

Asymptotic regime for impropriety tests of complex random vectors

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Abstract—Impropriety testing for complex-valued vectors has been considered lately due to potential applications ranging from digital communications to complex media imaging. This paper provides new results for such tests in the asymptotic regime, *i.e.* when the vector dimension and sample size grow commensurately to infinity. The studied tests are based on invariant statistics named *impropriety coefficients*. Limiting distributions for these statistics are derived, together with those of the Generalized Likelihood Ratio Test (GLRT) and Roy’s test, in the Gaussian case. This characterization in the asymptotic regime allows also to identify a phase transition in Roy’s test with potential application in detection of complex-valued low-rank subspace corrupted by proper noise in large datasets. Simulations illustrate the accuracy of the proposed asymptotic approximations.

I. INTRODUCTION

Testing for properness consists of deciding whether N -dimensional complex random vector $\mathbf{z} \in \mathbb{C}^N$ is *proper* or *improper*. Let us recall that a complex vector \mathbf{z} is called *circular* when \mathbf{z} is equal in distribution to $e^{i\theta}\mathbf{z}$ (*i.e.* the distribution of \mathbf{z} is invariant by a rotation of angle θ in the complex domain [1], [2]). Circularity for Gaussian complex vectors is actually called *properness*. The problem of testing whether a complex Gaussian vector is *proper* or not has several potential applications in signal processing and has been considered by several authors, using different means, including Generalized Likelihood Ratio Tests (GLRT) [3], [4], locally most powerful (LMP) test [5, Chapter 3] or frequency domain tests [6]. The asymptotic behavior of GLRT was studied for large sample sizes in the case of random variables ($N = 1$) [7] or small/fixed values of N [8]. The situation where both the dimension of the complex vector and the size of the sample tend to infinity was not considered until our recent preliminary study [9]. This article formalizes and extends [9], therefore filling a gap and provides insight into the asymptotic behavior of *impropriety* test when the vector dimension and sample size grow commensurately to infinity. Practical applications of this asymptotic regime occur for example in communications when a large number of dense arrays are deployed, *i.e.* for massive MIMO [10] and cognitive radio [11], as well as in fMRI [12] or phase retrieval and imaging in complex media [13].

Central to many of the tests available in the literature, the set of *invariant parameters* was first considered in [14], allowing for the derivation of *invariant statistics* used in [8]. Invariant parameters are in one-to-one correspondence with *canonical*

correlation coefficients [4] as explained in [5], and referred to as *impropriety coefficients* [15].

In this article, we consider N -dimensional complex-valued centered random vectors with Cartesian form such as $\mathbf{z} = \mathbf{u} + i\mathbf{v}$, *i.e.* \mathbf{u} and \mathbf{v} are N -dimensional real vectors with zero mean, *i.e.* $\mathbf{u} \in \mathbb{R}^N$, $\mathbf{v} \in \mathbb{R}^N$, and $\mathbb{E}[\mathbf{u}] = \mathbb{E}[\mathbf{v}] = \mathbf{0}$. Two *augmented representations* are classically used in the literature to study complex vectors $\mathbf{z} \in \mathbb{C}^N$, namely the *real augmented representation* \mathbf{x} and the *complex augmented representation* $\tilde{\mathbf{z}}$. The former consists of representing $\mathbf{z} \in \mathbb{C}^N$ by a twice larger real-valued vector $\mathbf{x} = [\mathbf{u}^T, \mathbf{v}^T]^T \in \mathbb{R}^{2N}$ made up from the real and imaginary parts of \mathbf{z} , while the latter consists of using a twice larger complex-valued vector $\tilde{\mathbf{z}} = [\mathbf{z}^T, \mathbf{z}^{*T}]^T \in \mathbb{C}^{2N}$ containing \mathbf{z} and its conjugate \mathbf{z}^* . Both representations are equivalent and easily connected using linear mappings given for example in [5]. In this paper, we make use of the real representation $\mathbf{x} \in \mathbb{R}^{2N}$.

Recalling that $\mathbb{E}[\mathbf{z}] = \mathbf{0}$, and thus that $\mathbb{E}[\mathbf{x}] = \mathbf{0}$, second order statistics of $\mathbf{z} \in \mathbb{C}^N$ are contained in the real-valued covariance matrix $\mathbf{C} \in \mathbb{R}^{2N \times 2N}$ of the real representation vector \mathbf{x} , which reads:

$$\mathbb{E}[\mathbf{x}\mathbf{x}^T] = \mathbf{C} = \begin{pmatrix} \mathbf{C}_{\mathbf{u}\mathbf{u}} & \mathbf{C}_{\mathbf{u}\mathbf{v}} \\ \mathbf{C}_{\mathbf{v}\mathbf{u}} & \mathbf{C}_{\mathbf{v}\mathbf{v}} \end{pmatrix} \quad (1)$$

where $\mathbf{C}_{\mathbf{a}\mathbf{b}} \in \mathbb{R}^{N \times N}$ denotes the real-valued (cross)covariance matrix between real vectors \mathbf{a} and \mathbf{b} , and with $\mathbf{C}_{\mathbf{b}\mathbf{a}} = \mathbf{C}_{\mathbf{a}\mathbf{b}}^T$. A complex-valued Gaussian vector $\mathbf{z} \in \mathbb{C}^N$ is called *proper iff* the following two conditions hold:

$$\mathbf{C}_{\mathbf{u}\mathbf{u}} = \mathbf{C}_{\mathbf{v}\mathbf{v}} \quad \text{and} \quad \mathbf{C}_{\mathbf{u}\mathbf{v}}^T = -\mathbf{C}_{\mathbf{u}\mathbf{v}}. \quad (2)$$

If these conditions are not fulfilled, then \mathbf{z} is called *improper*. *Properness* thus means that real and imaginary parts, *i.e.* \mathbf{u} and \mathbf{v} , have the same covariance matrix and their cross-covariance is skew-symmetric (2). When using the complex representation, *properness* is equivalent to having $\mathbb{E}[\mathbf{z}\mathbf{z}^T] = \mathbf{0}$, which means that \mathbf{z} and \mathbf{z}^* are uncorrelated. Note that even in the Gaussian case, \mathbf{z} and \mathbf{z}^* can not be independent (they are related by a one-to-one deterministic application) even being uncorrelated. Thus, the statistical problem of measuring the impropriety of complex-valued vectors is by nature different from standard canonical correlation analysis where the two Gaussian vectors are independent when their canonical correlation coefficients are zero. This emphasizes why state-of-the-art results from multivariate analysis such as canonical correlation analysis, or more recent works to test the independence between complex random vectors [16], [17] or the diagonality of the covariance matrix [18], cannot be extended in an easy manner to impropriety testing.

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The original contributions of the present work consist of the following results in the asymptotic high-dimensional regime: the limiting distribution of the maximal invariant statistics, and accurate approximations for standard test statistics used in multivariate analysis, namely the GLRT and Roy's test, are derived under the null hypothesis of properness. Moreover, a phase transition behavior is shown to exist, allowing for the detection of complex-valued low-rank signals (modeled as complex vectors) corrupted by proper noise.

The paper is organized as follows. In Section II, the maximal invariant statistics are introduced for impropriety testing. Their limiting distributions are also derived. Section III provides the exact and limiting distributions of GLRT, as well as the limiting distributions of Roy's test statistics. Some simulations are conducted in IV to appreciate the accuracy of the proposed approximations. The relation with canonical correlation analysis results is discussed in Section V. Some concluding remarks are given in the last section.

II. IMPROPRIETY COEFFICIENTS

In order to design tests based on second order statistics and that are not sensitive to reparametrization (by linear transformation), it is known [5], [14] that the impropriety coefficients should be the quantities to use. In this section, we introduce them and provide joint and marginal distributions in the asymptotic regime for their empirical estimates.

A. Testing problem

In many applications such as fMRI [19], DOA estimation [20] or communications [5]¹, it is common use to model the vector of interest, denoted \mathbf{z} , as being *improper* and corrupted by *proper* Gaussian noise. Consequently, statistical tests have been proposed to investigate the impropriety of a complex vector given a sample (of size M in the sequel), from which one needs to decide:

$$\begin{cases} H_0 : \mathbf{z} \text{ is proper iff condition (2) holds,} \\ H_1 : \mathbf{z} \text{ is improper otherwise.} \end{cases} \quad (3)$$

In order to design a statistical test which is invariant under linear transformation, and as explained in [14], one should use the eigenvalues of the augmented covariance matrix \mathbf{C} given in (1). Before introducing the invariant statistics to be derived from observed complex vectors, we first recall some results about the eigenvalues of real augmented PSD (Positive Semi-Definite) matrices.

