

Controllability results for stochastic coupled systems of fourth- and second-order parabolic equations

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Abstract

In this paper, we study some controllability and observability problems for stochastic systems coupling fourth- and second-order parabolic equations. The main goal is to control both equations with only one controller localized on the drift of the fourth-order equation. We analyze two cases: on one hand, we study the controllability of a backward system where the couplings are made through first-order terms. The key point is to use suitable Carleman estimates for the heat equation and the fourth-order operator with the same weight to deduce an observability inequality for the adjoint system. On the other hand, we study the controllability of a coupled model of forward equations. This case, which is well-known to be harder to solve, requires to prove an observability inequality for an adjoint backward system with initial datum in a finite dimensional space and employ the classical iterative method introduced in the seminal work by G. Lebeau and L. Robbiano.

Keywords: Null controllability, observability, coupled systems, forward and backward linear stochastic parabolic equations, Carleman estimates.

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1 Introduction

The stabilized Kuramoto-Sivashinsky system was proposed in [MFK01] as a model of front propagation in reaction-diffusion phenomena and combines dissipative features with dispersive

ones. This system consists of a Kuramoto-Sivashinsky-KdV (KS-KdV) equation linearly coupled to an extra dissipative equation. More precisely, the model has the form

$$\begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + \lambda u_{xx} + uu_x = v_x, \\ v_t - \Gamma v_{xx} + cv_x = u_x, \end{cases} \quad (1)$$

where γ, λ are positive coefficients accounting for the long-wave instability and the short-wave dissipation, respectively, $\Gamma > 0$ is the dissipative parameter and $c \in \mathbb{R} \setminus \{0\}$ is the group-velocity mismatch between wave modes.

This model applies to the description of surface waves on multilayered liquid films and serves as a one-dimensional model for turbulence and wave propagation, see [MFK01] for a more detailed discussion.

The controllability of coupled systems like (1) has attracted a lot of attention in the recent past (see for instance the survey [AKBGBdT11] and the references within). One of the common features among such works is the goal of controlling as many equations with the fewest number of controls. Roughly speaking, it is more difficult to deduce control properties for coupled systems than for single equations.

System (1) has been studied from the controllability point of view in the deterministic setting in various papers. In [CMP12], the authors address the boundary controllability when the control action is applied on both equations. Later, in [CMP15], it is proved that controllability with a single control supported in an interior open subset of the domain and acting on the fourth-order equation can be achieved, while [CnC16] studies the analogous problem but with an interior control acting only on the heat equation. Finally, in [CCM19], the problem is addressed with a single boundary control but only for a linearization of the system.

In this context, a natural question that arises is to what extent the controllability properties for the stochastic counterpart of (1) hold. Seen individually, the fourth- and second-order equations that compose this system have been studied from the control point of view in several works. We refer to [BRT03, TZ09, Liu14b] for some of the most representative works about the controllability of stochastic heat and parabolic-type equations, while we refer to [GCL15, Gao18] for some results about the linear stochastic fourth-order equation. Nevertheless, as a coupled system, to the best of the authors' knowledge, this kind of problem has not been studied yet in the literature.

1.1 Statement of the main results

In what follows, we fix $T > 0$, $\mathcal{D} = (0, 1)$, and denote \mathcal{D}_0 as any given nonempty open subset of \mathcal{D} . We will denote $Q = \mathcal{D} \times (0, T)$ and $\Sigma = \{0, 1\} \times (0, T)$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{W(t)\}_{t \geq 0}$ is defined such that $\{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration generated by $W(\cdot)$, augmented by all the P -null sets in \mathcal{F} . Let X be a Banach space. For $p \in [1, \infty]$, we define the space

$$L_{\mathcal{F}}^p(0, T; X) := \left\{ \phi : \begin{array}{l} \phi \text{ is an } X\text{-valued } \mathcal{F}_t\text{-adapted process} \\ \text{on } [0, T], \text{ and } \phi \in L^p([0, T] \times \Omega; X) \end{array} \right\},$$

endowed with the canonical norm and we denote by $L_{\mathcal{F}}^2(\Omega; C([0, T]; X))$ the Banach space consisting of all X -valued \mathcal{F}_t -adapted processes $\phi(\cdot)$ such that $E\left(\|\phi(\cdot)\|_{C([0, T]; X)}^2\right) < \infty$, also equipped with the canonical norm.

1.1.1 Controllability of the backward system

In the first part of the paper, we are interested in studying the null controllability for the following backward stochastic system

$$\begin{cases} dy - (\gamma y_{xxxx} - y_{xxx} + y_{xx})dt = (z_x - d_1 Y - d_2 Z + \chi_{\mathcal{D}_0} h)dt + YdW(t) & \text{in } Q, \\ dz + \Gamma z_{xx}dt = (z_x + y_x - d_3 Z)dt + ZdW(t) & \text{in } Q, \\ y = y_x = 0 & \text{on } \Sigma, \\ z = 0 & \text{on } \Sigma, \\ y(x, T) = y_T, \quad z(x, T) = z_T & \text{in } \mathcal{D}. \end{cases} \quad (2)$$

In (2), (y_T, z_T) is the terminal state, (y, z) is the state variable, h is the control variable, and d_i , $i = 1, 2, 3$, are suitable coefficients. As it is common in the theory of BSDE, the additional processes (Z, Y) are needed for the well-posedness of the system (see Proposition A.1).

The controllability problem we are interested can be formulated as follows.

Definition 1.1. System (2) is said to be null-controllable if for any given initial data $(y_T, z_T) \in L^2(\Omega, \mathcal{F}_T; L^2(\mathcal{D})^2)$, there exists a control $h \in L^2_{\mathcal{F}}(0, T; L^2(\mathcal{D}_0))$ such that the solution (y, z) to system (2) satisfies $(y(0), z(0)) = 0$ in \mathcal{D} , P -a.s.

Observe that the control h is applied only on the first equation of the system and acts indirectly through the coupling y_x in the drift of the second equation. As we have mentioned before, this situation is more complicated than for a single equation and in the stochastic setting even more difficulties appear. Indeed, there are only a handful of works studying controllability problems for coupled stochastic systems with less controls than equations, see, [LL12, Liu14a, LL18]. In particular, in [LL18], for controlling several parabolic equations with few controls, well-known facts such as Kalman-type conditions that are true in the deterministic setting (see e.g. [AKBGBdT11, Theorem 5.1]) are not longer valid for the stochastic setting.

Moreover, looking at system (2), we see that the equation is coupled by first-order terms only. This is of course more difficult than the case where only zero-order terms are used and classical methodologies for dealing with coupled systems are not longer valid (see e.g. [LL12, Theorem 1.2] and [LL18, Proposition 4]).

Here, inspired in the proof for the deterministic case (see [CMP15, Theorem 1.1]), we are able to prove that under suitable conditions for the coupling coefficients d_i , system (2) is null controllable. More precisely, we have the following.

Theorem 1.2. *Assume that $d_i \in L^\infty_{\mathcal{F}}(0, T; W^{2,\infty}(\mathcal{D}))$ for $i = 1, 2$ and $d_3 \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R})$. Then, system (2) is null-controllable.*

Using the classical equivalence between null-controllability and observability, the main ingredient of the proof consists in obtaining a suitable observability inequality for the corresponding adjoint system. Using stochastic versions of well-known Carleman estimates for the fourth-order operator and the heat equation with non-homogeneous Neumann boundary conditions, we adapt the methodology in [CMP15] to the stochastic framework.

1.1.2 Controllability of the forward system

Once Theorem 1.2 is established, a natural extension is to study the controllability of the corresponding forward equation. As it is known, the controllability of forward stochastic parabolic systems is a challenging topic and has been explored in different settings, see, for instance, [BRT03, TZ09, L11].

One of the main difficulties comes from the fact the adjoint of the forward system is precisely a backward stochastic equation (one can think for instance in (2) with $h \equiv 0$) and, as remarked in [BRT03], it is a very difficult task to establish observability inequalities for such kind of systems. For the case of the heat equation, this difficulty was overcome in [TZ09] by introducing two controls (one on the drift and one on the diffusion) for controlling a single equation and the same ideas were used in [GCL15] to study the controllability of a single forward fourth-order parabolic equation. The main tool used in those works is a suitable Carleman estimate with two observation terms.

If we follow this approach, we can obtain an observability inequality with four observations for the corresponding backward adjoint system. This leads us to have four controls for controlling just two equations. As we have mentioned, our goal is to control as many equations with the less number of controls. Therefore, we will focus on studying the controllability of the forward system

$$\begin{cases} dy + y_{xxxx} dt = \chi_{\mathcal{D}_0} h dt + (b_1 y + b_2 z) dW(t) & \text{in } Q, \\ dz - z_{xx} dt = y dt + b_3 z dW(t) & \text{in } Q, \\ y = y_{xx} = 0 & \text{on } \Sigma, \\ z = 0 & \text{on } \Sigma, \\ y(x, 0) = y_0, \quad z(x, 0) = z_0 & \text{in } \mathcal{D}. \end{cases} \quad (3)$$

As before, in (3), (y, z) is the state, h is the control variable and (y_0, z_0) is the initial datum. In this case, we assume that $b_i = b_i(t) \in L^\infty(0, T; \mathbb{R})$, $i = 1, 2, 3$.

At this point, the reader has already noticed that system (3) has a much less general structure than for instance (2) and the boundary conditions for the fourth-order equation are different. This comes from the tools employed to obtain our controllability result, which rely on the Lebeau-Robbiano strategy (see [LR95]) instead of Carleman estimates.

Our second main result is the following.

Theorem 1.3. *For any given initial data $(y_0, z_0) \in L^2(\Omega, \mathcal{F}_0; L^2(\mathcal{D})^2)$, system (3) is null controllable at time T , i.e., there exists $h \in L^2_{\mathcal{F}}(0, T; L^2(\mathcal{D}_0))$ such that $y(T) = z(T) = 0$ in \mathcal{D} , P -a.s.*

1.2 Outline of the paper

The rest of the paper is organized as follows. In Section 2, we prove in a first part the observability inequality for a suitable adjoint system and then, following well-known arguments, we obtain the proof of Theorem 1.2. In Section 3, we prove Theorem 1.3 by adapting the Lebeau-Robbiano strategy to the stochastic framework and following the previous works of Qi Lü [Lü1] and Xu Liu [Liu14a]. Finally, in Section 4, we make some final comments about our work.

2 Null controllability for the backward system

The goal of this section is to prove the null controllability of system (2). As already mentioned, this problem will be reformulated in terms of the observability of the adjoint system, which in

this case is given by

$$\begin{cases} du + (\gamma u_{xxxx} + u_{xxx} + u_{xx})dt = v_x dt + d_1 u dW(t) & \text{in } Q, \\ dv - \Gamma v_{xx} dt = (v_x + u_x)dt + (d_2 u + d_3 v)dW(t) & \text{in } Q, \\ u = u_x = 0 & \text{on } \Sigma, \\ v = 0 & \text{on } \Sigma, \\ u(x, 0) = u_0, \quad v(x, 0) = v_0 & \text{in } \mathcal{D}. \end{cases} \quad (4)$$

The main task is reduced to prove the following result.

Proposition 2.1. *Under the assumptions of Theorem 1.2, there exists a positive constant C such that for every $(u_0, v_0) \in L^2(\Omega, \mathcal{F}_0; L^2(\mathcal{D})^2)$, the solution (u, v) of (4) verifies*

$$\mathbb{E} \left(\int_{\mathcal{D}} (|u(T)|^2 + |v(T)|^2) dx \right) \leq C \mathbb{E} \left(\int_{Q_{\mathcal{D}_0}} |u|^2 dx dt \right). \quad (5)$$

To prove (5), we use Carleman estimates in the spirit of [CMP15] and adapt the procedure to the stochastic framework. As mentioned there, the key point is to have suitable estimates with the same weight functions allowing to obtain the observability inequality with only one observation term.

2.1 Preliminaries on stochastic Carleman estimates

We begin by recalling below two global Carleman estimates for the stochastic fourth-order equation and the stochastic heat equation with non-homogeneous boundary conditions. These will serve as the basis for obtaining our observability estimate.

Let us consider some $\mathcal{D}_1 \subset\subset \mathcal{D}_0$. Similar to [FI96], we show the following known result.

Lemma 2.2. *There is a $\psi \in C^\infty(\overline{\mathcal{D}})$ such that $\psi > 0$ in \mathcal{D} , $\psi(0) = \psi(1) = 0$ and $|\psi_x| > 0$ in $\overline{\mathcal{D}} \setminus \mathcal{D}_1$.*

Following [Gue07], for some positive constants k, m, μ , where $k > m$ and $m > 3$, we define

$$\alpha_m(x, t) = \frac{e^{\mu(\psi(x)+c_2)} - e^{\mu c_1}}{t^m (T-t)^m}, \quad \phi_m(x, t) = \frac{e^{\mu(c_2+\psi(x))}}{t^m (T-t)^m} \quad (6)$$

where

$$c_1 = k \left(\frac{m+1}{m} \right) \|\psi\|_\infty \quad \text{and} \quad c_2 = k \|\psi\|_\infty. \quad (7)$$

We define the weights in this way to fulfill the requirement that both estimates must have the same weight (cf. [CMP15]).

In the remainder of this section, we set $\theta = e^{\lambda \alpha_m}$ and in order to abridge the estimates, we use the following notation

$$I_{KS}(p) := \mathbb{E} \int_Q \theta^2 \lambda \phi_m (|p_{xxx}|^2 + \lambda^2 \phi_m^2 |p_{xx}|^2 + \lambda^4 \phi_m^4 |p_x|^2 + \lambda^6 \phi_m^6 |p|^2) dx dt, \quad (8)$$

$$I_H(q) := \mathbb{E} \int_Q \theta^2 \lambda \phi_m [|q_x|^2 + \lambda^2 \phi_m^2 |q|^2] dx dt. \quad (9)$$

The Carleman inequality for the forward stochastic KS system we shall use reads as follows.

Lemma 2.3. *Let $f \in L^2_{\mathcal{F}}(0, T; L^2(\mathcal{D}))$ and $F \in L^2_{\mathcal{F}}(0, T; H^2(\mathcal{D}))$ be given. There exist positive constants C , μ_1 and λ_0 , such that for any $\mu \geq \mu_0$, any $\lambda \geq \lambda_0(\mu)$ and any $p_0 \in L^2(\Omega; \mathcal{F}_0; L^2(\mathcal{D}))$, the solution p to*

$$\begin{cases} dp + p_{xxxx}dt = fdt + FdW(t) & \text{in } Q, \\ p = p_x = 0 & \text{in } \Sigma, \\ p(x, 0) = p_0 & \text{in } \mathcal{D}, \end{cases}$$

satisfies

$$\begin{aligned} I_{KS}(p) \leq C\mathbb{E} & \left(\int_Q \theta^2 |f|^2 dxdt + \int_Q \theta^2 \lambda^4 \phi_m^4 |F|^2 dxdt + \int_Q \theta^2 \lambda^2 \phi_m^2 |F_x|^2 dxdt \right. \\ & \left. + \int_Q \theta^2 |F_{xx}|^2 dxdt + \int_{Q_{\mathcal{D}_0}} \theta^2 \lambda^7 \phi_m^7 |p|^2 dxdt \right). \end{aligned} \quad (10)$$

The proof of Lemma 2.3 is essentially given in [GCL15]. Actually, the authors prove inequality (10) for slightly different weight functions, that is, they take $m = 1$, $c_2 = 3$ and $c_1 = 5$ in (6). However, a closer inspection shows that their proof can be adapted to our case just by considering that $|\partial_t \alpha_m| \leq CT\phi^{1+\frac{1}{m}}$ and $|\partial_{tt} \alpha_m| \leq CT^2\phi^{1+\frac{2}{m}}$ and changing a little bit the estimates regarding α_{xt} , α_{xxt} , α_{xxx} and so on in [GCL15, pp. 487]. It is also important to mention that we have kept the term containing p_{xxx} in the left-hand side of (10) (see eq. (8)) as it will be useful later.

