hat{X} \times \hat{U} \times \hat{W} \times V_\varsigma \to \hat{X}$ is defined as

$$\hat{f}(\hat{x}, \hat{v}, \hat{w}, \varsigma) = \Pi_x(f(\hat{x}, \hat{v}, \hat{w}, \varsigma)),$$

and $\Pi_x : X \to \hat{X}$ is the map that assigns to any $x \in X$, the representative point $\hat{x} \in \hat{X}$ of the corresponding partition set containing x. The initial state of $\widehat{\Sigma}$ is also selected according to $\hat{x}_0 := \Pi_x(x_0)$ with x_0 being the initial state of Σ.

Dynamical representation provided by Theorem 2.2 uses the map $\Pi_x : X \to \hat{X}$ that assigns to any $x \in X$, the representative point $\hat{x} \in \hat{X}$ of the corresponding partition set containing x. This map satisfies the inequality

$$\|\Pi_x(x) - x\| \leq \delta, \quad \forall x \in X, \tag{5}$$

where $\delta := \sup\{\|x - x'\|,\ x, x' \in \mathsf{X}_i,\ i = 1, 2, \ldots, n_x\}$ is the discretization parameter. We use this inequality in Section 5 for compositional construction of finite gMDPs.

Algorithm 1 is used in [14] for *compositional verification* of interconnected dt-SCS. In order to provide formal guarantee on the compositional approximation, [14] uses distance in probability as a metric. In other words, for a given specification φ and accuracy level ϵ, the discretization parameters for each subsystem can be selected a priori such that after composition

$$|\mathbb{P}(\Sigma \vDash \varphi) - \mathbb{P}(\widehat{\Sigma} \vDash \varphi)| \leq \epsilon, \tag{6}$$

where ϵ depends on the horizon of formula φ, Lipschitz constants of the stochastic kernels of subsystems, discretization parameters, and structure of the interconnection (cf. [14, Theorem 9]).

In the next sections, we provide an approach for *compositional synthesis* of interconnected dt-SCS. We first define the notions of stochastic storage and simulation functions for quantifying the error between two dt-SCS and two interconnected dt-SCS without internal signals, respectively. Then we establish an explicit dynamical representation of finite $\widehat{\Sigma}$ constructed in [14] and show how it can be used to compare interconnections of dt-SCS and those of their finite abstract counterparts based on these new notions. Finally, in the example section, we synthesize policies for abstract dt-SCS locally and refine them back to the original dt-SCS while providing guarantees on the quality of the synthesized policies with respect to satisfaction of local specifications. This guarantee is compared against the approach of [14] with the metric in (6) in the example section.

3 STOCHASTIC STORAGE AND SIMULATION FUNCTIONS

In this section, we first introduce a notion of so-called stochastic storage functions for dt-SCS with both internal and external inputs, which is adapted from the notion of storage functions from dissipativity theory. We then define a notion of so-called stochastic simulation functions for systems with only external inputs and outputs. We use these definitions to quantify closeness of two dt-SCS.

Definition 3.1. Consider dt-SCS $\Sigma = (X, U, W, \varsigma, f, Y_1, Y_2, h_1, h_2)$ and $\widehat{\Sigma} = (\hat{X}, \hat{U}, \hat{W}, \varsigma, \hat{f}, \hat{Y}_1, \hat{Y}_2, \hat{h}_1, \hat{h}_2)$ where $\hat{Y}_1 \subseteq Y_1$. A function $V : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ is called a stochastic storage function (SStF) from $\widehat{\Sigma}$ to Σ if there exist $\alpha \in \mathcal{K}_\infty$, $\kappa \in \mathcal{K}$, $\rho_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$, constant $\psi \in \mathbb{R}_{\geq 0}$, matrices $G, \hat{G}, H$ of appropriate dimensions, and symmetric matrix $\bar{X}$ with conformal block partitions $\bar{X}^{ij}$, $i, j \in \{1, 2\}$, such that for any $x \in X$ and $\hat{x} \in \hat{X}$ one has

$$\alpha(\|h_1(x) - \hat{h}_1(\hat{x})\|) \leq V(x, \hat{x}), \tag{7}$$

and $\forall \hat{v} \in \hat{U}\ \exists v \in U$ such that $\forall \hat{w} \in \hat{W}\ \forall w \in W$ one obtains

$$\begin{aligned}
&\mathbb{E}\Big[V(f(x, v, w, \varsigma), \hat{f}(\hat{x}, \hat{v}, \hat{w}, \varsigma)) \,\Big|\, x, \hat{x}, v, \hat{v}, w, \hat{w}\Big] - V(x, \hat{x}) \\
&\leq -\kappa(V(x, \hat{x})) + \rho_{\text{ext}}(\|\hat{v}\|) + \psi \\
&\quad + \begin{bmatrix} Gw - \hat{G}\hat{w} \\ h_2(x) - H\hat{h}_2(\hat{x}) \end{bmatrix}^T \underbrace{\begin{bmatrix} \bar{X}^{11} & \bar{X}^{12} \\ \bar{X}^{21} & \bar{X}^{22} \end{bmatrix}}_{\bar{X} :=} \begin{bmatrix} Gw - \hat{G}\hat{w} \\ h_2(x) - H\hat{h}_2(\hat{x}) \end{bmatrix}.
\end{aligned} \tag{8}$$

If there exists a SStF V from $\widehat{\Sigma}$ to Σ, this is denoted by $\widehat{\Sigma} \preceq_{\mathcal{S}} \Sigma$ and the control system $\widehat{\Sigma}$ is called an abstraction of concrete (original) system Σ. Note that $\widehat{\Sigma}$ may be finite or infinite depending on cardinalities of sets $\hat{X}, \hat{U}, \hat{W}$.

Remark 1. *The last term in inequality* (8) *is interpreted in dissipativity theory as the energy supply rate of the system [19]. Here we choose this function to be quadratic which results in tractable compositional conditions later in the form of linear matrix (in)equalities.*

Remark 2. *The second condition in Definition 3.1 implies implicitly the existence of a function $v = v_{\hat{v}}(x, \hat{x}, \hat{v})$ for the satisfaction of* (8)*. This function is called the* interface function *and can be used to refine a synthesized policy $\hat{v}$ for $\widehat{\Sigma}$ to a policy v for Σ.*

Now, we modify the above notion for the interconnected dt-SCS without internal inputs and outputs.

Definition 3.2. Consider two dt-SCS $\Sigma = (X, U, \varsigma, f, Y, h)$ and $\widehat{\Sigma} = (\hat{X}, \hat{U}, \varsigma, \hat{f}, \hat{Y}, \hat{h})$ without internal inputs and outputs, where

$\hat{Y} \subseteq Y$. A function $V : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ is called a *stochastic simulation function* (SSF) from $\widehat{\Sigma}$ to Σ if

- there exists $\alpha \in \mathcal{K}_\infty$ such that for all $x \in X$ and $\hat{x} \in \hat{X}$,
$$\alpha(\|h(x) - \hat{h}(\hat{x})\|) \leq V(x, \hat{x}), \tag{9}$$
- for all $x \in X$, $\hat{x} \in \hat{X}$, $\hat{v} \in \hat{U}$, there exists $v \in U$ such that
$$\begin{aligned}&\mathbb{E}\Big[V(f(x, v, \varsigma), \hat{f}(\hat{x}, \hat{v}, \varsigma)) \,\Big|\, x, \hat{x}, v, \hat{v}\Big] - V(x, \hat{x}) \\ &\leq -\kappa(V(x, \hat{x})) + \rho_{\text{ext}}(\|\hat{v}\|) + \psi,\end{aligned} \tag{10}$$
for some $\kappa \in \mathcal{K}$, $\rho_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$, and $\psi \in \mathbb{R}_{\geq 0}$.

If there exists a SSF V from $\widehat{\Sigma}$ to Σ, this is denoted by $\widehat{\Sigma} \preceq \Sigma$ and $\widehat{\Sigma}$ is called an abstraction of Σ.

Next theorem is borrowed from [15, Theorem 3.3], and shows how SSF can be used to compare output trajectories of two dt-SCS without internal inputs in a probabilistic setting.

