

COMPOSITIONAL CONSTRUCTION OF CONTROL BARRIER CERTIFICATES FOR LARGE-SCALE STOCHASTIC SWITCHED SYSTEMS

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ABSTRACT. In this paper, we propose a compositional framework for the construction of control barrier certificates for large-scale stochastic switched systems accepting *multiple* control barrier certificates with some *dwell-time* conditions. The proposed scheme is based on a notion of so-called *augmented pseudo-barrier certificates* computed for each switched subsystem, using which one can compositionally synthesize state-feedback controllers for interconnected systems enforcing safety specifications over a finite-time horizon. In particular, we first leverage sufficient max-type small-gain conditions to compositionally construct augmented control barrier certificates for interconnected systems based on the corresponding augmented pseudo-barrier certificates of subsystems. Then we quantify upper bounds on exit probabilities - the probability that an interconnected system reaches certain *unsafe* regions - in a finite-time horizon using the constructed augmented barrier certificates. We employ a technique based on a counter-example guided inductive synthesis (CEGIS) approach to search for control barrier certificates of each mode while synthesizing safety controllers providing switching signals. We demonstrate our proposed results by applying them first to a room temperature network containing 1000 rooms. Finally, we apply our techniques to a network of 500 switched subsystems (totally 1000 dimensions) accepting *multiple* barrier certificates with a *dwell-time* condition, and provide upper bounds on the probability that the interconnected system reaches some unsafe region in a finite-time horizon.

1. INTRODUCTION

Motivations. This paper is mainly motivated by the challenges emerging in the control of large-scale stochastic switched systems. In the past few years, stochastic switched systems have obtained considerable attentions among both control and computer scientists due to their broad applications in modeling many real-life systems. Since the closed-form solution of synthesized policies for stochastic switched systems is not in general obtainable, automated synthesis for this type of complex systems is naturally very challenging especially with respect to high-level logic properties, *e.g.*, linear temporal logic (LTL) formulae [Pnu77].

To alleviate the encountered computational complexity, approximation techniques have been proposed in relevant literatures, where original dynamics are approximated by simpler ones with finite state sets, *i.e.*, finite abstractions. However, one major bottleneck in the existing approximation techniques is the state-explosion problem. To tackle this issue, some recent works (*e.g.*, [LSMZ17, LSZ18a, LSZ18b, LZ19a, LSZ19c, LZ19b, LSZ19b, LSZ20a, LSZ20c, LSZ20b, LSZ19a, Lav19, NZ20, NSZ20a]) study different compositional schemes for the construction of (in)finite abstractions for complex stochastic systems via (in)finite abstractions of their smaller subsystems.

Control barrier certificates, as a discretization-free approach for controller synthesis of complex systems, have been introduced in recent years as another potential solution to mitigate the computational complexity arising in the analysis or synthesis of large-scale stochastic systems. In this respect, discretization-free techniques based on barrier certificates for stochastic hybrid systems are initially proposed in [PJP07]. Stochastic safety verification using barrier certificates for switched diffusion processes and stochastic hybrid systems is, respectively, proposed in [WB17] and [HCL⁺17]. A verification approach for stochastic switched systems against safe LTL objectives via barrier functions is proposed in [AJZ19]. Temporal logic verification of stochastic systems via control barrier certificates and its extension to formal synthesis are, respectively, proposed in [JSZ18] and [JSZ19]. Recently, compositional construction of control barrier certificates for large-scale stochastic discrete-time and continuous-time systems is respectively presented in [ALZ20] and [NSZ20b].

It should be noted that although [LSZ20a] provides a compositional approach for the same class of stochastic switched system as in this work, their proposed framework is based on the construction of finite abstractions which relies on the discretization of state and input sets and consequently suffers from the curse of dimensionality problem. In contrast, we propose here a compositional framework, for the first time for stochastic switched systems, based on control barrier certificates.

Contributions. In this paper, we propose a compositional framework for the construction of control barrier certificates for large-scale stochastic switched systems accepting *multiple* control barrier certificates with some *dwelling-time* conditions. To this end, we first provide an augmented framework for presenting each switched subsystem with several modes with a single system covering all modes (called augmented switched systems) whose output trajectories are exactly the same as those of original switched systems. We then compositionally construct augmented control barrier certificates for interconnected augmented systems based on so-called *augmented pseudo-barrier certificates* of subsystems by leveraging some max-type small-gain conditions. Afterwards, given the constructed augmented barrier certificates, we quantify upper bounds on the probability that interconnected systems reach certain unsafe regions in a finite-time horizon. We finally utilize a technique based on a counter-example guided inductive synthesis (CEGIS) approach to search for control barrier certificates of each mode.

To illustrate the effectiveness of our proposed results, we first apply them to a room temperature network in a circular building containing 1000 rooms and compositionally synthesize safety controllers to keep the temperature of each room in a comfort zone in a bounded time horizon. Eventually, to show the applicability of our results to switched systems accepting *multiple* barrier certificates with a *dwelling-time* condition, we apply our proposed techniques to a circular cascade network of 500 subsystems (totally 1000 dimensions) and provide upper bounds on the probability that the interconnected system reaches some unsafe region in a finite-time horizon.

Related Work. Although the proposed results in [WB17, HCL⁺17] deal with stochastic switched and a class of hybrid systems, their ultimate goal is to perform probabilistic safety verification via barrier certificates in a monolithic manner. In comparison, in this work, we propose a compositional framework for the construction of control barrier certificates for large-scale stochastic switched systems admitting *multiple* barrier certificates with some *dwelling-time* conditions. We utilize those barrier certificates and conditions to compositionally synthesize state-feedback controllers for interconnected systems enforcing safety specifications over a finite-time horizon.

2. DISCRETE-TIME STOCHASTIC SWITCHED SYSTEMS

2.1. Preliminaries. We consider a probability space $(\Omega, \mathcal{F}_\Omega, \mathbb{P}_\Omega)$, where Ω is the sample space, \mathcal{F}_Ω is a sigma-algebra on Ω comprising subsets of Ω as events, and \mathbb{P}_Ω is a probability measure that assigns probabilities to events. We assume that random variables introduced in this article are measurable functions of the form $X : (\Omega, \mathcal{F}_\Omega) \rightarrow (S_X, \mathcal{F}_X)$. Any random variable X induces a probability measure on its space (S_X, \mathcal{F}_X) as $\text{Prob}\{A\} = \mathbb{P}_\Omega\{X^{-1}(A)\}$ for any $A \in \mathcal{F}_X$. We often directly discuss the probability measure on (S_X, \mathcal{F}_X) without explicitly mentioning the underlying probability space and the function X itself.

A topological space S is called a Borel space if it is homeomorphic to a Borel subset of a Polish space (*i.e.*, a separable and completely metrizable space). Examples of a Borel space are the Euclidean spaces \mathbb{R}^n , its Borel subsets endowed with a subspace topology, as well as hybrid spaces. Any Borel space S is assumed to be endowed with a Borel sigma-algebra, which is denoted by $\mathbb{B}(S)$. We say that a map $f : S \rightarrow Y$ is measurable whenever it is Borel measurable.

2.2. Notation. The following notation is employed throughout the paper. We denote the set of real, positive and non-negative real numbers by $\mathbb{R}, \mathbb{R}_{>0}$, and $\mathbb{R}_{\geq 0}$, respectively. We use \mathbb{R}^n to denote a real space of n dimension. $\mathbb{N} := \{0, 1, 2, \dots\}$ represents the set of non-negative integers and $\mathbb{N}_{\geq 1} = \{1, 2, \dots\}$ is the set of positive integers. Given N vectors $x_i \in \mathbb{R}^{n_i}$, $x = [x_1; \dots; x_N]$ denotes the corresponding vector of dimension

$\sum_i n_i$. Given a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the infinity norm of x . Symbols \mathbb{I}_n , $\mathbf{0}_n$, and $\mathbf{1}_n$ denote the identity matrix in $\mathbb{R}^{n \times n}$ and the column vector in $\mathbb{R}^{n \times 1}$ with all elements equal to zero and one, respectively. The identity function and composition of functions are denoted by \mathcal{I}_d and symbol \circ , respectively. Given functions $f_i : X_i \rightarrow Y_i$, for any $i \in \{1, \dots, N\}$, their Cartesian product $\prod_{i=1}^N f_i : \prod_{i=1}^N X_i \rightarrow \prod_{i=1}^N Y_i$ is defined as $(\prod_{i=1}^N f_i)(x_1, \dots, x_N) = [f_1(x_1); \dots; f_N(x_N)]$. A function $\varphi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be a class \mathcal{K} function if it is continuous, strictly increasing, and $\varphi(0) = 0$. A class \mathcal{K} function $\varphi(\cdot)$ belongs to class \mathcal{K}_∞ if $\varphi(s) \rightarrow \infty$ as $s \rightarrow \infty$.

