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TOPOLOGICAL DUALITY IN HUMANOID ROBOT DYNAMICS

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Abstract

A humanoid robot system may be viewed as a collection of segments coupled at rotational joints which geometrically represent constrained rotational Lie groups. This allows a study of the dynamics of the motion of a humanoid robot. Several formulations are possible. In this paper, dual invariant topological structures are constructed and analyzed on the finite-dimensional manifolds associated with the humanoid motion. Both cohomology and homology structures are examined on the tangent (Lagrangian) as well as on the cotangent (Hamiltonian) bundles on the manifold of the humanoid motion configuration, represented by the toral Lie group. It is established all four topological structures give in essence the same description of humanoid dynamics. Practically this means that whichever of these approaches we use, ultimately we obtain the same mathematical results.

1. Introduction

Since the early papers of Vukobratovic [30, 31, 33–35], a vast body of research has evolved relating to the kinematics, dynamics and control of biped, anthropomorphic and humanoid robots [5, 10, 12, 13, 16, 23, 27–29, 32]. Some biped models admitted the ability of passive dynamic walking [19] and others powered walking [20]. The decade before last was dominated by work on solutions to the kinematic problems of redundancy and singularities [26, 36]. The last decade has been characterized mostly by the extensive use of intelligent, neuro-fuzzy-evolutionary control of humanoid dynamics [6, 9, 11, 21, 24, 25] and computer-graphics animation [15].

The purpose of the present study is to establish and analyze dual invariant algebraic-topological structures (homology and cohomology groups) on dual invariant differential-topological structures (tangent and cotangent bundles) of the finite-dimensional manifolds (Lie groups) arising in connection with humanoid dynamics [14]. The construction of the n-dimensional configuration manifold of humanoid motion makes

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use of the formalism of constrained rotational Lie groups [4, 14]. For the construction of momentum and velocity phase-spaces for humanoid dynamics we employ the differential topology of vector bundles [1, 7, 18]. The construction of the Lagrangian and Hamiltonian vector fields for humanoid dynamics is based on Riemannian and symplectic geometry [1, 2, 4, 7, 18, 22]. For the construction of invariant (co)homology structures involved in humanoid dynamics we use the basic structures of algebraic topology (see [3, 17] for details).

In this paper, we show that both cohomology and homology groups, operating on momentum—as well as on velocity-phase-space manifolds of humanoid motion (and therefore applying to both Hamiltonian and Lagrangian formalisms), give in essence the same mathematical description of humanoid dynamics.

2. Configuration, velocity and momentum manifolds

The kinematics of an n-segment humanoid chain are usually defined as a map between external (usually, end-effector) coordinates $x^r (r = 1, \dots, n)$ and internal (joint) coordinates $q^i (i = 1, \dots, N)$. The forward kinematics are defined as a nonlinear map $x^r = x^r(q^i)$ with corresponding linear vector functions $dx^r = \partial x^r / \partial q^i \, dq^i$ of differentials and $\dot{x}^r = \partial x^r / \partial q^i \, \dot{q}^i$ of velocities. Here and subsequently the summation convention is understood with repeated indices. When the rank of the configuration-dependent Jacobian matrix $J \equiv \partial x^r / \partial q^i$ is less than n kinematic singularities occur; the onset of this condition could be detected by the manipulability measure [24]. Inverse kinematics are defined conversely by a nonlinear map $q^i = q^i(x^r)$ with corresponding linear vector functions $dq^i = \partial q^i / \partial x^r \, dx^r$ of differentials and $\dot{q}^i = \partial q^i / \partial x^r \, \dot{x}^r$ of velocities. Again, in the case $n < N$ of redundancy, the inverse kinematic problem admits infinite solutions. Often pseudo-inverse configuration-control $\dot{q}^i = J^* \dot{x}^r$ is used instead (see [15]), where $J^* = J^T (J J^T)^{-1}$ denotes the Moore-Penrose pseudo-inverse of the Jacobian matrix J.

Humanoid joints, that is, internal coordinates q^i $(i = 1, \dots, N)$, constitute a smooth configuration manifold Q^N. Uniaxial 'hinge' joints represent constrained, classical, rotational Lie groups $SO(2)^i_{\mathrm{cnstr}}$, parameterized by constrained angles $q^i_{\mathrm{cnstr}} \equiv q^i \in [q^i_{\mathrm{min}}, q^i_{\mathrm{max}}]$. Three-axial 'ball-and-socket' joints represent constrained rotational groups $SO(3)^i_{\mathrm{cnstr}}$, parameterized by constrained Euler angles $q^i = q^{\phi_i}_{\mathrm{cnstr}}$. In the sequel the subscript 'cnstr' will be omitted, for the sake of simplicity, and always assumed in relation to internal coordinates q^i.