Theorem 3.3. *Let $\Sigma = (X, U, \varsigma, f, Y, h)$ and $\widehat{\Sigma} = (\hat{X}, \hat{U}, \varsigma, \hat{f}, \hat{Y}, \hat{h})$ be two dt-SCS without internal inputs and outputs, where $\hat{Y} \subseteq Y$. Suppose V is an SSF from $\widehat{\Sigma}$ to Σ, and there exists a constant $0 < \widehat{\kappa} < 1$ such that the function $\kappa \in \mathcal{K}$ in (10) satisfies $\kappa(r) \geq \widehat{\kappa} r$ $\forall r \in \mathbb{R}_{\geq 0}$. For any external input trajectory $\hat{v}(\cdot) \in \hat{\mathcal{U}}$ that preserves Markov property for the closed-loop $\widehat{\Sigma}$, and for any random variables a and $\hat{a}$ as the initial states of the two dt-SCS, there exists an input trajectory $v(\cdot) \in \mathcal{U}$ of Σ through the interface function associated with V such that the following inequality holds*

$$\begin{aligned}&\mathbb{P}\left\{\sup_{0 \leq k \leq T_d} \|y_{av}(k) - \hat{y}_{\hat{a}\hat{v}}(k)\| \geq \varepsilon \,|\, [a; \hat{a}]\right\} \\ &\leq \begin{cases} 1 - (1 - \frac{V(a,\hat{a})}{\alpha(\varepsilon)})(1 - \frac{\widehat{\psi}}{\alpha(\varepsilon)})^{T_d} & \text{if } \alpha(\varepsilon) \geq \frac{\widehat{\psi}}{\widehat{\kappa}}, \\ (\frac{V(a,\hat{a})}{\alpha(\varepsilon)})(1 - \widehat{\kappa})^{T_d} + (\frac{\widehat{\psi}}{\widehat{\kappa}\alpha(\varepsilon)})(1 - (1 - \widehat{\kappa})^{T_d}) & \text{if } \alpha(\varepsilon) < \frac{\widehat{\psi}}{\widehat{\kappa}}, \end{cases}\end{aligned} \tag{11}$$

where the constant $\widehat{\psi} \geq 0$ satisfies $\widehat{\psi} \geq \rho_{\text{ext}}(\|\hat{v}\|_\infty) + \psi$.

Remark 3. *Note that $\psi = 0$ if concrete and abstract systems are both continuous-space but possibly with different dimensions and share the same multiplicative noise (cf. Eqn. (2) in [15]). If $\rho_{\text{ext}}(\cdot)$ is also equal to zero, function V becomes a nonnegative supermartingale and, hence, one can readily extend the result of Theorem 3.3 to infinite-time horizon and quantify the distance between two systems by applying the results in [15, Corollary 3.4].*

4 COMPOSITIONAL ABSTRACTIONS FOR INTERCONNECTED SYSTEMS

In this section, we analyze networks of stochastic control subsystems and show how to construct their abstractions together with the corresponding simulation functions by using abstractions and stochastic storage functions of the subsystems.

4.1 Interconnected Stochastic Control Systems

We first provide a formal definition of interconnection of discrete-time stochastic control subsystems.

Definition 4.1. Consider $N \in \mathbb{N}_{\geq 1}$ stochastic control subsystems $\Sigma_i = (X_i, U_i, W_i, \varsigma_i, f_i, Y_{1i}, Y_{2i}, h_{1i}, h_{2i})$, $i \in \{1, \ldots, N\}$, and a matrix M defining the coupling between these subsystems. We require the condition $M \prod_{i=1}^N Y_{2i} \subseteq \prod_{i=1}^N W_i$ to have a well-posed interconnection. The interconnection of Σ_i, $\forall i \in \{1, \ldots, N\}$, is the dt-SCS $\Sigma = (X, U, \varsigma, f, Y, h)$, denoted by $\mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$, such that $X := \prod_{i=1}^N X_i$, $U := \prod_{i=1}^N U_i$, $f := \prod_{i=1}^N f_i$, $Y := \prod_{i=1}^N Y_{1i}$, and $h = \prod_{i=1}^N h_{1i}$, with the internal inputs constrained according to
$$[w_1; \ldots; w_N] = M[h_{21}(x_1); \ldots; h_{2N}(x_N)].$$

In the above definition we allow the interconnection matrix M to have real entries. This is a generalization of composition performed in [7] where the interconnection matrix takes only binary entries.

4.2 Compositional Abstractions

We assume that we are given N stochastic control subsystems $\Sigma_i = (X_i, U_i, W_i, \varsigma_i, f_i, Y_{1i}, Y_{2i}, h_{1i}, h_{2i})$ together with their corresponding abstractions $\widehat{\Sigma}_i = (\hat{X}_i, \hat{U}_i, \hat{W}_i, \varsigma_i, \hat{f}_i, \hat{Y}_{1i}, \hat{Y}_{2i}, \hat{h}_{1i}, \hat{h}_{2i})$ with SStF V_i from $\widehat{\Sigma}_i$ to Σ_i. Indicate by α_i, κ_i, $\rho_{i\text{ext}}$, H_i, G_i, $\hat{G}_i$, $\bar{X}_i$, $\bar{X}_i^{11}$, $\bar{X}_i^{12}$, $\bar{X}_i^{21}$, and $\bar{X}_i^{22}$, the corresponding functions, matrices, and the conformal block partitions appearing in Definition 3.1. In the next theorem, as one of the main results of the paper, we provide sufficient conditions for having an SSF from the interconnection of abstractions $\widehat{\Sigma} = \mathcal{I}(\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_N)$ to that of concrete ones $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$. This theorem enables us to quantify in probability the error between the interconnection of stochastic control subsystems and that of their abstractions in a compositional manner by leveraging Theorem 3.3.

Theorem 4.2. *Consider the interconnected stochastic control system $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ induced by $N \in \mathbb{N}_{\geq 1}$ stochastic control subsystems Σ_i and the coupling matrix M. Suppose that each stochastic control subsystem Σ_i admits an abstraction $\widehat{\Sigma}_i$ with the corresponding SStF V_i. Then the weighted sum*
$$V(x, \hat{x}) := \sum_{i=1}^N \mu_i V_i(x_i, \hat{x}_i) \tag{12}$$
is a stochastic simulation function from the interconnected control system $\widehat{\Sigma} = \mathcal{I}(\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_N)$, with coupling matrix $\hat{M}$, to $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ if $\mu_i > 0$, $i \in \{1, \ldots, N\}$, and $\hat{M}$ satisfy matrix (in)equality and inclusion
$$\begin{bmatrix} GM \\ I_{\tilde{q}} \end{bmatrix}^T \bar{X}_{cmp} \begin{bmatrix} GM \\ I_{\tilde{q}} \end{bmatrix} \preceq 0, \tag{13}$$
$$GMH = \hat{G}\hat{M}, \tag{14}$$
$$\hat{M} \prod_{i=1}^N \hat{Y}_{2i} \subseteq \prod_{i=1}^N \hat{W}_i, \tag{15}$$
where
$$\begin{aligned}&G := \text{diag}(G_1, \ldots, G_N),\ \hat{G} := \text{diag}(\hat{G}_1, \ldots, \hat{G}_N), \\ &H := \text{diag}(H_1, \ldots, H_N),\end{aligned}$$
$$\bar{X}_{cmp} := \begin{bmatrix} \mu_1\bar{X}_1^{11} & & & \mu_1\bar{X}_1^{12} & & \\ & \ddots & & & \ddots & \\ & & \mu_N\bar{X}_N^{11} & & & \mu_N\bar{X}_N^{12} \\ \mu_1\bar{X}_1^{21} & & & \mu_1\bar{X}_1^{22} & & \\ & \ddots & & & \ddots & \\ & & \mu_N\bar{X}_N^{21} & & & \mu_N\bar{X}_N^{22} \end{bmatrix}, \tag{16}$$
and $\tilde{q} = \sum_{i=1}^N q_{2i}$ with q_{2i} being the internal output dimensions of subsystems Σ_i.

Proof of Theorem 4.2 is provided in the Appendix. Figure 1 illustrates schematically the result of Theorem 4.2.