2.3. Discrete-Time Stochastic Switched Systems. We consider stochastic switched systems in discrete-time (dt-SS) defined formally as follows.

Definition 2.1. *A discrete-time stochastic switched system (dt-SS) is characterized by the tuple*

$$\Sigma = (X, P, \mathcal{P}, W, \varsigma, F, Y, h), \quad (2.1)$$

where:

- $X \subseteq \mathbb{R}^n$ is a Borel space as the state set of the system. We denote by $(X, \mathbb{B}(X))$ the measurable space with $\mathbb{B}(X)$ being the Borel sigma-algebra on the state space;
- $P = \{1, \dots, m\}$ is a finite set of modes;
- \mathcal{P} is a subset of $\mathcal{S}(\mathbb{N}, P)$ which denotes the set of functions from \mathbb{N} to P ;
- $W \subseteq \mathbb{R}^{\bar{p}}$ is a Borel space as the internal input set of the system;
- ς is a sequence of independent and identically distributed (i.i.d.) random variables on a set V_ς

$$\varsigma := \{\varsigma(k) : \Omega \rightarrow V_\varsigma, k \in \mathbb{N}\};$$

- $F = \{f_1, \dots, f_m\}$ is a collection of vector fields indexed by p . For all $p \in P$, the map $f_p : X \times W \times V_\varsigma \rightarrow X$ is a measurable function characterizing the state evolution of the system in mode p ;
- $Y \subseteq \mathbb{R}^q$ is a Borel space as the output set of the system;
- $h : X \rightarrow Y$ is a measurable function as the output map that maps a state $x \in X$ to its output $y = h(x)$.

For a given initial state $x(0) \in X$, an internal input sequence $w(\cdot) : \mathbb{N} \rightarrow W$, and a switching signal $\mathbf{p}(k) : \mathbb{N} \rightarrow P$, the evolution of the state of Σ is described as

$$\Sigma : \begin{cases} x(k+1) = f_{\mathbf{p}(k)}(x(k), w(k), \varsigma(k)), \\ y(k) = h(x(k)), \end{cases} \quad k \in \mathbb{N}. \quad (2.2)$$

We assume that the signal \mathbf{p} satisfies a *dwell-time* condition [Mor96] as defined in the next definition.

Definition 2.2. *Consider a switching signal $\mathbf{p} : \mathbb{N} \rightarrow P$ and define its switching time instants as*

$$\mathfrak{S}_{\mathbf{p}} := \{\mathfrak{s}_k : k \in \mathbb{N}_{\geq 1}\}.$$

Then, $\mathbf{p} : \mathbb{N} \rightarrow P$ has dwell-time $k_d \in \mathbb{N}$ [Mor96] if elements of $\mathfrak{S}_{\mathbf{p}}$ ordered as $\mathfrak{s}_1 \leq \mathfrak{s}_2 \leq \mathfrak{s}_3 \leq \dots$ satisfy $\mathfrak{s}_1 \geq k_d$ and $\mathfrak{s}_{k+1} - \mathfrak{s}_k \geq k_d, \forall k \in \mathbb{N}_{\geq 1}$.

For any $p \in P$, we use Σ_p to refer to system (2.2) with a constant switching signal $\mathbf{p}(k) = p$ for all $k \in \mathbb{N}$. We are ultimately interested in investigating interconnected dt-SS without internal inputs that result from the interconnection of dt-SS having both internal and external inputs. In this case, the interconnected dt-SS without internal inputs is indicated by the simplified tuple $(X, P, \mathcal{P}, \varsigma, F, Y, h)$ where $f_p : X \times V_\varsigma \rightarrow X, \forall p \in P$.

2.4. Augmented Stochastic Switched Systems. Here, given a dt-SS Σ , we introduce the notion of augmented dt-SS as in the next definition. Note that this notion is adapted from the definition of labeled transition systems defined in [BK08] and modified to capture the stochastic nature of the system. This provides an alternative description of switched systems enabling us to represent a switched system with a finite set of modes via an augmented system covering the whole modes.

Definition 2.3. Given a dt-SS $\Sigma = (X, P, \mathcal{P}, W, \varsigma, F, Y, h)$, we define the associated augmented dt-SS $\mathbb{A}(\Sigma) = (\mathbb{X}, \mathbb{P}, \mathbb{W}, \varsigma, \mathbb{F}, \mathbb{Y}, \mathbb{H})$, where:

- $\mathbb{X} = X \times P \times \{0, \dots, k_d - 1\}$ is the set of states. A state $(x, p, l) \in \mathbb{X}$ means that the current state of Σ is x , the current value of the switching signal is p , and the time elapsed since the latest switching time instant upper bounded by k_d is l ;
- $\mathbb{P} = P$ is the set of external inputs;
- $\mathbb{W} = W$ is the set of internal inputs;
- ς is a sequence of i.i.d. random variables;
- $\mathbb{F} : \mathbb{X} \times \mathbb{P} \times \mathbb{W} \times V_\varsigma \rightarrow \mathbb{X}$ is the one-step transition function given by $(x', p', l') = \mathbb{F}((x, p, l), p, w, \varsigma)$ if and only if $x' = f_p(x, w, \varsigma)$ and the following scenarios hold:
 - $l < k_d - 1, p' = p$, and $l' = l + 1$: switching is not allowed because the time elapsed since the latest switch is strictly smaller than the dwell-time;
 - $l = k_d - 1, p' = p$, and $l' = k_d - 1$: switching is allowed but no switch occurs;
 - $l = k_d - 1, p' \neq p$, and $l' = 0$: switching is allowed and a switch occurs;
- $\mathbb{Y} = Y$ is the output set;
- $\mathbb{H} : \mathbb{X} \rightarrow \mathbb{Y}$ is the output map defined as $\mathbb{H}(x, p, l) = h(x)$.

We associate respectively to \mathbb{P} and \mathbb{W} the sets \mathbf{P} and \mathbf{W} to be collections of sequences $\{p(k) : \Omega \rightarrow \mathbb{P}, k \in \mathbb{N}\}$ and $\{w(k) : \Omega \rightarrow \mathbb{W}, k \in \mathbb{N}\}$, in which $p(k)$ and $w(k)$ are independent of $\varsigma(t)$ for any $k, t \in \mathbb{N}$ and $t \geq k$. We also denote initial conditions of p and l by p_0 and $l_0 = 0$.

Remark 2.4. Note that in the augmented dt-SS $\mathbb{A}(\Sigma)$ in Definition 2.3, we added two additional variables p and l to the state tuple of the system Σ , in which l is a counter that depending on its value allows or prevents the system from switching, and p acts as a memory to record the latest mode.

Proposition 2.5. The output trajectory of the augmented dt-SS $\mathbb{A}(\Sigma)$ in Definition 2.3 can be uniquely mapped to an output trajectory of the switched system Σ defined in (2.2), and vice versa.

The proof is similar to that of [LSZ20a, Proposition 2.9] and is omitted here.

In the next section, in order to quantify upper bounds on the probability that the interconnected system reaches a certain unsafe region in a finite-time horizon, we first introduce notions of augmented control pseudo-barrier and barrier certificates for, respectively, augmented dt-SS (with both internal and external signals) and interconnected augmented dt-SS (without internal signals).

3. AUGMENTED CONTROL (PSEUDO-)BARRIER CERTIFICATES

Here, we first introduce a notion of augmented control pseudo-barrier certificates for augmented dt-SS with both internal and external inputs.