B. Invariant parameters

Let \mathcal{G} be the set of non-singular matrices $\mathbf{G} \in \mathbb{R}^{2N \times 2N}$ s.t.

$$\mathbf{G} = \begin{pmatrix} \mathbf{G}_1 & -\mathbf{G}_2 \\ \mathbf{G}_2 & \mathbf{G}_1 \end{pmatrix},$$

where $\mathbf{G}_1, \mathbf{G}_2 \in \mathbb{R}^{N \times N}$. Let \mathcal{S} be the set of all $2N \times 2N$ real positive definite symmetric matrices. According to the test

formulation (3) and condition (2), the null hypothesis H_0 is equivalent to $\mathbf{C} \in \mathcal{T} = \mathcal{S} \cap \mathcal{G}$.

As explained in [14], \mathcal{G} is a group (isomorphic to the group $GL_N(\mathbb{C})$ of non-singular $N \times N$ complex matrices under the mapping $\mathbf{G} \leftrightarrow \mathbf{G}_1 + i\mathbf{G}_2$), with the matrix multiplication as the group operation. Moreover \mathcal{G} acts transitively on \mathcal{T} under the action $(\mathbf{G}, \mathbf{T}) \in \mathcal{G} \times \mathcal{T} \mapsto \mathbf{G}\mathbf{T}\mathbf{G}^T \in \mathcal{T}$. Thus, a parametric characterization of H_0 should be invariant to this group action: the value of the parameters to be tested should be the same for \mathbf{C} and $\mathbf{G}\mathbf{C}\mathbf{G}^T$ for any $\mathbf{G} \in \mathcal{G}$.

Next, we introduce a decomposition for any $\mathbf{C} \in \mathcal{S}$ that was originally given in [14] and reads:

$$\mathbf{C} = \dot{\mathbf{C}} + \ddot{\mathbf{C}} \quad (4)$$

where

$$\begin{aligned} \dot{\mathbf{C}} &= \frac{1}{2} \begin{pmatrix} \mathbf{C}_{uu} + \mathbf{C}_{vv} & \mathbf{C}_{uv} - \mathbf{C}_{vu} \\ \mathbf{C}_{vu} - \mathbf{C}_{uv} & \mathbf{C}_{uu} + \mathbf{C}_{vv} \end{pmatrix} \in \mathcal{G}, \\ \ddot{\mathbf{C}} &= \frac{1}{2} \begin{pmatrix} \mathbf{C}_{uu} - \mathbf{C}_{vv} & \mathbf{C}_{uv} + \mathbf{C}_{vu} \\ \mathbf{C}_{uv} + \mathbf{C}_{vu} & \mathbf{C}_{vv} - \mathbf{C}_{uu} \end{pmatrix}. \end{aligned}$$

Using this decomposition, one can define the following $2N \times 2N$ real symmetric matrix

$$\mathbf{\Gamma}(\mathbf{C}) = \dot{\mathbf{C}}^{-\frac{1}{2}} \ddot{\mathbf{C}} \dot{\mathbf{C}}^{-\frac{1}{2}}. \quad (5)$$

It is now possible to give the following lemma about the parametrization of \mathbf{C} .

Lemma 1 (Invariant parameters [14]). *Any matrix $\mathbf{C} \in \mathcal{S}$ can be written as:*

$$\mathbf{C} = \mathbf{G} \begin{pmatrix} \mathbf{I}_N + \mathbf{D}_\lambda & 0 \\ 0 & \mathbf{I}_N - \mathbf{D}_\lambda \end{pmatrix} \mathbf{G}^T,$$

where $\mathbf{G} \in \mathcal{G}$, \mathbf{I}_N is the $N \times N$ identity matrix and $\mathbf{D}_\lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$ is an $N \times N$ diagonal matrix whose diagonal entries denoted as λ_n , for $1 \leq n \leq N$, are the non-negative eigenvalues of the $2N \times 2N$ matrix $\mathbf{\Gamma}(\mathbf{C})$ given in (5). They satisfy the following properties:

- 1) λ_n and $-\lambda_n$, for $1 \leq n \leq N$, form the set of eigenvalues of $\mathbf{\Gamma}(\mathbf{C})$,
- 2) $\lambda_n \in [0, 1]$ with, by convention, the following ordering $1 \geq \lambda_1 \geq \dots \geq \lambda_N \geq 0$.

Proof. See [14, lemma 5.1 and 5.2]. \square

Lemma 1 shows that any invariant parameterization of the covariance matrix \mathbf{C} for the group action of \mathcal{G} depends only on the N (non-negative) eigenvalues $1 \geq \lambda_1 \geq \dots \geq \lambda_N \geq 0$ of $\mathbf{\Gamma}(\mathbf{C})$. Thus these eigenvalues are termed *maximal invariant parameters* [21, Chapter 6]. Moreover under the null hypothesis H_0 , one has that $\lambda_1 = \dots = \lambda_N = 0$ as $\dot{\mathbf{C}}$ reduces to the zero matrix according to (2). Within the invariant parameterization, the testing problem in (3) becomes:

$$\begin{cases} H_0 : \lambda_1 = 0, \\ H_1 : \lambda_1 > 0, \end{cases} \quad (6)$$

where the alternative hypothesis H_1 means that there exists at least one positive eigenvalue. Note that the invariance property

¹See [5] and references therein for a larger list of applications involving complex-valued signals and impropriety related issues.

ensures that the test does not depend on the (common) representation basis of the real and imaginary parts of \mathbf{z} , *i.e.* vectors \mathbf{u} and \mathbf{v} . Also, λ_n , the eigenvalues of $\Gamma(\mathbf{C})$, are directly related to the ones obtained using the *complex augmented representation*, *i.e.* based on the complex covariance matrix:

$$\mathbb{E}[\tilde{\mathbf{z}}\tilde{\mathbf{z}}^\dagger] = \begin{pmatrix} \mathbf{C}_{\mathbf{z}\mathbf{z}} & \mathbf{C}_{\mathbf{z}\mathbf{z}^*} \\ \mathbf{C}_{\mathbf{z}^*\mathbf{z}} & \mathbf{C}_{\mathbf{z}^*\mathbf{z}^*} \end{pmatrix},$$

where \dagger stands for transposition and conjugation, and $\mathbf{C}_{\mathbf{r}\mathbf{s}} = \mathbb{E}[\mathbf{r}\mathbf{s}^\dagger]$ denotes the (complex) cross-covariance between the sized N complex vectors \mathbf{r} and \mathbf{s} . In fact, as detailed in [5, Chapter 3] and [8], [15], the eigenvalues λ_n , for $1 \leq n \leq N$, are also the square roots of the eigenvalues of the following $N \times N$ complex matrix:

$$\mathbf{C}_{\mathbf{z}^*\mathbf{z}^*}^{-1} \mathbf{C}_{\mathbf{z}^*\mathbf{z}} \mathbf{C}_{\mathbf{z}\mathbf{z}}^{-1} \mathbf{C}_{\mathbf{z}\mathbf{z}^*} \quad (7)$$

Matrix (7) corresponds to the usual population canonical correlation matrix to derive the canonical variables between two vectors, here the complex ones \mathbf{z} and \mathbf{z}^* . Hence the squared eigenvalues λ_n^2 are commonly referred to as the canonical correlation coefficients. In our specific framework of impropriety testing, these coefficients are also known as *circularity coefficients* [22], [23], or *impropriety coefficients* [15]. In the sequel, to better emphasize the statistical differences between our impropriety testing problem with the problem of testing the independence between two vectors², we will make use of the name *population impropriety coefficients* for the eigenvalues λ_n . The sample version of these eigenvalues (derived from the sample covariances) will be denoted as l_n and $r_n \equiv l_n^2$ for their squared values, and referred to as the *sample impropriety coefficients* and the *sample squared impropriety coefficients* respectively.