Remark 4. *Note that condition* (13) *with $G = I$ is exactly similar to the linear matrix inequality (LMI) appeared in [19] as compositional stability condition based on dissipativity theory. As discussed in [19], the LMI holds independently of the number of subsystems in many physical applications with specific interconnection structures including communication networks, flexible joint robots, and power generators.*

Remark 5. *For the compositional construction of finite gMDPs provided in the next section, condition* (14) *is satisfied by simply selecting $\hat{M} = M$. Notice that we have presented condition* (14) *in its general form without requiring the same dimensionality for the abstract and original systems. Existing results in the literature [16] leverage this condition in the context of model order reduction. Condition* (15) *is not also restrictive for the results provided in the next section since $\hat{W}_i$ and $\hat{Y}_{2i}$ are internal input and output sets of the abstract subsystems $\widehat{\Sigma}_i$, which are finite. Thus one can readily choose internal input sets $\hat{W}_i$ such that $\prod_{i=1}^{n} \hat{W}_i := \hat{M} \prod_{i=1}^{n} \hat{Y}_{2i}$ which implicitly implies a condition on the granularity of discretization for sets W_i and Y_{2i}. In other words, condition* (15) *is required for just having a well-posed interconnection and is automatically fulfilled by the proposed construction of finite MDP later in Section 5.*

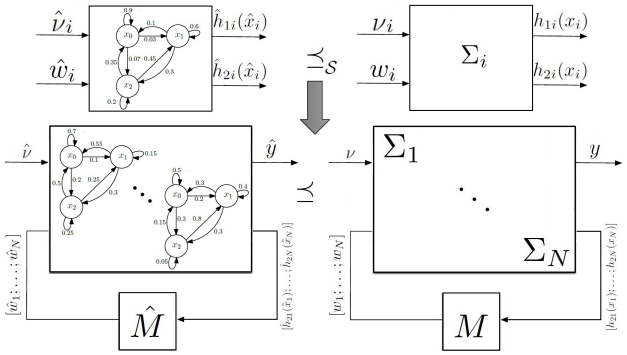

Figure 1: Compositionality results provided that conditions (13)**,** (14)**, and** (15) **are satisfied.**

5 CONSTRUCTION OF FINITE MARKOV DECISION PROCESSES

In the previous sections, Σ and $\widehat{\Sigma}$ were considered as general discrete-time stochastic control systems without discussing the cardinality of their state spaces. In this section, we consider Σ as an infinite dt-SCS and $\widehat{\Sigma}$ as its finite abstraction constructed as in Section 2.3. We impose conditions on the infinite dt-SCS Σ enabling us to find SStF from its finite abstraction $\widehat{\Sigma}$ to Σ. The required conditions are first presented in a general setting for nonlinear stochastic control systems in Section 5.1 and then represented via some matrix inequality for linear stochastic control systems in Section 5.2.

5.1 Discrete-Time Nonlinear Stochastic Control Systems

The stochastic storage function from finite MDP $\widehat{\Sigma}$ of Section 2.3 to Σ is established under the assumption that the original discrete-time stochastic control system Σ is so-called incrementally passivable as in Assumption 1.

Assumption 1. *A dt-SCS $\Sigma = (X, U, W, \varsigma, f, Y_1, Y_2, h_1, h_2)$ is called* incrementally passivable *if there exist functions $L : X \to U$ and $V : X \times X \to \mathbb{R}_{\geq 0}$ such that $\forall x, x' \in X$, $\forall \nu \in U$, $\forall w, w' \in W$, the inequalities:*

$$\underline{\alpha}(\|h_1(x) - h_1(x')\|) \leq V(x, x'), \tag{17}$$

and

$$\begin{aligned} &\mathbb{E}\Big[V(f(x,L(x)+\nu,w,\varsigma), f(x', L(x')+\nu, w', \varsigma))\Big|x, x', \nu, w, w'\Big] \\ &-V(x,x') \leq -\widehat{\kappa}(V(x,x')) + \\ &\overbrace{\begin{bmatrix} w-w' \\ h_2(x)-h_2(x') \end{bmatrix}^{T} \begin{bmatrix} \bar{X}^{11} & \bar{X}^{12} \\ \bar{X}^{21} & \bar{X}^{22} \end{bmatrix}}^{\bar{X}:=} \begin{bmatrix} w-w' \\ h_2(x)-h_2(x') \end{bmatrix}, \end{aligned} \tag{18}$$

hold for some $\underline{\alpha} \in \mathcal{K}_\infty$, $\widehat{\kappa} \in \mathcal{K}$, and matrix $\bar{X}$ of appropriate dimension.

Remark 6. *Note that Assumption 1 implies that V is a SStF from system Σ equipped with the state feedback controller L to itself. This type of property is closely related to the notion of so-called incremental stabilizability [20, 21].*

In Section 5.2, we show that inequalities (17)-(18) for a candidate quadratic function V and linear stochastic control systems boil down to some matrix inequality.

Under Assumption 1, the next theorem shows a relation between Σ and $\widehat{\Sigma}$, constructed as in Algorithm 1, via establishing a stochastic storage function between them.

Theorem 5.1. *Let Σ be an incrementally passivable dt-SCS via a function V as in Assumption 1 and $\widehat{\Sigma}$ be a finite MDP as in Algorithm 1. Assume that there exists a function $\gamma \in \mathcal{K}_\infty$ such that V satisfies*

$$V(x,x') - V(x,x'') \leq \gamma(\|x' - x''\|), \quad \forall x, x', x'' \in X. \tag{19}$$

Then V is a stochastic storage function from $\widehat{\Sigma}$ to Σ.

The proof of Theorem 5.1 is provided in the Appendix.

Remark 7. *As shown in [4] and by employing the mean value theorem, assumption* (19) *is always satisfied for any differentiable function V restricted to a compact subset of $X \times X$.*

Now we provide similar results as in Subsection 5.1 but tailored to linear stochastic control systems.

5.2 Discrete-Time Linear Stochastic Control Systems

In this subsection, we focus on the class of discrete-time linear stochastic control systems Σ and *quadratic* stochastic storage functions V. First, we formally define the class of discrete-time linear stochastic control systems. Afterwards, we construct their finite Markov decision processes $\widehat{\Sigma}$ as in Theorem 2.2, and then provide conditions under which a candidate V is an SStF from $\widehat{\Sigma}$ to Σ.

The class of linear stochastic control systems is given by

$$\Sigma: \begin{cases} x(k+1) = Ax(k) + Bv(k) + Dw(k) + \bar{N}\varsigma(k), \\ y_1(k) = C_1 x(k), \\ y_2(k) = C_2 x(k), \end{cases} \tag{20}$$

where the noise $\varsigma(k)$ is a sequence of independent random vectors with multivariate standard normal distributions. We use the tuple

$$\Sigma = (A, B, C_1, C_2, D, \bar{N}),$$

to refer to the class of discrete-time linear stochastic control systems of the form (20). Consider the following quadratic function

$$V(x,\hat{x}) = (x-\hat{x})^T \tilde{M}(x-\hat{x}), \tag{21}$$

where $\tilde{M}$ is a positive-definite matrix of appropriate dimension. In order to show that V in (21) is an SStF from $\widehat{\Sigma}$ to Σ, we require the following key assumption on Σ.

Assumption 2. *Let $\Sigma = (A, B, C_1, C_2, D, \bar{N})$. Assume that for some constant $0 < \widehat{\kappa} < 1$ and $\pi > 0$ there exist matrices $\tilde{M} \succ 0, K, \bar{X}^{11}, \bar{X}^{12}$, $\bar{X}^{21}$, and $\bar{X}^{22}$ of appropriate dimensions such that matrix inequality (22) holds.*

$$\begin{bmatrix} (1+\pi)(A+BK)^T\tilde{M}(A+BK) & (A+BK)^T\tilde{M}D \\ D^T\tilde{M}(A+BK) & (1+\pi)D^T\tilde{M}D \end{bmatrix} \preceq \begin{bmatrix} \widehat{\kappa}\tilde{M}+C_2^T\bar{X}^{22}C_2 & C_2^T\bar{X}^{21} \\ \bar{X}^{12}C_2 & \bar{X}^{11} \end{bmatrix} \tag{22}$$

Now, we provide another main result of this section showing that under some conditions V in (21) is an SStF from $\widehat{\Sigma}$ to Σ. The proof of this theorem is provided in the Appendix.

Theorem 5.2. *Let $\Sigma = (A, B, C_1, C_2, D, \bar{N})$ and $\widehat{\Sigma}$ be a finite Markov decision process with discretization parameter δ, and $\hat{Y}_1 \subseteq Y_1$. Suppose Assumption 2 holds, $C_1 = \hat{C}_1$, and $C_2 = \hat{C}_2$, then function V defined in (21) is an SStF from $\widehat{\Sigma}$ to Σ.*

Remark 8. *Note that for any linear stochastic control system $\Sigma = (A, B, C_1, C_2, D, \bar{N})$, stabilizability of the pair (A, B) is sufficient to satisfy Assumption 2.*

6 CASE STUDY

To demonstrate the effectiveness of our approach, we first apply our results to the temperature regulation in a circular building by constructing compositionally a finite abstraction of a network containing 200 rooms. Then, to show its applicability to strongly connected networks, the results are illustrated on a network with a fully-connected interconnection graph.