The functor **Lie** applied to the category $\mathcal{S}^\bullet[SO(n)^i]$ (for $n = 2, 3$ and $i = 1, \dots, N$) of rotational Lie groups $SO(n)^i$ (and their homomorphisms) gives the category $\mathcal{S}_\bullet[so(n)_i]$ of corresponding tangent Lie algebras $so(n)_i$ (and their homomorphisms). (A homomorphism of Lie groups is their homomorphism as abstract groups and their smooth map as manifolds, while a homomorphism of Lie algebras is a linear map of one Lie algebra into another that sends a product (Lie bracket) into a product.) Further applying the functor $\mathbf{Dual}_G$ to the category $\mathscr{S}_\bullet[so(n)_i]$ provides the category $\mathscr{S}_\bullet^*[so(n)_i^*]$ of cotangent or canonical Lie algebras $so(n)_i^*$ (and their homomorphisms). To go directly from $\mathscr{S}^\bullet[SO(n)^i]$ to $\mathscr{S}_\bullet^*[so(n)_i^*]$, we use the canonical functor $\mathbf{Can}$. Therefore we have the following commutative triangle.

$$\begin{tikzcd}
 & \mathscr{S}^\bullet[SO(n)^i] \arrow[dl, "\mathbf{Lie}"'] \arrow[dr, "\mathbf{Can}"] & \\
\mathscr{S}_\bullet[so(n)_i] \arrow[rr, "\overset{\cong}{\mathbf{Dual}_G}"] & & \mathscr{S}_\bullet^*[so(n)_i^*]
\end{tikzcd}$$

Both the tangent algebras $so(n)_i$ and the cotangent algebras $so(n)_i^*$ contain infinitesimal group generators, angular velocities $\dot{q}^i = \dot{q}^{\phi_i}$ in the first case and canonical angular momenta $p_i = p_{\phi_i}$ in the second [18]. As Lie group generators, angular velocities and angular momenta satisfy the respective commutation relations $[\dot{q}^{\phi_i}, \dot{q}^{\psi_i}] = \epsilon_\theta^{\phi\psi}\dot{q}^{\theta_i}$ and $[p_{\phi_i}, p_{\psi_i}] = \epsilon_{\phi\psi}^\theta p_{\theta_i}$, where the structure constants $\epsilon_\theta^{\phi\psi}$ and $\epsilon_{\phi\psi}^\theta$ constitute totally antisymmetric third-order tensors.

In this way, the functor $\mathbf{Dual}_G : \mathbf{Lie} \cong \mathbf{Can}$ establishes a geometrical duality between kinematics of angular velocities $\dot{q}^i$ (involved in Lagrangian formalism on the tangent bundle of Q^N) and that of angular momenta p_i (involved in Hamiltonian formalism on the cotangent bundle of Q^N). This is analyzed below. In other words, we have two functors $\mathbf{Lie}$ and $\mathbf{Can}$ from a category of Lie groups (of which $\mathscr{S}^\bullet[SO(n)^i]$ is a subcategory) into a category of their Lie algebras (of which $\mathscr{S}_\bullet[so(n)_i]$ and $\mathscr{S}_\bullet^*[so(n)_i^*]$ are subcategories), and a natural equivalence (functor isomorphism) between them defined by the functor $\mathbf{Dual}_G$. (As angular momenta p_i are in a bijective correspondence with angular velocities $\dot{q}^i$, every component of the functor $\mathbf{Dual}_G$ is invertible.)

If G_1 and G_2 are two Lie groups, their tensor product $G_1 \otimes G_2$ is also a Lie group [18]. The configuration manifold Q^N can be constructed as a constrained Lie group 'product-tree' depicted in Figure 1.

Here a line segment '–' corresponds to the tensor product operation '$\otimes$'. Naturally-distributed spine kinematics are represented by the lumped 'lumbo-sacral' $SO(3)$-joint. Finger and toe motions are not included.

