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Leader-follower Consensus of Multi-Agent Systems

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Abstract— **This paper addresses the leader-follower consensus problem of multi-agent systems with a time-invariant communication topology consisting of general linear node dynamics. A distributed observer-type consensus protocol based on relative output measurements is proposed. It is shown that the leader-follower consensus of multi-agent systems can be cast equivalently into the stability of a set of matrices of the same dimension as a single agent. The notion of consensus region is then introduced and analyzed by using tools from the stability of matrix pencils. It is further demonstrated that there exists an observer-type protocol that solves the leader-follower consensus problem, and meanwhile yields an unbounded consensus region, if and only if the agent dynamics are stabilizable and detectable. A multi-step consensus protocol design procedure is finally presented. The effectiveness of the theoretical results is demonstrated through numerical simulations.**

I. INTRODUCTION

In recent years, coordination of multi-agent systems has received compelling attentions from scientific communities. It has broad applications in satellite formation flying, cooperative search of unmanned air vehicles, scheduling of automated highway systems, air traffic control, and distributed optimization of multiple mobile robotic systems. One critical issue arising in multi-agent systems is to develop distributed control policies based on local information that enable all agents to reach an agreement on certain quantities of interest, which is known as the consensus problem.

Consensus problems have a long-standing tradition in computer science. In the context of multi-agent systems, recent years have witnessed dramatic advances of various distributed strategies that achieve agreement. [1] proposed a simple model for phase transition of a group of self-driven particles and numerically depicts the complexity of the model. [2] provided a theoretical explanation for the behavior observed in [1] by using graph theory. In [3], a general framework of consensus problem for networks of dynamic agents with fixed or switching topologies and communication time-delays was addressed. The conditions given by [3] were further relaxed in [4]. [5] proposed a passivity-based design framework to treat the group coordination problem. [6] and [7] considered tracking control for multi-agent consensus with an active leader and gave a local controller together with a neighbor-based state-estimation rule. A differential game approach was proposed in [8] to address the consensus problem. The consensus problem of multi-vehicle systems with respect to a time-varying reference state was considered in [9]. For a complete coverage of the literatures on consensus, readers are referred to the recent surveys [10], [11]. One well-known problem in most existing works is that the agent dynamics are restricted to be an integrator or high-order integrators. Another common problem is that most proposed consensus protocols are based on the relative states between neighboring agents, which in many cases are not available.

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In this paper, one concerns the leader-follower consensus problem of multi-agent systems under a fixed (time-invariant) communication topology, with each agent having general liner dynamics, which may also be considered as the linearized model of a nonlinear network. A distributed observer-type consensus protocol based on relative output measurements is proposed, which can be regarded as extension of the traditional observer-based controller for a single system. Other kinds of dynamic protocols have been suggested, e.g., in [12], [13]. The protocol used in this paper differs from that in [12] on that the distributed protocols here maintain the same communication structure as the agent network. Besides, the dynamic coupling law in [13] requires the absolute output measurement of each agent to be available, which is impractical in many cases. A typical instance is the deep-space formation flying [14].

It is shown that the leader-follower consensus of multi-agent systems can be cast equivalently into the stability of a set of matrices of the same dimension as a single agent. By introducing a positive scalar denoting the coupling strength between neighboring agents, the notion of consensus region is brought forward and analyzed with the help of the stability of matrix pencils. It is demonstrated that there exists an observer-type protocol that solves the leader-follower consensus problem, and meanwhile yields an unbounded consensus region, if and only if the agent dynamics are stabilizable and detectable. Based on the consensus region analysis, a three-step consensus protocol design procedure is further proposed, in which steps 1) and 2) deal only with the agent dynamics and feedback gain matrices of the consensus protocol, leaving the communication topology of the agent network to be handled in step 3) by manipulating the coupling strength. For completeness, relative-state consensus protocols are also considered as a special case at the end of the paper.