On the other hand, we have the following inequality for the heat equation with non-homogeneous boundary conditions.

Lemma 2.4. *Let $g_1, G \in L^2_{\mathcal{F}}(0, T; L^2(\mathcal{D}))$ and $g_2 \in L^2_{\mathcal{F}}(0, T; L^2(\partial\mathcal{D}))$ be given. There exist positive constants μ_1 , λ_1 and C , such that for any $\mu \geq \mu_1$, any $\lambda \geq \lambda_1(\mu)$ and any $q_0 \in L^2(\Omega, \mathcal{F}_0; L^2(\mathcal{D}))$, the solution q to*

$$\begin{cases} dq - q_{xx}dt = g_1dt + GdW(t) & \text{in } Q, \\ q_x = g_2 & \text{on } \Sigma, \\ q(x, 0) = q_0 & \text{in } \mathcal{D}, \end{cases}$$

satisfies

$$\begin{aligned} I_H(q) \leq C\mathbb{E} & \left(\int_{\Sigma} \theta^2 \lambda \phi_m |g_2|^2 d\sigma dt + \int_{Q_{\mathcal{D}_0}} \theta^2 \lambda^3 \phi_m^3 |q|^2 dxdt \right. \\ & \left. + \int_Q \theta^2 (|g_1|^2 + \lambda^2 \phi_m^2 |G|^2) dxdt \right). \end{aligned} \quad (11)$$

The proof of Lemma 2.4 can be found in [Yan18]. As for the case of the KS equation, the proof can be adapted by making the corresponding changes. Notice that in this case, there are not spatial derivatives of the diffusion term G on the right-hand side of estimate (11). This comes from the fact that the proof is done by means of a duality argument, instead of a pointwise estimate technique as used, for instance, in [TZ09].

2.2 Carleman estimate for the adjoint system

Now, we are in position to prove the main result of this section, that is to say, a Carleman estimate for system (4) with only one observation on the right-hand side. In more detail, we have the following result.

Theorem 2.5. *Assume that $d_i \in L^2_{\mathcal{F}}(0, T; W^{2,\infty}(\mathcal{D}))$ for $i = 1, 2$, $d_3 \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R})$ and let $m > 3$ be given. Then, there exists $C > 0$ and two constants $\mu_2, \lambda_2 > 0$ such that for any $\mu \geq \mu_2$, $\lambda \geq \lambda_2$ and any $(u_0, v_0) \in L^2(\Omega, \mathcal{F}_0; L^2(\Omega)^2)$, the solution (u, v) to (4) verifies*

$$\mathbb{E} \left(\int_Q \theta^2 \lambda^3 \phi_m^3 |v_x|^2 dx dt + \int_Q \theta^2 \lambda^7 \phi_m^7 |u|^2 dx dt \right) \leq C \mathbb{E} \left(\int_{Q_{\mathcal{D}_0}} \theta^2 \lambda^{23} \phi_m^{23} |u|^2 dx dt \right). \quad (12)$$

The outline of the proof follows the strategy of [CMP15], but since we are dealing with stochastic PDEs, some extra arguments are needed to conclude. For clarity, we have divided the proof in four steps which can be summarized as follows:

- First part: we look for the equation satisfied by v_x . As we will see, this equation has not prescribed boundary conditions, but we can apply estimate (11) to deduce an inequality with some boundary terms and a local estimate of v_x . Using trace and interpolation estimates, we can get rid of the boundary terms.
- Second part: we apply Carleman inequality (10) to the first equation of system (4) and add it to the estimate in the previous step. By using the variable λ we will absorb the lower order terms.
- Third part: here, we will estimate the local term of v_x by using the first equation (4) and leading to several local terms depending on u and their spatial derivatives.
- Fourth part: this step is divided in two: first, using the equation verified by u , we estimate the local term corresponding to the the highest-order derivative coming from the previous stage. Then, integrating by parts several times, we deduce the desired result.

Proof of Theorem 2.5. Consider sets $\mathcal{D}_i \subset \mathcal{D}$, $i = 4, 5$, such that $\mathcal{D}_5 \subset \subset \mathcal{D}_4 \subset \subset \mathcal{D}_0$. In the following, C stands for a generic positive constant that may vary from line to line. We also consider that the initial datum (u_0, v_0) is smooth enough (as usual, the general case follows from a density argument).

Step 1. Carleman estimate for v_x

A direct computation shows that v_x verifies the equation

$$dv_x - \Gamma(v_x)_{xx} dt = [u_{xx} + (v_x)_x] dt + [(d_2 u)_x + d_3 v_x] dW(t) \quad \text{in } Q,$$

with no prescribed boundary conditions. Hence, applying estimate (11) (for the set \mathcal{D}_4) to this equation yields

$$\begin{aligned} I_H(v_x) &\leq C \mathbb{E} \left(\int_{\Sigma} \theta^2 \lambda \phi_m |v_{xx}|^2 d\sigma dt + \int_{Q_{\mathcal{D}_4}} \theta^2 \lambda^3 \phi_m^3 |v_x|^2 dx dt \right) \\ &\quad + C \mathbb{E} \left(\int_Q \theta^2 |u_{xx} + v_{xx}|^2 dx dt + \int_Q \theta^2 \lambda^2 \phi_m^2 |(d_2 u)_x + d_3 v_x|^2 dx dt \right). \end{aligned} \quad (13)$$

Observe that

$$\begin{aligned} \mathbb{E} \left(\int_{\Sigma} \theta^2 \lambda \phi_m |v_{xx}|^2 d\sigma dt \right) &= \mathbb{E} \left(\int_0^T \theta^2(1, t) \lambda \phi_m(1, t) |v_{xx}(1, t)|^2 dt \right) \\ &\quad - \mathbb{E} \left(\int_0^T \theta^2(0, t) \lambda \phi_m(0, t) |v_{xx}(0, t)|^2 dt \right). \end{aligned}$$

For compactness, we will maintain the notation in (13).

Using that $d_3 \in L^\infty(0, T; \mathbb{R})$ and taking λ_1 large enough, we can simplify the above expression as follows

$$\begin{aligned} I_H(v_x) \leq & C\mathbb{E} \left(\int_{\Sigma} \theta^2 \lambda \phi_m |v_{xx}|^2 d\sigma dt + \int_{Q_{\mathcal{D}_4}} \theta^2 \lambda^3 \phi_m^3 |v_x|^2 dx dt \right) \\ & + C\mathbb{E} \left(\int_Q \theta^2 |u_{xx}|^2 dx dt + \int_Q \theta^2 \lambda^2 \phi_m^2 |(d_2 u)_x|^2 dx dt \right). \end{aligned} \quad (14)$$

for any $\lambda \geq \lambda_1$.

Let us focus now on the first term in the right-hand side of the previous inequality. Observe that thanks to the properties of the weight function ψ (see Lemma 2.2), the functions θ and ϕ_m achieve its minimum at the boundary points, that is

$$\phi_m(0, t) = \phi_m(1, t) = \min_{x \in \overline{\mathcal{D}}} \phi_m =: \phi_m^*(t), \quad (15)$$

$$\theta(0, t) = \theta(1, t) = \min_{x \in \overline{\mathcal{D}}} \theta =: \theta^*(t). \quad (16)$$

Using this notation and employing classical trace and Sobolev embedding theorems, we can obtain that

$$\mathbb{E} \left(\int_{\Sigma} (\theta^*)^2 \lambda \phi_m^* |v_{xx}|^2 d\sigma dt \right) \leq C\mathbb{E} \left(\int_0^T (\theta^*)^2 \lambda \phi_m^* \|v\|_{H^{5/2+\epsilon}(\mathcal{D})}^2 dt \right), \quad \forall \epsilon > 0.$$

Let us fix $0 < \epsilon < 1/2$. Applying the classical interpolation inequality in Sobolev spaces

$$\|v\|_{H^{t\sigma}} \leq \|v\|_{H^{t_0}}^{1-\sigma} \|v\|_{H^{t_1}}^{\sigma},$$

where $t_0, t_1 \in \mathbb{R}$ and $t_\sigma = (1-\sigma)t_0 + \sigma t_1$, $\sigma \in [0, 1]$, with $t_0 = 3$ and $t_1 = 1$ we get

$$\mathbb{E} \left(\int_{\Sigma} (\theta^*)^2 \lambda \phi_m^* |v_{xx}|^2 d\sigma dt \right) \leq C\mathbb{E} \left(\int_0^T (\theta^*)^2 \lambda \phi_m^* \left(\|v\|_{H^3(\mathcal{D})}^{1-\sigma} \|v\|_{H^1(\mathcal{D})}^{\sigma} \right)^2 dt \right)$$

for some $\sigma = \sigma(\epsilon) \in (0, \frac{1}{4})$.

We can conveniently rewrite the right-hand side of the above expression as

$$\mathbb{E} \left(\int_0^T \left[(\theta^*)^{2\sigma} (\lambda \phi_m^*)^{3\sigma} \|v\|_{H^1(\mathcal{D})}^{2\sigma} \right] \left[(\theta^*)^{2-2\sigma} (\lambda \phi_m^*)^{1-3\sigma} \|v\|_{H^3(\mathcal{D})}^{2(1-\sigma)} \right] dt \right)$$

and using Young inequality with $p = 1/\sigma$ and $q = 1/(1-\sigma)$, we get for any $\delta > 0$

$$\begin{aligned} & \mathbb{E} \left(\int_{\Sigma} (\theta^*)^2 \lambda \phi_m^* |v_{xx}|^2 d\sigma dt \right) \\ & \leq \delta \mathbb{E} \left(\int_0^T (\theta^*)^2 (\lambda \phi_m^*)^3 \|v\|_{H^1(\mathcal{D})}^2 dt \right) + C_\delta \mathbb{E} \left(\int_0^T (\theta^*)^2 (\lambda \phi_m^*)^{\frac{1-3\sigma}{1-\sigma}} \|v\|_{H^3(\mathcal{D})}^2 dt \right) \\ & \leq \delta \mathbb{E} \left(\int_Q \theta^2 \lambda^3 \phi_m^3 |v_x|^2 dx dt \right) + C_\delta \mathbb{E} \left(\int_0^T (\theta^*)^2 (\lambda \phi_m^*)^{\frac{1-3\sigma}{1-\sigma}} \|v\|_{H^3(\mathcal{D})}^2 dt \right), \end{aligned} \quad (17)$$

where we have used the fact that $v = 0$ on Σ and definitions (15)–(16).

Using estimate (17) in (14) and taking $\delta > 0$ small enough yields

$$\begin{aligned} I_H(v_x) \leq & C\mathbb{E} \left(\int_0^T (\theta^*)^2 (\lambda \phi_m^*)^{\frac{1-3\sigma}{1-\sigma}} \|v\|_{H^3(\mathcal{D})}^2 dt + \int_{Q_{\mathcal{D}_4}} \theta^2 \lambda^3 \phi_m^3 |v_x|^2 dx dt \right) \\ & + C\mathbb{E} \left(\int_Q \theta^2 |u_{xx}|^2 dx dt + \mathbb{E} \int_Q \theta^2 \lambda^2 \phi_m^2 |(d_2 u)_x|^2 dx dt \right). \end{aligned} \quad (18)$$

Now, the task is to absorb the global term containing the H^3 -norm. To this end, consider the function

$$g(t) = \lambda^{\frac{1}{2} - \frac{1}{m}} \theta^* (\phi_m^*)^{\frac{1}{2} - \frac{1}{m}}$$

and define the change of variables $\tilde{v} := g(t)v$. Then, using Itô's formula, we see that \tilde{v} satisfies the equation

$$\begin{cases} d\tilde{v} = (\Gamma \tilde{v}_{xx} + \tilde{v}_x + g_t v + g u_x) dt + (d_2 g u + d_3 \tilde{v}) dW(t) & \text{in } Q, \\ \tilde{v} = 0 & \text{on } \Sigma, \\ \tilde{v}(0) = 0 & \text{in } \mathcal{D}, \end{cases}$$

where we have used that $\lim_{t \rightarrow 0} g(t) = 0$. Using classical energy estimates for stochastic parabolic equations (see, for instance, [Zho92, Proposition 2.1]), we have that \tilde{v} satisfies

$$\begin{aligned} & \sup_{0 \leq t \leq T} \mathbb{E} \left(\|\tilde{v}(t)\|_{H^2(\mathcal{D})}^2 \right) + \mathbb{E} \left(\int_0^T \|\tilde{v}(t)\|_{H^3(\mathcal{D})}^2 dt \right) \\ & \leq C \mathbb{E} \left(\int_0^T \|g_t v + g u_x\|_{H^1(\mathcal{D})}^2 dt + \int_0^T \|d_2 g u\|_{H^2(\mathcal{D})}^2 dt \right). \end{aligned} \quad (19)$$

We observe that if we choose $m > \frac{1-\sigma}{\sigma}$ then $\frac{1-3\sigma}{1-\sigma} < 1 - \frac{2}{m}$, thus, from the definition of \tilde{v} , we obtain

$$\mathbb{E} \left(\int_0^T (\lambda \phi_m^*)^{\frac{1-3\sigma}{1-\sigma}} (\theta^*)^2 \|v\|_{H^3(\mathcal{D})}^2 dt \right) \leq \mathbb{E} \left(\int_0^T \|\tilde{v}\|_{H^3(\mathcal{D})}^2 dt \right) \quad (20)$$

whence we have from (19)

$$\mathbb{E} \left(\int_0^T (\lambda \phi_m^*)^{\frac{1-3\sigma}{1-\sigma}} (\theta^*)^2 \|v\|_{H^3(\mathcal{D})}^2 dt \right) \leq C \mathbb{E} \left(\int_0^T \|g_t v + g u_x\|_{H^1(\mathcal{D})}^2 dt + \int_0^T \|d_2 g u\|_{H^2(\mathcal{D})}^2 dt \right). \quad (21)$$

We remark that choosing $m > \frac{1-\sigma}{\sigma}$ with $\sigma \in (0, \frac{1}{4})$ implies that $m > 3$ which is consistent with the construction of the weights (6).