For the sake of the cerebellum-like motor control [21], the humanoid configuration manifold Q^N could be reduced to an N-torus as follows. Denote by S^1 the constrained unit circle in the complex plane. This is an Abelian Lie group [16]. We have the homeomorphisms $SO(3) \approx SO(2) \otimes SO(2) \otimes SO(2)$ and $SO(2) \approx S^1$. Let I^N be the unit cube $[0, 1]^N$ in $\mathbb{R}^N$ and '$\sim$' an equivalence relation on $\mathbb{R}^N$ obtained by 'glueing' together the opposite sides of I^N and preserving their orientation. The manifold of humanoid configurations depicted in Figure 1 can be represented as the quotient

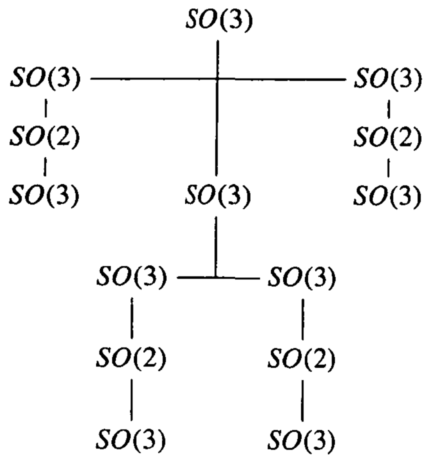

FIGURE 1. A constrained Lie group 'product-tree' as a configuration manifold of humanoid robot dynamics.

space of $\mathbb{R}^N$ by the space of the integral lattice points in $\mathbb{R}^N$, that is, a constrained N-dimensional torus

$$\mathbb{R}^N/Z^N = I^N/\!\sim\ \cong \prod_{i=1}^{N} S^1_i \equiv \{(q^i, i = 1, \dots, N) :\ \mathrm{mod}\ 2\pi\} = T^N.$$

Since S^1 is an Abelian Lie group, its N-fold tensor product T^N is also an Abelian Lie group, the toral group, of all nondegenerate diagonal $N \times N$ matrices. As a Lie group, the configuration space $Q^N = T^N$ of humanoid dynamics has a natural Banach manifold structure with local internal coordinates $q^i \in U$, U being an open set (chart) in T^N [1, 17].

Conversely by 'unglueing' the configuration space we obtain the primary unit cube. Let '$\sim^*$' denote an equivalent decomposition or 'unglueing' relation. By the Tychonoff product-topology theorem, for every such quotient space there exists a selector such that their quotient models are homeomorphic, that is, $T^N/\!\sim^* \approx A^N/\!\sim^*$ [1, 17]. Therefore I^N represents a selector for the configuration torus T^N and can be used as an N-directional 'command-space' for the topological control of humanoid motion. Any subset of degrees of freedom on the configuration torus T^N representing the joints included in humanoid motion has its simple, rectangular image in the command-space selector I^N. Operationally, this resembles what the brain-motor controller, the cerebellum, actually performs on the highest level of human motor control [14, 21].

We refer to the tangent bundle TQ^N of the humanoid configuration manifold Q^N (Figure 1) as the velocity phase-space manifold, and to its cotangent bundle T^*Q^N

as the momentum phase-space manifold. We shall prove that humanoid dynamics can be described equivalently in terms of two topologically dual functors $\mathbf{Lag}_T$ and $\mathbf{Ham}_{T^*}$ from $\mathbf{Diff}$, the category of smooth manifolds (and their smooth maps) of class C^p, into $\mathbf{Bund}$, the category of vector bundles (and vector-bundle maps) of class C^{p-1}, with $p \geq 1$. The functor $\mathbf{Lag}_T$ is represented physically by the second-order Lagrangian formalism on $TQ^N \in \mathbf{Bund}$, while $\mathbf{Ham}_{T^*}$ is represented by a first-order Hamiltonian formalism on $T^*Q^N \in \mathbf{Bund}$. We shall prove the existence of the topological functor isomorphism $\mathbf{Dual}_T : \mathbf{Lag}_T \cong \mathbf{Ham}_{T^*}$.

The Riemannian metric $g = \langle \, , \rangle$ on the configuration manifold Q^N is a positive-definite quadratic form $g : TQ^N \to \mathbb{R}$, given in local coordinates $q^i \in U$ (U open in Q^N), as

$$g_{ij} \mapsto g_{ij}(q, m) dq^i dq^j$$

(see [8]). Here

$$g_{ij}(q, m) = \sum_{\mu=1}^{n} m_\mu \delta_{rs} \frac{\partial x^r}{\partial q^i} \frac{\partial x^s}{\partial q^j}$$

is the covariant material metric tensor [8] defining a relation between internal and external coordinates and including n segmental masses m_μ. The quantities x^r are external coordinates ($r, s = 1, \ldots, 6n$) and $i, j = 1, \ldots, N \equiv 6n - h$, where h denotes the number of holonomic constraints.