II. NOTATION AND PRELIMINARIES

Let $\mathbf{R}^{n\times n}$ and $\mathbf{C}^{n\times n}$ are the set of $n\times n$ real matrices and complex matrices, respectively. The superscript T means

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transpose for real matrices and $*$ means conjugate transpose for complex matrices. I_N represents the identity matrix of dimension N. Let $\mathbf{1} \in \mathbf{R}^p$ denote the vector with all entries equal to one. Matrices, if not explicitly stated, are assumed to have compatible dimensions. $\otimes$ denotes the Kronecker product. For $\zeta \in \mathbf{C}$, denote by $\mathrm{Re}(\zeta)$ the real part of ζ and by $\mathrm{Im}(\zeta)$ the imaginary part ζ. A matrix $H \in \mathbf{C}^{n\times n}$ is Hurwitz (or stable), if all of its eigenvalues have strictly negative real parts.

A directed graph $\mathcal{G}$ consists of a node set $\mathcal{V}$ and an edge set $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$, where an edge is denoted as a pair of distinct nodes of $\mathcal{G}$. If $(i,j) \in \mathcal{E}$, then i is called the parent node, j the child node, and j is neighboring to i. A graph with the property that $(i,j) \in \mathcal{E}$ implies $(j,i) \in \mathcal{E}$ is said to be undirected. A path on $\mathcal{G}$ from vertices i_1 to i_l is a sequence of ordered edges of the form (i_k, i_{k+1}), $k = 1,\cdots,l-1$. A directed graph is called to be strongly connected if there exists a path between every pair of distinct nodes. A directed graph has or contains a directed spanning tree if there exists a node called root such that there exists a directed path from this node to every other node.

Suppose that there are m nodes in the graph. The adjacency matrix $A \in \mathbf{R}^{m\times m}$ is defined as $a_{ii} = 0$, a_{ij} is 1 if $(j,i) \in \mathcal{E}$ and 0 otherwise. The Laplacian matrix $\mathcal{L} \in \mathbf{R}^{m\times m}$ is defined as $\mathcal{L}_{ii} = \sum_{j\neq i} a_{ij}$, $\mathcal{L}_{ij} = -a_{ij}$ for $i \neq j$. Clearly, matrix $\mathcal{L}$ is symmetric if the graph is undirected.

Lemma 1: [4], [15] (i) All of the eigenvalues of $\mathcal{L}$ have nonnegative real parts; (ii) Zero is an eigenvalue of $\mathcal{L}$ with $\mathbf{1}$ as the corresponding right eigenvector. Furthermore, zero is a simple eigenvalue of $\mathcal{L}$ if and only if graph $\mathcal{G}$ has a directed spanning tree.

Lemma 2: [16] If graph $\mathcal{G}$ contains a directed spanning tree, then with proper permutation, the Laplacian matrix $\mathcal{L}$ can be reduced to the Frobenius normal form

$$\mathcal{L} = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1k} \\ 0 & L_{22} & \cdots & L_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_{kk} \end{bmatrix},$$

where L_{ii}, $i = 1,\cdots,k-1$, are irreducible, each L_{ii} has at least one row with positive row sum, and L_{kk} is irreducible or is zero matrix of dimension 1.

III. Observer-Type Consensus Protocol

Consider a group of $N+1$ identical agents with general linear dynamics, where an agent indexed by 0 is assigned as the leader and the agents indexed by $1,\cdots,N$, are referred to as followers. The dynamics of the i-th agent are denoted by

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i, \\ y_i &= Cx_i, \end{aligned} \tag{1}$$

where $x_i = [x_{i,1},\cdots,x_{i,n}]^T \in \mathbf{R}^n$ is the state, $u_i \in \mathbf{R}^p$ the control input, and $y_i \in \mathbf{R}^q$ the measured output. It is assumed that (A, B, C) is stabilizable and detectable.

The communication topology among the N follower agents is represented by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1,\cdots,N\}$ is the set of nodes (i.e., agents), and $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$ is the set of edges. An edge (i,j) in graph $\mathcal{G}$ denotes that follower j can obtain information from follower i, but not conversely. The leader agent receives no information from any other agent. It is assumed that only a subset of the follower agents has access to the output variable of the leader agent, whereas all the followers know the input of the leader.