6.1 Room Temperature Control

In this subsection, we apply our results to the temperature regulation of $n \geq 3$ rooms each equipped with a heater and connected on a circle. The model of this case study is adapted from [22] by including stochasticity in the model as additive noise. The evolution of temperature T can be described by the interconnected discrete-time stochastic control system

$$\Sigma: \begin{cases} T(k+1) = \bar{A}T(k) + \gamma T_h \nu(k) + \beta T_E + \varsigma(k), \\ y(k) = T(k), \end{cases}$$

where A is a matrix with diagonal elements $\bar{a}_{ii} = (1-2\eta-\beta-\gamma\nu_i(k))$, $i \in \{1,\dots,n\}$, off-diagonal elements $\bar{a}_{i,i+1} = \bar{a}_{i+1,i} = \bar{a}_{1,n} = \bar{a}_{n,1} = \eta$, $i \in \{1,\dots,n-1\}$, and all other elements are identically zero. Parameters η, β, and γ are conduction factors respectively between the rooms $i \pm 1$ and the room i, between the external environment and the room i, and between the heater and the room i. Moreover, $T(k) = [T_1(k);\dots;T_n(k)]$, $\nu(k) = [\nu_1(k);\dots;\nu_n(k)]$, $\varsigma(k) = [\varsigma_1(k);\dots;\varsigma_n(k)]$, $T_E = [T_{e1};\dots;T_{en}]$, where $T_i(k)$ and $\nu_i(k)$ are taking values in $[19, 21]$ and $[0, 0.6]$, respectively, for all $i \in \{1,\dots,n\}$. The parameter $T_{ei} = -1\,^\circ C$ are the outside temperature $\forall i \in \{1,\dots,n\}$, and $T_h = 50\,^\circ C$ is the heater temperature. Now, by introducing Σ_i described as

$$\Sigma_i: \begin{cases} T_i(k+1) = (1-2\eta-\beta-\gamma\nu_i(k))T_i(k) + \gamma T_h \nu_i(k) + \eta w_i(k) + \beta T_{ei} + \varsigma_i(k), \\ y_{1i}(k) = T_i(k), \\ y_{2i}(k) = T_i(k), \end{cases}$$

one can readily verify that $\Sigma = \mathcal{I}(\Sigma_1,\dots,\Sigma_N)$ where the coupling matrix M is such that $m_{i,i+1} = m_{i+1,i} = m_{1,n} = m_{n,1} = 1$, $i \in \{1,\dots,n-1\}$, and all other elements are identically zero. One can also verify that, $\forall i \in \{1,\dots,n\}$, condition (22) is satisfied with $\tilde{M}_i = 1$, $K_i = 0$, $\bar{X}_i^{11} = \eta^2(1+\pi_i)$, $\bar{X}_i^{22} = -3.38\eta(1+\pi_i)$, $\bar{X}_i^{12} = \bar{X}_i^{21} = \eta\lambda_i$, where $\lambda_i = 1-2\eta-\beta-\gamma\nu_i(k)$, and selecting some appropriate values for $\eta, \beta, \gamma, \widehat{\kappa}_i, \pi_i, \forall i \in \{1,\dots,n\}$. Hence, function $V_i(T_i,\hat{T}_i) = (T_i-\hat{T}_i)^2$ is an SStF from $\widehat{\Sigma}_i$ to Σ_i satisfying condition (7) with $\alpha_i(s) = s^2$ and condition (8) with $\kappa_i(s) := (1-\widehat{\kappa}_i)s$, $\rho_{\text{iext}}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, $\psi_i = (1+2/\pi_i)\delta_i^2$, $G_i = \hat{G}_i = H_i = 1$, and

$$\bar{X}_i = \begin{bmatrix} \eta^2(1+\pi_i) & \eta\lambda_i \\ \eta\lambda_i & -3.38\eta(1+\pi_i) \end{bmatrix}, \tag{23}$$

where the input ν_i is given via the interface function in (26) as $\nu_i = \hat{\nu}_i$. Now, we look at $\widehat{\Sigma} = \mathcal{I}(\widehat{\Sigma}_1,\dots,\widehat{\Sigma}_N)$ with a coupling matrix $\hat{M}$ satisfying condition (14) as $\hat{M} = M$. Choosing $\mu_1 = \cdots = \mu_N = 1$ and using $\bar{X}_i$ in (23), matrix $\bar{X}_{cmp}$ in (16) reduces to

$$\bar{X}_{cmp} = \begin{bmatrix} \eta^2(1+\pi)I_n & \eta\lambda I_n \\ \eta\lambda I_n & -3.38\eta(1+\pi)I_n \end{bmatrix},$$

where $\lambda = \lambda_1 = \cdots = \lambda_N$, $\pi = \pi_1 = \cdots = \pi_N$, and condition (13) reduces to

$$\begin{bmatrix} M \\ I_n \end{bmatrix}^T \bar{X}_{cmp} \begin{bmatrix} M \\ I_n \end{bmatrix} = \eta^2(1+\pi)M^TM + \eta\lambda M + \eta\lambda M^T - 3.38\eta(1+\pi)I_n \preceq 0,$$

without requiring any restrictions on the number or gains of the subsystems. In order to satisfy the above inequality, we used $M = M^T$, and $4\eta^2(1+\pi) + 4\eta\lambda - 3.38\eta(1+\pi) \leq 0$ employing Gershgorin circle theorem [23] which can be satisfied for the appropriate values of η, π and λ. By choosing finite internal input sets $\hat{W}_i$ of $\widehat{\Sigma}$ such that $\prod_{i=1}^n \hat{W}_i = \hat{M}\prod_{i=1}^n \hat{X}_i$, condition (15) is also satisfied. Now, one can verify that $V(T,\hat{T}) = \sum_{i=1}^n (T_i-\hat{T}_i)^2$ is an SSF from $\widehat{\Sigma}$ to Σ satisfying conditions (9) and (10) with $\alpha(s) = s^2$, $\kappa(s) := (1-\widehat{\kappa})s$, $\rho_{\text{ext}}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi = n(1+2/\pi)\delta^2$.

To demonstrate the effectiveness of proposed approach, we fix $n = 15$. By taking the state set discretization parameter $\delta_i = 0.005$, $\widehat{\kappa}_i = 0.99, \pi_i = 0.05, \forall i \in \{1,\dots,n\}, \eta = 0.1, \beta = 0.022, \gamma = 0.05$, one can readily verify that conditions (13) and (22) are satisfied. Accordingly, by using the stochastic simulation function V as in inequality (11) and taking the initial states of the interconnected systems Σ and $\widehat{\Sigma}$ as $20\mathbf{1}_{15}$, we guarantee that the distance between outputs of Σ and of $\widehat{\Sigma}$ will not exceed $\varepsilon = 0.63$ during the time horizon $T_d = 10$ with probability at least 90%, i.e.

$$\mathbb{P}(\|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\| \leq 0.63,\ \forall k \in [0,10]) \geq 0.9\ .$$

Note that for the construction of finite gMDP, we have selected the center of partition sets as representative points. This choice has further tightened the above inequality.

Let us now synthesize a controller for Σ via the abstraction $\widehat{\Sigma}$ such that the controller maintains the temperature of any room in the safe set [19,21]. The idea here is to first design a local control for abstraction $\widehat{\Sigma}_i$, and then refine it to system Σ_i using interface function. Consequently, controller for the interconnected system Σ would be a vector such that each of its components is the controller for the interconnected system Σ_i. We employ here software tool

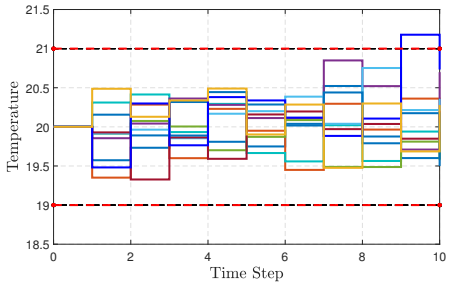

Figure 2: Closed-loop trajectories of a representative room with different noise realizations in a network of 15 rooms.

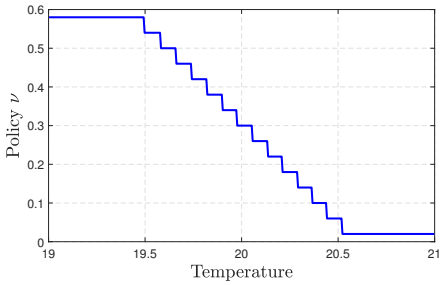

Figure 3: Policy ν for a representative room in a network of 15 rooms.