C. Invariant statistics

Consider a sample of size M , denoted $\mathbf{X} = \{\mathbf{x}_m\}_{m=1}^M$, where $\mathbf{x}_m = [\mathbf{u}_m^T, \mathbf{v}_m^T]^T$ are $2N$ -dimensional i.i.d. Gaussian real vectors with zero mean and covariance matrix \mathbf{C} . In the Gaussian framework, a sufficient statistics is given by the $2N \times 2N$ sample covariance matrix:

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_{\mathbf{u}\mathbf{u}} & \mathbf{S}_{\mathbf{u}\mathbf{v}} \\ \mathbf{S}_{\mathbf{v}\mathbf{u}} & \mathbf{S}_{\mathbf{v}\mathbf{v}} \end{pmatrix}, \quad (8)$$

with $\mathbf{S}_{\mathbf{a}\mathbf{b}} \in \mathbb{R}^{N \times N}$ the real-valued sample (cross)covariance matrix of real vectors $\{\mathbf{a}_m\}_{m=1}^M$ and $\{\mathbf{b}_m\}_{m=1}^M$ such that:

$$\mathbf{S}_{\mathbf{a}\mathbf{b}} = \frac{1}{M} \sum_{m=1}^M \mathbf{a}_m \mathbf{b}_m^T. \quad (9)$$

We assume here that $M \geq 2N$, thus \mathbf{S} belongs to the real symmetric positive definite matrices set \mathcal{S} . According to the previous section, since H_0 is invariant under the action of the group \mathcal{G} , an invariant test statistic must only depend on the N non-negative eigenvalues l_n , $1 \leq n \leq N$, of

$$\Gamma(\mathbf{S}) = \dot{\mathbf{S}}^{-\frac{1}{2}} \ddot{\mathbf{S}} \dot{\mathbf{S}}^{-\frac{1}{2}}. \quad (10)$$

²Note again that even under H_0 where the \mathbf{z} and \mathbf{z}^* are uncorrelated, *i.e.* $\lambda_1 = \dots = \lambda_N = 0$, they cannot be independent since they are deduced from one another in a deterministic way. The probability measure of the impropriety test statistics will therefore be different from that of the independent case.

These *sample impropriety coefficients* obey $1 \geq l_1 \geq \dots \geq l_N \geq 0$ according to Lemma 1, and are an estimate of the population impropriety coefficients λ_n obtained from the population covariance \mathbf{C} . Note that all the λ_n are zero under the null hypothesis H_0 , and at least one is non-negative otherwise. As a consequence, the distribution of the l_n should be stochastically greater under H_1 than under H_0 . All invariant test can be derived from this property. A key point to derive now a tractable statistical test procedure is to characterize the null distribution of these sample impropriety coefficients.

D. Eigenvalue distribution under H_0

Let $\mathcal{B}_N(\frac{1}{2}n_1, \frac{1}{2}n_2)$ denote the $N \times N$ -dimensional matrix variate beta distribution with parameters n_1 and n_2 as defined for instance in [24, definition 3.3.2, p. 110]. It is possible to obtain, under H_0 , the joint probability density function (pdf) of the squared eigenvalues of $\Gamma(\mathbf{S})$ in terms of this matrix-variate beta distribution.

Proposition 2 (Joint distribution of impropriety coefficients). *Under H_0 , the vector (r_1, \dots, r_N) of the sample squared impropriety coefficients $r_n \equiv l_n^2$, for $1 \leq n \leq N$, is distributed as the eigenvalues of the matrix-variate beta distribution $\mathcal{B}_N(\frac{1}{2}n_1, \frac{1}{2}n_2)$, with parameters $n_1 = N+1$ and $n_2 = M-N$. Moreover, the joint pdf of (r_1, \dots, r_N) is expressed as:*

$$p(r_1, \dots, r_N) \propto \prod_{n=1}^N (1 - r_n)^{(M-2N-1)/2} \prod_{k < n}^N (r_k - r_n), \quad (11)$$

where $1 \geq r_1 \geq \dots \geq r_N \geq 0$.

Proof. As shown in [14, pp. 39-41], the sample eigenvalue vector (l_1, \dots, l_N) is characterized by the following pdf:

$$p(l_1, \dots, l_N) \propto \prod_{n=1}^N (2l_n)(1 - l_n^2)^{(M-2N-1)/2} \prod_{k < n}^N (l_k^2 - l_n^2).$$

A simple change of variables yields the pdf of (r_1, \dots, r_N) given in (11). Moreover, according to [24, Theorem 3.3.4, p. 112], (11) is the pdf of the eigenvalues of the matrix variate beta distribution $\mathcal{B}_N(\frac{N+1}{2}, \frac{M-N}{2})$, which concludes the proof. \square

It is interesting to note that the pdf given in Proposition 2 is close to what would be obtained if one would perform a canonical correlation analysis on \mathbf{z} and \mathbf{z}^* considered as N -dimensional real Gaussian independent vectors. Here vectors \mathbf{z} and \mathbf{z}^* are actually complex valued and fully dependent. This is further discussed in Section V.

Expression (11) gives, under the H_0 hypothesis, the joint distribution of the squared sample impropriety coefficients (r_1, \dots, r_N) . In the general case, obtaining an analytic expression of marginal distributions of individual eigenvalues is a complicated task. However, in the asymptotic regime, *i.e.* when the dimension N and the number of samples M go to infinity while their ratio stays commensurable, one can obtain those marginal laws. The following theorem gives the distribution of one (unordered) sample impropriety coefficient in this regime.

Theorem 3 (Limiting empirical distribution). *As $M, N \rightarrow \infty$ with the ratio $M/N \rightarrow \gamma \in [2, +\infty)$ being finite, the marginal empirical distribution of the unordered sample squared impropriety coefficients (i.e. the squared eigenvalues of $\Gamma(\mathbf{S})$) converges, under the H_0 hypothesis, to the probability measure with density:*

$$f(r) = \frac{1}{2\pi(1-r)} \sqrt{4(\gamma-1)\frac{1-r}{r} - (\gamma-2)^2}, \quad (12)$$

on its support $r \in (0, c)$, with $c = \frac{4(\gamma-1)}{\gamma^2} \in (0, 1]$.

Proof. See Appendix A. \square

Corollary 4 (Moments). *The mean and variance of the limiting distribution under H_0 of a sample squared impropriety coefficients are expressed respectively as $1/\gamma$ and $(\gamma-1)/\gamma^3$.*

Proof. Expressions of these limiting moments can be derived directly from the pdf (12) by symbolic computation. \square

A few remarks are in order:

- When $\gamma \rightarrow +\infty$, the expression of the mean and variance emphasizes that the sample impropriety coefficients converge to zero, which are the population values under H_0 . This is the usual behavior in small dimension when N is fixed while M tends to infinity.
- Conversely, in the special case where $\gamma = 2$, the asymptotic null distribution of the squared sample impropriety coefficients is $\mathcal{B}(\frac{1}{2}, \frac{1}{2})$, known as the arcsine law. In this limiting case, the sample impropriety coefficients are symmetrically distributed on $[0, 1]$ around $1/2$ with two symmetric modes at the edges (even if the population impropriety coefficients are zero).

III. TESTING FOR IMPROPRIETY

In this section, we make use of the results from Proposition 2 and Theorem 3 to introduce the asymptotic behavior of two impropriety tests: the classical GLRT and Roy's test (based on the largest eigenvalue of the $\Gamma(\mathbf{S})$ matrix).

A. GLRT

1) *Expression of the GLRT statistic:* A very classical procedure to test for impropriety is obtained from the Generalized Likelihood Ratio Test (GLRT) statistic defined as:

$$T \propto \frac{\sup_{\mathbf{C} \text{ s.t. } H_0} p(\mathbf{X}; \mathbf{C})}{\sup_{\mathbf{C} \text{ s.t. } H_1} p(\mathbf{X}; \mathbf{C})},$$

where $p(\mathbf{X}; \mathbf{C})$ is the multivariate normal pdf of the sample \mathbf{X} composed of M i.i.d. $2N$ -dimension real Gaussian vectors with zero mean and covariance matrix \mathbf{C} . Under H_1 , $\mathbf{C} \in \mathcal{S}$ is a symmetric definite positive matrix. Its maximum likelihood (ML) estimate is the sample covariance \mathbf{S} . Under H_0 , one has that $\mathbf{C} = \dot{\mathbf{C}}$ so that $\mathbf{C} \in \mathcal{T}$. Then the ML estimate of \mathbf{C} under H_0 reduces to $\dot{\mathbf{S}}$, as shown for instance in [14]. The testing problem in (3) can thus be rephrased:

$$\begin{cases} H_0: & \mathbf{C} \in \mathcal{T}, \\ H_1: & \mathbf{C} \in \mathcal{S}. \end{cases} \quad (13)$$

Actually, the GLRT statistic is expressed as:

$$\begin{aligned} T &= |\mathbf{S}|/|\dot{\mathbf{S}}|, \\ &= |\dot{\mathbf{S}}^{\frac{1}{2}} (\mathbf{I}_{2N} + \Gamma(\mathbf{S})) \dot{\mathbf{S}}^{\frac{1}{2}}|/|\dot{\mathbf{S}}| = |\mathbf{I}_{2N} + \Gamma(\mathbf{S})|, \\ &= \prod_{n=1}^N (1 + l_n)(1 - l_n) = \prod_{n=1}^N (1 - r_n), \end{aligned} \quad (14)$$

where the first line is due to the Gaussian pdf expression; the second line comes from the decomposition $\mathbf{S} = \dot{\mathbf{S}} + \ddot{\mathbf{S}}$ and the expression (10) of $\Gamma(\mathbf{S})$; the third line comes from Lemma 1, where $r_n = l_n^2$, $1 \leq n \leq N$, are the sample squared impropriety coefficients. As explained previously, it is important to note that the GLRT is invariant: the resulting statistics given in (14) only depends on the eigenvalues of $\Gamma(\mathbf{S})$.

