

A Note on Passivity Based Stability Conditions for Bilateral Teleoperation [★]

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Abstract: In this paper, passivity based stability conditions that are widely used in bilateral teleoperation context, are derived via Integral Quadratic Constraints (IQCs). It is shown that the results can be carried over to robust control setting in a lossless fashion. With slight modifications, many important results in the particular engineering domains can be collected under this framework.

Keywords: Bilateral Teleoperation, Integral Quadratic Constraints, Unconditional Stability

1. INTRODUCTION

In the last two decades, passivity based approaches for stability analysis dominated the teleoperation literature and many variations around this central theme have been proposed (See Hokayem and Spong (2006) for a detailed survey and Hashtrudi-Zaad and Salcudean (2001) for passivity based controller architecture analysis). Starting from the celebrated stability theorem of interconnected passive systems (Llewellyn (1952)), the 2-port network interpretation of bilateral teleoperation systems became the standard of the field. In particular, contemporary frequency domain analysis of these systems almost always relies on the Llewellyn's conditions (or unconditional stability, absolute stability etc.) theorem. Equivalently, after performing a loop transformation such as scattering transformations or wave variables (e.g. Anderson and Spong (1989); Niemeyer and Slotine (1991); Desoer and Vidyasagar (1975)) Structured Singular Value (SSV) arguments are used to assess stability.

This paper serves to present the stability analysis of bilateral teleoperation systems via IQC framework and to provide a link to the classical results. By this, we would like to point out that seemingly different frequency domain approaches can be put into one general framework and more importantly, the problem can be cast in a pure system theoretical point of view. Due to the flexibility of the framework, the proofs are, unlike their original sources, rather straightforward and easy to generalize.

Besides the two classical results, we provide one simple exercise for the 3-port version of unconditional stability theorem for which there is no obvious analog from the corresponding engineering domains. Hence, this paper can also be viewed as providing simplifications of the proofs of the stability analysis tests in network theory, although we will not attempt to include a detailed discussion of the relevance. Finally, by the use of the Kalman-Yakubovich-

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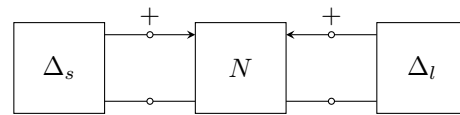


Fig. 1. A 2-port network.

Popov (KYP) Lemma, the analysis problem can be converted to an Linear Matrix Inequality (LMI) problem.

The notation is standard. For a hermitian matrix M , $M > 0$ ($M < 0$) denotes positive (negative) definiteness. Positive (negative) semi-definiteness is also denoted by $M \succeq 0$ ($M \preceq 0$). The set of real rational, proper and stable functions are denoted by \mathcal{RH}_∞ . Consequently $\mathcal{RH}_\infty^{\bullet \times \bullet}$ symbol denotes the real rational proper and stable transfer matrices, where the size of the matrix does not play a role in the discussion. For a signal $f(t)$ the hat denotes the Fourier transform i.e. $\hat{f}(i\omega)$. $\mathcal{L}_2^{\mathbb{R}}$ denotes the space of real-vector valued square summable functions. The norm on $\mathcal{L}_2^{\mathbb{R}}$ is defined as $\|f\| = \sqrt{\int_0^\infty f(t)^T f(t) dt}$. $\|G\|_\infty$ denotes the infinity norm for the linear system G .

2. PRELIMINARIES

The ubiquitous passivity framework for the analysis of a 2-port network is shown in Figure 1 where the Δ_s, Δ_l are passive LTI operators, representing the load and the source immittances. In this framework, the goal is to investigate the properties of the network N to achieve interconnection stability. In relation to teleoperation, the Δ blocks refer to the human operator and the unknown environment.

In general, we consider system interconnections as depicted in Figure 3 instead of the 2-port network setting. In this terminology, G models the nominal teleoperation system and Δ is a block diagonal concatenation of uncertainties e.g. human, environment, delays etc. Arriving to such an interconnection from a given immittance representation is straightforward by signal manipulations.