The relative measurements of other agents with respect to follower agent i will be synthesized into a single signal as follows

$$\zeta_i = c\left(\sum_{j=1}^N a_{ij}(y_i - y_j) + d_i(y_i - y_0)\right), \tag{2}$$

where $c > 0$ denotes the coupling strength, $a_{ii} = 0$, a_{ij} is 1 if follower agent i can obtain information from follower j and 0 otherwise, and $d_i > 0$ if follower i has access to the leader agent and 0 otherwise. A observer-type consensus protocol is proposed as

$$\begin{aligned} \dot{v}_i =& (A+BK)v_i \\ &+ F\left(c\sum_{j=1}^N a_{ij}C(v_i - v_j) + cd_iC(v_i - v_0) - \zeta_i\right), \\ u_i =& Kv_i + u_0, \end{aligned} \tag{3}$$

where $v_i \in \mathbf{R}^n$ is the protocol state, $i = 1,\cdots,N$, $F \in \mathbf{R}^{q\times n}$ and $K \in \mathbb{R}^{p\times n}$ are feedback gain matrices to be determined, and $v_0 \in \mathbb{R}^n$ is the state of system $\dot{v}_0 = (A+BK)v_0$. The term $\sum_{j=1}^N a_{ij}C(v_i - v_j)$ in (3) denotes the information exchanges between consensus protocols. It is observed that the protocols in (3) maintain the same communication structure as the agents.

With protocol (3), the leader-follower dynamic consensus problem is said to be solved if

$$x_i(t) \to x_0(t),\ v_i(t) \to 0,\ \forall i = 1,\cdots,N,\ \text{as } t \to \infty.$$

Let $x_{e,i} = x_i - x_r$, $v_{e,i} = v_i - v_r$, $\xi_i = [x_{e,i}^T, v_{e,i}^T]^T$, $\xi = [\xi_1^T,\cdots,\xi_N^T]^T$. Then, the closed-loop network dynamics can be written in the following form

$$\dot{\xi} = (I_N \otimes \mathcal{A} + c\widehat{\mathcal{L}} \otimes \mathcal{H})\xi, \tag{4}$$

where $\widehat{\mathcal{L}} = \mathcal{L} + \widehat{D}$, $\mathcal{L} \in \mathbf{R}^{N\times N}$ is the Laplacian matrix of $\mathcal{G}$, $\widehat{D} = \mathrm{diag}(d_1, d_2,\cdots,d_N)$, and

$$\mathcal{A} = \begin{bmatrix} A & BK \\ 0 & A+BK \end{bmatrix},\quad \mathcal{H} = \begin{bmatrix} 0 & 0 \\ -FC & FC \end{bmatrix}.$$

It is easy to see that the leader-follower consensus problem is solved if and only if the state of (4) converges to zero.

Lemma 3: Suppose that the directed communication graph $\mathcal{G}$ has a spanning tree, and the root agent of such tree has access to the leader. Then, all the eigenvalues of $\widehat{\mathcal{L}}$ have positive real parts.

Proof: Without loss of generality, one can rearrange, if necessary, the order of the followers in the network such that the Laplacian matrix $\mathcal{L}$ takes the Frobenius normal form.

Since at least the root follower agent has access to the leader agent, in light of Lemma 2, all the submatrices along the diagonal of $\widehat{\mathcal{L}}$ is irreducible and has at least one row with positive row sum. Then, by following the steps in proving Lemma 1 in [17], it is not difficult to obtain that all the eigenvalues of $\widehat{\mathcal{L}}$ have positive real parts. ■

Theorem 1: Assume that the communication topology $\mathcal{G}$ has a directed spanning tree, and the root agent of such tree has access to the leader. Then, protocol (3) solves the leader-follower consensus problem, if and only if the matrices $A+BK$, $A+c\hat{\lambda}_i FC$, $i=1,\cdots,N$, are Hurwitz, where $\hat{\lambda}_i, i=1,\cdots,N$, denote the eigenvalues of $\widehat{\mathcal{L}}$.