2) *Distribution under the hypothesis H_0 :* Let $\Lambda(d, m, n)$ denote Wilks lambda distribution, with dimension parameter d and degrees of freedom parameters m and n , as defined for instance in [25, definition 3.7.1, p. 81].

Theorem 5. *The GLRT statistics T given in (14) is distributed under H_0 as the following Wilks lambda distribution:*

$$T \sim \Lambda(N, M - N, N + 1).$$

Moreover this statistics can be expressed under H_0 as:

$$T = \prod_{n=1}^N u_n, \quad (15)$$

where the u_n are independent beta-distributed random variables such that $u_n \sim \mathcal{B}(\frac{M-N-n+1}{2}, \frac{N+1}{2})$, for $1 \leq n \leq N$.

Proof. According to Proposition 2, the r_n in (14) are distributed as the eigenvalues of the matrix variate beta distribution $\mathcal{B}_N(\frac{1}{2}n_1, \frac{1}{2}n_2)$ with parameters $n_1 = N + 1$ and $n_2 = M - N$. Using the mirror symmetry property of the beta distribution, the $(1 - r_n)$ are distributed as the eigenvalues of the random matrix $\mathbf{U} \sim \mathcal{B}_N(\frac{1}{2}n_2, \frac{1}{2}n_1)$. According now to [24, Theorem 3.3.3, p. 110], \mathbf{U} can be decomposed as $\mathbf{U} = \Theta^T \Theta$ where Θ is upper triangular with diagonal entries θ_{nn} that are independent and where $u_n \equiv \theta_{nn}^2 \sim \mathcal{B}(\frac{n_2-n+1}{2}, \frac{n_1}{2})$ for $1 \leq n \leq N$. This concludes the proof since $T = |\mathbf{U}| = \prod_{n=1}^N \theta_{nn}^2$. \square

Equation (15) gives also a more efficient way to sample from the null distribution of T in $O(N)$ independent draws, as it is actually not required to generate the $2N \times 2N$ sample covariance matrix \mathbf{S} , nor to compute the eigenvalues of $\Gamma(\mathbf{S})$.

3) *High-dimensional asymptotic distribution under H_0 :* The characterization given in (15) allows us to derive, under the null hypothesis H_0 , an asymptotic distribution for the GLRT statistic T in the high dimensional (i.e. large N) case. This yields a simple tractable closed form approximation of the considered Wilks lambda distribution when both the dimension N and the sample size M are large.

Theorem 6 (Central limit theorem in high dimension). *Let $T' = -\ln T$ where T is the GLRT statistic given in (14). Assume that $M, N \rightarrow \infty$ so that the ratio $M/N \rightarrow$*

$\gamma \in (2, +\infty)$. Under H_0 , the following asymptotic normal distribution is obtained for T' :

$$\frac{1}{s}(T' - m) \xrightarrow{d} \mathcal{N}(0, 1) \quad (16)$$

where

$$m = M \left[\ln \frac{\gamma}{\gamma-1} + \frac{\gamma-2}{\gamma} \ln \frac{\gamma-2}{\gamma-1} \right] + \frac{1}{2} \ln \frac{\gamma}{\gamma-2},$$

$$s^2 = 2 \left[\ln \frac{(\gamma-1)^2}{\gamma(\gamma-2)} + \frac{1}{M} \frac{1}{\gamma-2} \right].$$

Proof. See Appendix B. \square

Bartlett derived a classical approximation for Wilks lambda distribution [25, p. 94] in a low-dimensional setting. This gives, when the dimension N is fixed while M goes to infinity, the same asymptotic distribution as obtained in [8]:

$$-(M - N) \ln T \xrightarrow{d} \chi_{N(N+1)}^2, \quad (17)$$

where $\chi_{N(N+1)}^2$ denotes the chi-squared distribution with $N(N+1)$ degrees of freedom. Note that this approximation was used recently in [26], in a high-dimensional setting in the absence of better approximation.

Using Theorem 6, the Bartlett approximation can now be adjusted to cover both the low and high-dimensional cases. Let $\mathcal{G}(q, p)$ denote the gamma distribution with pdf $f(x) \propto x^{q-1} e^{-x/p}$, where q and p are the shape and scale parameters respectively.

Corollary 7 (Adjusted Bartlett approximation). *Let $\gamma = M/N > 2$, then the log-GLRT statistics $T' = -\ln(T)$ can be approximated as a shifted gamma distribution:*

$$\frac{1}{s}(T' - \alpha) \approx \mathcal{G}(q, p), \quad (18)$$

with

$$q = N(N+1)/2, \quad p = \sqrt{1/q}, \quad \alpha = m - pq s,$$

and m and s^2 are defined in Theorem 6, and where \approx stands for pointwise equivalence of distribution functions for large M under both the low dimensional, i.e. N is fixed and small w.r.t. M , or the high dimensional, i.e. N has order of M , regime.

Proof. In the high dimensional setting, under the assumptions of Theorem 6, the gamma distribution $\mathcal{G}(q, p)$ converges towards the normal one as the shape parameter q goes to infinity. Since the mean and variance of T' are the same for the shifted gamma approximation (18) and the normal one (16), they are asymptotically equivalent.

In the low-dimensional setting where N is fixed while M goes to infinity, one gets that p and q are fixed, $\alpha/s \rightarrow 0$, and $ps = \frac{2}{N(\gamma-1)} + O(1/\gamma^2) = \frac{2}{M-N} + O(1/M^2)$. Moreover, in this limiting case, $T' \xrightarrow{d} 0$ according to the decomposition given in (15). Thus $\frac{1}{s}(T' - \alpha) = \frac{p}{2}(M - N)T' + o(1)$. Because $\chi_q^2 = \frac{2}{p}\mathcal{G}(q, p)$, (18) means that $(M - N)T'$ is asymptotically χ_q^2 distributed. This is the Bartlett limiting distribution (17), which is known to be valid in this low dimensional asymptotic regime \square

B. Roy's test

In multivariate statistics, Roy's test is a well known procedure to detect the alternate hypothesis H_1 for which at least one eigenvalue is non-zero. This test relies on the statistics of the largest eigenvalue [25, p. 84] or, equivalently in our case, the statistics of the largest squared impropriety coefficients $r_1 = l_1^2$. The principle is to reject the H_0 hypothesis as soon as $r_1 > \eta_\alpha$, where the threshold η_α is tuned according to the law of r_1 under the H_0 hypothesis together with the nominal control level α (probability of false alarm).

Theorem 8 (Limiting null distribution for Roy's test). *As $M, N \rightarrow \infty$ such that the ratio $M/N \rightarrow \gamma \in [2, +\infty)$ is finite, let $W = \log(r_1/(1-r_1))$ be the logit transform of the largest impropriety coefficient r_1 . Under H_0 , the asymptotic law of W converges towards a first order Tracy-Widom law denoted as \mathcal{TW}_1 :*

$$\frac{W - \mu}{\sigma} \rightarrow \mathcal{TW}_1, \quad (19)$$

with

$$\mu = 2 \log \tan \left(\frac{\varphi + \psi}{2} \right),$$

$$\sigma^3 = \frac{16}{M^2} \frac{1}{\sin^2(\varphi + \psi) \sin \psi \sin \varphi},$$

$$\psi = \arccos \left(\frac{M-2N+1}{M} \right),$$

$$\varphi = \arccos \left(\frac{M-2N-1}{M} \right).$$

Proof. This is a direct result of proposition 2 and the asymptotic law of the largest eigenvalue of a $\mathcal{B}_N(\frac{1}{2}n_1, \frac{1}{2}n_2)$ matrix-variate distribution given in [27]. \square

The variable W being expressed as an increasing function of r_1 , Roy's test is equivalent to W and the Theorem 8 allows for the calibration of the test. It should be noted that [28] proposes a procedure to evaluate the exact (not asymptotic) law of W . Nevertheless, the simple approximation by the \mathcal{TW}_1 law is in practice sufficiently precise for most cases as long as the dimension is large enough (e.g. typically for $N \geq 10$).