Definition 3.1. Consider an augmented dt-SS $\mathbb{A}(\Sigma) = (\mathbb{X}, \mathbb{P}, \mathbb{W}, \varsigma, \mathbb{F}, \mathbb{Y}, \mathbb{H})$, and initial and unsafe sets $X_0, X_1 \subseteq X$ for the dt-SS Σ . Let us define $\mathbb{X}_0 = X_0 \times P \times \{0\}$, $\mathbb{X}_1 = X_1 \times P \times \{0, \dots, k_d - 1\}$, as initial and unsafe sets of the augmented system, respectively. A function $\mathcal{B} : \mathbb{X} \rightarrow \mathbb{R}_{\geq 0}$ is called an augmented control pseudo-barrier certificate (APBC) for $\mathbb{A}(\Sigma)$ if there exist functions $\alpha \in \mathcal{K}_\infty$, $\rho_{\text{int}} \in \mathcal{K}_\infty \cup \{0\}$, and constants $0 < \kappa < 1$, $\gamma, \psi \in \mathbb{R}_{\geq 0}$ and $\lambda \in \mathbb{R}_{> 0}$, such that

$$\mathcal{B}(x, p, l) \geq \alpha(\|\mathbb{H}(x, p, l)\|), \quad \forall (x, p, l) \in \mathbb{X}, \quad (3.1)$$

$$\mathcal{B}(x, p, l) \leq \gamma, \quad \forall (x, p, l) \in \mathbb{X}_0, \quad (3.2)$$

$$\mathcal{B}(x, p, l) \geq \lambda, \quad \forall (x, p, l) \in \mathbb{X}_1, \quad (3.3)$$

and $\forall (x, p, l) \in \mathbb{X}$, $\exists p' \in \mathbb{P}$, such that $\forall w \in \mathbb{W}$, one has $(x', p', l') = \mathbb{F}((x, p, l), p, w, \varsigma)$, and

$$\mathbb{E} \left[\mathcal{B}((x', p', l')) \mid x, p, l, w \right] \leq \max \left\{ \kappa \mathcal{B}(x, p, l), \rho_{\text{int}}(\|w\|), \psi \right\}, \quad (3.4)$$

where the expectation operator \mathbb{E} is with respect to ς under the one-step transition of the augmented dt-SS $\mathbb{A}(\Sigma)$.

Now, we modify the above notion for augmented dt-SS without internal inputs by eliminating all the terms related to w which will be employed later for relating interconnected augmented switched systems.

Definition 3.2. Consider an (interconnected) augmented dt-SS $\mathbb{A}(\Sigma) = (\mathbb{X}, \mathbb{P}, \varsigma, \mathbb{F}, \mathbb{Y}, \mathbb{H})$ without internal inputs, with initial and unsafe sets $X_0, X_1 \subseteq X$ for the dt-SS Σ . Let us define sets $\mathbb{X}_0, \mathbb{X}_1 \subseteq \mathbb{X}$ as respectively initial and unsafe sets of the augmented system. A function $\mathcal{B} : \mathbb{X} \rightarrow \mathbb{R}_{\geq 0}$ is called an augmented control barrier certificate (ABC) for $\mathbb{A}(\Sigma)$ if

$$\mathcal{B}(x, p, l) \leq \gamma, \quad \forall (x, p, l) \in \mathbb{X}_0, \quad (3.5)$$

$$\mathcal{B}(x, p, l) \geq \lambda, \quad \forall (x, p, l) \in \mathbb{X}_1, \quad (3.6)$$

and $\forall (x, p, l) \in \mathbb{X}, \exists p' \in \mathbb{P}$, such that one has $(x', p', l') = \mathbb{F}((x, p, l), p, \varsigma)$, and

$$\mathbb{E}[\mathcal{B}((x', p', l')) \mid x, p, l] \leq \max\{\kappa\mathcal{B}(x, p, l), \psi\}, \quad (3.7)$$

for some constants $0 < \kappa < 1$, $\gamma, \psi \in \mathbb{R}_{\geq 0}$ and $\lambda \in \mathbb{R}_{> 0}$ with $\gamma < \lambda$, where the expectation operator \mathbb{E} is with respect to ς under the one-step transition of the augmented dt-SS $\mathbb{A}(\Sigma)$.

Now we employ Definition 3.2 and propose an upper bound on the probability that an (interconnected) augmented dt-SS reaches an unsafe region via the next theorem.

Theorem 3.3. Let $\mathbb{A}(\Sigma) = (\mathbb{X}, \mathbb{P}, \varsigma, \mathbb{F}, \mathbb{Y}, \mathbb{H})$ be an (interconnected) augmented dt-SS without internal inputs. Suppose \mathcal{B} is an ABC for $\mathbb{A}(\Sigma)$. Then for any random variable a as the initial state, any initial mode p_0 , and $l_0 = 0$ as the initial counter, the probability that the interconnected augmented dt-SS reaches an unsafe set \mathbb{X}_1 within the time step $k \in [0, T_d]$ is upper bounded by δ as

$$\mathbb{P}\left\{\sup_{0 \leq k \leq T_d} \mathcal{B}(x(k), p(k), l(k)) \geq \lambda \mid a, p_0, l_0\right\} \leq \delta, \quad (3.8)$$

where

$$\delta = \begin{cases} 1 - (1 - \frac{\gamma}{\lambda})(1 - \frac{\psi}{\lambda})^{T_d}, & \text{if } \lambda \geq \frac{\psi}{\kappa}, \\ (\frac{\gamma}{\lambda})(1 - \kappa)^{T_d} + (\frac{\psi}{\kappa\lambda})(1 - (1 - \kappa)^{T_d}), & \text{if } \lambda < \frac{\psi}{\kappa}. \end{cases}$$

The proof of Theorem 3.3 is provided in Appendix.

4. COMPOSITIONAL CONSTRUCTION OF ABC

In this section, we analyze networks of stochastic switched subsystems by driving a max-type small-gain condition and discuss how to construct an ABC of the augmented dt-SS via the corresponding APBC of subsystems.

Suppose we are given N stochastic switched subsystems

$$\Sigma_i = (X_i, P_i, \mathcal{P}_i, W_i, \varsigma_i, F_i, Y_i, h_i), \quad i \in \{1, \dots, N\}, \quad (4.1)$$

where $F_i = \{f_1^i, \dots, f_{m_i}^i\}$, with its equivalent augmented dt-SS $\mathbb{A}(\Sigma_i) = (\mathbb{X}_i, \mathbb{P}_i, \mathbb{W}_i, \varsigma_i, \mathbb{F}_i, \mathbb{Y}_i, \mathbb{H}_i)$, in which their internal inputs and outputs are partitioned as

$$w_i = [w_{i1}; \dots; w_{i(i-1)}; w_{i(i+1)}; \dots; w_{iN}], \quad y_i = [y_{i1}; \dots; y_{iN}], \quad (4.2)$$

and their output sets and functions are of the form

$$Y_i = \prod_{j=1}^N Y_{ij}, \quad h_i(x_i) = [h_{i1}(x_i); \dots; h_{iN}(x_i)]. \quad (4.3)$$

We interpret outputs y_{ii} as *external* ones, whereas outputs y_{ij} with $i \neq j$ are *internal* ones which are utilized to interconnect these stochastic switched subsystems. For the interconnection, we assume that w_{ij} is equal to y_{ji} if there is a connection from Σ_j to Σ_i , otherwise we put the connecting output function identically zero, *i.e.*, $h_{ji} \equiv 0$.

Now, we are ready to define the *interconnection* of dt-SS $\Sigma_i = (X_i, P_i, \mathcal{P}_i, W_i, \varsigma_i, F_i, Y_i, h_i)$.

Definition 4.1. Consider $N \in \mathbb{N}_{\geq 1}$ dt-SS $\Sigma_i = (X_i, P_i, \mathcal{P}_i, W_i, \varsigma_i, F_i, Y_i, h_i)$, with the input-output configuration as in (4.2)-(4.3). The interconnection of $\Sigma_i, \forall i \in \{1, \dots, N\}$, is the interconnected dt-SS $\Sigma = (X, P, \mathcal{P}, \varsigma, F, Y, h)$, denoted by $\mathcal{I}(\Sigma_1, \dots, \Sigma_N)$, such that $X := \prod_{i=1}^N X_i$, $P := \prod_{i=1}^N P_i$, $\mathcal{P} := \prod_{i=1}^N \mathcal{P}_i$, $F := \prod_{i=1}^N F_i$, $Y := \prod_{i=1}^N Y_{ii}$, and $h = \prod_{i=1}^N h_{ii}$, subjected to the following constraint:

$$\forall i, j \in \{1, \dots, N\}, i \neq j: \quad w_{ji} = y_{ij}, \quad Y_{ij} \subseteq W_{ji}.$$

An example of the interconnection of two stochastic subsystems Σ_1 and Σ_2 is illustrated in Figure 1.

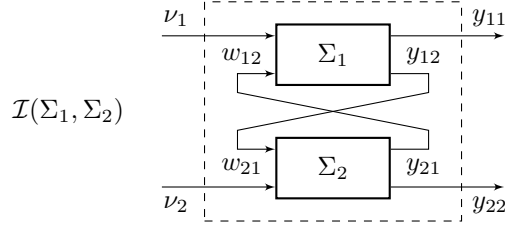


FIGURE 1. An interconnection of two stochastic subsystems Σ_1 and Σ_2 .