FAUST^2 [11] to synthesize a controller for Σ by taking the external input discretization parameter as 0.04, and standard deviation of the noise $\sigma_i = 0.28, \forall i \in \{1, \ldots, n\}$. Closed-loop state trajectories of the representative room with different noise realizations are illustrated in Figure 2. Policy ν and the associated safety probability for a representative room in the network are respectively plotted in Figures 3-4 as a function of initial temperature of the room. Policy ν is *locally sub-optimal* for each subsystem and is obtained by assuming that other subsystems do not violate safety specification. The synthesized policy ν is smoothly decreasing from the maximum input 0.6 to the minimum 0 as temperature increases. The maximum safety probability is around the center of the interval $[19, 21]$, and its minimums are at the two boundaries. Note that the oscillations appeared in Figures 3-4 are due to the state and input discretization. We now compare the guarantees provided by our approach and by [14]. Note that our result is based on finite gMDP while [14] uses Dynamic Bayesian Network (DBN) to capture the dependencies between subsystems. The comparison is shown in Figures 5-6 in logarithmic scale. In Figure 5 we have fixed $\varepsilon = 0.2$ (cf. (11)) and plotted the error as a function of discretization parameter δ and standard deviation of the noise σ. Our error of (11) is independent of σ while the error of [14] converges to infinity when σ goes to zero. Thus our new approach outperforms [14] for smaller standard deviation of noise. In Figure 6 we have fixed $\sigma = 0.28$ and plotted the error as a function of discretization parameter δ and ε. The error in [14] is independent of ε while our error increases when ε goes to zero. Thus there is a trade-off between ε and δ to get better bounds in comparison with [14].

In order to show scalability of our approach, we increase the number of rooms to $n = 200$. If we take the state set discretization parameter $\delta_i = 0.005, \widehat{\kappa}_i = 0.99, \pi_i = 0.98, \forall i \in \{1, \ldots, n\}, \eta = 0.1, \beta = 0.4, \gamma = 0.5$, conditions (13) and (22) are readily met. Moreover, if the initial states of the interconnected systems Σ and $\widehat{\Sigma}$ are

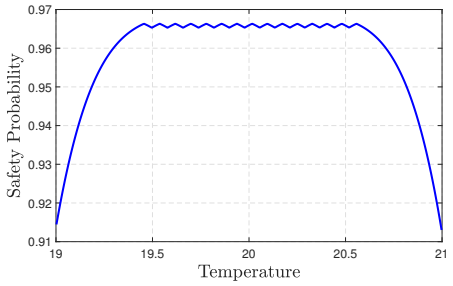

Figure 4: Closed-loop safety probability of a representative room with time horizon $T_d = 10$ in a network of 15 rooms.

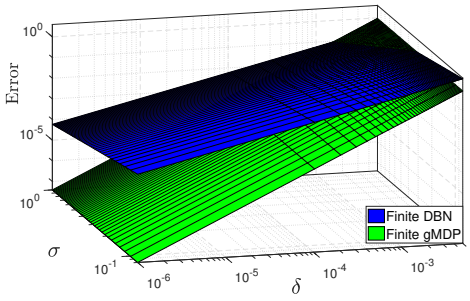

Figure 5: Comparison of error bound provided by the approach of this paper based on finite gMDP with that of [14] based on finite DBN. Plots are in logarithmic scale for a fixed $\varepsilon = 0.2$ (cf. (11)).

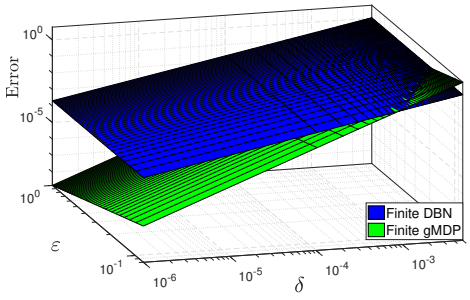

Figure 6: Comparison of error bound provided by the approach of this paper based on finite gMDP with that of [14] based on finite DBN. Plots are in logarithmic scale for a fixed noise standard deviation $\sigma = 0.28$.

started from 201_{200}, one can readily verify that the norm of error between outputs of Σ and of $\widehat{\Sigma}$ will not exceed 0.63 with probability at least 90% computed by the stochastic simulation function V as in inequality (11) for $T_d = 10$. Similarly, we synthesize a controller for Σ via the abstraction $\widehat{\Sigma}$ by taking the external input discretization parameter as 0.04, and $\sigma_i = 0.21, \forall i \in \{1, \ldots, n\}$. Closed-loop state trajectories of the representative room with different noise realizations are illustrated in Figure 7.

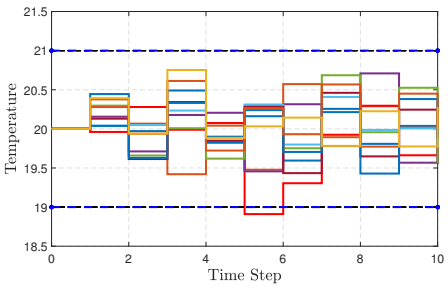

Figure 7: Closed-loop trajectories of a representative room with different noise realizations in a network of 200 rooms.

6.2 Fully Interconnected Network

In order to show applicability of our approach to strongly connected networks, we consider interconnected linear dt-SCS

$$\Sigma : \begin{cases} x(k+1) = \bar{G}x(k) + \nu(k) + \varsigma(k), \\ y(k) = x(k), \end{cases}$$

with matrix $\bar{G} = (I_n - \tau L) \in \mathbb{R}^{n\times n}$ where L is the Laplacian matrix of an undirected graph, and $0 < \tau < 1/\Delta$ with Δ being the maximum degree of the graph [24]. We expand state $x(k) = [x_1(k); \ldots; x_n(k)]$, input $\nu(k) = [\nu_1(k); \ldots; \nu_n(k)]$, and noise $\varsigma(k) = [\varsigma_1(k); \ldots; \varsigma_n(k)]$. Now, by defining Σ_i as

$$\Sigma_i : \begin{cases} x_i(k+1) = x_i(k) + \nu_i(k) + w_i(k) + \varsigma_i(k), \\ y_{1i}(k) = x_i(k), \\ y_{2i}(k) = x_i(k), \end{cases}$$

one can verify that $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ where the coupling matrix M is given by $M = -\tau L$. One can also verify that, $\forall i \in \{1, \ldots, n\}$, condition (22) is satisfied with $\tilde{M}_i = 1$, $K_i = -0.2$, $\bar{X}^{11} = (1 + \pi_i)$, $\bar{X}^{22} = 0$, $\bar{X}^{12} = \bar{X}^{21} = \lambda_i$, where $\lambda_i = 1 + K_i$, and $\widehat{\kappa}_i = 0.99$, $\pi_i = 0.55, \forall i \in \{1, \ldots, n\}$. Hence, function $V_i(x_i, \hat{x}_i) = (x_i - \hat{x}_i)^2$ is an SStF from $\widehat{\Sigma}_i$ to Σ_i satisfying condition (7) with $\alpha_i(s) = s^2$ and condition (8) with $\kappa_i(s) := (1 - \widehat{\kappa}_i)s$, $\rho_{\text{iext}}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, $\psi_i = (1 + 2/\pi_i)\delta_i^2$, and $G_i = \hat{G}_i = H_i = 1$. Now, we look at $\widehat{\Sigma} = \mathcal{I}(\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_N)$ with a coupling matrix $\hat{M}$ satisfying condition (14) by $\hat{M} = M$. Choosing $\mu_1 = \cdots = \mu_N = 1$, matrix $\bar{X}_{cmp}$ in (16) reduces to

$$\bar{X}_{cmp} = \begin{bmatrix} (1+\pi)I_n & \lambda I_n \\ \lambda I_n & 0 \end{bmatrix},$$

where $\lambda = \lambda_1 = \cdots = \lambda_N$, $\pi = \pi_1 = \cdots = \pi_N$, and condition (13) reduces to

$$\begin{bmatrix} -\tau L \\ I_n \end{bmatrix}^T \bar{X}_{cmp} \begin{bmatrix} -\tau L \\ I_n \end{bmatrix} = (1+\pi)\tau^2 L^T L - \lambda\tau L - \lambda\tau L^T = \tau L((1+\pi)\tau L - 2\lambda I_n) \preceq 0,$$

which is always satisfied without requiring any restrictions on the number or gains of the subsystems. In order to show the above inequality, we used $\tau L = \tau L^T \geq 0$ which are always true for Laplacian matrices of undirected graphs. By choosing finite internal input sets $\hat{W}_i$ of $\widehat{\Sigma}$ such that $\prod_{i=1}^n \hat{W}_i = \hat{M} \prod_{i=1}^n \hat{X}_i$, condition (15) is also satisfied. Now, one can verify that $V(x, \hat{x}) = \sum_{i=1}^n (x_i - \hat{x}_i)^2$ is an SSF from $\widehat{\Sigma}$ to Σ satisfying conditions (9) and (10) with $\alpha(s) = s^2$, $\kappa(s) := (1 - \widehat{\kappa})s$, $\rho_{\text{ext}}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi = n(1 + 2/\pi)\delta^2$.