C. Spiked impropriety model

Recently, spiked models involving complex valued vectors have been found to nicely describe multiplexed phase retrieval problems in complex media imaging [13]. Here, we provide theoretical results related to the phase transition behavior occurring in such models. Spiked models are special sparse cases for the alternative hypothesis H_1 . They assume that the rank of the population matrix is low and remains fixed in the high-dimensional asymptotic regime. For the impropriety test setting, this means that the number k of non-zero eigenvalues of $\Gamma(\mathbf{S})$, or equivalently $\tilde{\mathbf{C}}$, is fixed. An example, which corresponds to a low-rank improper signal corrupted by proper noise, is given in Section IV-B2.

Theorem 9 (Phase transition threshold). *Assume that there exist k non-zero population impropriety coefficients $\lambda_1 \geq \dots \geq \lambda_k > 0$, and $\lambda_{k+1} = \dots = \lambda_N = 0$ where k*

is fixed. Under the assumptions of Theorem 3, we have the following convergence for the square of the largest impropriety coefficient r_n with $1 \leq n \leq k$,

$$\begin{aligned} \text{if } \lambda_n^2 \leq \rho_c, & \quad r_n \xrightarrow{a.s.} c, \\ \text{If } \lambda_n^2 > \rho_c, & \quad r_n \xrightarrow{a.s.} \bar{\rho}_n, \end{aligned}$$

where

$$\rho_c = \frac{1}{\gamma - 1}, \quad \bar{\rho}_n = \lambda_n^2 \left(\frac{\gamma - 1}{\gamma} + \frac{1}{\gamma \lambda_n^2} \right)^2, \quad (20)$$

are respectively the phase transition threshold and the limiting values, and c is the edge of the limiting distribution of the bulk defined in Theorem 3.

Proof. This follows from Proposition 2 and from results for high dimensional limiting distribution of spiked models with Beta distributed matrices given in [29, see Theorem 1.8], or [30]. \square

Theorem 9 shows that when the spikes are weaker than a given phase transition threshold ρ_c , none of the sample impropriety coefficients separate from the bulk. This makes the testing problem challenging, and Roy's test would be powerless in this case. Conversely, for spikes λ_n^2 larger than ρ_c , it is easy to check that $\bar{\rho}_n > c$. These sample impropriety coefficients separate now from the bulk, and Roy's test is expected to be very powerful.

IV. SIMULATIONS

This section starts with simulations validating the accuracy of the asymptotic distributions derived under the properness hypothesis H_0 . Then, impropriety testing is illustrated under different alternative H_1 hypotheses: *i) equi-correlated* model (N non-zero identical impropriety coefficients $\lambda_1 = \dots = \lambda_N = \rho > 0$), *ii) spiked* model ($\lambda_1 > 0$ while $\lambda_2 = \dots = \lambda_N = 0$) and *iii) a mixed* model (first half of impropriety coefficients $\lambda_1, \dots, \lambda_{\lfloor N/2 \rfloor} > 0$ gradually decreases, and the other half is zero). Note that, in the following simulations, we make use, for the sake of readability, of the following abuse of notation: γ will now denote the ratio M/N in the considered approximations rather than its limit.

A. Empirical distribution of impropriety coefficients

1) *Empirical vs limiting distributions:* In order to illustrate the accuracy of Theorem 3 in various settings (including small vector dimensions), Fig. 1 displays, for different values of N and γ , the empirical distribution of the squares of the sample impropriety coefficients under the properness assumption H_0 . This shows the very good agreement with the limiting empirical distribution derived in Theorem 3. Note that, when $N = 10$, small fluctuations can be observed (gray bars) around the right edge c of the limiting empirical distribution. But in a larger dimension ($N = 100$), the greatest coefficients converged well towards this edge.

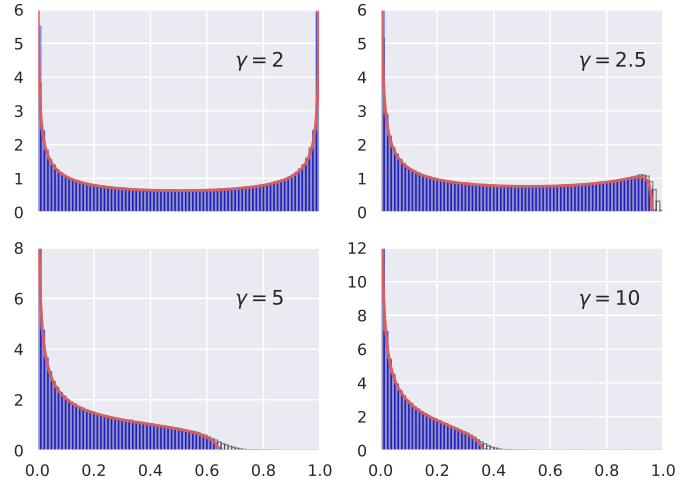


Fig. 1: Histograms of the squared sample impropriety coefficients under H_0 for different values of γ : blue bars are for $N = 100$, gray bars are for $N = 10$. The pdf of the limiting distribution given in theorem 3 is shown in solid red line. Histograms obtained with 1000 Monte-Carlo runs.

2) *Distribution of the GLRT statistic:* Fig. 2 depicts, for different values of M and γ , a probability-probability plot of the theoretical null distribution of T against each one of these asymptotic approximations. A deviation from the $y = x$ line indicates a difference between the theoretical and the asymptotic distributions. This shows that, as expected for high-dimensional setting (e.g., $\gamma \leq 5$) and/or large sample sizes (e.g., $M \geq 1000$), the asymptotic log-normal distribution derived in Theorem 6 becomes very accurate and much better than the Bartlett approximation. In addition, the adjusted Bartlett approximation obtained in Corollary 7 is very accurate in all cases (low/high-dimension or small/large sample size). This latter corrects and generalizes the classical Bartlett approximation that may behave poorly even for small (e.g. $N = 4$) dimensional vectors, as we can see in the top-left subplot ($M = 10$, $\gamma = 2.5$) of Fig. 2. The adjusted approximation is therefore of practical interest to calibrate the GLRT procedure according to a nominal significance level.

3) *Distribution of the Roy's statistic:* Fig. 3 depicts, for different values of N and γ , a probability-probability plot of the theoretical null distribution for the largest impropriety coefficients statistics r_1 , or equivalently its logit-transform W , against the asymptotic approximation given in Theorem 8. This shows that even for a moderate dimension ($N = 10$), the Tracy-Widom approximation is quite accurate, and becomes very accurate for a larger dimension ($N = 100$).

B. Some impropriety tests scenarios

1) *Equal impropriety coefficients:* The case where the population impropriety coefficients are all non-zero and equal, hereinafter referred to as *equi-correlated* model, can be obtained when the real and imaginary parts have a common

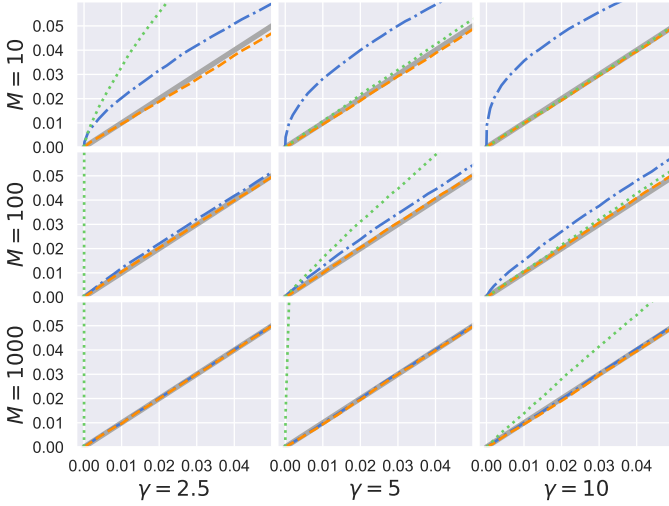


Fig. 2: Comparison, for different values of γ and M , of asymptotic approximations for the GLRT statistics T : $\Pr(T > q_\alpha)$ under H_0 vs the nominal control level α in $[0, 0.05]$ where q_α is the $1 - \alpha$ quantile either for the log-normal approximation given in Theorem 6, shown in dashdotted blue line, or the adjusted Bartlett approximation given in Corollary 7, shown in dashed orange line, or the standard Bartlett approximation (17), shown in dotted green line. The solid gray line represents the $y = x$ values. Probabilities estimated with 10^6 Monte-Carlo runs.