Proof: Let T be such a unitary matrix that $T^{-1}\widehat{\mathcal{L}}T=U$, where U is a upper-triangle matrix with the eigenvalues of $\widehat{\mathcal{L}}$ on the diagonal. Introduce the variable transformations $\xi=(T\otimes I_n)\tilde{\xi}$ with $\tilde{\xi}=[\tilde{\xi}_1^T,\cdots,\tilde{\xi}_N^T]^T$. Then, system (4) can be recast in terms of $\tilde{\xi}$ as

$$\dot{\tilde{\xi}}=(I_N\otimes\mathcal{A}+cU\otimes\mathcal{H})\tilde{\xi}. \tag{5}$$

Since the elements of the state matrix of (5) are either block diagonal or block upper-triangular, the stability of system (5) is equivalent to the simultaneous stability of the following N subsystems along the diagonal:

$$\dot{\tilde{\xi}}_i=(I_N\otimes\mathcal{A}+c\hat{\lambda}_i\otimes\mathcal{H})\tilde{\xi}_i,\quad i=1,\cdots,N. \tag{6}$$

It is easy to check that matrices $\mathcal{A}+c\hat{\lambda}_i\mathcal{H}$ are similar to

$$\begin{bmatrix} A+c\hat{\lambda}_iFC & 0\\ -c\hat{\lambda}_iFC & A+BK\end{bmatrix},\quad 1,\cdots,N.$$

Therefore, the stability of the matrices $A+BK$, $A+c\hat{\lambda}_iFC$, $i=1,\cdots,N$, is equivalent to that the state ξ of (4) converges asymptotically to zero, i.e., the leader-follower consensus problem is solved. ■

Remark 1: The importance of this theorem lies in that it converts the consensus problem of a large-scale multi-agent network under the observer-type protocol (3) into the stability of a set of matrices with the same dimension as a single agent, thereby significantly reducing the computational complexity. The observer-type consensus protocol (3) can be seen as an extension of the traditional observer-based controller for a single system to one of the multi-agent systems. The Separation Principle of the traditional observer-based controller still holds in the multi-agent setting. The effects of the communication topology on the consensus problem are represented by the eigenvalues of the corresponding modified Laplacian matrix $\widehat{\mathcal{L}}$. According to Lemma 3, the leader-follower consensus can be possibly reached for some protocols (3), even when only the root follower agent can have access to the output variable of the leader agent.

Remark 2: Theorem 1 generalizes the existing results on the consensus problem in at least two aspects. First, the agent dynamics are extended to be general linear, but not limited to single-integrator, double-integrators, or structural high-order linear systems as usually assumed in most existing papers [3], [4], [9]. Second, an observer-type consensus protocol is proposed, which is based only on relative output measurements between neighboring agents, in contrast to [13] where the dynamic protocol requires the absolute output of each agent to be available. Compared to the existing consensus protocols, a unique feature of the consensus protocol (3) is that a positive scalar called the coupling strength is introduced, similar to the complex network models as studied in [18], [19]. With this parameter, the notion of consensus region is brought forward, as detailed in the following subsection.

A. Consensus Region Analysis

Given a protocol of the form (3), the consensus problem can be cast into analyzing the following system:

$$\dot{\varsigma}=(\mathcal{A}+\sigma\mathcal{H})\varsigma=\begin{bmatrix} A & BK\\ -\sigma FC & A+BK+\sigma FC\end{bmatrix}\varsigma, \tag{7}$$

where $\varsigma\in\mathbf{R}^{2n}$, $\sigma\in\mathbf{C}$.

The stability of systems (7) depends on the parameter σ. The region $\mathcal{S}$ of the complex parameter σ, such that (7) is asymptotically stable, is called the consensus region of network (4) in this paper. It follows from Theorem 1 that the leader-follower consensus is reached if and only if

$$c(\alpha_k+\mathrm{i}\beta_k)\in\mathcal{S},\quad k=2,3,\cdots,N,$$

where $\mathrm{i}=\sqrt{-1}$, $\alpha_k=\mathrm{Re}(\lambda_k)$ and $\beta_k=\mathrm{Im}(\lambda_k)$. For an undirected communication graph, its consensus region $\mathcal{S}$ is an interval or a union of several intervals on the real axis. But for a directed graph, where the eigenvalues of $\mathcal{L}$ are generally complex numbers, its consensus region $\mathcal{S}$ is a region or a union of several regions on the complex plane.