The Lagrangian of the system is a quadratic form $L : TQ^N \to \mathbb{R}$ dependent on the velocity v and such that $L(v) = \langle v, v \rangle / 2$. It is given by

$$L(v) = \frac{1}{2} g_{ij}(q, m) v^i v^j$$

in local coordinates q^i, $v^i = \dot{q}^i \in U_v$ (U_v open in TQ^N) (see [2, 18]). The Hamiltonian of the system is a quadratic form $H : T^*Q^N \to \mathbb{R}$ dependent on momentum p and such that $H(p) = \langle p, p \rangle / 2$. It is given by

$$H(p) = \frac{1}{2} g^{ij}(q, m) p_i p_j$$

in local canonical coordinates $q^i, p_i \in U_p$ (U_p open in T^*Q^N) (see [2, 18]). The inverse (contravariant) material metric tensor is defined as (see [8])

$$g^{ij}(q, m) = \sum_{\mu=1}^{n} m_\mu \delta_{rs} \frac{\partial q^i}{\partial x^r} \frac{\partial q^j}{\partial x^s}.$$

For any smooth function L on TQ^N, the fibre derivative or Legendre transformation is a diffeomorphism $FL : TQ^N \to T^*Q^N$ with $F(w) \cdot v = \langle w, v \rangle$ from the momentum phase-space manifold to the velocity phase-space manifold associated with the metric $g = \langle\, ,\rangle$. In local coordinates $q^i, v^i = \dot{q}^i \in U_v$ (U_v open in TQ^N), FL is given by $(q^i, v^i) \mapsto (q^i, p_i)$ [18].

For the momentum phase-space manifold T^*Q^N we have the following (see [22]):

(i) There exists a unique canonical one-form θ_H with the property that, for any one-form β on the configuration manifold Q^N, we have $\beta^*\theta_H = \beta$. In local canonical coordinates $q^i, p_i \in U_p$ (U_p open in T^*Q^N) it is given by $\theta_H = p_i dq^i$.

(ii) There exists a unique nondegenerate symplectic (or Hamiltonian) two-form ω_H, which is closed ($d\omega_H = 0$) and exact ($\omega_H = d\theta_H = dp_i \wedge dq^i$). Each body segment has, in the general $SO(3)$ case, a sub-phase-space manifold $T^*SO(3)$ with

$$\omega_H^{(\mathrm{sub})} = dp_\phi \wedge d\phi + dp_\psi \wedge d\psi + dp_\theta \wedge d\theta.$$

Analogously, for the velocity phase-space manifold TQ^N we have the following (see [18]):

(i) There exists a unique one-form θ_L, defined by the *pull-back* $\theta_L = (FL) * \theta_H$ of θ_H by FL. In local coordinates $q^i, v^i = \dot{q}^i \in U_v$ (U_v open in TQ^N) it is given by $\theta_L = L_{v^i} dq^i$, where $L_{v^i} \equiv \partial L/\partial v^i$.

(ii) There exists a unique Lagrangian two-form ω_L, defined by the pull-back $\omega_L = (FL) * \omega_H$ of ω_H by FL, which is closed ($d\omega_L = 0$) and exact ($\omega_L = d\theta_L = dL_{v^i} \wedge dq^i$).

Both T^*Q^N and TQ^N are orientable manifolds, admitting the standard volumes given respectively by

$$\Omega_{\omega_H} = \frac{(-1)^{N(N+1)/2}}{N!}\,\omega_H^N, \qquad \Omega_{\omega_L} = \frac{(-1)^{N(N+1)/2}}{N!}\,\omega_L^N$$

in local coordinates $q^i, p_i \in U_p$ (U_p open in T^*Q^N), (resp. $q^i, v^i = \dot{q}^i \in U_v$ (U_v open in TQ^N)). They are given by

$$\Omega_H = dq^1 \wedge \cdots \wedge dq^N \wedge dp_1 \wedge \cdots \wedge dp_N,$$
$$\Omega_L = dq^1 \wedge \cdots \wedge dq^N \wedge dv^1 \wedge \cdots \wedge dv^N.$$

On the velocity phase-space manifold TQ^N we can define also the action $A : TQ^N \to \mathbb{R}$ by $A(v) = FL(v) \cdot v$ and the energy $E = A - L$. In local coordinates $q^i, v^i = \dot{q}^i \in U_v$ (U_v open in TQ^N) we have $A = v^i L_{v^i}$, so $E = v^i L_{v^i} - L$. The Lagrangian vector field X_L on TQ^N is determined by the condition $i_{X_L}\omega_L = dE$. Classically it is given by the second-order Lagrange equations (see [2, 18])

$$\frac{d}{dt}\frac{\partial L}{\partial v^i} = \frac{\partial L}{\partial q^i}. \tag{2.1}$$