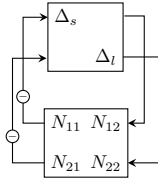


Fig. 2. Uncertain plant representation of a 2-port network.

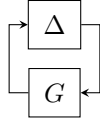


Fig. 3. The general interconnection

For example, admittance type of representations of 2-port networks are, in general, compatible with such $G - \Delta$ interconnections. The existence of such realizations are guaranteed for nominally stable systems. One can also use other types of immittance representations, but the resulting G often does not admit a proper transfer function which will be required later in this paper.

Definition 2.1. For a $G - \Delta$ interconnection where G, Δ are stable causal LTI operators with compatible dimensions, the interconnection is said to be *well-posed* if $(I - G\Delta)$ has a proper inverse. Moreover, the interconnection is said to be *stable* if it is well posed and $(I - G\Delta)$ has a stable inverse.

Definition 2.2. Let a bounded, causal operator $\Delta : \mathcal{L}_2^m[0, \infty) \rightarrow \mathcal{L}_2^n[0, \infty)$ be given. Then Δ is said to satisfy the Integral Quadratic Constraint defined by $\Pi(i\omega)$ if (1) holds for all $v \in \mathcal{L}_2^m$.

$$\int_{-\infty}^{\infty} \begin{pmatrix} \widehat{\Delta(v)}(i\omega) \\ \widehat{v}(i\omega) \end{pmatrix}^* \Pi(\omega) \begin{pmatrix} \widehat{\Delta(v)}(i\omega) \\ \widehat{v}(i\omega) \end{pmatrix} d\omega \succeq 0 \quad (1)$$

Many classical results from multiplier theory such as Circle-, Popov- and Zames-Falb criteria (with a brief historical account given in Megretski and Rantzer (1997)) have been incorporated in this framework.

Theorem 1. (Megretski and Rantzer (1997)). Let the system model $G \in \mathcal{RH}_\infty^{n \times m}$ be given and let $\Delta : \mathcal{L}_2^m \rightarrow \mathcal{L}_2^n$ be a bounded causal operator. Suppose that

- (1) for every $\tau \in [0, 1]$, the interconnection of G and $\tau\Delta$ is well-posed;
- (2) for every $\tau \in [0, 1]$, $\tau\Delta$ satisfies the IQC defined by $\Pi(i\omega)$ which is bounded as a function of $\omega \in \mathbb{R}$;
- (3) there exists some $\epsilon > 0$ such that

$$\begin{pmatrix} I \\ G(i\omega) \end{pmatrix}^* \Pi(i\omega) \begin{pmatrix} I \\ G(i\omega) \end{pmatrix} \preceq -\epsilon I \text{ for all } \omega \in \mathbb{R}. \quad (2)$$

Then the $G - \Delta$ interconnection in Figure 3 is stable.

Hence, if we are given with an IQC constraint that is satisfied by the uncertainty set of interest, the robust stability guarantee in the face of this particular uncertainty set boils down to the feasibility of (2). As quite frequently used in control theory, one can use the Kalman-Yakubovich-Popov (KYP) Lemma (See Rantzer (1996)) to convert these frequency dependent inequalities into Linear Matrix Inequalities (LMIs) by using the state space representations.

We use a simplified version of the general IQC theorem due to the passive LTI assumption of the involving operators:

Theorem 2. Assume that $G, \Delta \in \mathcal{RH}_\infty^{\bullet \times \bullet}$. Under these assumptions, the $G - \Delta$ interconnection is well-posed and stable if there exist a Hermitian matrix Π such that

$$\begin{pmatrix} \Delta(i\omega) \\ I \end{pmatrix}^* \Pi \begin{pmatrix} \Delta(i\omega) \\ I \end{pmatrix} \succeq 0 \quad (3)$$

$$\begin{pmatrix} I \\ G(i\omega) \end{pmatrix}^* \Pi \begin{pmatrix} I \\ G(i\omega) \end{pmatrix} \prec 0 \quad (4)$$

hold for all $\omega \in \mathbb{R} \cup \{\infty\}$.