Using once again that v has homogeneous Dirichlet boundary conditions and since

$$|\partial_t (\theta^* [\phi_m^*]^q)| \leq C \lambda (\phi_m^*)^{1 + \frac{1}{m}} (\theta^* [\phi_m^*]^q), \quad (22)$$

for any integer $q > 0$, we can bound in the right-hand side of (21) as follows

$$\begin{aligned} & \mathbb{E} \left(\int_0^T (\lambda \phi_m^*)^{\frac{1-3\sigma}{1-\sigma}} (\theta^*)^2 \|v\|_{H^3(\mathcal{D})}^2 dt \right) \\ & \leq C \mathbb{E} \left(\int_0^T \lambda^{3-\frac{2}{m}} (\phi_m^*)^3 (\theta^*)^2 \|v_x(t)\|_{L^2(\mathcal{D})}^2 dt + \int_0^T \|g u_x\|_{H^1(\mathcal{D})}^2 dt + \int_0^T \|d_2 g u\|_{H^2(\mathcal{D})}^2 dt \right) \\ & \leq C \mathbb{E} \left(\int_Q \lambda^{3-\frac{2}{m}} \phi_m^3 \theta^2 |v_x|^2 dx dt + \int_0^T \|g u_x\|_{H^1(\mathcal{D})}^2 dt + \int_0^T \|d_2 g u\|_{H^2(\mathcal{D})}^2 dt \right), \end{aligned}$$

where we have recalled (15)–(16).

For the last two terms in the above expression, it can be readily seen that since $u \in H_0^2(\mathcal{D})$ and $d_2 \in L_{\mathcal{F}}^\infty(0, T; W^{2,\infty}(\mathcal{D}))$, we have

$$\begin{aligned} & \mathbb{E} \left(\int_0^T (\lambda \phi_m^*)^{\frac{1-3\sigma}{1-\sigma}} (\theta^*)^2 \|v\|_{H^3(\mathcal{D})}^2 dt \right) \leq C \mathbb{E} \left(\int_Q \lambda^{3-2/m} \phi_m^3 \theta^2 |v_x|^2 dx dt \right) \\ & + C \|d_2\|_{L_{\mathcal{F}}^\infty(0, T; W^{2,\infty}(\mathcal{D}))}^2 \mathbb{E} \left(\int_Q \lambda^{1-2/m} \theta^2 \phi_m^{1-2/m} |u_{xx}|^2 dx dt \right). \end{aligned} \quad (23)$$

Observe that the power of λ in the first term of the right-hand side is lower than its counterpart in the left-hand side of (18). Hence, combining estimates (23) and (18) and taking λ large enough, we get

$$I_H(v_x) \leq C\mathbb{E} \left(\int_{Q_{\mathcal{D}_4}} \theta^2 \lambda^3 \phi_m^3 |v_x|^2 \, dxdt + \int_Q \lambda^{1-2/m} \theta^2 \phi_m^{1-2/m} |u_{xx}|^2 \, dxdt \right. \\ \left. + \int_Q \theta^2 |u_{xx}|^2 \, dxdt + \int_Q \theta^2 \lambda^2 \phi_m^2 |(d_2 u)_x|^2 \, dxdt \right), \quad (24)$$

for any $\lambda \geq \lambda_1$.

Step 2. Carleman estimate for u

Fix $m > 3$ coming from the previous step. We apply inequality (10) to the first equation of system (4), note that both estimates have the same weight. We readily see that

$$I_{KS}(u) \leq C\mathbb{E} \left(\int_Q \theta^2 |v_x - u_{xx} - u_{xxx}|^2 \, dxdt + \int_Q \theta^2 \lambda^4 \phi_m^4 |d_1 u|^2 \, dxdt \right. \\ \left. + \int_Q \theta^2 \lambda^2 \phi_m^2 |(d_1 u)_x|^2 + \int_Q \theta^2 |(d_1 u)_{xx}|^2 \, dxdt + \int_{Q_{\mathcal{D}_4}} \theta^2 \lambda^7 \phi_m^7 |u|^2 \, dxdt \right).$$

Using that $d_1 \in L_{\mathcal{F}}^\infty(0, T; W^{2, \infty}(\mathcal{D}))$ and taking $\lambda \geq \lambda_0$ large enough, we can absorb the lower order terms corresponding to the variable u , more precisely,

$$I_{KS}(u) \leq C\mathbb{E} \left(\int_Q \theta^2 |v_x|^2 \, dxdt + \int_{Q_{\mathcal{D}_4}} \theta^2 \lambda^7 \phi_m^7 |u|^2 \, dxdt \right). \quad (25)$$

Adding up inequalities (24) and (25), we take $\mu \geq \mu_2 = \max\{\mu_0, \mu_1\}$ and $\lambda \geq \lambda_2 = \max\{\lambda_0, \lambda_1\}$, we can absorb the remaining lower order terms to finally obtain

$$I_H(v_x) + I_{KS}(u) \leq C\mathbb{E} \left(\int_{Q_{\mathcal{D}_4}} \theta^2 \lambda^3 \phi_m^3 |v_x|^2 \, dxdt + \int_{Q_{\mathcal{D}_4}} \theta^2 \lambda^7 \phi_m^7 |u|^2 \, dxdt \right) \quad (26)$$

for all λ sufficiently large.

Step 3. Local energy estimate for v

The main goal of this step is to estimate the local term corresponding to v_x . We follow the classical methodology introduced in [dT00] but adapted to the stochastic setting.

Let us consider an open set \mathcal{D}_3 such that $\mathcal{D}_4 \subset\subset \mathcal{D}_3 \subset\subset \mathcal{D}_0$ and take a function $\eta \in C_0^\infty(\mathcal{D}_3)$ such that $\eta \equiv 1$ in \mathcal{D}_4 . We define $\zeta := \lambda^3 \phi_m^3 \theta^2 \eta$ and apply Itô's formula to compute $d(\zeta u v_x)$, from which we deduce

$$\mathbb{E} \left(\int_Q \zeta |v_x|^2 \, dxdt \right) = -\mathbb{E} \left(\int_Q \zeta_t u v_x \, dxdt \right) + \mathbb{E} \left(\int_Q \zeta (\gamma v_x u_{xxxx} + v_x u_{xxx} + v_x u_{xx}) \, dxdt \right) \\ - \mathbb{E} \left(\int_Q \zeta (\Gamma u v_{xxx} + u u_{xx} + u v_{xx}) \, dxdt \right) \\ - \mathbb{E} \left(\int_Q \zeta [d_1 u (d_2 u)_x + d_1 u d_3 v_x] \, dxdt \right) \\ =: \sum_{i=1}^9 K_i. \quad (27)$$

Let us estimate each K_i , $1 \leq i \leq 9$. For $i = 1$, we can use (22) (which is also valid for θ and ϕ_m) and Cauchy-Schwarz and Young inequalities to obtain

$$|K_1| \leq \epsilon \mathbb{E} \left(\int_Q \eta \lambda^3 \phi_m^3 \theta^2 |v_x|^2 dx dt \right) + C_\epsilon \mathbb{E} \left(\int_Q \eta \lambda^5 \phi_m^{5+\frac{2}{m}} \theta^2 |u|^2 dx dt \right) \quad (28)$$

for any $\epsilon > 0$.

Integrating by parts in the space variable, we have

$$\begin{aligned} K_2 &= -\mathbb{E} \left(\int_Q \gamma \eta \lambda^3 \phi_m^3 \theta^2 v_{xx} u_{xxx} dx dt \right) - \mathbb{E} \left(\int_Q \gamma \lambda^3 \phi_m^3 \theta^2 v_x \eta_x u_{xxx} dx dt \right) \\ &\quad - \mathbb{E} \left(\int_Q \gamma \lambda^3 \eta v_x (\phi_m^3 \theta^2)_x u_{xxx} dx dt \right). \end{aligned} \quad (29)$$

Noting that

$$|\partial_x(\theta^2 \phi_m^q)| \leq C \lambda \phi_m \theta^2 \phi_m^q, \quad \forall q \in \mathbb{Z}_+, \quad (30)$$

it is not difficult to see that

$$\begin{aligned} |K_2| &\leq \delta \mathbb{E} \left(\int_Q \theta^2 \lambda \phi_m |v_{xx}|^2 dx dt \right) + 2\epsilon \mathbb{E} \left(\int_Q \theta^2 \lambda^3 \phi_m^3 |v_x|^2 dx dt \right) \\ &\quad + C_{\epsilon, \delta} \left(\mathbb{E} \int_{Q_{\mathcal{D}_3}} \theta^2 \lambda^5 \phi_m^5 |u_{xxx}|^2 dx dt \right) \end{aligned} \quad (31)$$

for any $\delta, \epsilon > 0$. In this part, we have used that $\text{supp } \eta_x \subset \mathcal{D}_3$ for estimating the second term of (29).

For the third term, we have from Cauchy-Schwarz and Young inequalities

$$|K_3| \leq \epsilon \mathbb{E} \left(\int_Q \eta \lambda^3 \phi_m^3 \theta^2 |v_x|^2 dx dt \right) + C_\epsilon \mathbb{E} \left(\int_Q \lambda^3 \phi_m^3 \theta^2 \eta |u_{xxx}|^2 dx dt \right). \quad (32)$$

In the same fashion, we easily have

$$|K_4| \leq \epsilon \mathbb{E} \left(\int_Q \eta \lambda^3 \phi_m^3 \theta^2 |v_x|^2 dx dt \right) + C_\epsilon \mathbb{E} \left(\int_Q \eta \lambda^3 \phi_m^3 \theta^2 |u_{xx}|^2 dx dt \right). \quad (33)$$

For the term K_5 , we integrate by parts in the space variable to get

$$\begin{aligned} K_5 &= \mathbb{E} \left(\int_Q \Gamma \lambda^3 \phi_m^3 \theta^2 \eta u_x v_{xx} dx dt \right) + \mathbb{E} \left(\int_Q \Gamma \lambda^3 \phi_m^3 \theta^2 \eta_x u v_{xx} dx dt \right) \\ &\quad + \mathbb{E} \left(\int_Q \Gamma \lambda^3 \eta (\phi_m^3 \theta^2)_x u v_{xx} dx dt \right). \end{aligned}$$

Using (30) and the properties of the function η , we get after successive application of Cauchy-Schwarz and Young inequalities

$$\begin{aligned} |K_5| &\leq 3\mathbb{E} \left(\delta \int_Q \lambda \phi_m \theta^2 |v_{xx}|^2 dx dt \right) + C_\delta \mathbb{E} \left(\int_{Q_{\mathcal{D}_3}} \lambda^7 \phi_m^7 \theta^2 |u|^2 dx dt \right) \\ &\quad + C_\delta \mathbb{E} \left(\int_{Q_{\mathcal{D}_3}} \lambda^5 \phi_m^5 \theta^2 |u_x|^2 dx dt \right) \end{aligned} \quad (34)$$

for any $\delta > 0$.

For the sixth and seventh terms, we have

$$|K_6| \leq \frac{1}{2} \mathbb{E} \left(\int_Q \eta \lambda^3 \phi_m^3 \theta^2 |u|^2 \, dx dt \right) + \frac{1}{2} \mathbb{E} \left(\int_Q \eta \lambda^3 \phi_m^3 \theta^2 |u_{xx}|^2 \, dx dt \right), \quad (35)$$

and

$$|K_7| \leq \delta \mathbb{E} \left(\int_Q \eta \lambda \phi_m \theta^2 |v_{xx}|^2 \, dx dt \right) + C_\delta \mathbb{E} \left(\int_Q \eta \lambda^5 \phi_m^5 \theta^2 |u|^2 \, dx dt \right). \quad (36)$$

The last two terms can be estimated as

$$|K_8| + |K_9| \leq \epsilon \mathbb{E} \left(\int_Q \eta \lambda^3 \phi_m^3 \theta^2 |v_x|^2 \, dx dt \right) + C \mathbb{E} \left(\int_Q \eta \lambda^3 \phi_m^3 \theta^2 (|u|^2 + |u_x|^2) \, dx dt \right), \quad (37)$$

where the constant C depends on $\|d_i\|_{L^\infty(0,T;W^{2,\infty}(\mathcal{D}))}$, $i = 1, 2$ and $\|d_3\|_{L^\infty(0,T;\mathbb{R})}$.

Summarizing, we collect estimates (28) and (31)–(37), use the properties for η and employ them on identity (27) to obtain

$$\begin{aligned} \mathbb{E} \left(\int_{Q_{\mathcal{D}_4}} \lambda^3 \phi_m^3 \theta^2 |v_x|^2 \, dx dt \right) &\leq 5\delta \mathbb{E} \left(\int_Q \lambda \phi_m \theta^2 |v_{xx}|^2 \, dx dt \right) + 6\epsilon \mathbb{E} \left(\int_Q \lambda^3 \phi_m^3 \theta^2 |v_x|^2 \, dx dt \right) \\ &\quad + C_{\delta,\epsilon} \mathbb{E} \left(\int_{Q_{\mathcal{D}_3}} \theta^2 \lambda^7 \phi_m^7 |u|^2 \, dx dt + \int_{Q_{\mathcal{D}_3}} \theta^2 \lambda^5 \phi_m^5 |u_x|^2 \, dx dt \right) \\ &\quad + C_{\delta,\epsilon} \mathbb{E} \left(\int_{Q_{\mathcal{D}_3}} \theta^2 \lambda^3 \phi_m^3 |u_{xx}|^2 \, dx dt + \int_{Q_{\mathcal{D}_3}} \theta^2 \lambda^5 \phi_m^5 |u_{xxx}|^2 \, dx dt \right). \end{aligned}$$

Then, using the above inequality in (26) and taking ϵ and δ small enough, we get

$$\begin{aligned} I_H(v_x) + I_{KS}(u) &\leq C \mathbb{E} \left(\int_{Q_{\mathcal{D}_3}} \theta^2 \lambda^7 \phi_m^7 |u|^2 \, dx dt + \int_{Q_{\mathcal{D}_3}} \theta^2 \lambda^5 \phi_m^5 |u_x|^2 \, dx dt \right) \\ &\quad + C \mathbb{E} \left(\int_{Q_{\mathcal{D}_3}} \theta^2 \lambda^3 \phi_m^3 |u_{xx}|^2 \, dx dt + \int_{Q_{\mathcal{D}_3}} \theta^2 \lambda^5 \phi_m^5 |u_{xxx}|^2 \, dx dt \right). \end{aligned} \quad (38)$$

Step 4: Local energy estimates for u and their derivatives

In the previous step, we have estimated v_x in terms of local integrals of u and its derivatives, so the variable v does not longer appear on the right-hand side.

At this point, estimate (38) looks quite similar to its deterministic counterpart (cf. [CMP15, Proof of Theorem 3.1]). However, unlike that case, we cannot estimate the local term of u_{xxx} just by integrating by parts since we do not have a global term of u_{xxxx} on the left-hand side of (38).

Instead, we have the following result.