The Hamiltonian vector field X_H is defined on the momentum phase-space manifold T^*Q^N by the condition $i_{X_H}\omega = dH$. The condition may be expressed equivalently as $X_H = J\nabla H$, where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

In local canonical coordinates $q^i, p_i \in U_p$ (U_p open in T^*Q^N) the vector field X_H is classically given by the first-order Hamilton canonical equations (see [2, 18, 22])

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{2.2}$$

As a toral Lie group, the configuration manifold Q^N is Hausdorff. Therefore for $x = (q^i, p_i) \in U_p$ (U_p open in T^*Q^N), there exists a unique one-parameter group of diffeomorphisms $\phi_t : T^*Q^N \to T^*Q^N$ such that $\frac{d}{dt}\big|_{t=0}\phi_t x = J\nabla H(x)$. This is termed Hamiltonian phase flow and represents the maximal integral curve $t \mapsto (q^i(t), p_i(t))$ of the Hamiltonian vector field X_H passing through the point x for $t = 0$.

The flow ϕ_t is symplectic if ω_H is constant along it (that is, $\phi_t^*\omega_H = \omega_H$) if and only if its Lie derivative vanishes (that is, $L_{X_H}\omega_H = 0$). A symplectic flow consists of canonical transformations on T^*Q^N, that is, local diffeomorphisms that leave ω_H invariant. By Liouville's theorem, a symplectic flow ϕ_t preserves the phase volume on T^*Q^N. Also, the total energy $H = E$ of the system is conserved along ϕ_t, that is, $H \circ \phi_t = \phi_t$.

Lagrangian flow can be defined analogously (see [18]).

For a Lagrangian (resp. a Hamiltonian) vector field X_L (resp. X_H) on Q^N, there is a base integral curve $c_0(t) = (q^i(t), v^i(t))$ (resp. $c_0(t) = (q^i(t), p_i(t))$) if and only if $c_0(t)$ is a geodesic. This is given by the contravariant velocity equation

$$\dot{q}^i = v^i, \qquad \dot{v}^i + \Gamma^i_{jk} v^j v^k = 0 \tag{2.3}$$

in the former case and by the covariant momentum equation

$$\dot{q}^k = g^{ki} p_i, \qquad \dot{p}_i + \Gamma^i_{jk} g^{jl} g^{km} p_l p_m = 0 \tag{2.4}$$

in the latter [18]. Here Γ^i_{jk} denote the Christoffel symbols of an affine connection in an open chart U on Q^N, defined on the Riemannian metric $g = \langle\,,\rangle$ by

$$\Gamma^i_{jk} = g^{il}\Gamma_{jkl}, \qquad \Gamma_{jkl} = \frac{1}{2}\left(\frac{\partial g_{kl}}{\partial q^j} + \frac{\partial g_{jl}}{\partial q^k} - \frac{\partial g_{jk}}{\partial q^l}\right).$$

The left-hand sides $\dot{\bar{v}}^i = \dot{v}^i + \Gamma^i_{jk} v^j v^k$ (resp. $\dot{\bar{p}}_i = \dot{p}_i + \Gamma^i_{jk} g^{jl} g^{km} p_l p_m$) in the second parts of (2.3) and (2.4) represent the intrinsic or Bianchi covariant derivative of the velocity (resp. momentum) with respect to t. Parallel transport on Q^N is defined by $\dot{\bar{v}}^i = 0$, (resp. $\dot{\bar{p}}_i = 0$). When this applies X_L (resp. X_H) is called the geodesic spray and its flow the geodesic flow [2].

For the dynamics in the gravitational potential field $V : Q^N \to \mathbb{R}$, the Lagrangian $L : TQ^N \to \mathbb{R}$ (resp. the Hamiltonian $H : T^*Q^N \to \mathbb{R}$) has an extended form (see [18])

$$L(v, q) = g_{ij} v^i v^j/2 - V(q), \quad \big(\text{resp. } H(p, q) = g^{ij} p_i p_j/2 + V(q)\big).$$

A Lagrangian vector field X_L (resp. Hamiltonian vector field X_H) is still defined by the second-order Lagrangian equations (2.3) (resp. first-order Hamiltonian equations (2.4)) [2, 18].