It is worth noting that the consensus region $\mathcal{S}$ can be seen as the stability region of the matrix pencil $\mathcal{A}+\sigma\mathcal{H}$ with respect to the complex parameter σ. Thus, tools from the stability of matrix pencils will be utilized to analyze the consensus region problem. Before moving on, the lemma below is needed.

Lemma 4: [20] Given a complex-coefficient polynomial,

$$p(s)=s^2+(a+\mathrm{i}b)s+c+\mathrm{i}d, \tag{8}$$

where $a,b,c,d\in\mathbf{R}$, $p(s)$ is stable if and only if $a>0$, $abd+a^2c-d^2>0$.

In the above lemma, only second-order polynomials are considered. Similar results for high-order complex-coefficient polynomials can also be given (see [20]). However, in the latter case, the analysis will be more complicated.

The following example has a bounded consensus region.

Example 1: The agent dynamics and the consensus protocol are given by (1) and (3), respectively, with

$$A=\begin{bmatrix}-2 & 2\\ -1 & 1\end{bmatrix},\quad B=\begin{bmatrix}1\\0\end{bmatrix},\quad C=\begin{bmatrix}1 & 0\end{bmatrix},$$

$$F=\begin{bmatrix}1\\-1\end{bmatrix},\quad K=\begin{bmatrix}-1 & 2\end{bmatrix},\quad \sigma=x+\mathrm{i}y.$$

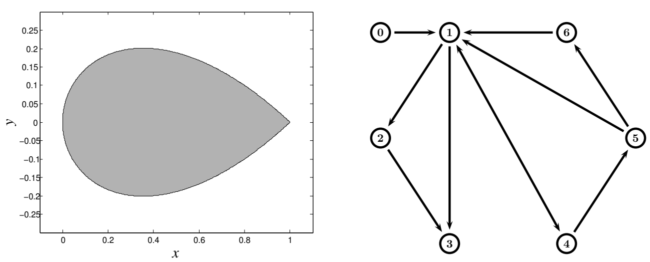
It is assumed that the input u_0 of the leader is zero. Obviously, $A+BK$ is Hurwitz. The characteristic polynomial of $A+\sigma FC$ is

$$\det(sI-(A+\sigma FC))=s^2+(1-x-\mathrm{i}y)s+3(x+\mathrm{i}y).$$

By Lemma 4, $A + (x + \mathrm{i}y)FC$ is stable if and only if $1 - x > 0$ and $-3y^2(1 - x) + 3x(1 - x)^2 - 9y^2 > 0$. The consensus region $\mathcal{S}$ in this case is depicted in Fig. 1 (a), which is bounded. Assume that the communication graph is given by Fig. 1 (b), where only agent 1 has access to the leader with $d_1 = 1$. The corresponding modified Laplacian matrix $\widehat{\mathcal{L}}$ is

$$\widehat{\mathcal{L}} = \begin{bmatrix} 4 & 0 & 0 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix},$$

with eigenvalues $0.1185, 1, 1.3194 \pm 0.4978i, 2, 4.2426$. It can be verified that consensus is achieved if and only if $0 < c \leq 0.2357$. Fig. 2 depicts the states of network (4) with the coupling strengths $c = 0.2$ for this example.



(a) (b)

Fig. 1. (a) Bounded consensus region; (b) the communication topology.

Remark 3: The consensus region discussed in this subsection is similar to the synchronization region studied in [18], [19], [21]. The consensus region serves in certain sense as a measurement for the robustness of the protocol (3) to parametric uncertainty. An example can be easily constructed such that the bound of the consensus region on the real axis for certain protocol is quite narrow, e.g., $[1, 1.005]$, and all the eigenvalues of $\widehat{\mathcal{L}}$ are 1. Assume that the coupling strength c in (2) are subjected to multiplicative uncertainties, e.g., (2) is changed to $\hat{\zeta}_i = (1 + \epsilon)\zeta_i$, where ϵ denotes the uncertainty. Clearly, if $|\epsilon| > 0.005$, then this protocol will fails to solve the consensus problem. Therefore, given a consensus protocol, the consensus region should be large enough in both real and imaginary parts for the protocol to maintain a desirable robustness margin.