Lemma 2.6. *Consider an open set \mathcal{D}_2 such that $\mathcal{D}_3 \subset\subset \mathcal{D}_2 \subset\subset \mathcal{D}_0$. Then, there exists $C > 0$*

such that

$$\begin{aligned}
& \mathbb{E} \left(\int_{Q_{\mathcal{D}_3}} \lambda^5 \phi_m^5 \theta^2 |u_{xxx}|^2 dx dt \right) \\
& \leq 4\epsilon \mathbb{E} \left(\int_Q \lambda \phi_m \theta^2 |u_{xxx}|^2 dx dt \right) + 2\delta \mathbb{E} \left(\int_Q \lambda \phi_m \theta^2 |v_{xx}|^2 dx dt \right) \\
& + \rho \mathbb{E} \left(\int_Q \lambda^3 \phi_m^3 \theta^2 |v_x|^2 dx dt \right) + C \mathbb{E} \left(\int_{Q_{\mathcal{D}_2}} \lambda^{13} \phi_m^{13} \theta^2 |u_x|^2 dx dt \right) \\
& + C \mathbb{E} \left(\int_{Q_{\mathcal{D}_2}} \lambda^{15} \phi_m^{15} \theta^2 |u|^2 dx dt \right) + C \mathbb{E} \left(\int_{Q_{\mathcal{D}_2}} \lambda^7 \phi_m^7 \theta^2 |u_{xx}|^2 dx dt \right) \tag{39}
\end{aligned}$$

for any $\epsilon, \delta, \rho > 0$.

The idea of the proof is to obtain the differential of a suitable product and argue as in the previous step. To avoid too much word repetition, we present a brief proof on Appendix B.

Using (39) in (38) and taking ϵ, δ and ρ small enough we get

$$\begin{aligned}
I_H(v_x) + I_{KS}(u) & \leq C \mathbb{E} \left(\int_{Q_{\mathcal{D}_2}} \theta^2 \lambda^{15} \phi_m^{15} |u|^2 dx dt + \int_{Q_{\mathcal{D}_2}} \theta^2 \lambda^{13} \phi_m^{13} |u_x|^2 dx dt \right) \\
& + C \mathbb{E} \left(\int_{Q_{\mathcal{D}_2}} \theta^2 \lambda^7 \phi_m^7 |u_{xx}|^2 dx dt \right). \tag{40}
\end{aligned}$$

Now, taking \mathcal{D}_1 with $\mathcal{D}_2 \subset\subset \mathcal{D}_1 \subset\subset \mathcal{D}_0$ and constructing a cut-off function $\eta_2 \in C_0^\infty(\mathcal{D}_1)$ such that $\eta_2 \equiv 1$ in \mathcal{D}_2 , we estimate

$$\begin{aligned}
\mathbb{E} \left(\int_{Q_{\mathcal{D}_2}} \theta^2 \lambda^7 \phi_m^7 |u_{xx}|^2 dx dt \right) & \leq \mathbb{E} \left(\int_Q \eta_2 \theta^2 \lambda^7 \phi_m^7 |u_{xx}|^2 dx dt \right) \\
& = -\mathbb{E} \left(\int_Q \eta_2 \theta^2 \lambda^7 \phi_m^7 u_{xxx} u_x dx dt \right) \\
& + \frac{1}{2} \mathbb{E} \left(\int_Q (\eta_2 \lambda^7 \phi_m^7 \theta^2)_{xx} |u_x|^2 dx dt \right) \\
& \leq \epsilon \mathbb{E} \left(\int_Q \theta^2 \lambda \phi_m |u_{xxx}|^2 dx dt \right) \\
& + C_\epsilon \mathbb{E} \left(\iint_{Q_{\mathcal{D}_1}} \theta^2 \lambda^{13} \phi_m^{13} |u_x|^2 dx dt \right). \tag{41}
\end{aligned}$$

Moreover, taking $\eta_3 \in C_0^\infty(\mathcal{D}_0)$ such that $\eta_3 \equiv 1$ in \mathcal{D}_1 , we can argue in the same way to obtain

$$\begin{aligned}
\mathbb{E} \left(\int_{Q_{\mathcal{D}_1}} \theta^2 \lambda^{13} \phi_m^{13} |u_x|^2 dx dt \right) & \leq \mathbb{E} \left(\int_Q \eta_3 \lambda^{13} \phi_m^{13} |u_x|^2 dx dt \right) \\
& \leq \epsilon \mathbb{E} \left(\int_Q \theta^2 \lambda^3 \phi_m^3 |u_{xx}|^2 dx dt \right) \\
& + C_\epsilon \mathbb{E} \left(\int_{Q_{\mathcal{D}_0}} \theta^2 \lambda^{23} \phi_m^{23} |u|^2 dx dt \right). \tag{42}
\end{aligned}$$

Putting together (40), (41) and (42) and taking $\epsilon > 0$ sufficiently small, we obtain the desired result. This ends the proof. \square

2.3 The observability inequality

Once we have obtained the Carleman estimate (12), the observability inequality (5) follows immediately.

Proof of Proposition 2.1. The proof is classical and follows well-known arguments (see, e.g., [FCG06] in the deterministic setting). For completeness, we sketch it briefly.

Using the properties of the weight functions θ and ϕ_m , it is not difficult to see that

$$\begin{aligned} (\lambda\phi_m)^{23}\theta^2 &\leq C, \quad \forall(x, t) \in Q, \\ (\lambda\phi_m)^j\theta^2 &\geq C_j, \quad \forall(x, t) \in \mathcal{D} \times \left(\frac{T}{4}, \frac{3T}{4}\right), \quad j = 3, 7, \end{aligned}$$

where the constant $C > 0$ only depends on \mathcal{D} , \mathcal{D}_0 , m and T . Therefore, we get from (12)

$$\mathbb{E} \left(\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\mathcal{D}} (|v_x|^2 + |u|^2) \, dxdt \right) \leq C\mathbb{E} \left(\iint_{Q_{\mathcal{D}_0}} |u|^2 \, dxdt \right). \quad (43)$$

Using Itô's formula, we compute $dv^2 = 2vdv + (dv)^2$ and using the equation verified by v , we deduce

$$\begin{aligned} \mathbb{E} \left(\int_{\mathcal{D}} |v(t_2)|^2 \, dx \right) - \mathbb{E} \left(\int_{\mathcal{D}} |v(t_1)|^2 \, dx \right) &= 2\mathbb{E} \left(\int_{t_1}^{t_2} \int_{\mathcal{D}} (\Gamma v_{xx} + u_x + v_x) v \, dxdt \right) \\ &\quad + \mathbb{E} \left(\int_{t_1}^{t_2} \int_{\mathcal{D}} |d_2u + d_3v|^2 \, dxdt \right) \end{aligned}$$

for all $0 \leq t_1 < t_2 \leq T$. Integrating by parts in the space variable, we get

$$\begin{aligned} \mathbb{E} \left(\int_{\mathcal{D}} |v(t_2)|^2 \, dx \right) + 2\Gamma\mathbb{E} \left(\int_{t_1}^{t_2} \int_{\mathcal{D}} |v_x|^2 \, dxdt \right) \\ = \mathbb{E} \left(\int_{\mathcal{D}} |v(t_1)|^2 \, dx \right) - 2\mathbb{E} \left(\int_{t_1}^{t_2} \int_{\mathcal{D}} (u - v) v_x \, dxdt \right) \\ + \mathbb{E} \left(\int_{t_1}^{t_2} \int_{\mathcal{D}} |d_2u + d_3v|^2 \, dxdt \right). \end{aligned} \quad (44)$$

Arguing in the same way for the variable u , we may obtain

$$\begin{aligned} \mathbb{E} \left(\int_{\mathcal{D}} |u(t_2)|^2 \, dx \right) + 2\gamma\mathbb{E} \left(\int_{t_1}^{t_2} \int_{\mathcal{D}} |u_{xx}|^2 \, dxdt \right) \\ = \mathbb{E} \left(\int_{\mathcal{D}} |u(t_1)|^2 \, dx \right) + 2\mathbb{E} \left(\int_{t_1}^{t_2} \int_{\mathcal{D}} u_x u_{xx} \, dxdt \right) + 2\mathbb{E} \left(\int_{t_1}^{t_2} \int_{\mathcal{D}} |u_x|^2 \, dxdt \right) \\ + 2\mathbb{E} \left(\int_{t_1}^{t_2} \int_{\mathcal{D}} uv_x \, dxdt \right) + \mathbb{E} \left(\int_{t_1}^{t_2} \int_{\mathcal{D}} |d_1u|^2 \, dxdt \right). \end{aligned} \quad (45)$$

Combining estimates (44)–(45) and using Cauchy-Schwarz and Young inequalities we get

$$\begin{aligned} \mathbb{E} \left(\int_{\mathcal{D}} (|u(t_2)|^2 + |v(t_2)|^2) \, dx \right) + \frac{\gamma}{2}\mathbb{E} \left(\int_{t_1}^{t_2} \int_{\mathcal{D}} |u_{xx}|^2 \, dxdt \right) + \frac{\Gamma}{2}\mathbb{E} \left(\int_{t_1}^{t_2} \int_{\mathcal{D}} |v_x|^2 \, dxdt \right) \\ \leq C\mathbb{E} \left(\int_{t_1}^{t_2} \int_{\mathcal{D}} (|u|^2 + |v|^2) \, dxdt \right) + C\mathbb{E} \left(\int_{\mathcal{D}} (|u(t_1)|^2 + |v(t_1)|^2) \, dx \right), \end{aligned} \quad (46)$$

where $C > 0$ depends on γ , Γ and the norms of d_i , $i = 1, 2, 3$. Here, we also used the inequality

$$\int_{\mathcal{D}} |u_x|^2 dx \leq \epsilon \int_{\mathcal{D}} |u_{xx}|^2 dx + C_\epsilon \int_{\mathcal{D}} |u|^2 dx, \quad \text{for all } \epsilon > 0.$$

Using Gronwall inequality and then integrating from $(\frac{T}{4}, \frac{3T}{4})$, we get from (46)

$$\frac{T}{2} \mathbb{E} \left(\int_{\mathcal{D}} (|v(T)|^2 + |u(T)|^2) dx \right) \leq C \mathbb{E} \left(\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\mathcal{D}} (|v|^2 + |u|^2) dx dt \right). \quad (47)$$

The result follows by combining (43) with (47) and employing Poincaré inequality. \square

2.4 Null controllability result

The proof is standard and follows well-known arguments, see, for instance, [TZ09, LL12].

Proof of Theorem 1.2. We introduce the linear subspace of $L^2_{\mathcal{F}}(0, T; L^2(\mathcal{D}_0))$

$$\mathcal{X} = \left\{ u|_{Q_{\mathcal{D}_0} \times \Omega} \mid (u, v) \text{ solve (4) with some } (u_0, v_0) \in L^2(\Omega, \mathcal{F}_0; L^2(\mathcal{D})^2) \right\}$$

and define the linear functional on \mathcal{X} as

$$\mathcal{L}(u|_{Q_{\mathcal{D}_0} \times \Omega}) := \mathbb{E} \left(\int_{\mathcal{D}} (u(T)y_T + v(T)z_T) dx \right).$$

Note that \mathcal{L} is a bounded linear functional on \mathcal{X} . Indeed, by means of Cauchy-Schwarz inequality and Proposition 2.1, we have

$$|\mathcal{L}(u|_{Q_{\mathcal{D}_0} \times \Omega})| \leq \sqrt{C} \left(\mathbb{E} \int_{Q_{\mathcal{D}_0}} |u|^2 dx dt \right)^{1/2} \left(\mathbb{E} \int_{\mathcal{D}} (|y_T|^2 + |z_T|^2) dx \right)^{1/2}.$$

where C is the constant appearing in (5). Using Hahn-Banach theorem, \mathcal{L} can be extended to a bounded linear function of $L^2_{\mathcal{F}}(0, T; L^2(\mathcal{D}_0))$ and, for the sake of simplicity, we use the same notation for the extension. From Riesz representation theorem we can find a random field $h \in L^2_{\mathcal{F}}(0, T; L^2(\mathcal{D}_0))$ such that

$$\mathbb{E} \left(\int_{\mathcal{D}} (u(T)y_T + v(T)z_T) dx \right) = \mathbb{E} \left(\int_{Q_{\mathcal{D}_0}} hu dx dt \right). \quad (48)$$

We claim that this h is exactly the control that drives the solution of (y, z) to zero. Using Itô's formula, we compute both $d(yu)$ and $d(zv)$ and after integration by parts, we get

$$\mathbb{E} \left(\int_{\mathcal{D}} y_T u(T) dx \right) - \mathbb{E} \left(\int_{\mathcal{D}} y(0)u_0 dx \right) = \mathbb{E} \left(\int_Q (z_x - d_2 Z + h\chi_{\mathcal{D}_0})u dx dt \right) - \mathbb{E} \left(\int_Q v y_x dx dt \right)$$

and

$$\mathbb{E} \left(\int_{\mathcal{D}} z_T v(T) dx \right) - \mathbb{E} \left(\int_{\mathcal{D}} z(0)v_0 dx \right) = \mathbb{E} \left(\int_Q y_x v dx dt \right) + \mathbb{E} \left(\int_Q (d_2 Z - z_x)u dx dt \right).$$

Adding the previous expressions, we obtain

$$\mathbb{E} \left(\int_{\mathcal{D}} (y_T u(T) + z_T v(T)) dx \right) - \mathbb{E} \left(\int_{\mathcal{D}} (y(0)u_0 + z(0)v_0) dx \right) = \mathbb{E} \left(\int_{Q_{\mathcal{D}_0}} hu dx dt \right) \quad (49)$$

and comparing (49) and (48) yields

$$\mathbb{E} \left(\int_{\mathcal{D}} (y(0)u_0 + z(0)v_0) dx \right) = 0.$$

Since (u_0, v_0) can be chosen arbitrarily in $L^2(\Omega, \mathcal{F}_0; L^2(\mathcal{D})^2)$, we have $(y(0), z(0)) = 0$ in \mathcal{D} , P -a.s. This ends the proof. \square

3 Null controllability of the forward system

3.1 An observability inequality in a finite dimensional space and some auxiliary results

This section is devoted to prove Theorem 1.3. The proof is based on the Lebeau-Robbiano method and the first step is to prove an observability inequality for the adjoint system in a finite dimensional space. The results below are inspired and follow the presentation of [Lř1] and [Liu14a].

We consider the linear operator \mathcal{A} in $L^2(\mathcal{D})$

$$\mathcal{A}z := -z_{xx}, \quad \forall z \in H^2(\mathcal{D}) \cap H_0^1(\mathcal{D}).$$

Let $\{\lambda_i\}_{i=1}^\infty$ be the eigenvalues of \mathcal{A} and $\{\varphi_i\}_{i=1}^\infty$ be the corresponding (normalized) eigenfunctions. The family $\{\varphi_i\}_{i=1}^\infty$ is an orthonormal basis of $L^2(\mathcal{D})$. Also, in this simple case, one can see that $\lambda_i = (i\pi)^2$ and $\varphi_i = \sqrt{2} \sin(i\pi x)$.

According to [LRR19], if $\mathcal{A}^2 u := u_{xxxx}$ on \mathcal{D} together with the boundary conditions $u = u_{xx} = 0$ on $\{0, 1\}$, the family $\{\varphi_i\}_{i=1}^\infty$ is actually composed of eigenfunctions of \mathcal{A}^2 associated to the eigenvalues $\mu_i = \lambda_i^2$. Moreover, this operator satisfies the following spectral inequality.