The Legendre transformation or fibre derivative $FL : TQ^N \to T^*Q^N$ thus maps Lagrange's equations (2.1) and (2.3) into Hamilton's equations (2.2) and (2.4) (see [28, 30]). Clearly there exists a diffeomorphism $FH : T^*Q^N \to TQ^N$ such that $FL = (FH)^{-1}$. In local canonical coordinates $q^i, p_i \in U_p$ (U_p open in T^*Q^N) this is given by $(q^i, p_i) \mapsto (q^i, v^i)$ and thus maps Hamilton's equations (2.4) into Lagrange's equations (2.3).

A general form of the forced, non-conservative Hamilton's equations (resp. Lagrange's equations) is given as

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} + F_i(t, q^i, p_i), \quad \left(\text{resp. } \frac{d}{dt}\frac{\partial L}{\partial v^i} - \frac{\partial L}{\partial q^i} = F_i(t, q^i, v^i)\right)$$

(see [14, 18]). Here the $F_i(t, q^i, p_i)$ (resp. $F_i(t, q^i, v^i)$) represent any kind of covariant forces, including dissipative and elastic joint forces, as well as actuator drives and control forces, as a function of time, coordinates and momenta. In covariant form we have

$$\begin{aligned} &\dot{q}^k = g^{ki} p_i, \qquad \dot{p}_i + \Gamma^i_{jk} g^{jl} g^{km} p_l p_m = F_i(t, q^i, p_i), \\ &\big(\text{resp. } \quad \dot{q}^i = v^i, \qquad \dot{v}^i + \Gamma^i_{jk} v^j v^k = g^{ij} F_j(t, q^i, v^i)\big). \end{aligned}$$

3. (Co)Homologies of velocity and momentum manifolds

We now search for invariant topological duality in the humanoid dynamics on the configuration manifold $M \equiv Q^N$ (Figure 1) in the form of (co)homology structures [3, 8] on the momentum T^*M and velocity TM phase-space manifolds.

A. Cohomology If $\mathscr{C} = \mathscr{H}^\bullet\mathscr{M}$ (resp. $\mathscr{C} = \mathscr{L}^\bullet\mathscr{M}$) represents the Abelian category of cochains on the momentum phase-space manifold T^*M (resp. the velocity phase-space manifold TM), we have the category $\mathscr{S}^\bullet(\mathscr{H}^\bullet\mathscr{M})$ (resp. $\mathscr{S}^\bullet(\mathscr{L}^\bullet\mathscr{M})$) of generalized cochain complexes $A^\bullet$ in $\mathcal{H}^\bullet\mathcal{M}$ (resp. $\mathcal{L}^\bullet\mathcal{M}$) and if $A'_n = 0$ for $n < 0$ we have a subcategory $\mathcal{S}^\bullet_{\mathcal{DR}}(\mathcal{H}^\bullet\mathcal{M})$ (resp. $\mathcal{S}^\bullet_{\mathcal{DR}}(\mathcal{L}^\bullet\mathcal{M})$) of De Rham differential complexes in $\mathcal{S}^\bullet(\mathcal{H}^\bullet\mathcal{M})$ (resp. $\mathcal{S}^\bullet(\mathcal{L}^\bullet\mathcal{M})$)

$$A^\bullet_{DR} : 0 \to \Omega^0(T^*M) \xrightarrow{d} \Omega^1(T^*M) \xrightarrow{d} \Omega^2(T^*M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^N(T^*M) \xrightarrow{d} \cdots$$
$$\big(\text{resp. } A^\bullet_{DR} : 0 \to \Omega^0(TM) \xrightarrow{d} \Omega^1(TM) \xrightarrow{d} \Omega^2(TM) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^N(TM) \xrightarrow{d} \cdots\big),$$

where $A'_N = \Omega^N(T^*M)$ (resp. $A'_N = \Omega^N(TM)$) is the vector space of all N-forms on T^*M (resp. TM) over $\mathbb{R}$.