Similarly, we define a notion of the *interconnection* for augmented dt-SS $\mathbb{A}(\Sigma_i) = (\mathbb{X}_i, \mathbb{P}_i, \mathbb{W}_i, \varsigma_i, \mathbb{F}_i, \mathbb{Y}_i, \mathbb{H}_i)$.

Definition 4.2. Consider $N \in \mathbb{N}_{\geq 1}$ augmented dt-SS $\mathbb{A}(\Sigma_i) = (\mathbb{X}_i, \mathbb{P}_i, \mathbb{W}_i, \varsigma_i, \mathbb{F}_i, \mathbb{Y}_i, \mathbb{H}_i)$, with the input-output configuration as in (4.2)-(4.3). The interconnection of $\mathbb{A}(\Sigma_i), \forall i \in \{1, \dots, N\}$, is the interconnected augmented dt-SS $\mathbb{A}(\Sigma) = (\mathbb{X}, \mathbb{P}, \varsigma, \mathbb{F}, \mathbb{Y}, \mathbb{H})$, denoted by $\mathcal{I}(\mathbb{A}(\Sigma_1), \dots, \mathbb{A}(\Sigma_N))$, such that $\mathbb{X} := \prod_{i=1}^N \mathbb{X}_i$, $\mathbb{P} := \prod_{i=1}^N \mathbb{P}_i$, $\mathbb{Y} := \prod_{i=1}^N \mathbb{Y}_{ii}$, $\mathbb{H} = \prod_{i=1}^N \mathbb{H}_{ii}$, and the map $\mathbb{F} = \prod_{i=1}^N \mathbb{F}_i$ is the transition function given by $(x', p', l') = \mathbb{F}((x, p, l), p, \varsigma)$ if and only if $x' = f_p(x, w, \varsigma)$, where $f_p = \prod_{i=1}^N f_{p_i}^i$, and the following scenarios hold for any $i \in \{1, \dots, N\}$:

- $l_i < k_{d_i} - 1$, $p'_i = p_i$, and $l'_i = l_i + 1$;
- $l_i = k_{d_i} - 1$, $p'_i = p_i$, and $l'_i = k_{d_i} - 1$;
- $l_i = k_{d_i} - 1$, $p'_i \neq p_i$, and $l'_i = 0$;

where $x = [x_1; \dots; x_N]$, $p = [p_1; \dots; p_N]$, $l = [l_1; \dots; l_N]$, $\varsigma = [\varsigma_1; \dots; \varsigma_N]$, and subjected to the following constraint:

$$\forall i, j \in \{1, \dots, N\}, i \neq j: \quad w_{ji} = y_{ij}, \quad \mathbb{Y}_{ij} \subseteq \mathbb{W}_{ji}.$$

Assume for the augmented dt-SS $\mathbb{A}(\Sigma_i) = (\mathbb{X}_i, \mathbb{P}_i, \mathbb{W}_i, \varsigma_i, \mathbb{F}_i, \mathbb{Y}_i, \mathbb{H}_i), i \in \{1, \dots, N\}$, there exists an APBC \mathcal{B}_i with the corresponding functions and constants denoted by $\alpha_i, \rho_{\text{inti}}, \kappa_i, \gamma_i, \lambda_i$ and ψ_i as in Definition 3.1. Now we raise the following max small-gain assumption to establish the main compositionality result of the paper.

Assumption 4.3. Assume that \mathcal{K}_∞ functions κ_{ij} defined as

$$\kappa_{ij}(s) := \begin{cases} \kappa_i s, & \text{if } i = j, \\ \rho_{\text{inti}}(\alpha_j^{-1}(s)), & \text{if } i \neq j, \end{cases}$$

satisfy

$$\kappa_{i_1 i_2} \circ \kappa_{i_2 i_3} \circ \dots \circ \kappa_{i_{r-1} i_r} \circ \kappa_{i_r i_1} < \mathcal{I}_d, \quad (4.4)$$

for all sequences $(i_1, \dots, i_r) \in \{1, \dots, N\}^r$ and $r \in \{1, \dots, N\}$.

The small-gain condition (4.4) implies the existence of \mathcal{K}_∞ functions $\sigma_i > 0$ [Rüf10, Theorem 5.5], satisfying

$$\max_{i,j} \left\{ \sigma_i^{-1} \circ \kappa_{ij} \circ \sigma_j \right\} < \mathcal{I}_d, \quad i, j = \{1, \dots, N\}. \quad (4.5)$$

Remark 4.4. Note that the small-gain condition (4.4) is a standard one in studying the stability of large-scale interconnected systems via ISS Lyapunov functions [DRW07, DRW10]. This condition is automatically satisfied if each κ_{ij} is less than identity (i.e., $\kappa_{ij} < \mathcal{I}_d, \forall i, j \in \{1, \dots, N\}$).

In the next theorem, we show that if Assumption 4.3 holds and $\max_i \sigma_i^{-1}$ is concave (in order to employ Jensen's inequality), then we can construct an ABC of $\mathbb{A}(\Sigma)$ using the APBC of $\mathbb{A}(\Sigma_i)$.

Theorem 4.5. Consider the interconnected augmented dt-SS $\mathbb{A}(\Sigma) = (\mathbb{X}, \mathbb{P}, \varsigma, \mathbb{F}, \mathbb{Y}, \mathbb{H})$ induced by $N \in \mathbb{N}_{\geq 1}$ augmented dt-SS $\mathbb{A}(\Sigma_i)$. Suppose that each $\mathbb{A}(\Sigma_i)$ admits an APBC \mathcal{B}_i as defined in Definition 3.1. If Assumption 4.3 holds and

$$\max_i \left\{ \sigma_i^{-1}(\lambda_i) \right\} > \max_i \left\{ \sigma_i^{-1}(\gamma_i) \right\}, \quad (4.6)$$

then the function $\mathcal{B}(x, p, l)$ defined as

$$\mathcal{B}(x, p, l) := \max_i \left\{ \sigma_i^{-1}(\mathcal{B}_i(x_i, p_i, l_i)) \right\}, \quad (4.7)$$

is an ABC for the interconnected augmented dt-SS $\mathcal{I}(\mathbb{A}(\Sigma_1), \dots, \mathbb{A}(\Sigma_N))$ provided that $\max_i \sigma_i^{-1}$ for σ_i as in (4.5) is concave.

The proof of Theorem 4.5 is provided in Appendix.

Remark 4.6. Note that the condition (4.6) in general is not very restrictive since functions σ_i in (4.5) play an important role in rescaling APBC for subsystems while normalizing the effect of internal gains of other subsystems (cf. [DRW10] for a similar argument but in the context of stability analysis via ISS Lyapunov functions). Then one can expect that the condition (4.6) holds in many applications due to this rescaling.

5. CONSTRUCTION OF APBC

In this section, we impose conditions on the dt-SS Σ_p enabling us to find an APBC for $\mathbb{A}(\Sigma)$. The APBC for the augmented dt-SS $\mathbb{A}(\Sigma)$ is established under the assumption that the given dt-SS Σ_p has control barrier certificates (CBC) for all modes as in the following definition.

Definition 5.1. Consider a dt-SS Σ_p , and sets $X_0, X_1 \subseteq X$ as initial and unsafe sets of the given dt-SS, respectively. A function $\mathcal{B}_p : X \rightarrow \mathbb{R}_{\geq 0}$ is said to be a control barrier certificate (CBC) for Σ_p if there exist functions $\alpha_p \in \mathcal{K}_\infty$, $\rho_{\text{int}p} \in \mathcal{K}_\infty \cup \{0\}$, and constants $0 < \kappa_p < 1$, $\gamma_p, \psi_p \in \mathbb{R}_{\geq 0}$ and $\lambda_p \in \mathbb{R}_{> 0}$, such that

$$\mathcal{B}_p(x) \geq \alpha_p(\|h(x)\|), \quad \forall x \in X, \quad (5.1)$$

$$\mathcal{B}_p(x) \leq \gamma_p, \quad \forall x \in X_0, \quad (5.2)$$

$$\mathcal{B}_p(x) \geq \lambda_p, \quad \forall x \in X_1, \quad (5.3)$$

and $\forall x \in X, \forall w \in W$, one has

$$\mathbb{E} \left[\mathcal{B}_p(x(k+1)) \mid x, w \right] \leq \max \left\{ \kappa_p \mathcal{B}_p(x), \rho_{\text{int}p}(\|w\|), \psi_p \right\}. \quad (5.4)$$

In order to construct an APBC for the augmented dt-SS $\mathbb{A}(\Sigma)$, we need also to raise the following assumption.