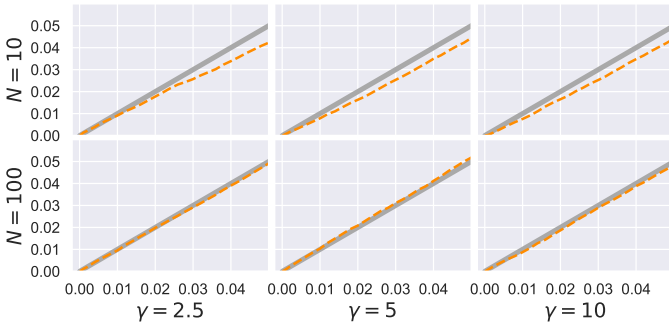


Fig. 3: Comparison, for different values of γ , of asymptotic approximations for the Roy's statistics W : $\Pr(W > q_\alpha)$ under H_0 vs the nominal control level α in $[0, 0.05]$ where q_α is the $1 - \alpha$ quantile for the Tracy-Widom approximation given in Theorem 8, shown in dashed orange line. The solid gray line represents the $y = x$ values. Probabilities estimated with 10^6 Monte-Carlo runs.

contribution:

$$\begin{aligned} \mathbf{u}_m &= \mathbf{s}_m + \sqrt{\theta} \mathbf{q}_m, \\ \mathbf{v}_m &= \mathbf{t}_m + \sqrt{\theta} \mathbf{q}_m, \end{aligned} \quad (21)$$

where $\theta > 0$, \mathbf{s}_m , \mathbf{t}_m and \mathbf{q}_m are i.i.d. Gaussian vectors in \mathbb{R}^N , for $1 \leq m \leq M$. Straightforward computations show that the non-negative roots of $\Gamma(\mathbf{C})$, i.e the population impropriety coefficients $\lambda_1, \dots, \lambda_N$, are all equal to $\lambda \equiv \frac{\theta}{1+\theta}$.

Fig. 4 displays the power of both GLRT and Roy's test, under the alternative H_1 obtained for this equi-correlated model, as a function of the impropriety level λ^2 . As expected,

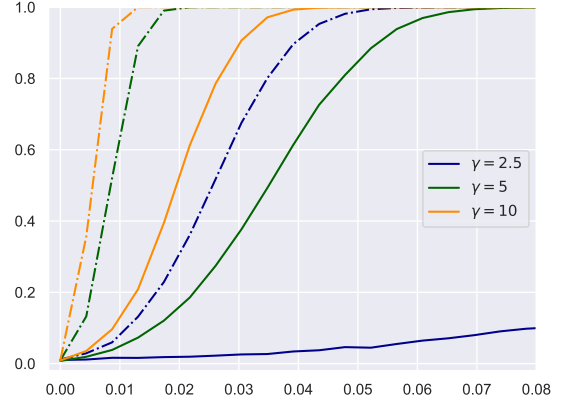


Fig. 4: Power of Roy's test, solid line, and GLRT, dashdotted line, vs the squared population impropriety coefficients λ^2 under the equi-correlated model described in (21), for different values of γ : blue lines for $\gamma = 2.5$, green for $\gamma = 5$, orange for $\gamma = 10$ (dimension $N = 100$, nominal value of false alarm probability $\alpha = 0.01$). Powers estimated with 1000 Monte-Carlo runs.

the GLRT, which uses information from all sample impropriety coefficients, is here much more powerful than Roy's test, especially in the high dimension case ($\gamma = 2.5$) where M and N are close.

2) *Spiked model*: A Gaussian spike model with a single non-zero impropriety coefficient can be obtained when the real and imaginary parts have a common contribution of rank one:

$$\begin{aligned} \mathbf{u}_m &= \mathbf{s}_m + \sqrt{\theta} w_m \boldsymbol{\varphi}, \\ \mathbf{v}_m &= \mathbf{t}_m + \sqrt{\theta} w_m \boldsymbol{\varphi}, \end{aligned} \quad (22)$$

where $\theta > 0$, $\boldsymbol{\varphi} \in \mathbb{R}^N$ is a normed deterministic vector $\|\boldsymbol{\varphi}\|_2 = 1$, w_m are i.i.d. Gaussian centered random variables with unit variance, and \mathbf{s}_m , \mathbf{t}_m are Gaussian i.i.d. vectors in \mathbb{R}^N , for $1 \leq m \leq M$. This scenario depicts a case where a low-rank improper signal is corrupted by proper noise. Straightforward computations show that there is a single non-zero population impropriety coefficient, a *spike*, which expresses as $\lambda_1 = \frac{\theta}{1+\theta}$.

Fig. 5 displays the empirical distribution of the squares of the sample impropriety coefficients under alternative H_1 spiked model for different spike level λ_1 . Again, the bulk of these coefficients matches very well the limiting distribution derived under the properness hypothesis H_0 , whatever the spike level. In addition, for "weak" spikes, i.e. when λ_1^2 is small relative to the phase transition threshold ρ_c defined in Theorem 9, the greatest sample impropriety coefficient r_1 , which is an estimator of the spike power λ_1^2 , does not separate from this bulk and is stuck around the edge c of the limiting distribution. Conversely, for stronger spikes where $\lambda_1^2 > \rho_c$, r_1 clearly separates from the bulk and concentrates around the limiting value $\bar{\rho}_1$. This numerically supports Theorem 9.

In Fig. 6 the power of both GLRT and Roy's tests are displayed as a function of the spike power λ_1^2 . This shows that for "weak" spikes, i.e. when $\lambda_1^2 < \rho_c$, the two tests have a very low power. In fact, the largest sample impropriety

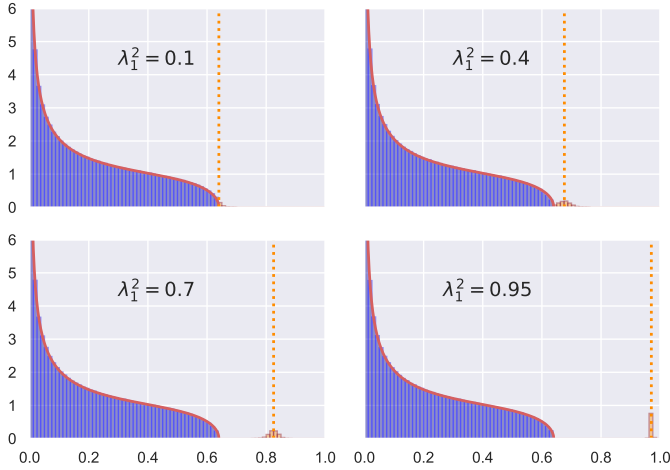


Fig. 5: Histogram of the squared sample impropriety coefficients under the spiked impropriety model described in (22) with $N = 100$ and $\gamma = 5$. Blue bars are for the bulk of the squared impropriety coefficients lower than the edge of the limiting distribution c , orange bars are for the ones greater than c . The pdf of the limiting distribution under H_0 given in Theorem 3 is shown in solid red line. The limiting spike values \bar{p}_1 defined in Theorem 9 are depicted as vertical dotted orange lines. The phase transition threshold is here $\rho_c = 0.25$. Top row sub-figures are for a squared population impropriety coefficient $\lambda_1^2 = 0.1 < \rho_c$ (left) and $\lambda_1^2 = 0.4 > \rho_c$ (right), bottom row is likewise for $\lambda_1^2 = 0.7 > \rho_c$ and $\lambda_1^2 = 0.95 > \rho_c$ respectively. Histograms obtained with 1000 Monte-Carlo runs.

coefficient r_1 does not separate from the bulk and cannot be detected correctly using Roy’s test. It is interesting to note that GLRT, which uses the information in all the coefficients, is here slightly “more powerful” to detect such weak spikes. Nevertheless, as soon as r_1 separates from the bulk, *i.e.* for stronger spikes where $\lambda_1^2 > \rho_c$, Roy’s test becomes much more powerful than the GLRT, with a power that converges quickly towards 1 as expected.