In particular, we recover the small-gain and passivity (under negative feedback sign) theorems respectively, if one replaces the constant symmetric matrix $\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$ or $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ as Π multiplier e.g.

$$\begin{aligned} \text{Re}\{\Delta(i\omega)\} \geq 0 \\ \text{Re}\{G(i\omega)\} > 0 \end{aligned} \iff \begin{aligned} \begin{pmatrix} \Delta(i\omega) \\ 1 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta(i\omega) \\ 1 \end{pmatrix} \geq 0 \\ \begin{pmatrix} 1 \\ -G(i\omega) \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -G(i\omega) \end{pmatrix} < 0 \end{aligned} \quad (5)$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$.

3. EQUIVALENT IQC STABILITY TESTS FOR COMMON STABILITY APPROACHES

In this section, we give the equivalent tests for the widely used results from network theory. It is shown that frequently used stability tests in bilateral teleoperation context, can be seen as particular cases of robustness tests. The conditions follow from a straightforward application of the IQC theorem and the emphasis is on the resulting conditions and to the best of our knowledge, has not been stated elsewhere. Our main motivation is explicitly show the underlying basic connection to the systems theory and moreover to demonstrate that the analysis can be done without confining the problem into a particular network theoretical setting.

We point out the fact that in the following enumerated cases, we only prove the sufficiency of the results, and leave the necessity direction to a later section where we give some brief remarks about the exactness of the tests.

3.1 Llewellyn Stability Criteria

The well known conditions for stability of a two-port network, formulated in Llewellyn (1952); Bolinder (1957); Rollett (1962), are recalled in the following theorem. An explicit indication of the frequency dependence is omitted for notational convenience.

Theorem 3. [Llewellyn's 2-port Stability Theorem] A 2-port network, as depicted in Figure 1, described by its immittance matrix

$$N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}$$

and interconnected to passive termination immittances, is stable if and only if

$$R_{11} > 0 \text{ or } R_{22} > 0, \quad (6)$$

and

$$4(R_{11}R_{22} + X_{12}X_{21})(R_{11}R_{22} - R_{12}R_{21}) - (R_{12}X_{21} - R_{21}X_{12})^2 > 0 \quad (7)$$

or

$$2R_{11}R_{22} - |N_{12}N_{21}| - \Re N_{12}N_{21} > 0 \quad (7')$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$, where R_{ij} and X_{ij} denotes the real and imaginary parts of the corresponding block of N_{ij} respectively.

As shown in Rollett (1962), the conditions stated in this theorem are invariant under immittance substitution. Hence, without loss of generality, we assume that the network and the terminations are setup in accordance with a $G - \Delta$ interconnection as in Figure 2. Otherwise, one can reshuffle the signals by elementary signal manipulations. This particular choice greatly simplifies the presentation.

The stability conditions of Theorem 3 can be reproduced via the IQC theorem as follows. If Δ_l and Δ_s are passive and stable LTI systems, they satisfy

$$\Delta_l + \Delta_l^* \geq 0 \quad \text{and} \quad \Delta_s + \Delta_s^* \geq 0$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$. If we choose arbitrary $\lambda_1(\omega) > 0$ and $\lambda_2(\omega) > 0$, it is clear that the inequalities

$$\begin{aligned} \lambda_1(\Delta_l + \Delta_l^*) &\geq 0 \\ \lambda_2(\Delta_s + \Delta_s^*) &\geq 0 \end{aligned}$$

persist to hold, which in turn can be combined into

$$\begin{pmatrix} \Delta_l & 0 \\ 0 & \Delta_s \\ 1 & 0 \\ 0 & 1 \end{pmatrix}^* \begin{pmatrix} 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Delta_l & 0 \\ 0 & \Delta_s \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \succeq 0.$$