Assumption 5.2. Suppose there exists $\mu \geq 1$ such that

$$\forall x \in X, \forall p, p' \in P, \quad \mathcal{B}_p(x) \leq \mu \mathcal{B}_{p'}(x). \quad (5.5)$$

Remark 5.3. Assumption 5.2 is a standard one in the literature for switched systems accepting multiple Lyapunov functions with dwell-time similar to the one appeared in [Lib03, equation (3.6)].

Under Definition 5.1 and Assumption 5.2, the next theorem lays the foundations for constructing an APBC for $\mathbb{A}(\Sigma)$.

Theorem 5.4. *Let $\Sigma = (X, P, \mathcal{P}, W, \varsigma, F, Y, h)$ be a switched subsystem with its equivalent augmented system $\mathbb{A}(\Sigma) = (\mathbb{X}, \mathbb{P}, \mathbb{W}, \varsigma, \mathbb{F}, \mathbb{Y}, \mathbb{H})$. Let \mathcal{B}_p be a CBC for Σ_p , $\forall p \in P$, as in Definition 5.1, and assume Assumption 5.2 holds, and consider $\epsilon > 1$. If $\forall p \in P$, $k_d \geq \epsilon \frac{\ln(\mu)}{\ln(1/\kappa_p)} + 1$, then*

$$\mathcal{B}(x, p, l) = \frac{1}{\kappa_p^{l/\epsilon}} \mathcal{B}_p(x), \quad (5.6)$$

is an APBC for $\mathbb{A}(\Sigma)$.

The proof of Theorem 5.4 is provided in Appendix.

Remark 5.5. *Note that if there exists a common CBC $\mathcal{B} : X \rightarrow \mathbb{R}_{\geq 0}$ for all switching modes $p \in P$ satisfying conditions of Definition 5.1 and Assumption 5.2 (with $\mu = 1$), then $\mathcal{B}(x, p, l) = \mathcal{B}(x)$ (cf. the first case study).*

6. COMPUTATION OF CBC

We employ an approach based on a counter-example guided inductive synthesis (CEGIS) framework to find a CBC for each mode of subsystems Σ_{p_i} with a finite set of modes P_i , $i \in \{1, \dots, N\}$. The approach employs satisfiability (feasibility) solvers for finding CBC of a given parametric form employing existing Satisfiability Modulo Theories (SMT) solvers such as Z3 [DMB08], dReal [GKC13], and OptiMathSAT [ST15]. In order to use CEGIS framework, we raise the following assumption.

Assumption 6.1. *Each mode of switched subsystem Σ_{p_i} , $\forall p_i \in P$, $\forall i \in \{1, \dots, N\}$, has a compact state set X_i , and a compact internal input set W_i .*

Under Assumption 6.1, conditions (5.1)-(5.4) can be rephrased as a satisfiability problem which can be searched for a parametric CBC using CEGIS approach. The feasibility condition that is required to be satisfied for the existence of a CBC \mathcal{B}_{p_i} is given in the following lemma.

Lemma 6.2. *Consider a mode of switched subsystem Σ_{p_i} satisfying Assumption 6.1. Suppose there exist functions \mathcal{B}_{p_i} , $\alpha_{p_i} \in \mathcal{K}_{\infty}$, $\rho_{\text{int}p_i} \in \mathcal{K}_{\infty} \cup \{0\}$, and constants $0 < \kappa_{p_i} < 1$, $\gamma_{p_i}, \psi_{p_i} \in \mathbb{R}_{\geq 0}$ and $\lambda_{p_i} \in \mathbb{R}_{>0}$, such that the following expression is true:*

$$\begin{aligned} \bigwedge_{x_i \in X_i} \mathcal{B}_{p_i} &\geq \alpha_{p_i}(\|h_i(x_i)\|) \bigwedge_{x \in X_{0_i}} \mathcal{B}_{p_i} \leq \gamma_{p_i} \bigwedge_{x \in X_{1_i}} \mathcal{B}_{p_i} \geq \lambda_{p_i} \bigwedge_{x_i \in X_i} \bigwedge_{w_i \in W_i} (\mathbb{E}[\mathcal{B}_{p_i}(x_i(k+1)) \mid x_i, w_i]) \\ &\leq \max \left\{ \kappa_{p_i} \mathcal{B}_{p_i}(x_i), \rho_{\text{int}p_i}(\|w_i\|), \psi_{p_i} \right\}. \end{aligned} \quad (6.1)$$

Then, \mathcal{B}_{p_i} satisfies conditions (5.1)-(5.4) in Definition 5.1.

Now one can utilize the CEGIS approach to search for a parametric CBC satisfying (6.1) (implying original conditions (5.1)-(5.4)). For a detailed discussion on the CEGIS approach, we refer the interested reader to [JSZ19, Subsection 5.3.2].

7. CASE STUDY

To demonstrate the effectiveness of the proposed results, we first apply our approaches to a room temperature network in a circular building containing 1000 rooms. We compositionally synthesize safety controllers to maintain the temperature of each room in a comfort zone in a bounded time horizon. Moreover, to show the applicability of our results to switched systems accepting *multiple* barrier certificates with a *dwell-time* condition, we apply our technique to a circular cascade network of 500 subsystems (totally 1000 dimensions) and provide upper bounds on the probability that the interconnected system reaches some unsafe region in a finite-time horizon.

7.1. Room Temperature Network. The model of this case study is borrowed from [MGW18] by including a stochasticity in the model as an additive noise. The evolution of the temperature $T(\cdot)$ in the interconnected system is governed by the following dynamics:

$$\Sigma : \begin{cases} T(k+1) = AT(k) + \theta T_h B_{\mathbf{p}(k)} + \beta T_E + 0.25\varsigma(k), \\ y(k) = T(k), \end{cases}$$

where $A \in \mathbb{R}^{n \times n}$ is a matrix with diagonal elements given by $\bar{a}_{ii} = (1 - 2\eta - \beta - \theta b_{ip_i})$, off-diagonal elements $\bar{a}_{i,i+1} = \bar{a}_{i+1,i} = \bar{a}_{1,n} = \bar{a}_{n,1} = \eta$, $i \in \{1, \dots, n-1\}$, and all other elements are identically zero. Parameters $\eta = 0.005$, $\beta = 0.022$, and $\theta = 0.05$ are conduction factors, respectively, between the rooms $i \pm 1$ and i , the external environment and the room i , and the heater and the room i . Outside temperatures are the same for all rooms: $T_{ei} = -1^\circ\text{C}$, $\forall i \in \{1, \dots, n\}$, and the heater temperature is $T_h = 50^\circ\text{C}$. Moreover, $T(k) = [T_1(k); \dots; T_n(k)]$, $\varsigma = [\varsigma_1(k); \dots; \varsigma_n(k)]$, $T_E = [T_{e_1}; \dots; T_{e_n}]$, and $B_p = [b_{1p_1}; \dots; b_{np_n}]$, such that

$$b_{ip_i} = \begin{cases} 0, & \text{if } p_i = 1, \\ 0.1, & \text{if } p_i = 2, \\ 0.2, & \text{if } p_i = 3, \\ 0.3, & \text{if } p_i = 4, \\ 0.4, & \text{if } p_i = 5, \\ 0.5, & \text{if } p_i = 6, \\ 0.6, & \text{if } p_i = 7, \end{cases}$$

with the finite set of modes $P_i = \{1, \dots, 7\}$, $i \in \{1, \dots, n\}$. Now by considering the individual rooms as Σ_i represented by

$$\Sigma_i : \begin{cases} T_i(k+1) = \bar{a}_{ii}T_i(k) + \theta T_h b_{ip_i(k)} + \eta w_i(k) + \beta T_{ei}(k) + 0.25\varsigma_i(k), \\ y_i(k) = T_i(k), \end{cases}$$

one can readily verify that $\Sigma = \mathcal{I}(\Sigma_1, \dots, \Sigma_N)$, equivalently $\Sigma = \mathcal{I}(\mathbb{A}(\Sigma_1), \dots, \mathbb{A}(\Sigma_N))$, where $w_i(k) = [T_{i-1}(k); T_{i+1}(k)]$ (with $T_0 = T_n$ and $T_{n+1} = T_1$).

The regions of interest in this example are $X_i \in [1, 50]$, $X_{0i} \in [19, 21]$, $X_{1i} = [1, 17] \cup [23, 50]$, $\forall i \in \{1, \dots, n\}$. The main goal is to find an ABC for the interconnected system such that a switching signal is synthesized for Σ keeping the temperature of rooms in the comfort zone $[17, 23]^{1000}$.