Lemma 3.1. *Let \mathcal{D}_0 be an open subset of \mathcal{D} . There exists $C > 0$ such that for any $r > 0$ it holds*

$$\|z\|_{L^2(\mathcal{D})} \leq C e^{Cr^{1/4}} \|z\|_{L^2(\mathcal{D}_0)}$$

for every $z \in \text{span}\{\varphi_i\}_{\mu_i \leq r}$.

Remark 3.2. As we anticipated in Section 1, we are using here the boundary conditions $u = u_{xx} = 0$ on $\{0, 1\}$ to exploit the facts that both operators share eigenfunctions and that $\mu_i = \lambda_i^2$, which is not the case for the fourth-order operator with clamped boundary conditions, i.e., $u = u_x = 0$ on $\{0, 1\}$. We refer the reader to [AE13, Gao16, LRR19] for variants of Lemma 3.1 corresponding to the clamped boundary conditions.

Now, for any $\tau > 0$, consider the the following backward stochastic system

$$\begin{cases} du - u_{xxxx} dt = (-v - b_1 \bar{u}) dt + \bar{u} dW(t) & \text{in } Q_\tau, \\ dv + v_{xx} dt = (-b_2 \bar{u} - b_3 \bar{v}) dt + \bar{v} dW(t) & \text{in } Q_\tau, \\ u = u_{xx} = 0 & \text{on } \Sigma_\tau, \\ v = 0 & \text{on } \Sigma_\tau, \\ u(\tau) = u_\tau, \quad v(\tau) = v_\tau & \text{in } \mathcal{D}. \end{cases} \quad (50)$$

For any terminal data $(u_\tau, v_\tau) \in L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{D})^2)$, we know thanks to Proposition A.2 and Remark A.3 that system (50) has a unique solution

$$(u, v, \bar{u}, \bar{v}) \in \left[L_{\mathcal{F}}^2(0, \tau; H^2(\mathcal{D}) \times H_0^1(\mathcal{D})) \cap L^2(\Omega; C([0, \tau]; L^2(\mathcal{D})^2)) \right] \times L_{\mathcal{F}}^2(0, \tau; L^2(\mathcal{D})^2).$$

For each $r > 0$, consider the space $\mathcal{X}_r = \text{span}\{\varphi_i\}_{\lambda_i \leq \sqrt{r}}$ and denote by Π_r the orthogonal projection from $L^2(\mathcal{D})$ to \mathcal{X}_r . The first result of this section is an observability inequality with only one observation for system (50) with final data in \mathcal{X}_r . The result reads as follows.

Proposition 3.3. *There exists a positive constant C independent of τ such that for any $\sqrt{r} \geq \lambda_1$ and $(u_\tau, v_\tau) \in L^2(\Omega, \mathcal{F}_\tau; (\mathcal{X}_r)^2)$, the corresponding solution (u, v, \bar{u}, \bar{v}) of (50) satisfies:*

1) if $2\lambda_1 > \sigma$

$$\mathbb{E} \left(|u(0)|_{L^2(\mathcal{D})}^2 \right) + \mathbb{E} \left(|v(0)|_{L^2(\mathcal{D})}^2 \right) \leq C \frac{e^{C\sqrt{r}}}{\tau^5} \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}_0} |u|^2 dx dt \right), \quad (51)$$

2) otherwise

$$\mathbb{E} \left(|u(0)|_{L^2(\mathcal{D})}^2 \right) + \mathbb{E} \left(|v(0)|_{L^2(\mathcal{D})}^2 \right) \leq C \frac{e^{\sigma\tau + C\sqrt{r}}}{\tau^5} \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}_0} |u|^2 dx dt \right), \quad (52)$$

where $\sigma := 1 + 4 \left(\sum_{i=1}^3 \|b_i\|_{L^\infty(0, T; \mathbb{R})}^2 \right)$.

Proof. For the sake of clarity, the proof has been divided in several steps. In what follows, C denotes a generic positive constant independent of τ that may vary from line to line.

Since we are assuming that (u_τ, v_τ) belongs to the space $L^2(\Omega, \mathcal{F}_\tau; (\mathcal{X}_r)^2)$, we have that in fact the initial data can be written as

$$u_\tau = \sum_{\lambda_i \leq \sqrt{r}} u_{\tau, i} \varphi_i(x), \quad v_\tau = \sum_{\lambda_i \leq \sqrt{r}} v_{\tau, i} \varphi_i(x),$$

for some sequences $\{u_{\tau, i}\}_i, \{v_{\tau, i}\}_i$ of \mathcal{F}_τ -measurable random variables, whose elements can be computed as $u_{\tau, i} = (u_\tau, \varphi_i)_{L^2(\mathcal{D})}$ and $v_{\tau, i} = (v_\tau, \varphi_i)_{L^2(\mathcal{D})}$.

In this case, it is not difficult to see that system (50) can be reduced to a backward system of SDEs. Indeed, the solution (u, v, \bar{u}, \bar{v}) to (50) can be expressed as

$$\begin{aligned} u &= \sum_{\mu_i \leq r} u_i(t) \varphi_i(x), & \bar{u} &= \sum_{\mu_i \leq r} \bar{u}_i(t) \varphi_i(x), \\ v &= \sum_{\lambda_i \leq \sqrt{r}} v_i(t) \varphi_i(x), & \bar{v} &= \sum_{\lambda_i \leq \sqrt{r}} \bar{v}_i(t) \varphi_i(x), \end{aligned}$$

where $u_i, v_i \in L^2_{\mathcal{F}}(\Omega; C([0, T]))$ and $\bar{u}_i, \bar{v}_i \in L^2_{\mathcal{F}}(0, T; \mathbb{R})$ verify

$$\begin{cases} du_i = (\mu_i u_i - v_i - b_1 \bar{u}_i) dt + \bar{u}_i dW(t) & \text{in } (0, \tau), \\ dv_i = (\lambda_i v_i - b_2 \bar{u}_i - b_3 \bar{v}_i) dt + \bar{v}_i dW(t) & \text{in } (0, \tau), \\ u_i(\tau) = u_{\tau, i}, \quad v_i(\tau) = v_{\tau, i}. \end{cases} \quad (53)$$

Here, we have used that

$$\mathcal{A}^2 u = \sum_{\mu_i \leq r} \mu_i \varphi_i u_i, \quad \mathcal{A} v = \sum_{\lambda_i \leq \sqrt{r}} \lambda_i \varphi_i v_i, \quad (54)$$

with $u_i(t) = (u, \varphi_i)_{L^2(\mathcal{D})}$ and $v_i(t) = (v, \varphi_i)_{L^2(\mathcal{D})}$.

Step 1. First estimates

The first step is to obtain a suitable estimate for the solution to (50). We concentrate on the estimate needed to prove case 2). For the case 1), the proof follows almost by the same procedure. We give a brief comment in Remark 3.4.

Using Itô's formula, we compute $d(e^{\sigma t} u^2)$ and obtain

$$\begin{aligned} \mathbb{E} \left(\int_{\mathcal{D}} e^{\sigma t} |u(t)|^2 dx \right) &= \mathbb{E} \left(\int_{\mathcal{D}} |u(0)|^2 dx \right) + \mathbb{E} \left(\int_0^t \int_{\mathcal{D}} e^{\sigma s} (\sigma |u|^2 + |\bar{u}|^2) dx ds \right) \\ &\quad + 2\mathbb{E} \left(\int_0^t e^{\sigma s} \sum_{\mu_i \leq r} \mu_i |u_i(s)|^2 ds \right) - 2\mathbb{E} \left(\int_0^t \int_{\mathcal{D}} e^{\sigma s} (uv + ub_1 \bar{u}) dx ds \right). \end{aligned}$$

Here, we have used (54) and the orthogonality of the functions φ_i .

Applying Cauchy-Schwarz and Young inequalities on the last term of the previous expression, we obtain

$$\begin{aligned} \mathbb{E} \left(\int_{\mathcal{D}} e^{\sigma t} |u(t)|^2 dx \right) &\geq \mathbb{E} \left(\int_{\mathcal{D}} |u(0)|^2 dx \right) + \mathbb{E} \left(\int_0^t \int_{\mathcal{D}} e^{\sigma s} (\sigma |u|^2 + |\bar{u}|^2) dx ds \right) \\ &\quad + 2\mathbb{E} \left(\int_0^t e^{\sigma s} \sum_{\mu_i \leq r} \mu_i |u_i(s)|^2 ds \right) - \mathbb{E} \left(\int_0^t \int_{\mathcal{D}} e^{\sigma s} (|u|^2 + |v|^2) dx ds \right) \\ &\quad - \mathbb{E} \left(\int_0^t \int_{\mathcal{D}} \left(\epsilon |b_1 u|^2 + \frac{|\bar{u}|^2}{\epsilon} \right) dx ds \right) \end{aligned} \quad (55)$$

for any $\epsilon > 0$ and any $t \in (0, \tau)$.

Applying the same tricks to $d(e^{\sigma t} v^2)$, it can be readily seen that

$$\begin{aligned} \mathbb{E} \left(\int_{\mathcal{D}} e^{\sigma t} |v(t)|^2 dx \right) &\geq \mathbb{E} \left(\int_{\mathcal{D}} |v(0)|^2 dx \right) + \mathbb{E} \left(\int_0^t \int_{\mathcal{D}} e^{\sigma s} (\sigma |v|^2 + |\bar{v}|^2) dx ds \right) \\ &\quad + 2\mathbb{E} \left(\int_0^t e^{\sigma s} \sum_{\lambda_i \leq \sqrt{r}} \lambda_i |v_i(s)|^2 ds \right) - \mathbb{E} \left(\int_0^t \int_{\mathcal{D}} \left(\epsilon |b_2 v|^2 + \frac{|\bar{v}|^2}{\epsilon} \right) dx ds \right) \\ &\quad - \mathbb{E} \left(\int_0^t \int_{\mathcal{D}} \left(\epsilon |b_3 v|^2 + \frac{|\bar{v}|^2}{\epsilon} \right) dx ds \right). \end{aligned} \quad (56)$$

Adding up (55) and (56) and taking $\epsilon = 4$ yield

$$\begin{aligned} &\mathbb{E} \left(\int_{\mathcal{D}} e^{\sigma t} |u(t)|^2 dx + \int_{\mathcal{D}} e^{\sigma t} |v(t)|^2 dx \right) \\ &\geq \mathbb{E} \left(\int_{\mathcal{D}} |u(0)|^2 dx + \int_{\mathcal{D}} |v(0)|^2 dx \right) + \mathbb{E} \left(\int_0^t \int_{\mathcal{D}} e^{\sigma s} (\sigma |v|^2 + \sigma |v|^2) dx ds \right) \\ &\quad + \frac{1}{2} \mathbb{E} \left(\int_0^t \int_{\mathcal{D}} e^{\sigma s} |\bar{u}|^2 dx ds \right) + \frac{3}{4} \mathbb{E} \left(\int_0^t \int_{\mathcal{D}} e^{\sigma s} |\bar{v}|^2 dx ds \right) - 4\mathbb{E} \left(\int_0^t \int_{\mathcal{D}} e^{\sigma s} |b_1 u|^2 dx ds \right) \\ &\quad - 4\mathbb{E} \left(\int_0^t \int_{\mathcal{D}} e^{\sigma s} |b_2 v|^2 dx ds \right) - 4\mathbb{E} \left(\int_0^t \int_{\mathcal{D}} e^{\sigma s} |b_3 v|^2 dx ds \right) - \mathbb{E} \left(\int_0^t \int_{\mathcal{D}} e^{\sigma s} (|u|^2 + |v|^2) dx ds \right) \\ &\quad + 2\mathbb{E} \left(\int_0^t e^{\sigma s} \sum_{\mu_i \leq r} \mu_i |u_i(s)|^2 ds \right) + 2\mathbb{E} \left(\int_0^t e^{\sigma s} \sum_{\lambda_i \leq \sqrt{r}} \lambda_i |v_i(s)|^2 ds \right). \end{aligned} \quad (57)$$

Dropping the positive terms in the above expression and recalling the definition of σ (see Proposition 3.3), we get

$$e^{\sigma\tau} \mathbb{E} \left(\int_{\mathcal{D}} |u(t)|^2 dx + \int_{\mathcal{D}} |v(t)|^2 dx \right) \geq \mathbb{E} \left(\int_{\mathcal{D}} |u(0)|^2 dx + \int_{\mathcal{D}} |v(0)|^2 dx \right) \quad (58)$$

for all $t \in (0, \tau)$.

Step 2. A weighted energy inequality

Assume without loss of generality that $\tau < 1$ and introduce the function $\xi(t) = t^2(\tau - t)^2$ for $t \in (0, \tau)$. Note that by construction $0 \leq \xi(t) \leq 1$ and

$$\xi(0) = \xi(\tau) = 0, \quad (59)$$

$$\left| \frac{\xi'}{\sqrt{\xi}} \right| \leq 4. \quad (60)$$

Applying Itô's formula to compute $d(\xi uv)$ and using (59), we obtain

$$\begin{aligned} \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi |v|^2 dx dt \right) &= \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi' uv dx ds \right) + \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi \bar{u} \bar{v} dx dt \right) \\ &\quad - \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} (b_1 \bar{u} v + b_2 \bar{u} u + b_3 \bar{v} u) dx dt \right) \\ &\quad + \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} (\xi u_{xxxx} v - \xi v_{xx} u) dx dt \right) \\ &=: \sum_{i=1}^4 I_i. \end{aligned} \quad (61)$$

Now, we estimate each I_i , $i = 1, \dots, 4$. For the first one, using (60) and Cauchy-Schwarz and Young inequalities, we easily get

$$|I_1| \leq \frac{\epsilon}{2} \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi |v|^2 dx dt \right) + \frac{8}{\epsilon} \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} |u|^2 dx dt \right), \quad (62)$$

for any $0 < \epsilon < 1$. In the same way, we have

$$|I_2| \leq \frac{\epsilon}{2} \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{3/2} |\bar{v}|^2 dx dt \right) + \frac{1}{2\epsilon} \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{1/2} |\bar{u}|^2 dx dt \right). \quad (63)$$

For the third term, applying successively Cauchy-Schwarz and Young inequalities, we get

$$\begin{aligned} |I_3| &\leq \epsilon \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi |v|^2 dx dt \right) + \epsilon \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{3/2} |\bar{v}|^2 dx dt \right) \\ &\quad + \frac{C}{\epsilon} \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{1/2} |\bar{u}|^2 dx dt + \int_0^\tau \int_{\mathcal{D}} |u|^2 dx dt \right), \end{aligned} \quad (64)$$

where the constant $C > 0$ depends on L^∞ -norms of b_i , $i = 1, 2, 3$. In this step, we have used repeatedly that $0 \leq \xi(t) \leq 1$ to adjust the powers of ξ appearing above.