After division by $\lambda_2(\omega)$ and with $\lambda(\omega) = \frac{\lambda_1(\omega)}{\lambda_2(\omega)}$ we obtain

$$\begin{pmatrix} \Delta_l & 0 \\ 0 & \Delta_s \\ 1 & 0 \\ 0 & 1 \end{pmatrix}^* \begin{pmatrix} 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \\ \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Delta_l & 0 \\ 0 & \Delta_s \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \succeq 0. \quad (8)$$

Therefore, we have constructed a whole family of multipliers, parameterized by $\lambda(\omega) > 0$, such that the quadratic constraint (8) holds for all passive $\Delta_l, \Delta_s \in \mathcal{RH}_\infty$. Stability of the $G - \Delta$ interconnection is then guaranteed if one can find a positive $\lambda(\omega)$ such that the corresponding frequency domain inequality (FDI) is also satisfied at each frequency. Without indicating frequency dependency, this FDI reads as

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -N_{11} & -N_{12} \\ -N_{21} & -N_{22} \end{pmatrix}^* \begin{pmatrix} 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \\ \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -N_{11} & -N_{12} \\ -N_{21} & -N_{22} \end{pmatrix} \prec 0. \quad (9)$$

where the negation of N is due to the negative feedback sign that is required for passive interconnections. Therefore, the resulting condition is reduced to checking whether there exists a $\lambda > 0$ such that

$$H = \begin{pmatrix} -2\lambda R_{11} & -\lambda N_{12} - N_{21}^* \\ -\lambda N_{12}^* - N_{21} & -2R_{22} \end{pmatrix} \prec 0$$

holds at each frequency. This leads us to the relation with the classical results. Indeed, the 2×2 matrix H is negative definite if and only if

$$R_{11} > 0 \quad \text{or} \quad R_{22} > 0 \quad (10)$$

and

$$\begin{aligned} \det H &= (-R_{12}^2 - X_{12}^2)\lambda^2 - R_{21}^2 - X_{21}^2 \\ &+ (4R_{11}R_{22} - 2R_{12}R_{21} + 2X_{12}X_{21})\lambda > 0 \end{aligned} \quad (11)$$

Since the coefficient of the quadratic term of this polynomial is negative, the apex of this parabola should stay above the λ -axis for some $\lambda > 0$ for all frequencies. Using the apex coordinates of a concave parabola one can show that this is equivalent to (7). Symmetry of the resulting conditions with respect to the indices is shown by simply switching the roles of λ_1 and λ_2 in our derivation.

Remark 4. In the previous FDI condition (9), if one assumes $\lambda = 1$ over all frequencies, we also recover the strict version of Raisbeck's conditions (Raisbeck (1954)). Comparison of Raisbeck's and Llewellyn's criteria shows that the use of frequency dependent multipliers demonstrate the possibility of a substantial decrease of conservatism in the stability analysis. In fact, the difference between Llewellyn's remarkable conditions and Raisbeck's conditions is a dynamic multiplier instead of the static $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Remark 5. One should also note that Llewellyn's conditions are both sufficient and necessary, hence there is no conservatism regarding the stability test. The conservatism that is often associated with this test is due to the assumptions that are made on the properties of human and environment. Thus, if one wishes to reduce the conservatism, additional structural information about the operators should be included.

Remark 6. If we assume further that the uncertainties are passive but not necessarily LTI, then we resort to a static (constant over all frequencies) λ and sufficient conditions follow. Nevertheless it is less conservative than the classical passivity theorem with $\lambda = 1$.

3.2 Unconditional Stability Analysis of 3-port Networks

Recently, in Khademian and Hashtrudi-Zaad (2010), the Llewellyn's analysis method is applied to three port networks. The main idea is to obtain a 2-port network by terminating one of the ports with a known environment model and then applying the test on the resulting particular 2-port network. We obtain the exact conditions for the unconditional stability of a 3-port in a similar fashion from the previous case without a port termination. The practicality of this test is limited and it is a demonstration of the simplicity to obtain stability conditions for the scenarios for which there are no obvious circuit theoretical analogies. As we should expect, this test is more conservative than of Khademian and Hashtrudi-Zaad (2010) since they include additional information about the model of the port that is terminated, therefore the uncertainty set modeled by the problem is significantly smaller. This is exactly in accordance with our claim that we have to include more information about the uncertainty sets to deal with robustness with reduced conservatism. Nevertheless, it might be of use for certain network analysis problems where stability of the network is the only concern.