Note that in this example $\mathcal{B}_p = \mathcal{B}_{p'}$, $\forall p, p' \in P$ (i.e., there exists a common barrier certificate with $\mu = 1$). Then $\mathcal{B}(x, p, l) = \mathcal{B}(x)$ as discussed in Remark 5.5. We employ the SMT solver Z3 and CEGIS approach to compute an APBC of an order 4 based on Lemma 6.2 as $\mathcal{B}_i(T_i) = -0.00012T_i^4 + 0.01045T_i^3 - 0.19932T_i^2 - 0.64538T_i + 28.68175$. Furthermore, the corresponding constants and functions in Definition 3.1 satisfying conditions (3.1)-(3.4) are quantified as $\gamma_i = 0.16$, $\lambda_i = 1.2$, $\psi_i = 7.07 \times 10^{-4}$, $\kappa_i = 0.99$, $\alpha_i(s) = 4.5 \times 10^{-5}s^2$, and $\rho_i(s) = 9.3 \times 10^{-6}s^2$, $\forall s \in \mathbb{R}_{\geq 0}$. Then $\mathcal{B}_i(x_i)$ is an APBC for $\mathbb{A}(\Sigma_i)$,

We now proceed with constructing an ABC for the interconnected system using APBC of subsystems. We check the small-gain condition (4.4) that is required for the compositionality result. By taking $\sigma_i(s) = s$, $\forall i \in \{1, \dots, n\}$, the condition (4.4) and as a result the condition (4.5) are always satisfied. Moreover, the compositionality condition (4.6) is met since $\lambda_i > \gamma_i$, $\forall i \in \{1, \dots, n\}$. Then one can conclude that $\mathcal{B}(T) = \max_i \left\{ -0.00012T_i^4 + 0.01045T_i^3 - 0.19932T_i^2 - 0.64538T_i + 28.68175 \right\}$ is an ABC for $\mathbb{A}(\Sigma)$ satisfying conditions (3.5)-(3.7) with $\gamma = 0.16$, $\lambda = 1.2$, $\kappa = 0.99$, and $\psi = 7.07 \times 10^{-4}$.

By employing Theorem 3.3, one can guarantee that the temperature of the interconnected system Σ starting from the initial condition $a \in [19, 21]^{1000}$ remains in the comfort region $[17, 23]^{1000}$ during the time horizon $T_d = 10$ with a probability at least 87%, i.e.,

$$\mathbb{P}\left\{ \mathcal{B}(T(k)) < 1.2 \mid a, \forall k \in [0, 10] \right\} \geq 0.87. \quad (7.1)$$

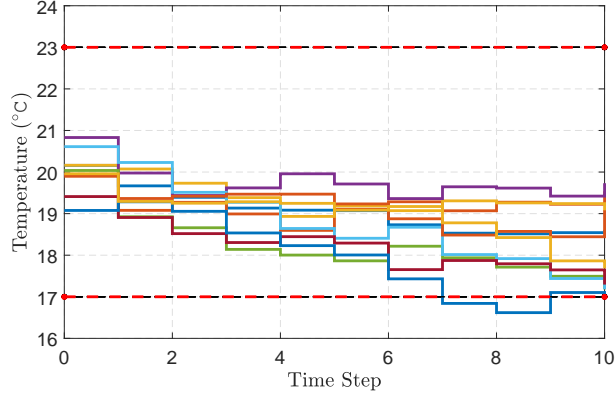


FIGURE 2. Closed-loop state trajectories of a representative room with 10 noise realizations in a network of 1000 rooms.

State trajectories of the closed-loop system in a network of 1000 rooms for a representative room with 10 noise realizations are illustrated in Figure 2.

7.2. Switched Systems Accepting Multiple Barrier Certificates with Dwell-Time. In order to show the applicability of our results to switched systems accepting *multiple* barrier certificates with a *dwell-time* condition, we apply our techniques to a circular cascade network of 500 subsystems (totally 1000 dimensions). The model of the system does not have a common barrier certificate because it exhibits unstable behaviors for different switching signals [Lib03] (*i.e.*, if one periodically switches between different modes, the trajectory goes to infinity). The dynamics of the interconnected system are described by

$$\Sigma : \begin{cases} x(k+1) = A_{\mathbf{p}(k)}x(k) + B_{\mathbf{p}(k)} + R\zeta(k), \\ y(k) = x(k), \end{cases}$$

where

$$A_{\mathbf{p}(k)} = \begin{bmatrix} \bar{A}_{p_i} & 0 & \cdots & \cdots & \tilde{A} \\ \tilde{A} & \bar{A}_{p_i} & 0 & \cdots & 0 \\ 0 & \tilde{A} & \bar{A}_{p_i} & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \tilde{A} & \bar{A}_{p_i} \end{bmatrix}_{n \times n},$$

$$\bar{A}_{p_i} = \begin{cases} \begin{bmatrix} 0.05 & 0 \\ 0.9 & 0.03 \end{bmatrix}, & \text{if } p_i = 1, \\ \begin{bmatrix} 0.02 & -1.2 \\ 0 & 0.05 \end{bmatrix}, & \text{if } p_i = 2, \end{cases} \quad \tilde{A} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}.$$

We choose $R = \text{diag}(0.1\mathbf{1}_2, \dots, 0.1\mathbf{1}_2)$ and fix here $N = 500$. Furthermore, $B_p = [b_{1p_1}; \dots; b_{Np_N}]$ such that

$$b_{ip_i} = \begin{cases} \begin{bmatrix} -0.9 \\ 0.5 \end{bmatrix}, & \text{if } p_i = 1, \\ \begin{bmatrix} 0.9 \\ -0.2 \end{bmatrix}, & \text{if } p_i = 2. \end{cases}$$

We partition $x(k)$ as $x(k) = [x_1(k); \dots; x_N(k)]$ and $\varsigma(k)$ as $\varsigma(k) = [\varsigma_1(k); \dots; \varsigma_N(k)]$, where $x_i(k), \varsigma_i(k) \in \mathbb{R}^2$, *i.e.*, $x_i = [x_{i1}; x_{i2}]$, $\varsigma_i = [\varsigma_{i1}; \varsigma_{i2}]$. Now, by introducing the individual subsystems Σ_i described as

$$\Sigma_i : \begin{cases} x_i(k+1) = \bar{A}_{\mathbf{p}_i(k)} x_i(k) + \tilde{A}_i w_i(k) + b_{i\mathbf{p}_i(k)} + 0.1\mathbf{1}_2 \varsigma_i(k), \\ y_i(k) = x_i(k), \end{cases}$$

where $w_i(k) = y_{i-1}$, $i \in \{1, \dots, N\}$, with $y_0 = y_N$, one can readily verify that $\Sigma = \mathcal{I}(\Sigma_1, \dots, \Sigma_N)$, equivalently $\Sigma = \mathcal{I}(\mathbb{A}(\Sigma_1), \dots, \mathbb{A}(\Sigma_N))$.

The regions of interest here are $X_i \in [-6, 6]^2$, $X_{0_i} \in [-0.5, 0.5]^2$, $X_{1_i} = [-6, -2]^2 \cup [2, 6]^2$, $\forall i \in \{1, \dots, N\}$. The main goal is to find an ABC for the interconnected system such that a switching signal is synthesized for Σ regulating the state of subsystems in a safe zone $[-2, 2]^{1000}$. We first find a CBC for each mode based on Definitions 5.1 using software tool SOSTOOLS [PAV⁺13] and the SDP solver SeDuMi [Stu99]. One can verify that, $\forall i \in \{1, \dots, N\}$, conditions (5.1)-(5.4) are satisfied by

$$\begin{aligned} \text{for } p_i = 1: \quad & \gamma_{p_i} = 0.15, \lambda_{p_i} = 2.4, \kappa_{p_i} = 0.469, \psi_{p_i} = 5.42 \times 10^{-6}, \\ & \alpha_{p_i}(s) = 4 \times 10^{-5} s^2, \rho_{\text{int}p_i}(s) = 2.71 \times 10^{-6} s^2, \forall s \in \mathbb{R}_{\geq 0}, \\ \text{for } p_i = 2: \quad & \gamma_{p_i} = 0.16, \lambda_{p_i} = 2.3, \kappa_{p_i} = 0.498, \psi_{p_i} = 6.88 \times 10^{-6}, \\ & \alpha_{p_i}(s) = 5 \times 10^{-5} s^2, \rho_{\text{int}p_i}(s) = 3.44 \times 10^{-6} s^2, \forall s \in \mathbb{R}_{\geq 0}, \end{aligned}$$