The following example has a disconnected the consensus region.

Example 2: The agent dynamics and the consensus protocol are given by (1) and (3), respectively, with

$$A = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$F = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} -1 & 2 \end{bmatrix}, \quad \sigma = x + \mathrm{i}y.$$

Obviously, $A + BK$ is Hurwitz. The characteristic polynomial of $A + \sigma FC$ is

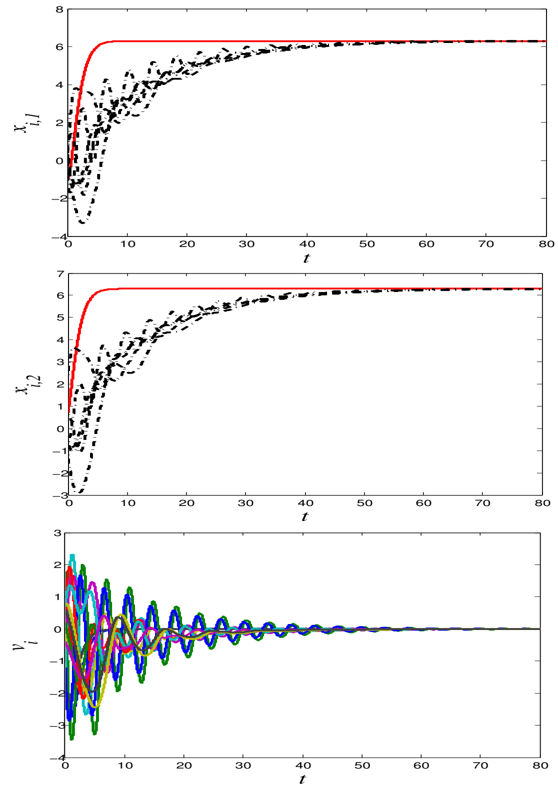


Fig. 2. The states of the agent network (4) when $c = 0.2$.

$$\det(sI - (A + \sigma FC)) = s^2 + s - 2 + (x+1)(x+2) - y^2 + \mathrm{i}(2y+3)y.$$

By Lemma 4, $A + (x + \mathrm{i}y)BF$ is stable if and only if $x(x + 3) - y^2 - (2x + 3)^2y^2 > 0$. The consensus region $\mathcal{S}$ in this case is depicted in Fig. 3, which is composed of two disjoint subregions, both with unbounded real part and bounded imaginary part. If the communication graph is still given by Fig. 1 (b) with $d_1 = 1$, then consensus is achieved if and only if $0 < c \leq 0.4802$. If the coupling strength c can be negative, then the consensus will be achieved also when $c < -25.3165$.

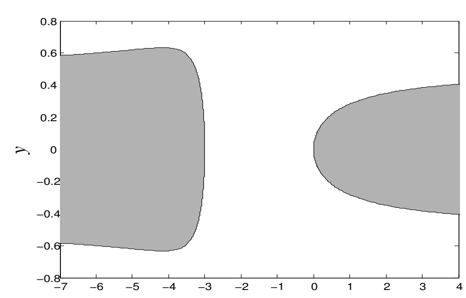


Fig. 3. Disconnected consensus region.

B. Consensus Protocol Design

The consensus region problem has been discussed in the last subsection, when the protocol is known. On the other hand, in many cases, the protocols have to be designed so as to solve the consensus problem for various given communication topologies.

From the above subsection, it can be seen that the cases with bounded consensus regions are more complicated than the cases with unbounded consensus regions. Hence, it is convenient to design a protocol such that the consensus region is unbounded.

Proposition 1: Given the agent dynamics (1), there exists a matrix F such that $A+(x+\mathrm{i}y)FC$ is Hurwitz for all $x \in [1,\infty)$, $y \in (-\infty,\infty)$, if and only if (A,C) is detectable.