Let us see the term I_4 .

$$\begin{aligned}
I_4 &= \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} (\xi u_{xxxx} v - \xi v_{xx} u) \, dx dt \right) \\
&= \mathbb{E} \left(\int_0^\tau \xi \left((\mathcal{A}^2 u, v)_{L^2(\mathcal{D})} + (\mathcal{A} v, u)_{L^2(\mathcal{D})} \right) dt \right) \\
&= \mathbb{E} \left(\int_0^\tau \xi \left(\sum_{\mu_i \leq r} \mu_i u_i(t) v_i(t) + \sum_{\lambda_i \leq \sqrt{r}} \lambda_i v_i(t) u_i(t) \right) dt \right).
\end{aligned}$$

Using that $\mu_i = \lambda_i^2$ and applying Cauchy-Schwarz and Young inequalities we readily obtain

$$|I_4| \leq \mathbb{E} \left(\epsilon \int_0^\tau \xi^{3/2} \sum_{\lambda_i \leq \sqrt{r}} \lambda_i |v_i(t)|^2 dt + \frac{C}{\epsilon} \int_0^\tau \xi^{1/2} \sum_{\lambda_i \leq \sqrt{r}} (\lambda_i + \lambda_i^3) |u_i(t)|^2 dt \right). \quad (65)$$

Putting (61) and (62)–(65) together, we get

$$\begin{aligned}
&\mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi |v|^2 \, dx dt \right) \\
&\leq \epsilon \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi |v|^2 \, dx dt \right) + \epsilon \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{3/2} |\bar{v}|^2 \, dx dt + \int_0^\tau \xi^{3/2} \sum_{\lambda_i \leq \sqrt{r}} \lambda_i |v_i(t)|^2 dt \right) \\
&\quad + \frac{C}{\epsilon} \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} |u|^2 \, dx dt + \int_0^\tau \int_{\mathcal{D}} \xi^{1/2} |\bar{u}|^2 \, dx dt + \int_0^\tau \xi^{1/2} \sum_{\lambda_i \leq \sqrt{r}} (\lambda_i + \lambda_i^3) |u_i(t)|^2 dt \right), \quad (66)
\end{aligned}$$

for any $0 < \epsilon < 1$ and where C depends only on $\|b_i\|_{L^\infty_{\mathcal{F}}(0, \tau; \mathbb{R})}^2$, $i = 1, 2, 3$.

Step 3. Estimates for \bar{v} and \bar{u}

In this step, we estimate some of the terms in the right-hand side of (66). The idea is to use the parameter ϵ to absorb all the terms except the one containing $|u|^2$.

Computing $d(\xi^{3/2} v^2)$, one can obtain

$$\begin{aligned}
\mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{3/2} \bar{v}^2 \, dx dt \right) &= -\frac{3}{2} \left(\mathbb{E} \int_0^\tau \int_{\mathcal{D}} \xi^{1/2} \xi_t v^2 \, dx dt \right) + 2 \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{3/2} v v_{xx} dt dx \right) \\
&\quad + 2 \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{3/2} b_2 v \bar{u} \, dx dt \right) + 2 \left(\mathbb{E} \int_0^\tau \int_{\mathcal{D}} \xi^{3/2} b_3 v \bar{v} \, dx dt \right).
\end{aligned}$$

Using property (60) on the first term of the right-hand side and arguing as before, we get

$$\begin{aligned}
\mathbb{E} \int_0^\tau \int_{\mathcal{D}} \xi^{3/2} \bar{v}^2 \, dx dt &\leq 6 \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi |v|^2 \, dx dt \right) - 2 \left(\mathbb{E} \int_0^\tau \xi^{3/2} \sum_{\lambda_i \leq \sqrt{r}} \lambda_i |v_i(t)|^2 dt \right) \\
&\quad + \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{5/2} |v|^2 \, dx dt \right) + \|b_2\|_{L^\infty_{\mathcal{F}}(0, \tau; \mathbb{R})}^2 \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{1/2} |\bar{u}|^2 \, dx dt \right) \\
&\quad + \delta \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{3/2} |\bar{v}|^2 \, dx dt \right) + \frac{\|b_3\|_{L^\infty_{\mathcal{F}}(0, \tau; \mathbb{R})}^2}{\delta} \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{3/2} |v|^2 \, dx dt \right)
\end{aligned}$$

for any $\delta > 0$. Therefore, using that $\xi(t) \leq 1$ and taking δ small enough, we get

$$\begin{aligned} & \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{3/2} \bar{v}^2 \, dx dt + 2 \int_0^\tau \xi^{3/2} \sum_{\lambda_i \leq \sqrt{r}} \lambda_i |v_i(t)|^2 dt \right) \\ & \leq C \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi |v|^2 \, dx dt + \int_0^\tau \int_{\mathcal{D}} \xi^{1/2} |\bar{u}|^2 \, dx dt \right) \end{aligned} \quad (67)$$

where $C > 0$ only depends on the norms of b_2 and b_3 .

We proceed to estimate the last term of the above expression. We compute $d(\xi^{1/2} u^2)$ and thus we obtain

$$\begin{aligned} \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{1/2} \bar{u}^2 \, dx dt \right) &= -\frac{1}{2} \left(\mathbb{E} \int_0^\tau \int_{\mathcal{D}} \xi^{-1/2} \xi_t u^2 \, dx dt \right) - 2 \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{1/2} u u_{xxxx} \, dx dt \right) \\ & \quad + 2 \left(\mathbb{E} \int_0^\tau \int_{\mathcal{D}} \xi^{1/2} u v \, dx dt \right) + 2 \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{1/2} b_1 u \bar{u} \, dx dt \right). \end{aligned}$$

Following the arguments above, we can obtain by means of Cauchy-Schwarz and Young inequalities the following

$$\begin{aligned} & \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{1/2} |\bar{u}|^2 \, dx dt + 2 \int_0^\tau \xi^{1/2} \sum_{\mu_i \leq r} \mu_i |u_i(t)|^2 dt \right) \\ & \leq 2 \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} |u|^2 \, dx dt \right) + \mathbb{E} \left(\rho \int_0^\tau \int_{\mathcal{D}} \xi |v|^2 \, dx dt + \frac{1}{\rho} \int_0^\tau \int_{\mathcal{D}} |u|^2 \, dx dt \right) \\ & \quad + \delta_2 \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{1/2} |\bar{u}|^2 \, dx dt \right) + \frac{\|b_1\|_{L^\infty(0,\tau;\mathbb{R})}^2}{\delta_2} \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} |u|^2 \, dx dt \right) \end{aligned}$$

for some constants $\rho, \delta_2 > 0$ and where we have used that $\xi(t) \leq 1$ in the last term. Choosing δ_2 small enough, we get

$$\begin{aligned} & \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{1/2} |\bar{u}|^2 \, dx dt + 2 \int_0^\tau \xi^{1/2} \sum_{\mu_i \leq r} \mu_i |u_i(t)|^2 dt \right) \\ & \leq C \left(1 + \frac{1}{\rho} \right) \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} |u|^2 \, dx dt \right) + \rho \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi |v|^2 \, dx dt \right) \end{aligned} \quad (68)$$

for some positive constant C and some $\rho > 0$ to be chosen.

Noting that $(\lambda_i + \lambda_i^3) \leq 2\sqrt{r}\mu_i$ for all $\lambda_i \leq \sqrt{r}$, we obtain from estimate (66)

$$\begin{aligned} & \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi |v|^2 \, dx dt \right) \\ & \leq \epsilon \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi |v|^2 \, dx dt \right) + \epsilon \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{3/2} |\bar{v}|^2 \, dx dt + \int_0^\tau \xi^{3/2} \sum_{\lambda_i \leq \sqrt{r}} \lambda_i |v_i(t)|^2 dt \right) \\ & \quad + \frac{C}{\epsilon} \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} |u|^2 \, dx dt \right) + \frac{C}{\epsilon} \mathbb{E} \left(\sqrt{r} \int_0^\tau \int_{\mathcal{D}} \xi^{1/2} |\bar{u}|^2 \, dx dt + 2\sqrt{r} \int_0^\tau \xi^{1/2} \sum_{\mu_i \leq r} \mu_i |u_i(t)|^2 dt \right), \end{aligned}$$

where we have used that $\sqrt{r} > 1$ in the second to last term.

Multiplying by ϵ on both sides of (67) and using the result in the second term of the right-hand side of the above inequality, we get

$$\begin{aligned}
& \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi |v|^2 \, dx dt \right) \\
& \leq \epsilon(1+C) \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi |v|^2 \, dx dt \right) + \frac{C}{\epsilon} \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} |u|^2 \, dx dt \right) \\
& \quad + C \left(\epsilon + \frac{1}{\epsilon} \right) \mathbb{E} \left(\sqrt{r} \int_0^\tau \int_{\mathcal{D}} \xi^{1/2} |\bar{u}|^2 \, dx dt \right) + \frac{2C}{\epsilon} \mathbb{E} \left(\sqrt{r} \int_0^\tau \xi^{1/2} \sum_{\mu_i \leq r} \mu_i |u_i(t)|^2 dt \right) \\
& \leq \epsilon C \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi |v|^2 \, dx dt \right) + \frac{C}{\epsilon} \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} |u|^2 \, dx dt \right) \\
& \quad + \frac{C}{\epsilon} \sqrt{r} \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi^{1/2} |\bar{u}|^2 \, dx dt + \int_0^\tau \xi^{1/2} \sum_{\mu_i \leq r} \mu_i |u_i(t)|^2 dt \right).
\end{aligned}$$

where we used that $\epsilon < 1$.

Finally, multiplying by \sqrt{r} in both sides of (68), setting $\rho = \epsilon^2/\sqrt{r}$ and after a straightforward computation, we get

$$\mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi |v|^2 \, dx dt \right) \leq \frac{Cr}{\epsilon^3} \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} |u|^2 \, dx dt \right) + \epsilon C \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi |v|^2 \, dx dt \right),$$

and taking $\epsilon > 0$ small enough, we get

$$\begin{aligned}
\mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} \xi |v|^2 \, dx dt \right) & \leq Cr \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} |u|^2 \, dx dt \right) \\
& \leq C e^{2r^{1/4}} \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} |u|^2 \, dx dt \right),
\end{aligned} \tag{69}$$

for some $C > 0$ uniform with respect to r and τ . In the last line we have used that $x < e^{2x^{1/4}}$ for all $x > 0$.

Step 4. Last arrangements and conclusion

Now, we are in position to proof (52). Integrating inequality (58) in $(\frac{\tau}{4}, \frac{3\tau}{4})$ and using that $\xi(t)^{-1} \leq (\frac{4}{\tau})^4$ for $t \in (\frac{\tau}{4}, \frac{3\tau}{4})$, we obtain

$$\begin{aligned}
\mathbb{E} \left(\int_{\mathcal{D}} |u(0)|^2 \, dx + \int_{\mathcal{D}} |v(0)|^2 \, dx \right) & \leq \frac{2e^{\sigma\tau}}{\tau} \mathbb{E} \left(\int_{\frac{\tau}{4}}^{\frac{3\tau}{4}} \int_{\mathcal{D}} |u|^2 \, dx dt \right) \\
& \quad + \frac{2e^{\sigma\tau}}{\tau} \left(\frac{4}{\tau} \right)^4 \mathbb{E} \left(\int_{\frac{\tau}{4}}^{\frac{3\tau}{4}} \int_{\mathcal{D}} \xi |v|^2 \, dx dt \right).
\end{aligned}$$

Then, applying estimate (69) to the last term of the above equation, we have

$$\begin{aligned}
\mathbb{E} \left(\int_{\mathcal{D}} |u(0)|^2 \, dx + \int_{\mathcal{D}} |v(0)|^2 \, dx \right) & \leq \frac{2e^{\sigma\tau}}{\tau} \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} |u|^2 \, dx dt \right) \\
& \quad + \frac{C e^{\sigma\tau}}{\tau^5} e^{2r^{1/4}} \mathbb{E} \left(\int_0^\tau \int_{\mathcal{D}} |u|^2 \, dx dt \right).
\end{aligned} \tag{70}$$

Finally, applying Lemma 3.1 to u , the solution to the first equation of system (50) we have

$$\mathbb{E} \left(\int_0^\tau \|u\|_{L^2(\mathcal{D})}^2 dt \right) \leq C e^{Cr^{1/4}} \mathbb{E} \left(\int_0^\tau \|u\|_{L^2(\mathcal{D}_0)}^2 dt \right)$$

and combining it with (70), we obtain the desired result. \square

Remark 3.4. To obtain the observability inequality (51), we just need to change a little bit Step 1 in the above proof. We have to prove an estimate similar to (58) but with $\sigma = 0$. To do this, it is enough to compute $d(u^2)$ and $d(v^2)$ and arrive to an estimate similar to (57) and then notice that the only way to absorb the negative terms is by considering the hypothesis $2\lambda_1 > \sigma$. The rest of the proof can be followed exactly.

Once we have proved the observability inequalities in Proposition 3.3, we can establish the following controllability result.

Proposition 3.5. *For each $\sqrt{r} \geq \lambda_1$ and any $\tau > 0$, there exists a control $h_r \in L^2_{\mathcal{F}}(0, \tau; L^2(\mathcal{D}_0))$ such that the corresponding controlled solution (y, z) to (3) satisfies*

$$\Pi_r(y(\tau)) = \Pi_r(z(\tau)) = 0 \quad \text{in } \mathcal{D}, \text{ P-a.s.}$$

Moreover, we can estimate the control cost and the size of the controlled solution as

1) if $2\lambda_1 > \sigma$

$$\|h_r\|_{L^2_{\mathcal{F}}(0, \tau; L^2(\mathcal{D}_0))}^2 \leq C \frac{e^{Cr^{1/4}}}{\tau^5} \mathbb{E} \left(|y_0|_{L^2(\mathcal{D})}^2 + |z_0|_{L^2(\mathcal{D})}^2 \right) \quad (71)$$

and

$$\mathbb{E} \left(|y(\tau)|_{L^2(\Omega)}^2 + |z(\tau)|_{L^2(\Omega)}^2 \right) \leq C_2 \frac{e^{C_2 r^{1/4}}}{\tau^5} \mathbb{E} \left(|y_0|_{L^2(\mathcal{D})}^2 + |z_0|_{L^2(\mathcal{D})}^2 \right); \quad (72)$$

2) in the general case,

$$\|h_r\|_{L^2_{\mathcal{F}}(0, \tau; L^2(\mathcal{D}_0))}^2 \leq C \frac{e^{\sigma\tau + Cr^{1/4}}}{\tau^5} \mathbb{E} \left(|z_0|_{L^2(\mathcal{D})}^2 + |y_0|_{L^2(\mathcal{D})}^2 \right)$$

and

$$\mathbb{E} \left(|y(\tau)|_{L^2(\Omega)}^2 + |z(\tau)|_{L^2(\Omega)}^2 \right) \leq \left(C_2 \frac{e^{\sigma\tau + C_2 r^{1/4}}}{\tau^5} + 1 \right) e^{\sigma\tau} \mathbb{E} \left(|y_0|_{L^2(\mathcal{D})}^2 + |z_0|_{L^2(\mathcal{D})}^2 \right).$$

The arguments needed to prove this proposition are similar to those presented in the proof of Theorem 1.2 and only minor adaptations are required, see Section 2.4. For brevity, we omit the proof.

We also need a dissipation result for the uncontrolled solution. We present the following.