with $\mathcal{B}_{1_i}(x_i) = 0.2309x_{i1}^2 + 0.1160x_{i1}x_{i2} + 0.000001x_{i1} + 0.2529x_{i2}^2 - 0.000001x_{i2} + 0.000000002$, $\mathcal{B}_{2_i}(x_i) = 0.2394x_{i1}^2 + 0.1101x_{i1}x_{i2} - 0.000002x_{i1} + 0.2588x_{i2}^2 - 0.000008x_{i2} + 0.000000005$. One can also verify that the condition (5.5) is met with $\mu = 2$. By taking $\epsilon = 2$, one can get the dwell-time $k_d = 3$. Hence, $\mathcal{B}_i(x_i, p_i, l_i) = \frac{1}{\kappa_{p_i}^{1/2}} \mathcal{B}_i(x_i)$ is an APBC for $\mathbb{A}(\Sigma_i)$ satisfying conditions (3.1)-(3.4) with $\alpha_i(s) = 4 \times 10^{-5} s^2$, $\forall s \in \mathbb{R}_{\geq 0}$, $\gamma_i = 0.321$, $\lambda_i = 2.3$, $\kappa_i = 0.706$, $\rho_{\text{int}i}(s) = 9.78 \times 10^{-6} s^2$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi_i = 1.95 \times 10^{-5}$.

We now proceed with constructing an ABC for the interconnected system using APBC of subsystems. We check the small-gain condition (4.4). By taking $\sigma_i(s) = s$, $\forall i \in \{1, \dots, N\}$, the condition (4.4) and as a result the condition (4.5) are always satisfied. Moreover, the compositionality condition (4.6) is met since $\lambda_i > \gamma_i$, $\forall i \in \{1, \dots, N\}$. Hence, $\mathcal{B}(x, p, l) = \max_i \left\{ \frac{1}{\kappa_{p_i}^{1/2}} \mathcal{B}_i(x_i) \right\}$ is an ABC for the interconnected $\mathbb{A}(\Sigma)$ satisfying conditions (3.5)-(3.7) with $\gamma = 0.321$, $\lambda = 2.3$, $\kappa = 0.706$, and $\psi = 1.95 \times 10^{-5}$.

By employing Theorem 3.3, one can guarantee that the state of the interconnected system Σ starting from the initial condition $a \in [-0.5, 0.5]^{1000}$, with any initial mode p_0 and $l_0 = 0$, remains in the safe set $[-2, 2]^{1000}$ during the time horizon $T_d = 100$ with a probability at least 86%, *i.e.*,

$$\mathbb{P} \left\{ \mathcal{B}(x(k), p(k), l(k)) < 2.3 \mid a, p_0, l_0, \forall k \in [0, 100] \right\} \geq 0.86.$$

8. DISCUSSION

In this work, we proposed a compositional scheme for constructing control barrier certificates for large-scale stochastic switched systems accepting *multiple* barrier certificates with some *dwell-time* conditions. Those barrier certificates provide upper bounds on the probability that interconnected systems reach certain unsafe regions in finite-time horizons. The main goal was to synthesize control policies driving switching signals satisfying safety properties for interconnected systems by utilizing so-called augmented pseudo-barrier certificates of subsystems. We constructed augmented barrier certificates for interconnected switched systems using augmented pseudo-barrier certificates of subsystems as long as some max-type small-gain conditions hold. We employed a systematic technique based on a counter-example guided inductive synthesis (CEGIS) approach and computed control barrier certificates for each mode of a subsystem. We illustrated our proposed results by applying them to two different case studies.

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10. APPENDIX

Proof. (Theorem 3.3) According to the condition (3.6), $\mathbb{X}_1 \subseteq \{(x, p, l) \in \mathbb{X} \mid \mathcal{B}(x, p, l) \geq \lambda\}$. Then we have

$$\begin{aligned} & \mathbb{P}\left\{(x(k), p(k), l(k)) \in \mathbb{X}_1 \text{ for } 0 \leq k \leq T_d \mid a, p_0, l_0\right\} \\ & \leq \mathbb{P}\left\{\sup_{0 \leq k \leq T_d} \mathcal{B}(x(k), p(k), l(k)) \geq \lambda \mid a, p_0, l_0\right\}. \end{aligned} \quad (10.1)$$

The proposed bounds in (3.8) follows directly by applying [Kus67, Theorem 3, Chapter III] to (10.1) (but adapted to stochastic switched systems) and employing respectively conditions (3.7) and (3.5). \square

Proof. (Theorem 4.5) We first show that conditions (3.5) and (3.6) in Definition 3.2 hold. For any $(x, p, l) \in \mathbb{X}_0 = \prod_{i=0}^N \mathbb{X}_{0_i}$ and from (3.2), we have

$$\mathcal{B}(x, p, l) = \max_i \left\{ \sigma_i^{-1}(\mathcal{B}_i(x_i, p_i, l_i)) \right\} \leq \max_i \left\{ \sigma_i^{-1}(\gamma_i) \right\} = \gamma,$$

and similarly for any $(x, p, l) \in \mathbb{X}_1 = \prod_{i=1}^N \mathbb{X}_{1_i}$ and from (3.3), one has

$$\mathcal{B}(x, p, l) = \max_i \left\{ \sigma_i^{-1}(\mathcal{B}_i(x_i, p_i, l_i)) \right\} \geq \max_i \left\{ \sigma_i^{-1}(\lambda_i) \right\} = \lambda,$$

satisfying conditions (3.5) and (3.6) with $\gamma = \max_i \left\{ \sigma_i^{-1}(\gamma_i) \right\}$ and $\lambda = \max_i \left\{ \sigma_i^{-1}(\lambda_i) \right\}$.

Now we show that the condition (3.7) holds, as well. Let $\kappa(s) = \max_{i,j} \{\sigma_i^{-1} \circ \kappa_{ij} \circ \sigma_j(s)\}$. It follows from (4.5) that $\kappa < \mathcal{I}_d$. Moreover, $\lambda > \gamma$ according to (4.6). Since $\max_i \sigma_i^{-1}$ is concave, one can readily acquire the chain of inequalities in (10.2) using Jensen's inequality, and by defining the constant ψ as

$$\psi := \max_i \sigma_i^{-1}(\psi_i).$$

Hence $\mathcal{B}(x, p, l)$ is an ABC for the interconnected augmented dt-SS $\mathcal{I}(\mathbb{A}(\Sigma_1), \dots, \mathbb{A}(\Sigma_N))$ which completes the proof. \square

Proof. (Theorem 5.4) For any $(x, p, l) \in \mathbb{X}$, we get

$$\|\mathbb{H}(x, p, l)\| = \|h(x)\| \leq \alpha_p^{-1}(\mathcal{B}_p(x)) = \alpha_p^{-1}(\kappa_p^{l/\epsilon} \mathcal{B}((x, p, l))).$$

$$\begin{aligned}
\mathbb{E}[\mathcal{B}(x', p', l') \mid x, p, l] &= \mathbb{E}\left[\max_i \left\{ \sigma_i^{-1}(\mathcal{B}_i(x'_i, p'_i, l'_i)) \right\} \mid x, p, l\right] \\
&\leq \max_i \left\{ \sigma_i^{-1}(\mathbb{E}[\mathcal{B}_i(x'_i, p'_i, l'_i) \mid x, p, l]) \right\} = \max_i \left\{ \sigma_i^{-1}(\mathbb{E}[\mathcal{B}_i(x'_i, p'_i, l'_i) \mid x_i, p_i, l_i]) \right\} \\
&\leq \max_i \left\{ \sigma_i^{-1}(\max\{\kappa_i \mathcal{B}_i(x_i, p_i, l_i), \rho_{\text{inti}}(\|w_i\|), \psi_i\}) \right\} = \max_i \left\{ \sigma_i^{-1}(\max\{\kappa_i \mathcal{B}_i(x_i, p_i, l_i), \rho_{\text{inti}}(\max_{j, j \neq i} \{\|w_{ij}\|\}), \psi_i\}) \right\} \\
&= \max_i \left\{ \sigma_i^{-1}(\max\{\kappa_i \mathcal{B}_i(x_i, p_i, l_i), \rho_{\text{inti}}(\max_{j, j \neq i} \{\|y_{ji}\|\}), \psi_i\}) \right\} \\
&= \max_i \left\{ \sigma_i^{-1}(\max\{\kappa_i \mathcal{B}_i(x_i, p_i, l_i), \rho_{\text{inti}}(\max_{j, j \neq i} \{\|\mathbb{H}_j(x_j, p_j, l_j)\|\}), \psi_i\}) \right\} \\
&\leq \max_i \left\{ \sigma_i^{-1}(\max\{\kappa_i \mathcal{B}_i(x_i, p_i, l_i), \rho_{\text{inti}}(\max_{j, j \neq i} \{\alpha_j^{-1}(\mathcal{B}_j(x_j, p_j, l_j))\}), \psi_i\}) \right\} \\
&= \max_{i, j} \left\{ \sigma_i^{-1}(\max\{\kappa_{ij} \mathcal{B}_j(x_j, p_j, l_j), \psi_i\}) \right\} = \max_{i, j} \left\{ \sigma_i^{-1}(\max\{\kappa_{ij} \circ \sigma_j \circ \sigma_j^{-1}(\mathcal{B}_j(x_j, p_j, l_j)), \psi_i\}) \right\} \\
&\leq \max_{i, j, z} \left\{ \sigma_i^{-1}(\max\{\kappa_{ij} \circ \sigma_j \circ \sigma_z^{-1}(\mathcal{B}_z(x_z, p_z, l_z)), \psi_i\}) \right\} \\
&= \max_{i, j} \left\{ \sigma_i^{-1}(\max\{\kappa_{ij} \circ \sigma_j(\mathcal{B}(x, p, l)), \psi_i\}) \right\} = \max \left\{ \kappa \mathcal{B}(x, p, l), \psi \right\}. \tag{10.2}
\end{aligned}$$