Proof: (Necessity): It is trivial by letting $x=1$, $y=0$.

(Sufficiency): Since (A,C) is detectable, there exists a matrix F such that $A+FC$ is Hurwitz, i.e., there exists a matrix $P>0$ such that

$$(A+FC)^T P + P(A+FC) < 0.$$

Let $PF = Y$. Then, the above inequality becomes

$$A^T P + PA + YC + C^T Y^T < 0.$$

By Finsler's Lemma [22], there exists a matrix Y satisfying the above inequality if and only if there exists a scalar $\tau > 0$ such that

$$AP + PA^T - \tau C^T C < 0. \tag{9}$$

Since matrix P is to be determined, let $\tau = 2$ without loss of generality. Then,

$$AP + PA^T - 2C^T C < 0. \tag{10}$$

Obviously, when (10) holds, for any $\tau \geq 2$, (9) holds. Take $Y = -C^T$, i.e., $F = -P^{-1}C^T$. By the above inequalities, $A + xFC$ is Hurwitz for all $x \in [1,\infty)$. Thus, one has

$$\begin{aligned}&(A+(x+y\mathrm{i})FC)^* P + P(A+(x+y\mathrm{i})FC)\\ &\quad= (A+(x-y\mathrm{i})FC)^T P + P(A+(x+y\mathrm{i})FC)\\ &\quad= AP + PA^T - 2xC^T C < 0,\end{aligned}$$

for all $x \in [1,\infty), y \in (-\infty,\infty)$. ■

Theorem 1 and the above proposition together lead to the following result.

Theorem 2: Assume that the communication topology $\mathcal{G}$ has a directed spanning tree, and the root agent of such tree has access to the leader. Then, there exists a distributed protocol in the form of (3) that solves the consensus problem, and meanwhile yields an unbounded consensus region $\mathcal{S} \triangleq [1,\infty) \times (-\infty,\infty)$, if and only if (A,B,C) is stabilizable and detectable.

A multi-step consensus protocol design procedure based on the consensus region notion is presented.

Algorithm 1: Suppose that $\mathcal{G}$ contains a directed spanning tree, the root agent of such tree has access to the leader, and that (A,B,C) is stabilizable and detectable. A protocol of the form (3) solving the leader-follower consensus problem can be constructed according to the following steps.

1) Choose the matrix K such that $A+BK$ is Hurwitz.

2) Solve the LMI (10) to get one solution $P>0$. Then, choose the feedback gain matrix $F = -P^{-1}C^T$.

3) Select the coupling strength c larger than the threshold value c_{th}, given by

$$c_{th} = \frac{1}{\min\limits_{i=2,\cdots,N} \mathrm{Re}(\hat{\lambda}_i)}, \tag{11}$$

where $\hat{\lambda}_i$, $i=1,\cdots,N$, are the eigenvalues of the $\widehat{\mathcal{L}}$.

Remark 4: One distinct feature of Algorithm 1 is that it decouples the effects of the agent dynamics and the protocol dynamics on the consensus stability from that that of the communication topology. To be specific, steps 1) and 2) deal only with the agent dynamics and feedback gain matrices of the consensus protocol, leaving the communication topology of the agent network to be handled in step 3) by manipulating the coupling strength. This feature will be more desirable for the case when the agent number N is large, for which the eigenvalues of $\widehat{\mathcal{L}}$ are hard to determine or even troublesome to estimate. In this case, one only needs to choose the coupling strength c to be large enough.

Now, Example 1 in the above subsection is revisited.