Proposition 3.6. *Assume that $h \equiv 0$ in system (3). For any $(y_0, z_0) \in L^2_{\mathcal{F}}(\Omega, \mathcal{F}_0; L^2(\mathcal{D})^2)$ with $\Pi_{\lambda_k}(y_0) = \Pi_{\lambda_k}(z_0) = 0$ in \mathcal{D} , P-a.s., the corresponding solution (y, z) satisfies*

$$\mathbb{E} \left(|y(t)|_{L^2(\mathcal{D})}^2 + |z(t)|_{L^2(\mathcal{D})}^2 \right) \leq e^{-\gamma_{k+1}t} \mathbb{E} \left(|y_0|_{L^2(\mathcal{D})}^2 + |z_0|_{L^2(\mathcal{D})}^2 \right), \quad \forall t \in [0, T],$$

where $\gamma_{k+1} = 2\lambda_{k+1} - \sigma$ where σ is defined in Proposition 3.3.

The proof of this proposition is standard and can be done by following almost the same procedure as in [Lř1, Proposition 2.3] and [Liu14a, Proposition 4.1].

Using the above results, we prove the following corollary, which will be of interest during the proof of Theorem 1.3.

Corollary 3.7. *Assume that $2\lambda_1 > \sigma$. For each $\sqrt{r} \geq \lambda_1$, $0 < \tau < T$ and $(y_0, z_0) \in L^2(\Omega, \mathcal{F}_0; L^2(\mathcal{D})^2)$, there exists a control $h_r \in L^2_{\mathcal{F}}(0, \tau; L^2(\mathcal{D}_0))$ such that*

$$\|h_r\|_{L^2_{\mathcal{F}}(0, \tau; L^2(\mathcal{D}_0))}^2 \leq C \frac{e^{C\tau^{1/4}}}{\tau^5} \mathbb{E} \left(|y_0|_{L^2(\mathcal{D})}^2 + |z_0|_{L^2(\mathcal{D})}^2 \right)$$

and

$$\mathbb{E} \left(|y(\tau)|_{L^2(\Omega)}^2 + |z(\tau)|_{L^2(\Omega)}^2 \right) \leq \frac{C_2}{\tau^5} e^{C_2\tau^{1/4} - \frac{\sqrt{r}\tau}{2}} \mathbb{E} \left(|y_0|_{L^2(\mathcal{D})}^2 + |z_0|_{L^2(\mathcal{D})}^2 \right).$$

Proof. The proof is straightforward. Use Proposition 3.5 in the interval $(0, \frac{\tau}{2})$, this will give a control w_r such that $\Pi_r(y(\tau/2)) = \Pi_r(z(\tau/2)) = 0$ together with the estimates (71) and (72), but with τ replaced by $\tau/2$. Set

$$h_r = \begin{cases} w_r & \text{for } t \in (0, \tau/2), \\ 0 & \text{for } t \in (\tau/2, \tau). \end{cases}$$

Clearly, h_r and w_r have the same norm.

Now, note that there exists $k \in \mathbb{N}^*$ such that $\lambda_{k+1} > \sqrt{r}$. We can apply Proposition 3.6 in the interval $(\tau/2, \tau)$ to deduce

$$\mathbb{E} \left(|y(\tau)|_{L^2(\mathcal{D})}^2 + |z(\tau)|_{L^2(\mathcal{D})}^2 \right) \leq e^{-\gamma_{k+1}\frac{\tau}{2}} \mathbb{E} \left(|y(\tau/2)|_{L^2(\mathcal{D})}^2 + |z(\tau/2)|_{L^2(\mathcal{D})}^2 \right) \quad (73)$$

since $\Pi_r(y(\tau/2)) = \Pi_r(z(\tau/2)) = 0$ implies that the first k modes of the equations have been killed. Using that $\lambda_{k+1} > 2\lambda_1$ for any $k \in \mathbb{N}^*$, we get

$$\gamma_{k+1} > \lambda_{k+1} + 2\lambda_1 - \sigma > \sqrt{r}$$

where we have used the hypothesis $2\lambda_1 > \sigma$. Combining this with estimate (73) and (72) (with τ replaced by $\tau/2$), we obtain the desired result. Thus, the proof is complete. \square

3.2 Null controllability result

Here, we are going to prove Theorem 1.3. We follow the spirit of [Boy20, Section IV.2] and [Lř1]. Without loss of generality, we can always suppose that we are in the case 1) of Propositions (3.3) and (3.5). Indeed, since $\lambda_i \rightarrow \infty$ as $i \rightarrow \infty$, there exists some $k \in \mathbb{N}^*$ such that $2\lambda_k > \sigma$. Thus, we split the time interval in $(0, \bar{T}) \cup (\bar{T}, T)$ and design a control \bar{h} such that $\Pi_{\lambda_k}(y(\bar{T})) = \Pi_{\lambda_k}(z(\bar{T})) = 0$. This is possible, thanks to the general case 2) in Propositions (3.3) and (3.5).

Proof of Theorem 1.3. The idea is to split the time interval $(0, T)$ into subintervals of size τ_j , $j \geq 1$, with

$$\sum_{j=1}^{\infty} \tau_j = T$$

and apply successively a partial control as in Corollary 3.7 with a cut frequency r_j tending to infinity as $j \rightarrow \infty$. We set

$$\tau_j = \frac{T}{2^j} \quad \text{and} \quad r_j = \beta^2 (2^j)^4 \quad (74)$$

for some $\beta > 0$ to be determined.

Let $T_j = \sum_{k=1}^j \tau_k$, for $j \geq 1$. We proceed as follows.

1. During the interval $(0, \tau_1) = (0, T_1)$, we apply a control h_{r_1} as given in Corollary 3.7 with $r = r_1$, in such a way that

$$\|h_{r_1}\|_{L^2_{\mathcal{F}}(0, T_1; L^2(\mathcal{D}_0))}^2 \leq C \frac{e^{Cr_1^{1/4}}}{\tau_1^5} \mathbb{E} \left(|y_0|_{L^2(\mathcal{D})}^2 + |z_0|_{L^2(\mathcal{D})}^2 \right)$$

and

$$\mathbb{E} \left(|y(T_1)|_{L^2(\Omega)}^2 + |z(T_1)|_{L^2(\Omega)}^2 \right) \leq \frac{C_2}{\tau_1^5} e^{C_2 r_1^{1/4} - \frac{\sqrt{r_1} \tau_1}{2}} \mathbb{E} \left(|y_0|_{L^2(\mathcal{D})}^2 + |z_0|_{L^2(\mathcal{D})}^2 \right).$$

with

$$\Pi_{r_1}(y(T_1) = 0) = \Pi_{r_1}(z(T_1) = 0), \quad P\text{-a.s.}$$

2. During the interval $(\tau_1, \tau_1 + \tau_2)$, we apply a control h_{r_2} once again given by Corollary 3.7 with $r = r_2$ in such a way that

$$\|h_{r_2}\|_{L^2_{\mathcal{F}}(T_1, T_2; L^2(\mathcal{D}_0))}^2 \leq C \frac{e^{Cr_2^{1/4}}}{\tau_2^5} \mathbb{E} \left(|y(T_1)|_{L^2(\mathcal{D})}^2 + |z(T_1)|_{L^2(\mathcal{D})}^2 \right)$$

and

$$\mathbb{E} \left(|y(T_2)|_{L^2(\Omega)}^2 + |z(T_2)|_{L^2(\Omega)}^2 \right) \leq \frac{C_2^2}{\tau_1^5 \tau_2^5} e^{C_2(r_1^{1/4} + r_2^{1/4}) - \frac{\sqrt{r_1} \tau_1}{2} - \frac{\sqrt{r_2} \tau_2}{2}} \mathbb{E} \left(|y_0|_{L^2(\mathcal{D})}^2 + |z_0|_{L^2(\mathcal{D})}^2 \right).$$

with

$$\Pi_{r_2}(y(T_2) = 0) = \Pi_{r_2}(z(T_2) = 0), \quad P\text{-a.s.}$$

3. By an inductive procedure, we can build a control h_{r_j} on the time interval (T_{j-1}, T_j) such that

$$\|h_{r_j}\|_{L^2_{\mathcal{F}}(T_{j-1}, T_j; L^2(\mathcal{D}_0))}^2 \leq C \frac{e^{Cr_j^{1/4}}}{\tau_j^5} \mathbb{E} \left(|y(T_{j-1})|_{L^2(\mathcal{D})}^2 + |z(T_{j-1})|_{L^2(\mathcal{D})}^2 \right) \quad (75)$$

and

$$\begin{aligned} & \mathbb{E} \left(|y(T_j)|_{L^2(\mathcal{D})}^2 + |z(T_j)|_{L^2(\mathcal{D})}^2 \right) \\ & \leq \frac{C_2^j}{\prod_{k=1}^j \tau_k^5} e^{C_2 \left(\sum_{k=1}^j r_k^{1/4} \right) - \frac{1}{2} \sum_{k=1}^j \tau_k \sqrt{r_k}} \mathbb{E} \left(|y_0|_{L^2(\mathcal{D})}^2 + |z_0|_{L^2(\mathcal{D})}^2 \right). \end{aligned}$$

with

$$\Pi_{r_j}(y(T_j) = 0) = \Pi_{r_j}(z(T_j) = 0), \quad P\text{-a.s.} \quad (76)$$

4. By definition (74), we have

$$\begin{aligned} C_2 \sum_{k=1}^j r_k^{1/4} - \frac{1}{2} \sum_{k=1}^j \tau_k \sqrt{r_k} &= C_2 \sqrt{\beta} \sum_{k=1}^j 2^k - \frac{\beta}{2} T \sum_{k=1}^j 2^k \\ &= \left(C_2 \sqrt{\beta} - \frac{\beta}{2} T \right) (2^{j+1} - 2) \end{aligned}$$

and choosing β sufficiently large so that

$$\tilde{\beta} := \frac{\beta}{2} T - C_2 \sqrt{\beta} > 0$$

we can obtain the estimate

$$\begin{aligned} & \mathbb{E} \left(|y(T_j)|_{L^2(\mathcal{D})}^2 + |z(T_j)|_{L^2(\mathcal{D})}^2 \right) \\ & \leq C_3 C_2^j \left[\frac{2^{j(j+1)/2}}{T^{5j}} \right] e^{-\tilde{\beta}2^{j+1}} \mathbb{E} \left(|y_0|_{L^2(\mathcal{D})}^2 + |z_0|_{L^2(\mathcal{D})}^2 \right). \end{aligned} \quad (77)$$

5. Using estimate (77) in (75), we obtain

$$\|h_{r_j}\|_{L^2_{\mathcal{F}}(T_{j-1}, T_j; L^2(\mathcal{D}_0))}^2 \leq C C_3 C_2^{j-1} \left[\frac{2^{5j+j(j-1)/2}}{T^{5j}} \right] e^{(C\sqrt{\beta}-\tilde{\beta})2^j} \mathbb{E} \left(|y_0|_{L^2(\mathcal{D})}^2 + |z_0|_{L^2(\mathcal{D})}^2 \right),$$

and increasing the value of β to ensure that

$$\hat{\beta} := \tilde{\beta} - C\sqrt{\beta} > 0$$

we obtain

$$\|h_{r_j}\|_{L^2_{\mathcal{F}}(T_{j-1}, T_j; L^2(\mathcal{D}_0))}^2 \leq C C_3 C_2^{j-1} \left[\frac{2^{5j+j(j-1)/2}}{T^{5j}} \right] e^{-\hat{\beta}2^j} \mathbb{E} \left(|y_0|_{L^2(\mathcal{D})}^2 + |z_0|_{L^2(\mathcal{D})}^2 \right). \quad (78)$$

6. Estimate (78) shows that

$$\sum_{j=1}^{\infty} \|h_{r_j}\|_{L^2_{\mathcal{F}}(T_{j-1}, T_j; L^2(\mathcal{D}_0))}^2 < \infty$$

and in particular the control h that comes from gluing together all the $(h_{r_j})_{j \in \mathbb{N}^*}$ is an element of $L^2_{\mathcal{F}}(0, T; L^2(\mathcal{D}_0))$. Moreover, from (76) and (77) and since $T_j \rightarrow T$ as $j \rightarrow \infty$, we conclude that

$$y(T) = z(T) = 0 \text{ in } \mathcal{D}, \text{ } P\text{-a.s.}$$

This concludes the proof. \square

4 Further remarks and conclusions

We conclude our work by presenting some concluding remarks and two open problems regarding the controllability of fourth- and second-order parabolic systems.

1. *On the nonlinear system.* In the context of Theorem 1.2, we have presented a stochastic model that resembles very much the stabilized Kuramoto-Sivashinsky system (1). Nevertheless, system (2) is linear and therefore it would be interesting to treat the nonlinear case. However, as far as the author's knowledge, this is a difficult problem to treat in the stochastic setting (even for the simple case of the semilinear heat equation) due to the lack of compactness in the functional spaces where the equation is posed (see [TZ09]).
2. *Controlling from the heat equation.* The main results we have presented deal with the case where the systems are being controlled from the fourth-order equation and thus a natural question that arises is the possibility of controlling from the second-order parabolic equation. In this direction, it seems that Theorem 1.3 can be adapted without major modifications, since we have at hand a spectral inequality for the heat equation (see, e.g.

[LR95]) and the same methodology for proving Proposition 3.3 still applies. Nonetheless, a rigorous proof is still needed.

However, the case of the backward equation (2) requires a more delicate analysis. In the deterministic setting, this question was answered in [CnC16] by proving a Carleman estimate for the fourth-order equation with non-homogeneous boundary conditions (see Theorem 3.5 in the aforementioned reference). The proof is based on duality arguments and requires to define the solution of the corresponding equation by transposition. In this regard, we think that the results from [Gao18] to define the solution of a fourth-order stochastic equation by transposition can be combined with the well-known duality analysis of [Liu14b] to deduce the analogous result in the stochastic framework. This will be analyzed in a forthcoming paper [Per20].

3. *More general coupling for the forward equation.* For the case of Theorem 1.3, we can consider the more general coupled system

$$\begin{cases} dy + y_{xxxx} dt = (a_1 y + a_2 z + \chi_{\mathcal{D}_0} h) dt + (b_1 y + b_2 z) dW(t) & \text{in } Q, \\ dz - z_{xx} dt = (a_3 y + a_4 z) dt + b_3 z dW(t) & \text{in } Q, \end{cases} \quad (79)$$

where $a_i \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R})$ and $a_3(t) \geq d_0$ or $a_3(t) \leq -d_0$ for some positive constant d_0 and for all $t \in [0, T]$. Indeed, we just need to change the definition of σ to

$$\sigma = 1 + 4 \left(\sum_{i=1}^4 \|a_i\|_{L^\infty_{\mathcal{F}}(0, T; \mathbb{R})} \right) + 4 \left(\sum_{i=1}^3 \|b_i\|_{L^\infty_{\mathcal{F}}(0, T; \mathbb{R})}^2 \right)$$

which can be readily identified from Step 1 of the proof of Proposition 3.3. The rest of the proof can be followed exactly.