Let $Z^N(T^*M) = \operatorname{Ker}(d)$ (resp. $Z^N(TM) = \operatorname{Ker}(d)$) and $B^N(T^*M) = \operatorname{Im}(d)$ (resp. $B^N(TM) = \operatorname{Im}(d)$) denote respectively the real vector spaces of cocycles and coboundaries of degree N. Since $d_{N+1}d_N = d^2 = 0$, it follows that $B^N(T^*M) \subset Z^N(T^*M)$ (resp. $B^N(TM) \subset Z^N(TM)$). The quotient vector space

$$H^N_{DR}(T^*M) = \operatorname{Ker}(d)/\operatorname{Im}(d) = Z^N(T^*M)/B^N(T^*M)$$
$$\big(\text{resp.} \quad H^N_{DR}(TM) = \operatorname{Ker}(d)/\operatorname{Im}(d) = Z^N(TM)/B^N(TM)\big),$$

we refer to as the De Rham cohomology group (vector space) of humanoid dynamics on T^*M (resp. TM). The elements of $H^N_{DR}(T^*M)$ (resp. $H^N_{DR}(TM)$) are equivalence sets of cocycles. Two cocycles ω_1 and ω_2 are cohomologous, or belong to the same equivalence set (written $\omega_1 \sim \omega_2$) if and only if they differ by a coboundary $\omega_1 - \omega_2 = d\theta$. Any form $\omega_H \in \Omega^N(T^*M)$ (resp. $\omega_L \in \Omega^N(T^*M)$) has a De Rham cohomology class $[\omega_H] \in H^N_{DR}(T^*M)$ (resp. $[\omega_L] \in H^N_{DR}(TM)$) [3, 8].

The symplectic form ω_H on T^*M (resp. the Lagrangian form ω_L on TM) is by definition both a closed two-form or two-cocycle and an exact two-form or two-coboundary. Therefore the two-dimensional De Rham cohomology group of humanoid motion is defined as a quotient vector space

$$H^2_{DR}(T^*M) = Z^2(T^*M)/B^2(T^*M) \quad \big(\text{resp. } H^2_{DR}(TM) = Z^2(TM)/B^2(TM)\big).$$

As T^*M (resp. TM) is a compact symplectic (resp. Lagrangian) manifold of dimension $2N$, it follows that ω_H^N (resp. ω_L^N) is a volume element on T^*M (resp. TM) and the $2N$-dimensional De Rham cohomology class $[\omega_H^M] \in H^{2N}_{DR}(T^*M)$ (resp. $[\omega_L^N] \in H^{2N}_{DR}(TM)$) is nonzero. Since $[\omega_H^N] = [\omega_H]^N$ (resp. $[\omega_L^N] = [\omega_L]^N$), then $[\omega_H] \in H^2_{DR}(T^*M)$ (resp. $[\omega_H] \in H^2_{DR}(TM)$) and all of its powers up to the Nth must be zero as well [5]. The existence of such an element is a necessary condition for T^*M (resp. TM) to admit a symplectic structure $\omega_H = dp_i \wedge dq_i$ (resp. Lagrangian structure $\omega_L = dL_{v^i} \wedge dq^i$).

A De Rham complex A^*_{DR} on T^*M (resp. TM) can be considered as a system of second-order differential equations $d^2\theta_H = 0$, $\theta_H \in \Omega^N(T^*M)$ (resp. $d^2\theta_L = 0$, $\theta_L \in \Omega^N(TM)$) having a solution represented by $Z^N(T^*M)$ (resp. $Z^N(TM)$). In local coordinates $q^i, p_i \in U_p$ (U_p open in T^*M) (resp. $q^i, v^i \in U_v$ (U_v open in T^M)) we have $d^2\theta_H = d^2(p_i dq^i) = d(dp_i \wedge dq^i) = 0$, (resp. $d^2\theta_L = d^2(L_{v^i} dq^i) = d(dL_{v^i} \wedge dq^i) = 0$).

B. Homology If $\mathscr{C} = \mathscr{H}_\bullet\mathscr{M}$ (resp. $\mathscr{C} = \mathscr{L}_\bullet\mathscr{M}$) represents an Abelian category of chains on T^*M (resp. TM), we have a category $\mathscr{S}_\bullet(\mathscr{H}_\bullet\mathscr{M})$ (resp. $\mathscr{S}_\bullet(\mathscr{L}_\bullet\mathscr{M})$) of generalized chain complexes $\mathscr{A}_\bullet$ in $\mathscr{H}_\bullet\mathscr{M}$ (resp. $\mathscr{L}_\bullet\mathscr{M}$), and if $A = 0$ for $n < 0$ we have a sub-category $\mathscr{S}_\bullet^C(H_\bullet M)$ (resp. $\mathscr{S}_\bullet^C(L_\bullet M)$) of chain complexes in $\mathscr{H}_\bullet\mathscr{M}$ (resp. $\mathscr{L}_\bullet\mathscr{M}$)

$$A_\bullet : 0 \leftarrow C^0(T^*M) \xleftarrow{\partial} C^1(T^*M) \xleftarrow{\partial} C^2(T^*M) \xleftarrow{\partial} \cdots \xleftarrow{\partial} C^n(T^*M) \xleftarrow{\partial} \cdots$$
$$\left(\text{resp. } A_\bullet : 0 \leftarrow C^0(TM) \xleftarrow{\partial} C^1(TM) \xleftarrow{\partial} C^2(TM) \xleftarrow{\partial} \cdots \xleftarrow{\partial} C^n(TM) \xleftarrow{\partial} \cdots\right)$$