Example 3: The agent dynamics and the feedback gain matrix K of the protocol (3) remain the same as in Example 1, while matrix F will be redesigned via Algorithm 1. By solving LMI (10), a feasible solution F is obtained as $F = \left[\begin{smallmatrix}-7.0314\\ 8.4153\end{smallmatrix}\right]$. Differing from Example 1, an unbounded consensus region of the form $[1,\infty) \times (-\infty,\infty)$ can be obtained here. This can be verified in another way by noticing that the characteristic polynomial of $A + \sigma FC$ becomes

$$\begin{aligned}&\det(sI - (A+\sigma FC))\\ &\quad= s^2 + (1 + 7.0314(x+\mathrm{i}y))s - 23.862(x+\mathrm{i}y).\end{aligned}$$

Then, $A+(x+\mathrm{i}y)FC$ is Hurwitz, if and only if

$$\begin{aligned}&1 + 7.0314x > 0,\\ &(7.0314 + 49.44x)y^2 + x(1+7.0314x)^2 + 23.862y^2 > 0.\end{aligned}$$

The consensus region $\mathcal{S}$ in this case is the right-half plane by the vertical line $x = -\frac{1}{7.0314}$, except the white area depicted in Fig. 4, which obviously contains the region $[1,\infty) \times (-\infty,\infty)$. The protocol (3) with feedback gains F and K as above and any $c>0$ will solve the consensus problem for any communication graph containing a spanning tree.

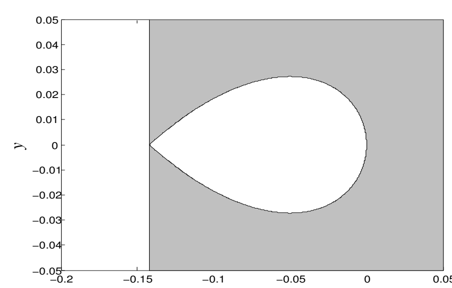


Fig. 4. The unbounded consensus region.

IV. Relative-State Consensus Protocol As A Special Case

In this subsection, a special case when the relative states between neighboring agents are available is considered. For this case, a distributed static protocol can be proposed as

$$u_i = cL\sum_{j=1}^{N} a_{ij}(x_i - x_j) + u_0, \tag{12}$$

where $c > 0$ and a_{ij} is the same as defined in (2), and $L \in \mathbf{R}^{p\times n}$ is the feedback gain matrix to be determined.

With protocol (12), the leader-follower consensus problem is then defined as

$$x_i(t) \to x_0(t),\ \forall i = 1, \cdots, N,\ \text{as } t \to \infty.$$

Corollary 1: Assume that the communication topology $\mathcal{G}$ has a directed spanning tree, and the root agent of such tree has access to the leader. Then, protocol (12) solves the leader-follower consensus problem, if and only if matrices $A + c\hat{\lambda}_i BF$ are Hurwitz for $i = 1, \cdots, N$, where $\hat{\lambda}_i$ are the same as in Theorem 1.

Similar to the observer-type protocol case, the consensus region of protocol (12) corresponds to the stability region of system $\dot{\tilde{\varsigma}} = (A + \sigma BL)\tilde{\varsigma}$ with respect to $\sigma \in \mathbf{C}$, where $\tilde{\varsigma} \in \mathbf{R}^n$.

The dual of Proposition 1 is obtained as follows.

Proposition 2: Given the agent dynamics (1), there exists a matrix L such that $A + (x + \mathrm{i}y)BL$ is Hurwitz for all $x \in [1, \infty)$, $y \in (-\infty, \infty)$, if and only if (A, B) is stabilizable.

Algorithm 2: Given (A, B) that is stabilizable, a protocol of the form (12) solving the consensus problem can be constructed according to the following steps.

1) Solve the following LMI

$$AP + PA^T - 2BB^T < 0 \tag{13}$$

to get one solution $P > 0$. Then, choose the feedback gain matrix $L = -B^T P^{-1}$.

2) Select the coupling strength $c > c_{th}$, with c_{th} given in (11).

V. Conclusions

In this paper, one addresses the leader-follower consensus problem of multi-agent systems under time-invariant communication topology, with each agent having general linear dynamics. A distributed observer-type consensus protocol based on relative output measurements is proposed. The notion of consensus region is introduced and analyzed by using tools from the stability of matrix pencils. It is shown that there exists an observer-type protocol that solves the leader-follower consensus problem, and meanwhile yields an unbounded consensus region, if and only if the agent dynamics are stabilizable and detectable. Build on the consensus region analysis, a multi-step consensus protocol designing procedure is finally presented.

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