To illustrate the results, we assume L is the Laplacian matrix of a complete graph and $\tau = 0.1$. We fix $n = 150$, and the state discretization parameter $\delta_i = 0.005, \forall i \in \{1, \ldots, n\}$. By using the stochastic simulation function V and inequality (11), and taking the initial states of the interconnected systems Σ and $\widehat{\Sigma}$ as $20\mathbf{1}_{150}$, we guarantee that the distance between outputs of Σ and of $\widehat{\Sigma}$ will not exceed $\varepsilon = 0.63$ during the time horizon $T_d = 10$ with probability at least 90%.

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8 APPENDIX

Proof. **(Theorem 2.2)** It is sufficient to show that (3) holds for dynamical representation of $\widehat{\Sigma}$ in (4) and that of Σ. For any $x, x' \in \hat{X}$, $v \in \hat{U}$ and $w \in \hat{W}$,

$$\begin{aligned}\hat{T}_x(x'|x,v,w) &= \mathbb{P}(x' = \hat{f}(x,v,w,\varsigma)) \\ &= \mathbb{P}(x' = \Pi_x(f(x,v,w,\varsigma))) = \mathbb{P}(f(x,v,w,\varsigma) \in \Xi(x')),\end{aligned}$$

where $\Xi(x')$ is the partition set with x' as its representative point as defined in Step 4 of Algorithm 1. Using the probability measure $\vartheta(\cdot)$ of random variable ς we can write

$$\hat{T}_x(x'|x,v,w) = \int_{\Xi(x')} f(x,v,w,\varsigma)d\vartheta(\varsigma) = T_x(\Xi(x')|x,v,w),$$

which completes the proof. □

Proof. **(Theorem 4.2)** We first show that SSF V in (12) satisfies the inequality (9) for some $\mathcal{K}_\infty$ function α. For any $x = [x_1;\dots;x_N] \in X$ and $\hat{x} = [\hat{x}_1;\dots;\hat{x}_N] \in \hat{X}$, one gets:

$$\begin{aligned}\|h(x) - \hat{h}(\hat{x})\| &= \|[h_{11}(x_1);\dots;h_{1N}(x_N)] - [\hat{h}_{11}(\hat{x}_1);\dots;\hat{h}_{1N}(\hat{x}_N)]\| \\ &\leq \sum_{i=1}^{N} \|h_{1i}(\hat{x}_i) - \hat{h}_{1i}(x_i)\| \leq \sum_{i=1}^{N} \alpha_i^{-1}(V_i(x_i,\hat{x}_i)) \leq \bar{\alpha}(V(x,\hat{x})),\end{aligned}$$

with function $\bar{\alpha} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined for all $r \in \mathbb{R}_{\geq 0}$ as

$$\bar{\alpha}(r) := \max\left\{\textstyle\sum_{i=1}^{N} \alpha_i^{-1}(s_i) \,\middle|\, s_i \geq 0,\ \sum_{i=1}^{N} \mu_i s_i = r\right\}.$$

It is not hard to verify that function $\bar{\alpha}(\cdot)$ defined above is a $\mathcal{K}_\infty$ function. By taking the $\mathcal{K}_\infty$ function $\alpha(r) := \bar{\alpha}^{-1}(r)$, $\forall r \in \mathbb{R}_{\geq 0}$, one obtains

$$\alpha(\|h(x) - \hat{h}(\hat{x})\|) \leq V(x,\hat{x}),$$

satisfying inequality (9). Now we prove that SSF V in (12) satisfies inequality (10). Consider any $x = [x_1;\dots;x_N] \in X$, $\hat{x} = [\hat{x}_1;\dots;\hat{x}_N] \in \hat{X}$, and $\hat{v} = [\hat{v}_1;\dots;\hat{v}_N] \in \hat{U}$. For any $i \in \{1,\dots,N\}$, there exists $v_i \in U_i$, consequently, a vector $v = [v_1;\dots;v_N] \in U$, satisfying (8) for each pair of subsystems Σ_i and $\widehat{\Sigma}_i$ with the internal inputs given by $[w_1;\dots;w_N] = M[h_{21}(x_1);\dots;h_{2N}(x_N)]$ and $[\hat{w}_1;\dots;\hat{w}_N] = \hat{M}[\hat{h}_{21}(\hat{x}_1);\dots;\hat{h}_{2N}(\hat{x}_N)]$. Then we have the chain of inequalities in (24) using conditions (13) and (14) and by defining $\kappa(\cdot), \psi, \rho_{\text{ext}}(\cdot)$ as

$$\kappa(r) := \min\left\{\sum_{i=1}^{N} \mu_i \kappa_i(s_i) \,\middle|\, s_i \geq 0,\ \sum_{i=1}^{N} \mu_i s_i = r\right\},\ \psi := \sum_{i=1}^{N} \mu_i \psi_i,$$

$$\rho_{\text{ext}}(r) := \max\left\{\sum_{i=1}^{N} \mu_i \rho_{i\text{ext}}(s_i) \,\middle|\, s_i \geq 0,\ \|[s_1;\dots;s_N]\| = r\right\}.$$

Note that κ and ρ_{ext} in (24) belong to $\mathcal{K}$ and $\mathcal{K}_\infty \cup \{0\}$, respectively, because of their definition provided above. Hence, we conclude that V is an SSF from $\widehat{\Sigma}$ to Σ. □

Proof. **(Theorem 5.1)** Since system Σ is incrementally passivable, from (17), $\forall x \in X$ and $\forall \hat{x} \in \hat{X}$, we have

$$\underline{\alpha}(\|h_1(x) - h_1(\hat{x})\|) = \underline{\alpha}(\|h_1(x) - \hat{h}_1(\hat{x})\|) \leq V(x,\hat{x}),$$

satisfying (7) with $\alpha(s) := \underline{\alpha}(s)\ \forall s \in \mathbb{R}_{\geq 0}$. Now by taking the conditional expectation from (19), $\forall x \in X, \forall \hat{x} \in \hat{X}, \forall \hat{v} \in \hat{U}, \forall w \in W, \forall \hat{w} \in \hat{W}$, we have

$$\begin{aligned}&\mathbb{E}\Big[V(f(x,L(x)+\hat{v},w,\varsigma),\hat{f}(\hat{x},\hat{v},\hat{w},\varsigma))\Big|x,\hat{x},\hat{v},w,\hat{w}\Big] - \\ &\mathbb{E}\Big[V(f(x,L(x)+\hat{v},w,\varsigma),f(\hat{x},L(\hat{x})+\hat{v},\hat{w},\varsigma))\Big|x,\hat{x},\hat{v},w,\hat{w}\Big] \\ &\leq \mathbb{E}\Big[\gamma(\|\hat{f}(\hat{x},\hat{v},\hat{w},\varsigma) - f(\hat{x},L(\hat{x})+\hat{v},\hat{w},\varsigma)\|)\Big|\hat{x},\hat{x},\hat{v},w,\hat{w}\Big],\end{aligned}$$

where $\hat{f}(\hat{x},\hat{v},\hat{w},\varsigma) = \Pi_x(f(\hat{x},L(\hat{x})+\hat{v},\hat{w},\varsigma))$. Using Theorem 2.2 and inequality (5), the above inequality reduces to

$$\begin{aligned}&\mathbb{E}\Big[V(f(x,L(x)+\hat{v},w,\varsigma),\hat{f}(\hat{x},\hat{v},\hat{w},\varsigma))\Big|x,\hat{x},\hat{v},w,\hat{w}\Big] - \\ &\mathbb{E}\Big[V(f(x,L(x)+\hat{v},w,\varsigma),f(\hat{x},L(\hat{x})+\hat{v},\hat{w},\varsigma))\Big|x,\hat{x},\hat{v},w,\hat{w}\Big] \\ &\leq \gamma(\delta).\end{aligned}$$