3) *Mixed scenario:* We consider now a mixed scenario with some highly improper, some less improper, and also some proper components. More precisely, we generate power imbalances between the real and imaginary parts as follows

$$\begin{aligned} \mathbf{u}_m &= \mathbf{s}_m, \\ \mathbf{v}_m &= \mathbf{t}_m + \sqrt{2\theta}\mathbf{w}_m, \end{aligned} \quad (23)$$

where $\theta > 0$, \mathbf{s}_m , \mathbf{t}_m are Gaussian i.i.d. vectors in \mathbb{R}^N , for $1 \leq m \leq M$, and \mathbf{w}_m are i.i.d. Gaussian centered vector where p_k denotes, in a principal component analysis, the fraction of variance explained by the k th principal component, hence $1 \geq p_1 \geq p_2 \geq \dots \geq p_N \geq 0$ and $p_1 + \dots + p_N = 1$, and such that \mathbf{w}_m have a unit total variance, *i.e.* $E[\|\mathbf{w}_m\|_2^2] = 1$. Thus the ordered population impropriety coefficients read $\lambda_k = \frac{\theta p_k}{1 + \theta p_k}$, for $1 \leq k \leq N$.

Fig. 7 shows the power of both GLRT and Roy’s test as a function of the principal spike power λ_1^2 . In this setting, the dimension is $N = 20$ and we have that:

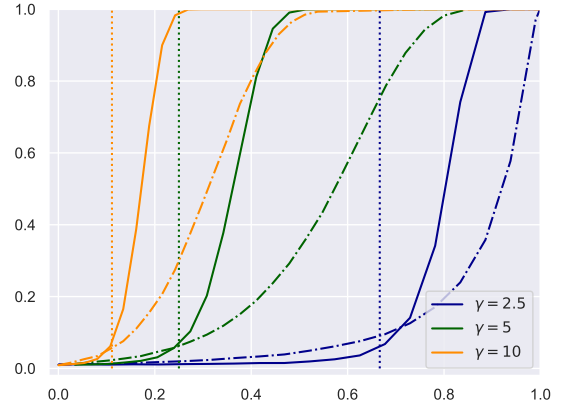


Fig. 6: Power of Roy’s test, solid line, and GLRT, dashdotted line, vs the largest squared population impropriety coefficient λ_1^2 under the spiked impropriety model described in (22). The phase transition thresholds ρ_c given in Theorem 9 are depicted as vertical dotted lines. Results are shown for different values of γ : blue lines for $\gamma = 2.5$, green for $\gamma = 5$, orange for $\gamma = 10$ (dimension $N = 100$, nominal value of false alarm probability $\alpha = 0.01$). Powers estimated with 1000 Monte-Carlo runs.

- 95% of the variance of the source \mathbf{w}_m is explained by its first five principal components ($p_1 = 50\%$, $p_2 = 20\%$, $p_3 = p_4 = 10\%$, $p_5 = 5\%$) which yield the most significant (*i.e.* the largest) impropriety coefficients
- the remaining 5% are explained by the following five components ($p_6 = \dots = p_{10} = 1\%$), which yield significantly smaller impropriety coefficients.

As a consequence, half of the impropriety coefficients are non zero, while the other half are zero: $\lambda_{11} = \dots = \lambda_{20} = 0$. This scenario mimics usual principal component analysis where the interesting source lives on a lower dimensional space and its explained variance gradually decreases with the order of the principal components. These curves emphasize that GLRT, which uses all the sample coefficients, can be much more powerful than Roy’s test for small sample size ($\gamma = 2.5$ and $\gamma = 5$). However for larger values of γ , Roy’s test becomes significantly more powerful than GLRT for stronger spikes λ_1^2 . Note also that even if half of the impropriety coefficients are non-zero, the phase transition behavior given in Thm. 9 is still present for Roy’s test.

V. RELATION WITH EXISTING WORK ON CCA

In CCA, one is classically interested in testing for independence between N -dimensional real or complex Gaussian random vectors \mathbf{x} and \mathbf{y} . Invariant statistics to reject the null hypothesis of independence consist in the squared canonical correlation coefficients. When \mathbf{x}_i and \mathbf{y}_i , $1 \leq i \leq M$, are M i.i.d. copies of N -dimensional real Gaussian independent vectors, the N squared sample canonical correlation coefficients are known to be jointly distributed as the eigenvalues of a matrix-variate beta distribution $\mathcal{B}_N(\frac{n_1}{2}, \frac{n_2}{2})$, with $n_1 = N$ and $n_2 = M - N$ as shown in [24, Section 11.3].

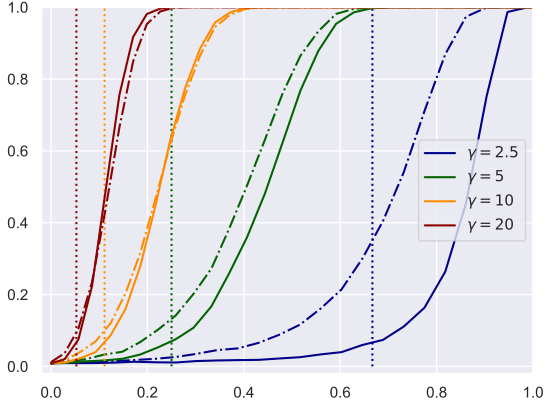


Fig. 7: Power of Roy's test, solid line, and GLRT, dashdotted line, vs the largest squared population impropriety coefficients λ_1^2 under the model described in (23) for dimension $N = 20$, with $p_1 = 50\%$, $p_2 = 20\%$, $p_3 = p_4 = 10\%$, $p_5 = 5\%$, $p_6 = \dots = p_{10} = 1\%$, and $p_k = 0$ for $11 \leq k \leq 20$. The phase transition thresholds ρ_c given in Theorem 9 are depicted as vertical dotted lines. Results are shown for different values of γ : blue lines for $\gamma = 2.5$, green for $\gamma = 5$, orange for $\gamma = 10$ and red for $\gamma = 20$ (nominal value of false alarm probability $\alpha = 0.01$). Powers estimated with 1000 Monte-Carlo runs.

For the impropriety case, Prop. 2 shows that the N squared impropriety coefficients are jointly distributed as the eigenvalues of a matrix-variate beta distribution $\mathcal{B}_N(\frac{n_1}{2}, \frac{n_2}{2})$ with parameters $n_1 = N + 1$ and $n_2 = M - N$. Surprisingly, this is quite close to the real and independent CCA case, the difference being a $+1$ offset in the n_1 parameter. Despite this similarity, it is important to note the following points.

Independence is not compatible with impropriety testing

The two regimes of parameters (in the matrix-variate beta distributions) obtained for the real and independent CCA case or the impropriety case remain incompatible. This is intrinsically due to the fact that the underlying assumptions are not compatible. As shown for instance by (7), the impropriety coefficients are the canonical correlation coefficients between vectors \mathbf{z} and \mathbf{z}^* which are complex-valued and fully dependent.

Impropriety testing is a more structured problem

Consequently we do not think that classical results for CCA between real, or complex, independent vectors can be readily extended or adapted to the impropriety detection problem. Our contribution is thus a way to overcome this and to provide new insights on the impropriety problem. In this paper we have precisely proposed a direct characterization of the usual impropriety testing statistics. Furthermore, to our knowledge, the numerous existing works on impropriety have never been able to easily adapt these CCA results to characterize impropriety coefficients in finite or even asymptotic regimes.

The $+1$ offset is non-negligible

In the high dimensional regime where both M and N go to infinity, it might be tempting to consider that the $+1$ offset in the n_1 parameter of the matrix-beta distribution becomes negligible, and thus that the distributions of the statistics for testing the independence between real Gaussian vectors or for the impropriety detection problem are asymptotically equivalent. This is actually incorrect. The $+1$ offset modifies the asymptotic distribution of any linear spectral statistic such as the GLRT, but also of the largest eigenvalue one, and therefore of Roy's test. For instance, Theorem 6 and its demonstration allows us to show that the asymptotic mean m' of the GLRT statistics derived for the parameters $n_1 = N$ and $n_2 = M - N$ (real and independent CCA case) is $m' = m + \ln \frac{\gamma-1}{\gamma}$ where m is the asymptotic mean of the impropriety GLRT statistics given in Theorem 6, while the asymptotic variance remains bounded. An illustration of this mismatch for both GLRT and Roy's test is depicted in Fig. 8.