The only modification needed here compared to the previous case, is to take a system description $N \in \mathcal{RH}_\infty^{3 \times 3}$ and three passive uncertainty blocks living in \mathcal{RH}_∞ . Then, from the following quadratic constraint which reflects the passivity requirement of these blocks

$$\begin{pmatrix} \Delta \\ I \end{pmatrix}^* \begin{pmatrix} 0 & \Lambda \\ \Lambda & 0 \end{pmatrix} \begin{pmatrix} \Delta \\ I \end{pmatrix} \succeq 0 \quad (12)$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$ where we omit the explicit frequency dependence and also we use the abbreviations

$$\Delta = \text{blkdiag}\{\Delta_1(i\omega), \Delta_2(i\omega), \Delta_3(i\omega)\} \quad (13)$$

$$\Lambda = \text{blkdiag}\{\lambda_1(\omega), \lambda_2(\omega), \lambda_3(\omega)\} \quad (14)$$

Then, we obtain the corresponding FDI as follows:

$$\begin{pmatrix} I \\ -N \end{pmatrix}^* \begin{pmatrix} 0 & \Lambda \\ \Lambda & 0 \end{pmatrix} \begin{pmatrix} I \\ -N \end{pmatrix} \prec 0 \quad (15)$$

Hence, we obtain the 3-port unconditional stability conditions as follows

Theorem 7. [Llewellyn's 3-port Stability Theorem] A 3-port network described by its transfer function $N \in \mathcal{RH}_\infty^{3 \times 3}$ and interconnected to the passive and block diagonal Δ as given in (13) is stable if and only if, there exists a $\Lambda \succ 0$ such that (15) holds for all $\omega \in \mathbb{R} \cup \{\infty\}$.

Now, if one studies this negativity condition in detail, the exact conditions for unconditional stability can be obtained from the negative definiteness of a 3×3 matrix by a symbolic computation. Consequently, if one wishes to obtain formulas similar to (6),(7), the resulting conditions are quite tedious. Moreover, the variation in presenting the resulting conditions would be again due to the different representations of this negativity condition e.g. symbolic eigenvalue computation, the sign definiteness of the cofactors etc. whereas in the IQC formulation this is completely avoided and via KYP lemma, it can be numerically checked via LMIs in an exact fashion, without frequency gridding. This represents the formulation power and simplicity, at the same time, provides an example of the theoretical restrictions imposed by confining the problem into one particular engineering domain.

3.3 Rollett's Stability Conditions

As we derived the Llewellyn's stability conditions, it is also similarly straightforward to derive unconditional stability when the network is represented with the scattering parameters. The corresponding interconnection is given by

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \Delta_l & 0 \\ 0 & \Delta_s \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \quad (16)$$

Rollett's conditions¹ (Rollett (1962); Kurokawa (1965)) for stability formulated as follows: At every frequency, the inequality

$$K = \frac{1 + |\nabla|^2 - |S_{11}|^2 - |S_{22}|^2}{2|S_{12}S_{21}|} > 1 \quad (17)$$

holds together with an auxiliary condition that can be stated in at least five different ways e.g.

$$1 - |S_{11}|^2 > |S_{12}S_{21}|, \quad (1 - |S_{22}|^2) > |S_{12}S_{21}|$$

(Woods (1976)), and where $\nabla = S_{11}S_{22} - S_{12}S_{21}$. The variable ∇ is used to avoid confusion with our general uncertainty variable Δ . With almost identical arguments, one derives the following quadratic constraints for stable LTI systems Δ_l and Δ_s whose gain is bounded by one (while omitting the frequency dependence):

$$\begin{pmatrix} \Delta_l & 0 \\ 0 & \Delta_s \end{pmatrix}^* \begin{pmatrix} -\lambda & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \Delta_l & 0 \\ 0 & \Delta_s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \succeq 0 \quad (18)$$