It is important to mention that here we are still considering only time-dependent coefficients. It is known that for general coupling coefficients, the problem is much harder to solve and very few results are known. In this direction, it would be interesting to see if the newer approach used in [DL19], which relies on Malliavin Calculus tools (see, e.g., [Nua06]) can be used to prove a Kalman-type condition for testing the partial-approximate controllability (in the spirit of [DL19]) for system (79).

A Well-posedness results

We devote this section to present some results and make some comments about the well-posedness of systems (2), (3), (4), and (50). For conciseness, we assume that the coefficients d_i and b_i have the same regularity as in Theorems 1.2 and 1.3.

We begin with the forward system. We have the following general result.

Proposition A.1. *Assume that $u_0, v_0 \in L^2(\Omega, \mathcal{F}_0; L^2(\mathcal{D}))$ and $f_i, g_i \in L^2_{\mathcal{F}}(0, T; L^2(\mathcal{D}))$, $i = 1, 2$. Then, the system*

$$\begin{cases} du + (\gamma u_{xxxx} + u_{xxx} + u_{xx}) dt = (f_1 + v_x) dt + (g_1 + d_1 u) dW(t) & \text{in } Q, \\ dv - \Gamma v_{xx} dt = (f_2 + v_x + u_x) dt + (g_2 + d_2 u + d_3 v) dW(t) & \text{in } Q, \\ u = u_x = 0 & \text{on } \Sigma, \\ v = 0 & \text{on } \Sigma, \\ u(x, 0) = u_0, \quad v(x, 0) = v_0 & \text{in } \mathcal{D}, \end{cases} \quad (80)$$

has a unique solution $(y, z) \in L^2_{\mathcal{F}}(0, T; H^2_0(\mathcal{D}) \times H^1_0(\mathcal{D})) \cap L^2(\Omega; C([0, T]; L^2(\mathcal{D})^2))$. Moreover, there exists some $C > 0$ only depending on T, Γ, γ and $d_i, i = 1, 2, 3$, such that

$$\begin{aligned} & \sup_{0 \leq t \leq T} \mathbb{E} \left(\|u\|_{L^2(\mathcal{D})}^2 + \|v\|_{L^2(\mathcal{D})}^2 \right) + \mathbb{E} \left(\int_0^T \|u(t)\|_{H^2_0(\mathcal{D})}^2 dt + \int_0^T \|v(t)\|_{H^1_0(\mathcal{D})}^2 dt \right) \\ & \leq C \mathbb{E} \left(\|u_0\|_{L^2(\mathcal{D})}^2 + \|v_0\|_{L^2(\mathcal{D})}^2 + \sum_{i=1}^2 \left\{ \int_0^T \|f_i(t)\|_{L^2(\mathcal{D})}^2 dt + \int_0^T \|g_i(t)\|_{L^2(\mathcal{D})}^2 dt \right\} \right). \end{aligned}$$

Sketch of the proof. Seeing individually, the existence and uniqueness of the solutions to the stochastic parabolic equation have been studied in [KR77] (see also [Zho92] for a more accessible reference), while the analysis of the fourth-order PDE has been made in [GCL15].

Adapting [Zho92, Proposition 2.1], we can obtain for the second equation of system (80)

$$\begin{aligned} & \mathbb{E} \left(\|v(t)\|_{L^2(\mathcal{D})}^2 \right) + \mathbb{E} \left(\int_0^t \|v(s)\|_{H^1_0(\mathcal{D})}^2 ds \right) \\ & \leq C \mathbb{E} \left(\|v_0\|_{L^2(\mathcal{D})}^2 + \int_0^t \|f_2(s)\|_{L^2(\mathcal{D})}^2 ds + \int_0^t \|g_2(s)\|_{L^2(\mathcal{D})}^2 ds \right) \\ & \quad + C \mathbb{E} \left(\int_0^t \left(\|u_x(s)\|_{L^2(\mathcal{D})}^2 + \|u(s)\|_{L^2(\Omega)}^2 + \|v(s)\|_{L^2(\mathcal{D})}^2 \right) ds \right), \end{aligned} \tag{81}$$

for some $C > 0$ only depending on T, γ, d_2 and d_3 .

In the same way, following [GCL15, Proposition 2.3] and using the interpolation inequality

$$\int_{\mathcal{D}} |u_x|^2 dx \leq \delta \int_{\mathcal{D}} |u_{xx}|^2 dx + C_\delta \int_{\mathcal{D}} |u|^2 dx, \quad \text{for all } \delta > 0, \tag{82}$$

to handle the term u_{xxx} (up to some integration by parts), we can derive

$$\begin{aligned} & \mathbb{E} \left(\|u(t)\|_{L^2(\mathcal{D})}^2 \right) + \mathbb{E} \left(\int_0^t \|u(s)\|_{H^2_0(\mathcal{D})}^2 ds \right) \\ & \leq C \mathbb{E} \left(\|u_0\|_{L^2(\mathcal{D})}^2 + \int_0^t \|f_1(s)\|_{L^2(\mathcal{D})}^2 ds + \int_0^t \|g_1(s)\|_{L^2(\mathcal{D})}^2 ds \right) + \epsilon \mathbb{E} \left(\int_0^t \|v_x(s)\|_{L^2(\mathcal{D})}^2 ds \right) \\ & \quad + C \mathbb{E} \left(\int_0^t \|u(s)\|_{L^2(\mathcal{D})}^2 ds \right) \end{aligned} \tag{83}$$

where $C > 0$ only depends on T, Γ and d_1 , and valid for any $\epsilon > 0$.

Adding up (81) and (83) and taking ϵ small enough we get

$$\begin{aligned} & \mathbb{E} \left(\|v(t)\|_{L^2(\mathcal{D})}^2 + \|u(t)\|_{L^2(\mathcal{D})}^2 \right) + \mathbb{E} \left(\int_0^t \|v(s)\|_{H^1_0(\mathcal{D})}^2 ds + \int_0^t \|u(s)\|_{H^2_0(\mathcal{D})}^2 ds \right) \\ & \leq C \mathbb{E} \left(\|v_0\|_{L^2(\mathcal{D})}^2 + \int_0^t \|f_2(s)\|_{L^2(\mathcal{D})}^2 ds + \int_0^t \|g_2(s)\|_{L^2(\mathcal{D})}^2 ds \right) \\ & \quad + C \mathbb{E} \left(\|u_0\|_{L^2(\mathcal{D})}^2 + \int_0^t \|f_1(s)\|_{L^2(\mathcal{D})}^2 ds + \int_0^t \|g_1(s)\|_{L^2(\mathcal{D})}^2 ds \right) \\ & \quad + C \mathbb{E} \left(\int_0^t \left(\|u_x(s)\|_{L^2(\mathcal{D})}^2 + \|u(s)\|_{L^2(\Omega)}^2 + \|v(s)\|_{L^2(\mathcal{D})}^2 \right) ds \right), \end{aligned}$$

Using the definition of the H^2_0 -norm and the interpolation inequality (82) we can absorb the term corresponding to u_x . Finally, employing Gronwall inequality and following classical arguments yields the existence of a solution in the class $L^2_{\mathcal{F}}(0, T; H^2_0(\mathcal{D}) \times H^1_0(\mathcal{D})) \cap L^2(\Omega; C([0, T]; L^2(\mathcal{D})^2))$. The uniqueness also follows by classical arguments. \square

In the same spirit, we can prove a general result for the backward system (2). The result reads as follows.

Proposition A.2. *Assume that $y_T, z_T \in L^2(\Omega, \mathcal{F}_T; L^2(\mathcal{D}))$ and $F_i \in L^2_{\mathcal{F}}(0, T; L^2(\mathcal{D}))$, $i = 1, 2$. Then, the system*

$$\begin{cases} dy - (\gamma y_{xxxx} - y_{xxx} + y_{xx})dt = (z_x - d_1 Y - d_2 Z + F_1)dt + YdW(t) & \text{in } Q, \\ dz + \Gamma z_{xx}dt = (z_x + y_x - d_3 Z + F_2)dt + ZdW(t) & \text{in } Q, \\ y = y_x = 0 & \text{on } \Sigma, \\ z = 0 & \text{on } \Sigma, \\ y(x, T) = y_T, \quad z(x, T) = z_T & \text{in } \mathcal{D}, \end{cases} \quad (84)$$

has a unique solution $(y, z, Y, Z) \in [L^2_{\mathcal{F}}(0, T; H^2_0(\mathcal{D}) \times H^1_0(\mathcal{D})) \cap L^2(\Omega; C([0, T]; L^2(\mathcal{D})^2))] \times L^2_{\mathcal{F}}(0, T; L^2(\mathcal{D})^2)$. Moreover, there exists some $C > 0$ only depending on T, Γ, γ and $d_i, i = 1, 2, 3$, such that

$$\begin{aligned} & \sup_{0 \leq t \leq T} \mathbb{E} \left(\|y\|_{L^2(\mathcal{D})}^2 + \|z\|_{L^2(\mathcal{D})}^2 \right) + \mathbb{E} \left(\int_0^T \|y(t)\|_{H^2_0(\mathcal{D})}^2 dt + \int_0^T \|z(t)\|_{H^1_0(\mathcal{D})}^2 dt \right) \\ & + \mathbb{E} \left(\int_0^T \|Y(t)\|_{L^2(\mathcal{D})}^2 dt + \int_0^T \|Z(t)\|_{L^2(\mathcal{D})}^2 dt \right) \\ & \leq C \mathbb{E} \left(\|y_T\|_{L^2(\mathcal{D})}^2 + \|z_T\|_{L^2(\mathcal{D})}^2 + \int_0^T \|F_1(t)\|_{L^2(\mathcal{D})}^2 dt + \int_0^T \|F_2(t)\|_{L^2(\mathcal{D})}^2 dt \right). \end{aligned}$$

The proof is totally analogous to the one of Proposition A.1. In fact, we just need to change the existence and uniqueness result for each individual equation, that is, we need to consider [Zho92, Theorem 3.1] for the parabolic equation and [GCL15, Proposition 2.4] for the fourth-order one. We omit it here.

We conclude this section by making the following comment about systems (3) and (50).

Remark A.3. Besides the coupling terms, the only difference with respect to systems (80) and (84) is the boundary condition, instead of having u_x we have to replace with u_{xx} . This can be handled without any problem by adapting the proofs of [GCL15, Proposition 2.3 and 2.4] for the forward and backward fourth-order systems (we just need to change the eigenvalue problem and the basis employed for the Galerkin method) and then argue as above. Therefore, roughly speaking, Propositions A.1 and A.2 are also valid for (3) and (50) by replacing the space $H^2_0(\mathcal{D})$ for $H^2(\mathcal{D})$ and taking the appropriate couplings.

B Sketch of the proof of Lemma 2.6

Since most of the arguments are similar to those in Step 3 of the proof of Theorem 2.5, we proceed briefly. Let us consider an open set \mathcal{D}_2 such that $\mathcal{D}_3 \subset\subset \mathcal{D}_2 \subset\subset \mathcal{D}_0$ and take $\hat{\eta} \in C^\infty_0(\mathcal{D}_2)$ satisfying $\hat{\eta} \equiv 1$ in \mathcal{D}_3 . Using Itô's formula, we compute $d(\hat{\zeta} u u_{xx})$, where $\hat{\zeta} := \hat{\eta} \lambda^5 \phi_m^5 \theta^2$. After

a long, but straightforward computation we get

$$\begin{aligned}
2\gamma\mathbb{E}\left(\int_Q \hat{\zeta}|u_{xxx}|^2 dxdt\right) &= -\gamma\mathbb{E}\left(\int_Q \left[3\hat{\zeta}_{xx}u_xu_{xxx} + \hat{\zeta}_{xxx}uu_{xxx} - 2\hat{\zeta}_{xx}|u_{xx}|^2\right] dxdt\right) \\
&+ \mathbb{E}\left(\int_Q \left[2\hat{\zeta}_x u_x u_{xxx} + \hat{\zeta}_{xx} u u_{xxx} + \hat{\zeta}_{xx} u u_{xx} - \hat{\zeta}_{xx}|u_x|^2\right] dxdt\right) \\
&+ \mathbb{E}\left(\int_Q \left[\hat{\zeta}u_x v_{xx} + \hat{\zeta}_x u v_{xx} - \hat{\zeta}_t u u_{xx} - \hat{\zeta}_x |u_{xx}|^2\right] dxdt\right) \\
&+ \mathbb{E}\left(\int_Q \left[2\hat{\zeta}|u_{xx}|^2 - \hat{\zeta}u_{xx}v_x\right] dxdt\right) \\
&+ \mathbb{E}\left(\int_Q \left[\hat{\zeta}|(d_1u)_x|^2 - \frac{1}{2}\hat{\zeta}_{xx}|d_1u|^2\right] dxdt\right). \tag{85}
\end{aligned}$$

Using the definition of ϕ_m and θ , we can see that

$$\begin{aligned}
|\partial_x^i(\theta^2\phi_m^p)| &\leq C_i\lambda^i\phi_m^{p+i}\theta^2, \quad i = 1, 2, 3, \\
|\partial_t(\theta^2\phi_m^p)| &\leq C\lambda\phi_m^{p+1+\frac{1}{m}}\theta^2, \quad \forall p \in \mathbb{N}^*,
\end{aligned}$$

from which we deduce

$$\begin{aligned}
|\hat{\zeta}_t| &\leq C\lambda^6\phi_m^{6+\frac{1}{m}}\theta^2\eta, \\
|\partial_x^i(\hat{\zeta})| &\leq C_i\lambda^{5+i}\phi_m^{5+i}\theta^2\sum_{j=0}^i(\partial_x^j\eta), \quad i = 1, 2, 3,
\end{aligned} \tag{86}$$

for all $(x, t) \in \mathcal{D}_2 \times (0, T)$.

Using Cauchy-Schwarz and Young inequalities together with (86) and taking into account the properties of the function $\hat{\eta}$, we can obtain from (85) the following estimate

$$\begin{aligned}
&\mathbb{E}\left(\int_{Q_{\mathcal{D}_3}} \lambda^5\phi_m^5\theta^2|u_{xxx}|^2 dxdt\right) \\
&\leq 4\epsilon\mathbb{E}\left(\int_Q \lambda\phi_m|u_{xxx}|^2 dxdt\right) + 2\delta\mathbb{E}\left(\int_Q \lambda\phi_m\theta^2|v_{xx}|^2 dxdt\right) \\
&+ \rho\mathbb{E}\left(\int_Q \lambda^3\phi_m^3\theta^2|v_x|^2 dxdt\right) + C\mathbb{E}\left(\int_{Q_{\mathcal{D}_2}} \lambda^{13}\phi_m^{13}\theta^2|u_x|^2 dxdt\right) \\
&+ C\mathbb{E}\left(\int_{Q_{\mathcal{D}_2}} \lambda^{15}\phi_m^{15}\theta^2|u|^2 dxdt\right) + C\mathbb{E}\left(\int_{Q_{\mathcal{D}_2}} \lambda^7\phi_m^7\theta^2|u_{xx}|^2 dxdt\right),
\end{aligned}$$

for any positive constants ϵ, δ, ρ and where $C > 0$ depends on $\|d_1\|_{L_{\mathcal{F}}^\infty(0, T; W^{2, \infty}(\mathcal{D}))}$. This concludes the proof.

Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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