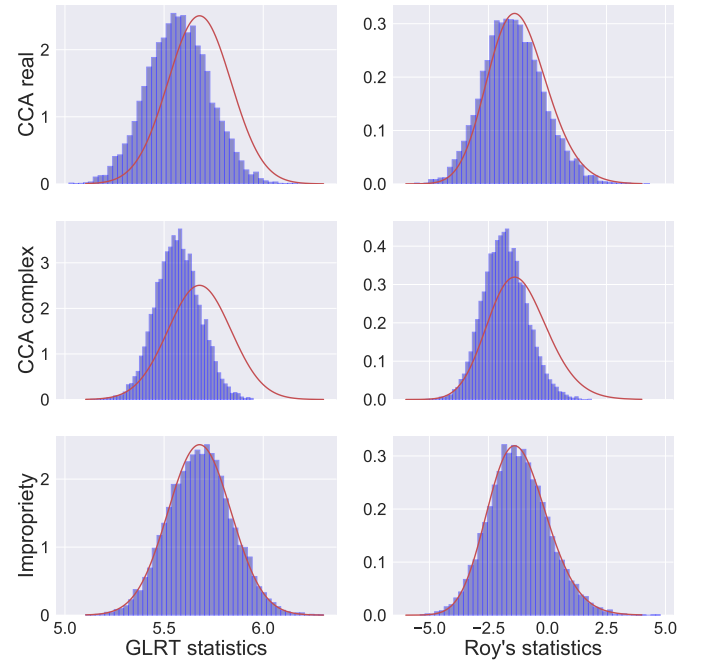


Fig. 8: Histograms for $M = 500$ and $N = 50$ ($\gamma = 10$) of the GLRT statistics (left column) and Roy's statistics (right column). The pdf of the GLRT and Roy's test limiting distribution for impropriety testing given in Thm. 6 and Thm. 8 are shown in solid red line in the left and right column respectively. Top row: CCA statistics for independent and real vectors. Middle row: CCA for independent and complex vectors. Bottom row: Impropriety statistics. Histograms obtained with 10000 Monte-Carlo runs.

VI. CONCLUDING REMARKS

Properness testing for complex Gaussian random vectors in the asymptotic regime relies on the characterization of sample impropriety coefficients. In particular, their limiting distributions give access to the behavior of classical GLRT and Roy's test. The results presented in this article demonstrate

that the asymptotic regime is actually reached quite rapidly in practice and the proposed original approximations are well-suited to a wide range of complex-valued datasets.

The phase transition highlighted in Roy's test has also potential applications in the search for complex-valued low-rank signals corrupted by proper noise in large datasets. In this context, the proposed high dimensional approximations can be extended to the sequential testing problem [31] to estimate the number of improper sources, *i.e.* of non-zero propriety coefficients. Another natural extension of the proposed work consists of considering the case of quaternion random vectors which possess several properness levels, thus trying to decipher their correlation symmetry patterns. This could be helpful in the spectral characterization of bivariate signals [32] among other quaternion signal processing applications.

ACKNOWLEDGMENT

The authors would like to thank Prof. Romain Couillet for his many valuable comments and suggestions.

APPENDIX A

PROOF OF THEOREM 3 (LIMITING EMPIRICAL DISTRIBUTION)

According to Proposition 2, the sample squared propriety coefficients are distributed under H_0 as the eigenvalues of a matrix-variate Beta distribution $\mathcal{B}_N(\frac{1}{2}n_1, \frac{1}{2}n_2)$. Moreover, [24, Thm 3.3.1, p. 109] shows that the eigenvalues of a $\mathcal{B}_N(\frac{1}{2}n_1, \frac{1}{2}n_2)$ random matrix are distributed as the eigenvalues of $\mathbf{A}(\mathbf{A} + \mathbf{B})^{-1}$, where $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{N \times N}$ are two independent random matrices with respective distributions the Wishart laws $W_N(n_1, \mathbf{I}_N)$ and $W_N(n_2, \mathbf{I}_N)$, where n_1 and n_2 are their respective number of *degrees of freedom* and where the Wishart *scaling matrix* parameter is set to the identity matrix \mathbf{I}_N . Thus the limiting distribution for the propriety coefficients can be derived as the limiting distribution of the eigenvalues of $\mathbf{A}(\mathbf{A} + \mathbf{B})^{-1}$.

Assume now that we are in the asymptotic regime where $N, n_1, n_2 \rightarrow +\infty$ with $N/n_1 \rightarrow d \in (0, 1]$ and $N/n_2 \rightarrow d' \in (0, 1)$.

Then, as demonstrated in [33], the empirical law of the eigenvalues of the following F -matrix $\frac{d}{d'}\mathbf{A}\mathbf{B}^{-1}$ converges to a distribution with density given as:

$$f(x) = \frac{(1-d')\sqrt{(x-a)(b-x)}}{2\pi x(xd+d')}, \quad (24)$$

on the interval $x \in [a, b]$, where $a = \left(\frac{1-\sqrt{1-(1-d)(1-d')}}{1-d'}\right)^2$

and $b = \left(\frac{1+\sqrt{1-(1-d)(1-d')}}{1-d'}\right)^2$.

Note that each eigenvalue r_i of $\mathbf{A}(\mathbf{A} + \mathbf{B})^{-1}$ can be deduced from each eigenvalue x_i of $\frac{d}{d'}\mathbf{A}\mathbf{B}^{-1}$ thanks to the relation $r_i = \frac{d'x_i}{d+d'x_i}$. The continuous mapping theorem ensures therefore that the asymptotic law of the eigenvalues of a $\mathcal{B}_N(\frac{1}{2}n_1, \frac{1}{2}n_2)$ random matrix can be directly deduced from (24) using the aforementioned change of variable.

Last, according to proposition 2, the parameters for the matrix-variate Beta distribution in our case are $n_1 = N + 1$

and $n_2 = M - N$. Due to the asymptotic regime stated above, one gets that $d = \lim N/n_1 = 1$ and $d' = \lim N/n_2 = \frac{1}{\gamma-1}$. Plugging this parameter values in the asymptotic distribution of the eigenvalues of the $\mathcal{B}_N(\frac{1}{2}n_1, \frac{1}{2}n_2)$ random matrix deduced from (24) gives finally the limiting density given in Theorem 3 and concludes the proof.

APPENDIX B

PROOF OF THEOREM 6 (CENTRAL LIMIT THEOREM)

According to theorem 5, $T' = \sum_{n=1}^N \zeta_n$ where the ζ_n are independent random variables such that $\zeta_n = -\ln u_n$ with u_n beta distributed s.t. $u_n \sim \mathcal{B}(a_n, b)$, $a_n = \frac{M-N-n+1}{2}$, $b = \frac{N+1}{2}$, for $1 \leq n \leq N$. Based on the centered moments of a logarithmically transformed beta-distributed variable as given in [34], then $E[\zeta_n] = \psi(a_n + b) - \psi(a_n)$ where $\psi(\cdot)$ is the digamma function, and $\text{var}[\zeta_n] = \psi_1(a_n) - \psi_1(a_n + b)$ where $\psi_1(\cdot)$ is the trigamma function. Using asymptotic expansions of the digamma and trigamma functions for large argument x ,

$$\psi(x) = \log x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right), \quad \psi_1(x) = \frac{1}{x} + \frac{1}{2x^2} + O\left(\frac{1}{x^3}\right),$$

plus Stirling formula $\log(n!) = n \log n - n + \frac{1}{2} \log(2\pi n) + O\left(\frac{1}{n}\right)$ for large n , straightforward computations omitted here for the sake of brevity yield that $E[T'] = \sum_{n=1}^N E[\zeta_n] = m_M + O(1/M)$ and $\text{var}(T') = \sum_{n=1}^N \text{var}(\zeta_n) = s_M^2 + O(1/M^2)$.

In order to apply Lyapunov central limit theorem [35, p. 362] to $T' = \sum_{n=1}^N \zeta_n$, it is sufficient to show that

$$\frac{1}{\text{var}(T')^2} \sum_{n=1}^N E\left[(\zeta_n - E[\zeta_n])^4\right] \rightarrow 0.$$

The expression of the fourth order centered moment of ζ_n gives that $E\left[(\zeta_n - E[\zeta_n])^4\right] = O(1/(M-n+2)^2)$ for $1 \leq n \leq N$. Then $\sum_{n=1}^N E\left[(\zeta_n - E[\zeta_n])^4\right] = O(1/M)$. As $\text{var}(T') = s_M^2 + O(1/M^2)$, it comes that $\text{var}(T')^{-2} = O(1)$ and the previous Lyapunov sufficient condition holds. Thus

$$Z \equiv \frac{1}{\sqrt{\text{var}(T')}} \sum_{n=1}^N (\zeta_n - E[\zeta_n]) \xrightarrow{d} \mathcal{N}(0, 1).$$

By noting finally that $\frac{1}{s_M}(T' - m_M) = Z + O(1/M)$, Slutsky's theorem allows us to conclude the proof.

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