¹ Interestingly, it is just Llewellyn's conditions in scattering coordinates.

via the inequalities,

$$\lambda_1(\omega)(-\Delta_l(i\omega)^* \Delta_l(i\omega) + 1) \geq 0$$

$$\lambda_2(\omega)(-\Delta_s(i\omega)^* \Delta_s(i\omega) + 1) \geq 0$$

Then, from the FDI, stability is assured if one can find a positive frequency dependent λ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}^* \begin{pmatrix} -\lambda & 0 \\ 0 & -1 \\ \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \prec 0 \quad (19)$$

or, equivalently,

$$H = \begin{pmatrix} |S_{21}|^2 + \lambda(|S_{11}|^2 - 1) & S_{22}S_{21}^* + \lambda S_{12}S_{11}^* \\ S_{22}^*S_{21} + \lambda S_{12}^*S_{11} & |S_{22}|^2 - 1 + \lambda|S_{12}|^2 \end{pmatrix} \prec 0$$

holds for all frequencies. Then, it is elementary to show the negative definiteness of H , say by showing $\det(H) > 0$ and any of the diagonal entries of H being negative for all $\omega \in \mathbb{R} \cup \{\infty\}$. Positivity of the determinant of H means

$$(1 - |S_{22}|^2 - |S_{11}|^2 + |\nabla|^2)\lambda - |S_{12}|^2\lambda^2 - |S_{21}|^2 > 0 \quad (20)$$

Using the shorthand notation, $f(\lambda) = -a\lambda^2 + b\lambda - c > 0$ with $a, c > 0$, for (20), we require the apex coordinates $\left(\frac{b}{2a}, \frac{b^2 - 4ac}{4a}\right)$ both be positive. Since $a > 0$, we have

$$(1 + |\nabla|^2 - |S_{22}|^2 - |S_{11}|^2)^2 > 4|S_{12}S_{21}|^2 \quad (21)$$

from $b^2 > 4ac$. Moreover, for the constraint $b > 0$, negativity of the diagonal terms leads to

$$\lambda(1 - |S_{11}|^2) > |S_{21}|^2 \quad \text{or} \quad 1 - |S_{22}|^2 > \lambda|S_{12}|^2 \quad (22)$$

To make the connection to the classical auxiliary conditions, observe that evaluating $f(\lambda)$ at $\lambda_0 = \sqrt{\frac{c}{a}} = \frac{|S_{21}|}{|S_{12}|}$ would lead to the condition $b > 0$ since $f(\lambda_0) = b\sqrt{\frac{c}{a}} - 2c > 0$, hence (22) becomes

$$1 - |S_{11}|^2 > |S_{12}S_{21}| \quad \text{or} \quad 1 - |S_{22}|^2 > |S_{12}S_{21}| \quad (23)$$

In the literature, λ_0 is denoted by "maximum stable power gain". Finally, now that we have included the condition $b > 0$ explicitly, one can take the square root of (21) and obtain

$$1 + |\nabla|^2 - |S_{22}|^2 - |S_{11}|^2 > 2|S_{12}S_{21}| \quad (24)$$

which is precisely Rollett's first condition.

This test formulated in terms of existence of a $\lambda > 0$ also recovers the stability parameter μ of Edwards and Sinsky (1992) in the sense that we have only one condition to be checked for the stability test. Recently, the μ parameter has been used in the context of teleoperation in Haddadi and Hashtrudi-Zaad (2009) and their results can also be obtained similarly by modifying the multiplier for output strictly passive uncertainties.