Employing (18) and since $h_2 = \hat{h}_2$, we get

$$\begin{aligned}&\mathbb{E}\Big[V(f(x,L(x)+\hat{v},w,\varsigma),f(\hat{x},L(\hat{x})+\hat{v},\hat{w},\varsigma))\Big|x,\hat{x},\hat{v},w,\hat{w}\Big] \\ &- V(x,\hat{x}) \leq -\widehat{\kappa}(V(x,\hat{x})) + \\ &\begin{bmatrix} w-\hat{w} \\ h_2(x)-\hat{h}_2(\hat{x})\end{bmatrix}^T \begin{bmatrix}\bar{X}^{11} & \bar{X}^{12} \\ \bar{X}^{21} & \bar{X}^{22}\end{bmatrix} \begin{bmatrix} w-\hat{w} \\ h_2(x)-\hat{h}_2(\hat{x})\end{bmatrix}.\end{aligned}$$

It follows that $\forall x \in X, \forall \hat{x} \in \hat{X}, \forall \hat{u} \in U$, and $\forall w \in W, \forall \hat{w} \in \hat{W}$,

$$\begin{aligned}&\mathbb{E}\Big[V(f(x,L(x)+\hat{v},w,\varsigma),\hat{f}(\hat{x},\hat{v},\hat{w},\varsigma)))\Big|x,\hat{x},\hat{v},w,\hat{w}\Big] - V(x,\hat{x}) \\ &\leq -\widehat{\kappa}(V(x,\hat{x})) + \gamma(\delta) + \\ &\begin{bmatrix} w-\hat{w} \\ h_2(x)-\hat{h}_2(\hat{x})\end{bmatrix}^T \begin{bmatrix}\bar{X}^{11} & \bar{X}^{12} \\ \bar{X}^{21} & \bar{X}^{22}\end{bmatrix} \begin{bmatrix} w-\hat{w} \\ h_2(x)-\hat{h}_2(\hat{x})\end{bmatrix},\end{aligned}$$

satisfying (8) with $\psi = \gamma(\delta)$, $\nu = L(x) + \hat{v}$, $\kappa = \widehat{\kappa}$, $\rho_{ext} \equiv 0$, and G, $\hat{G}$, H are identity matrices of appropriate dimensions. Hence, V is an SStF from $\widehat{\Sigma}$ to Σ, which completes the proof. □

Proof. **(Theorem 5.2)** Here, we show that $\forall x, \forall \hat{x}, \forall \hat{v}, \exists \nu, \forall \hat{w}, \forall w$, V satisfies $\frac{\lambda_{\min}(\tilde{M})}{\lambda_{\max}(C_1^T C_1)}\|C_1 x - \hat{C}_1 \hat{x}\|^2 \leq V(x,\hat{x})$ and

$$\begin{aligned}&\mathbb{E}\Big[V(f(x,\nu,w,\varsigma),\hat{f}(\hat{x},\hat{v},\hat{w},\varsigma))\,\Big|\,x,\hat{x},\hat{v},w,\hat{w}\Big] - V(x,\hat{x}) \\ &\leq -(1-\widehat{\kappa})(V(x,\hat{x})) + (1+2/\pi)\lambda_{\max}(\tilde{M})\delta^2 \\ &\quad + \begin{bmatrix} w-\hat{w} \\ h_2(x)-\hat{h}_2(\hat{x})\end{bmatrix}^T \begin{bmatrix}\bar{X}^{11} & \bar{X}^{12} \\ \bar{X}^{21} & \bar{X}^{22}\end{bmatrix} \begin{bmatrix} w-\hat{w} \\ h_2(x)-\hat{h}_2(\hat{x})\end{bmatrix}.\end{aligned}$$

Since $C_1 = \hat{C}_1$, we have $\|C_1 x - \hat{C}_1\hat{x}\|^2 = (x-\hat{x})^T C_1^T C_1 (x-\hat{x})$. Since $\lambda_{\min}(C_1^T C_1)\|x-\hat{x}\|^2 \leq (x-\hat{x})^T C_1^T C_1 (x-\hat{x}) \leq \lambda_{\max}(C_1^T C_1)\|x-\hat{x}\|^2$ and similarly $\lambda_{\min}(\tilde{M})\|x-\hat{x}\|^2 \leq (x-\hat{x})^T \tilde{M}(x-\hat{x}) \leq \lambda_{\max}(\tilde{M})\|x-\hat{x}\|^2$, it can be readily verified that $\frac{\lambda_{\min}(\tilde{M})}{\lambda_{\max}(C_1^T C_1)}\|C_1 x - \hat{C}_1\hat{x}\|^2 \leq V(x,\hat{x})$ holds $\forall x, \forall \hat{x}$, implying that inequality (7) holds with $\alpha(s) = \frac{\lambda_{\min}(\tilde{M})}{\lambda_{\max}(C_1^T C_1)} s^2$ for any $s \in \mathbb{R}_{\geq 0}$. We proceed with showing that the inequality (8) holds, as well. Given any x, $\hat{x}$, and $\hat{v}$, we choose ν via the following *interface* function:

$$\nu = \nu_{\hat{v}}(x,\hat{x},\hat{v}) := K(x-\hat{x}) + \hat{v}. \tag{26}$$

By employing the definition of the interface function, we simplify

$$Ax + B\nu_{\hat{v}}(x,\hat{x},\hat{v}) + Dw + \bar{N}\varsigma - \Pi_x(A\hat{x} + B\hat{v} + D\hat{w} + \bar{N}\varsigma)$$

to

$$(A+BK)(x-\hat{x}) + D(w-\hat{w}) + F,$$

$$\mathbb{E}\Big[\sum_{i=1}^{N}\mu_i\Big[V_i(f_i(x_i,\,\nu_i,\,w_i,\,\varsigma_i),\hat{f}_i(\hat{x}_i,\,\hat{\nu}_i,\,\hat{w}_i,\,\varsigma_i))\,|\,x,\,\hat{x},\,\hat{\nu}\Big]\Big]-\sum_{i=1}^{N}\mu_i V_i(x_i,\,\hat{x}_i)=\sum_{i=1}^{N}\mu_i\mathbb{E}\Big[V_i(f_i(x_i,\,\nu_i,\,w_i,\,\varsigma_i),\hat{f}_i(\hat{x}_i,\,\hat{\nu}_i,\,\hat{w}_i,\,\varsigma_i))\,|\,x,\,\hat{x},\,\hat{\nu}\Big]-\sum_{i=1}^{N}\mu_i V_i(x_i,\,\hat{x}_i)$$

$$=\sum_{i=1}^{N}\mu_i\mathbb{E}\Big[V_i(f_i(x_i,\,\nu_i,\,w_i,\,\varsigma_i),\hat{f}_i(\hat{x}_i,\,\hat{\nu}_i,\,\hat{w}_i,\,\varsigma_i))\,|\,x_i,\,\hat{x}_i,\,\hat{\nu}_i\Big]-\sum_{i=1}^{N}\mu_i V_i(x_i,\,\hat{x}_i)\le\sum_{i=1}^{N}\mu_i\Big(-\kappa_i(V_i(x_i,\,\hat{x}_i))+\rho_{i\text{ext}}(\|\hat{\nu}_i\|)+\psi_i+\begin{bmatrix}G_i w_i-\hat{G}_i\hat{w}_i\\ h_{2i}(x_i)-H_i\hat{h}_{2i}(\hat{x}_i)\end{bmatrix}^T$$

$$\begin{bmatrix}\bar{X}_i^{11} & \bar{X}_i^{12}\\ \bar{X}_i^{21} & \bar{X}_i^{22}\end{bmatrix}\begin{bmatrix}G_i w_i-\hat{G}_i\hat{w}_i\\ h_{2i}(x_i)-H_i\hat{h}_{2i}(\hat{x}_i)\end{bmatrix}\Big)=\sum_{i=1}^{N}-\mu_i\kappa_i(V_i(x_i,\,\hat{x}_i))+\sum_{i=1}^{N}\mu_i\rho_{i\text{ext}}(\|\hat{\nu}_i\|)+\sum_{i=1}^{N}\mu_i\psi_i$$