(see [3, 7]). Here $A_N = C^N(T^*M)$ (resp. $A_N = C^N(TM)$) is the vector space of all finite chains C on T^*M (resp. TM) over $\mathbb{R}$, and $\partial_N = \partial : C^{N+1}(T^*M) \to C^N(T^*M)$ (resp. $\partial_N = \partial : C^{N+1}(TM) \to C^N(TM)$). A finite chain C such that $\partial C = 0$ is an N-cycle. A finite chain C such that $C = \partial B$ is an N-boundary. Let $Z_N(T^*M) = Ker(\partial)$ (resp. $Z_N(TM) = Ker(\partial)$) and $B_N(T^*M) = Im(\partial)$ (resp. $B_N(TM) = Im(\partial)$) denote respectively real vector spaces of cycles and boundaries of degree N. Since $\partial_{N-1}\partial_N = \partial^2 = 0$, then $B_N(T^*M) \subset Z_N(T^*M)$ (resp. $B_N(TM) \subset Z_N(TM)$). The quotient vector space

$$H_N^C(T^*M) = Z_N(T^*M)/B_N(T^*M) \quad \left(\text{resp. } H_N^C(TM) = Z_N(TM)/B_N(TM)\right)$$

represents an N-dimensional homology group (vector space) of humanoid dynamics. The elements of $H_N^C(T^*M)$ (resp. $H_N^C(TM)$) are equivalence sets of cycles. Two cycles C_1 and C_2 are homologous, or belong to the same equivalence set (written $C_1 \sim C_2$), if and only if they differ by a boundary $C_1 - C_2 = \partial B$. The homology class of a finite chain $C \in C^N(T^*M)$ (resp. $C \in C^N(TM)$) is $[C] \in H_N^C(T^*M)$ (resp. $[C] \in H_N^C(TM)$).

In case of an N-torus ($M = Q^N = T^N$), the Betti numbers [3, 7] of the humanoid motion are given by

$$b^0 = 1,\ b^1 = N,\ \ldots,\ b^p = \binom{N}{p},\ \ldots,\ b^{N-1} = N,\ b^N = 1 \quad (0 \le p \le N). \tag{3.1}$$

From the homotopy axiom for De Rham cohomologies [3], it follows that $H_{DR}^\bullet(M) \approx H_{DR}^\bullet(TM) \approx H_{DR}^\bullet(T^*M)$. Also from the De Rham theorem it follows that $H_{DR}^\bullet(X) = H_\bullet(X)$ for any smooth manifold X. Therefore $b^N = b_N$ are given by (3.1) for all three manifolds $X = T^N, TT^N, T^*T^N$ of the humanoid dynamics.

Thus we conclude that both the De Rham N-cohomology groups $H_{DR}^N(T^*M)$ (resp. $H_{DR}^N(TM)$) and the N-homology groups $H_N^C(T^*M)$ (resp. $H_N^C(TM)$) of Hamiltonian (resp. Lagrangian) formalism give in essence the same description of humanoid dynamics. This proves the existence of a topological functor isomorphism, that is, a natural equivalence $\mathtt{Dual}_T : \mathtt{Lag}_T \cong \mathtt{Ham}_{T^*}$, between the Lagrangian and the Hamiltonian dynamics on the humanoid configuration manifold $M = Q^N$ (Figure 1).

4. Conclusion

In this article the four invariant topological structures founded on the configuration manifold of the humanoid robot dynamics are analyzed. Both the cohomology and the homology groups, complexes and categories are established and examined on the tangent bundle (humanoid velocity-phase-space) as well as on the cotangent bundle (humanoid momentum-phase-space). It is proved that all four analyzed topological structures give in essence equivalent mathematical description of the humanoid dynamics. In other words, there is a topological natural equivalence or functor isomorphism $\mathtt{Dual}_T : \mathtt{Lag}_T \cong \mathtt{Ham}_{T^*}$, between Lagrangian and Hamiltonian formalisms as pictured in Figure 1. Whichever of the four approaches to humanoid dynamics we employ, the system we obtain ultimately will therefore be essentially the same.

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