Remark 8. We can derive these results from the passivity point of view. This requires only to perform the congruence transformation

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2b}} & (1-b) \\ \frac{1}{\sqrt{2b}} & b \end{pmatrix}^T \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2b}} & (1-b) \\ \frac{1}{\sqrt{2b}} & b \end{pmatrix} \quad (25)$$

with the usual scattering transformation matrix, which turns the multiplier for small-gain conditions into the one imposing passivity constraints. One can see that plugging (25) into (5) provides the simplest relation between two interpretations. By this relation, we simply reemphasize that the scattering transformations (Anderson and Spong

(1989)) and wave variable (Niemeyer and Slotine (1991, 2004)) methods utilize the equivalence of stability implications of the small gain and passivity theorems under certain loop transformations (Desoer and Vidyasagar, 1975, Sec. VI.10). Therefore, loop transformations are not essential to analyze the stability of teleoperation problems since we can either use (8) or (18) with scattering transformation to characterize the passive LTI uncertainties.

For example, one can further perform delay robustness tests without any scattering transformation since the stability test does not rely on solely the passivity theorem. In fact, the uncertainties are characterized by their own quadratic constraints. Hence, no “passification” step is required. This approach has been used in (Polat (2011a,b)) for uncertain delays and time-varying parametric uncertainty scenarios.

There has been a discussion in various studies (e.g. Lombardi and Neri (1999); Edwards and Sinsky (1992); Woods (1976)) whether testing both or one of the conditions in (23) is sufficient. Note that, it just rolls out from our FDI condition that we need only one of these auxiliary conditions. In fact, (19) renders this discussion obsolete since we deal with a single matrix inequality to be tested at each frequency. Even further, we obtain a one-shot solution from the corresponding LMI problem.

Avoiding the Frequency Gridding: Either the existence of frequency dependent multipliers, say λ in the previous cases, or the classical conditions in their original form, one has to check sufficiently large number of points on the frequency axis. Even this might not directly guarantee stability since the decision is based on the gridding resolution. Instead, we can use the well-established semi-definite programming algorithms (Boyd et al. (1994)), by introducing suitable parameterizations of the frequency-dependent multiplier λ and using state-space descriptions. Then, KYP lemma allows to convert the FDI into one feasibility test of semi-definite programming without the need for any frequency-gridding. This is the line of reasoning which allows to handle substantially more complicated uncertainty structures and, to a certain extent, even permits to perform optimal controller synthesis. Due to space constraints, we only present the resulting LMI conditions for the Rollett test, since the details can be found elsewhere e.g. Megretski and Rantzer (1997).

Assume a multiplier $\lambda(\omega) \in \mathbb{R}$ factorized as

$$\lambda(\omega) = \Phi(i\omega)^* D \Phi(i\omega)$$

such that $D \in \mathbb{R}^{n \times n}$ is a symmetric matrix and

$$\Phi(s) = \left(1 \quad \frac{1}{s+a} \quad \frac{1}{(s+a)^2} \quad \cdots \quad \frac{1}{(s+a)^{n-1}} \right)^T$$

for some fixed $a > 0$ and sufficiently large $n \in \mathbb{Z}_+$. Then one can rewrite (19) as

$$\begin{pmatrix} I \\ S \end{pmatrix}^* \begin{pmatrix} \Phi \\ 1 \end{pmatrix}^* \underbrace{\begin{pmatrix} -D & 0 \\ 0 & -1 \\ \hline D & 0 \\ 0 & 1 \end{pmatrix}}_{\mathcal{P}} \begin{pmatrix} \Phi \\ 1 \end{pmatrix} \begin{pmatrix} I \\ S \end{pmatrix} \prec 0 \quad (26)$$

hence, by denoting $\Psi = \begin{pmatrix} \Phi \\ 1 \end{pmatrix}$, (19) becomes

$$\begin{pmatrix} \Psi \\ \Psi S \end{pmatrix}^* \underbrace{\begin{pmatrix} -D & 0 \\ 0 & -1 \\ \hline D & 0 \\ 0 & 1 \end{pmatrix}}_{\mathcal{P}} \begin{pmatrix} \Psi \\ \Psi S \end{pmatrix} \prec 0 \quad (27)$$