$$+\begin{bmatrix}G_1 w_1-\hat{G}_1\hat{w}_1\\ \vdots\\ G_N w_N-\hat{G}_N\hat{w}_N\\ h_{21}(x_1)-H_1\hat{h}_{21}(\hat{x}_1)\\ \vdots\\ h_{2N}(x_N)-H_N\hat{h}_{2N}(\hat{x}_N)\end{bmatrix}^T\begin{bmatrix}\mu_1\bar{X}_1^{11} & & & \mu_1\bar{X}_1^{12} & & \\ & \ddots & & & \ddots & \\ & & \mu_N\bar{X}_N^{11} & & & \mu_N\bar{X}_N^{12}\\ \mu_1\bar{X}_1^{21} & & & \mu_1\bar{X}_1^{22} & & \\ & \ddots & & & \ddots & \\ & & \mu_N\bar{X}_N^{21} & & & \mu_N\bar{X}_N^{22}\end{bmatrix}\begin{bmatrix}G_1 w_1-\hat{G}_1\hat{w}_1\\ \vdots\\ G_N w_N-\hat{G}_N\hat{w}_N\\ h_{21}(x_1)-H_1\hat{h}_{21}(\hat{x}_1)\\ \vdots\\ h_{2N}(x_N)-H_N\hat{h}_{2N}(\hat{x}_N)\end{bmatrix}=\sum_{i=1}^{N}-\mu_i\kappa_i(V_i(x_i,\,\hat{x}_i))$$

$$+\sum_{i=1}^{N}\mu_i\rho_{i\text{ext}}(\|\hat{\nu}_i\|)+\sum_{i=1}^{N}\mu_i\psi_i+\begin{bmatrix}GM\begin{bmatrix}h_{21}(x_1)\\ \vdots\\ h_{2N}(x_N)\end{bmatrix}-\hat{G}\hat{M}\begin{bmatrix}\hat{h}_{21}(\hat{x}_1)\\ \vdots\\ \hat{h}_{2N}(\hat{x}_N)\end{bmatrix}\\ h_{21}(x_1)-H_1\hat{h}_{21}(\hat{x}_1)\\ \vdots\\ h_{2N}(x_N)-H_N\hat{h}_{2N}(\hat{x}_N)\end{bmatrix}^T\bar{X}_{cmp}\begin{bmatrix}GM\begin{bmatrix}h_{21}(x_1)\\ \vdots\\ h_{2N}(x_N)\end{bmatrix}-\hat{G}\hat{M}\begin{bmatrix}\hat{h}_{21}(\hat{x}_1)\\ \vdots\\ \hat{h}_{2N}(\hat{x}_N)\end{bmatrix}\\ h_{21}(x_1)-H_1\hat{h}_{21}(\hat{x}_1)\\ \vdots\\ h_{2N}(x_N)-H_N\hat{h}_{2N}(\hat{x}_N)\end{bmatrix}=\sum_{i=1}^{N}-\mu_i\kappa_i(V_i(x_i,\,\hat{x}_i))$$

$$+\sum_{i=1}^{N}\mu_i\rho_{i\text{ext}}(\|\hat{\nu}_i\|)+\sum_{i=1}^{N}\mu_i\psi_i+\begin{bmatrix}h_{21}(x_1)-H_1\hat{h}_{21}(\hat{x}_1)\\ \vdots\\ h_{2N}(x_N)-H_N\hat{h}_{2N}(\hat{x}_N)\end{bmatrix}^T\begin{bmatrix}GM\\ I_{\tilde{q}}\end{bmatrix}^T\bar{X}_{cmp}\begin{bmatrix}GM\\ I_{\tilde{q}}\end{bmatrix}\begin{bmatrix}h_{21}(x_1)-H_1\hat{h}_{21}(\hat{x}_1)\\ \vdots\\ h_{2N}(x_N)-H_N\hat{h}_{2N}(\hat{x}_N)\end{bmatrix}$$

$$\le\sum_{i=1}^{N}-\mu_i\kappa_i(V_i(x_i,\,\hat{x}_i))+\sum_{i=1}^{N}\mu_i\rho_{i\text{ext}}(\|\hat{\nu}_i\|)+\sum_{i=1}^{N}\mu_i\psi_i\le-\kappa\,(V\,(x,\,\hat{x}))+\rho_{\text{ext}}(\|\hat{\nu}\|)+\psi\,.$$

(24)

$$\mathbb{E}\Big[V(f(x,\,\nu,\,w,\,\varsigma),\hat{f}(\hat{x},\,\hat{\nu},\,\hat{w},\,\varsigma))\,\Big|\,x,\,\hat{x},\,\hat{\nu},\,w,\,\hat{w}\Big]-V(x,\,\hat{x})=(x-\hat{x})^T(A+BK)^T\tilde{M}(A+BK)(x-\hat{x})+2(x-\hat{x})^T(A+BK)^T\tilde{M}D(w-\hat{w})$$

$$+(w-\hat{w})^T D^T\tilde{M}D(w-\hat{w})+2(x-\hat{x})^T(A+BK)^T\tilde{M}\mathbb{E}\Big[F\,|\,x,\,\hat{x},\,\hat{\nu},\,w,\,\hat{w}\Big]+2(w-\hat{w})^T D^T\tilde{M}\mathbb{E}\Big[F\,|\,x,\,\hat{x},\,\hat{\nu},\,w,\,\hat{w}\Big]+\mathbb{E}\Big[F^T\tilde{M}F\,|\,x,\,\hat{x},\,\hat{\nu},\,w,\,\hat{w}\Big]$$

$$-V(x,\,\hat{x})\le\begin{bmatrix}x-\hat{x}\\ w-\hat{w}\end{bmatrix}^T\begin{bmatrix}(1+\pi)(A+BK)^T\tilde{M}(A+BK) & (A+BK)^T\tilde{M}D\\ D^T\tilde{M}(A+BK) & (1+\pi)D^T\tilde{M}D\end{bmatrix}\begin{bmatrix}x-\hat{x}\\ w-\hat{w}\end{bmatrix}+(1+2/\pi)\lambda_{\max}(\tilde{M})\delta^2-V(x,\,\hat{x})$$

$$\le\begin{bmatrix}x-\hat{x}\\ w-\hat{w}\end{bmatrix}^T\begin{bmatrix}\widehat{\kappa}\tilde{M}+C_2^T\bar{X}^{22}C_2 & C_2^T\bar{X}^{21}\\ \bar{X}^{12}C_2 & \bar{X}^{11}\end{bmatrix}\begin{bmatrix}x-\hat{x}\\ w-\hat{w}\end{bmatrix}+(1+2/\pi)\lambda_{\max}(\tilde{M})\delta^2-V(x,\,\hat{x})$$

$$=-(1-\widehat{\kappa})(V(x,\,\hat{x}))+\begin{bmatrix}w-\hat{w}\\ C_2x-\hat{C}_2\hat{x}\end{bmatrix}^T\begin{bmatrix}\bar{X}^{11} & \bar{X}^{12}\\ \bar{X}^{21} & \bar{X}^{22}\end{bmatrix}\begin{bmatrix}w-\hat{w}\\ C_2x-\hat{C}_2\hat{x}\end{bmatrix}+(1+2/\pi)\lambda_{\max}(\tilde{M})\delta^2$$

(25)

where $F=A\hat{x}+B\hat{\nu}+D\hat{w}+\bar{N}\varsigma-\Pi_{\mathcal{X}}(A\hat{x}+B\hat{\nu}+D\hat{w}+\bar{N}\varsigma)$. Using Young's inequality [25] as $ab\le\frac{\pi}{2}a^2+\frac{1}{2\pi}b^2$, for any $a,b\ge0$ and any $\pi>0$, and by employing Cauchy-Schwarz inequality, $C_2=\hat{C}_2$, and since

$$\|F\|\le\delta,\quad F^T\tilde{M}F\le\lambda_{\max}(\tilde{M})\delta^2,$$

one can obtain the chain of inequalities in (25). Hence, the proposed V in (21) is an SStF from $\widehat{\Sigma}$ to Σ, which completes the proof. Note that functions $\alpha\in\mathcal{K}_\infty$, $\kappa\in\mathcal{K}$, $\rho_{\text{ext}}\in\mathcal{K}_\infty\cup\{0\}$, and matrix $\bar{X}$ in Definition 3.1 associated with V in (21) are defined as $\alpha(s)=\frac{\lambda_{\min}(\tilde{M})}{\lambda_{\max}(C_1^TC_1)}s^2$, $\kappa(s):=(1-\widehat{\kappa})s$, $\rho_{\text{ext}}(s):=0$, $\forall s\in\mathbb{R}_{\ge0}$, and $\bar{X}=\begin{bmatrix}\bar{X}^{11} & \bar{X}^{12}\\ \bar{X}^{21} & \bar{X}^{22}\end{bmatrix}$. Moreover, positive constant ψ in (8) is $\psi=(1+2/\pi)\lambda_{\max}(\tilde{M})\delta^2$. □