Define the following minimal state space realizations

$$\Phi(s) = \begin{bmatrix} A_\Phi & B_\Phi \\ C_\Phi & D_\Phi \end{bmatrix}, \quad \begin{pmatrix} \Psi \\ \Psi S \end{pmatrix}(s) = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$$

then via KYP Lemma, existence of a symmetric matrix D such that $\Phi(i\omega)^* D \Phi(i\omega) > 0$ and (19) holds for all $\omega \in \mathbb{R} \cup \{\infty\}$ is equivalent to the existence of symmetric matrices D, Z and \mathcal{X} such that following LMIs hold

$$\begin{pmatrix} I & 0 \\ \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}^T \begin{pmatrix} 0 & \mathcal{X} & 0 \\ \mathcal{X} & 0 & 0 \\ 0 & 0 & \mathcal{P} \end{pmatrix} \begin{pmatrix} I & 0 \\ \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \prec 0 \quad (28)$$

$$\begin{pmatrix} I & 0 \\ A_\Phi & B_\Phi \\ C_\Phi & D_\Phi \end{pmatrix}^T \begin{pmatrix} 0 & Z & 0 \\ Z & 0 & 0 \\ 0 & 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ A_\Phi & B_\Phi \\ C_\Phi & D_\Phi \end{pmatrix} \succ 0 \quad (29)$$

By slightly changing the multiplier \mathcal{P} , Llewellyn’s conditions can be obtained through identical steps.

3.4 Exactness of Robustness Tests

As we have mentioned before, the IQC tests presented in these cases only prove sufficiency of the stability conditions. Still, in all these cases necessity can be seen to be a specialization of a celebrated exactness result in structured singular-value theory as discussed in Packard and Doyle (1993); Scherer (2005). It is far beyond the scope of this paper to give the all possible cases where the robustness tests are exact, but for a particular class of the LTI uncertainty cases (and for a very limited cases of time varying real parametric uncertainties), it is well-known that the structured robust stability tests can be shown to be exact. Nevertheless, for our purposes, it is sufficient to mention that all cases presented above are in accordance with the uncertainty types for which the tests are exact (e.g. (Packard and Doyle, 1993, Table 1)). In particular, 3-port counterpart of the Llewellyn’s conditions are indeed sufficient and necessary. Another interesting important case is when the designer defines a performance channel and tests robust performance against two LTI passive uncertainty blocks. This problem also satisfies the exactness conditions and gives a direct opportunity to test different scenarios with respect to a performance index. For a detailed discussion, we refer to Packard and Doyle (1993); Fan et al. (1988); Scherer (2005).

4. CONCLUSIONS

In summary, applying IQC theory to the standard configurations as considered in a 2-port network analogy, allows to recover the existing stability results. On the other hand, the IQC framework offers interesting possibilities for generalizations that are worthwhile to be investigated in the context of stability analysis and controller synthesis for teleoperation systems. We have emphasized the relation between network stability results and techniques from multiplier theory, in order to substantiate that, network stability theorems can be viewed as specializations of

those obtained with the IQC framework of robust control. Although it is certainly not our intention to question the existing engineering insights and the related physical intuition brought in by network theory, microwave theory etc., it might be worthwhile to pursue the somewhat more abstract IQC approach for applications in bilateral teleoperation context.

This opens up the way either to apply existing or to develop new IQC-based multiplier results for handling nonlinearities, delays or sampling effects in order to address the special complications in teleoperation. As a particular strength of this framework, one can effectively include structural information about the involved uncertainty that goes beyond passivity. Moreover it is rather straightforward to include performance specifications as well. One could then model the human operator and the environment in a somewhat more refined fashion and directly analyze realistic performance specifications while still guaranteeing stability.

We strongly believe that this allows to investigate various possibilities for tuning haptic controllers that go beyond the manually tuned PID controllers in order to achieve optimal performance (and thus improving the quality of the teleoperator) without sacrificing stability, instead of solely relying on the mechanical properties of the devices.

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