Since $\frac{1}{\kappa_p^{l/\epsilon}} > 1$, one can conclude that the inequality (3.1) holds with $\alpha(s) = \min_p \{\alpha_p(s)\}$, $\forall s \in \mathbb{R}_{\geq 0}$. Now we show that inequalities (3.2) and (3.3) hold, as well. For any $(x, p, l) \in \mathbb{X}_0$, one has

$$\mathcal{B}(x, p, l) = \frac{1}{\kappa_p^{l/\epsilon}} \mathcal{B}_p(x) \leq \frac{1}{\kappa_p^{l/\epsilon}} \gamma_p,$$

and similarly for any $(x, p, l) \in \mathbb{X}_1$, one has

$$\mathcal{B}(x, p, l) = \frac{1}{\kappa_p^{l/\epsilon}} \mathcal{B}_p(x) \geq \frac{1}{\kappa_p^{l/\epsilon}} \lambda_p,$$

satisfying conditions (3.2) and (3.3) with $\gamma = \max_p \left\{ \frac{1}{\kappa_p^{(k_d-1)/\epsilon}} \gamma_p \right\}$ and $\lambda = \min_p \{\lambda_p\}$ (since $\frac{1}{\kappa_p^{l/\epsilon}} > 1$).

Now we proceed with showing the inequality (3.4). In order to show that the function $\mathcal{B}(x, p, l)$ in (5.6) satisfies (3.4), we should consider the three different scenarios as in Definition 2.3. For the first scenario ($l < k_d - 1, p' = p$, and $l' = l + 1$), we have:

$$\begin{aligned}
\mathbb{E}[\mathcal{B}(x', p', l') \mid x, p, l, w] &= \frac{1}{\kappa_{p'}^{l'/\epsilon}} \mathbb{E}[\mathcal{B}_{p'}(x') \mid x, p, w] \\
&= \frac{1}{\kappa_p^{(l+1)/\epsilon}} \mathbb{E}[\mathcal{B}_p(f_p(x, w, \varsigma)) \mid x, w] \\
&\leq \frac{1}{\kappa_p^{(l+1)/\epsilon}} \max \left\{ \kappa_p \mathcal{B}_p(x(k)), \rho_{\text{int}p}(\|w\|), \psi_p \right\} \\
&= \max \left\{ \kappa_p^{\frac{\epsilon-1}{\epsilon}} \mathcal{B}_p(x, p, l), \frac{1}{\kappa_p^{(l+1)/\epsilon}} \rho_{\text{int}p}(\|w\|), \frac{1}{\kappa_p^{(l+1)/\epsilon}} \psi_p \right\} \\
&\leq \max \left\{ \kappa_p^{\frac{\epsilon-1}{\epsilon}} \mathcal{B}_p(x, p, l), \frac{1}{\kappa_p^{k_d/\epsilon}} \rho_{\text{int}p}(\|w\|), \frac{1}{\kappa_p^{k_d/\epsilon}} \psi_p \right\};
\end{aligned}$$

Note that the last inequality holds since $l < k_d - 1$, and consequently, $l + 1 < k_d$.

For the second scenario ($l = k_d - 1, p' = p$, and $l' = k_d - 1$), we have:

$$\begin{aligned}
\mathbb{E}[\mathcal{B}(x', p', l') \mid x, p, l, w] &= \frac{1}{\kappa_{p'}^{l'/\epsilon}} \mathbb{E}[\mathcal{B}_{p'}(x') \mid x, p, w] \\
&= \frac{1}{\kappa_p^{l/\epsilon}} \mathbb{E}[\mathcal{B}_p(f_p(x, w, \varsigma)) \mid x, w] \\
&\leq \frac{1}{\kappa_p^{l/\epsilon}} \max \left\{ \kappa_p \mathcal{B}_p(x(k)), \rho_{\text{int}p}(\|w\|), \psi_p \right\} \\
&= \max \left\{ \kappa_p \mathcal{B}(x, p, l), \frac{1}{\kappa_p^{l/\epsilon}} \rho_{\text{int}p}(\|w\|), \frac{1}{\kappa_p^{l/\epsilon}} \psi_p \right\} \\
&\leq \max \left\{ \kappa_p^{\frac{\epsilon-1}{\epsilon}} \mathcal{B}(x, p, l), \frac{1}{\kappa_p^{k_d/\epsilon}} \rho_{\text{int}p}(\|w\|), \frac{1}{\kappa_p^{k_d/\epsilon}} \psi_p \right\};
\end{aligned}$$

Note that the last inequality holds since $\epsilon > 1$, and consequently, $0 < \frac{\epsilon-1}{\epsilon} < 1$.

For the last scenario ($l = k_d - 1, p' \neq p$, and $l' = 0$), using Assumption 5.2 we have:

$$\begin{aligned}
\mathbb{E}[\mathcal{B}((x', p', l')) \mid x, p, l, w] &= \frac{1}{\kappa_{p'}^{l'/\epsilon}} \mathbb{E}[\mathcal{B}_{p'}(x') \mid x, p, w] \\
&\leq \mu \mathbb{E}[\mathcal{B}_p(f_p(x, w, \varsigma)) \mid x, w] \\
&\leq \mu \max \left\{ \kappa_p \mathcal{B}_p(x(k)), \rho_{\text{int}p}(\|w\|), \psi_p \right\} \\
&= \mu \kappa_p^{(k_d-1)/\epsilon} \frac{1}{\kappa_p^{l/\epsilon}} \max \left\{ \kappa_p \mathcal{B}_p(x(k)), \rho_{\text{int}p}(\|w\|), \psi_p \right\} \\
&= \max \left\{ \mu \kappa_p^{(k_d-1)/\epsilon} \kappa_p \mathcal{B}((x, p, l)), \mu \rho_{\text{int}p}(\|w\|), \mu \psi_p \right\} \\
&\leq \max \left\{ \kappa_p \mathcal{B}((x, p, l)), \mu \rho_{\text{int}p}(\|w\|), \mu \psi_p \right\} \\
&\leq \max \left\{ \kappa_p^{\frac{\epsilon-1}{\epsilon}} \mathcal{B}((x, p, l)), \frac{1}{\kappa_p^{k_d/\epsilon}} \rho_{\text{int}p}(\|w\|), \frac{1}{\kappa_p^{k_d/\epsilon}} \psi_p \right\};
\end{aligned}$$

Note that the last scenario holds since $\forall p \in P, k_d \geq \epsilon \frac{\ln(\mu)}{\ln(1/\kappa_p)} + 1$, and equivalently $\forall p \in P, \mu \kappa_p^{(k_d-1)/\epsilon} \leq 1$. By defining $\kappa = \max_p \left\{ \kappa_p^{\frac{\epsilon-1}{\epsilon}} \right\}$, $\rho_{\text{int}}(s) = \max_p \left\{ \frac{1}{\kappa_p^{k_d/\epsilon}} \rho_{\text{int}p}(s) \right\}$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi = \max_p \left\{ \frac{1}{\kappa_p^{k_d/\epsilon}} \psi_p \right\}$, the inequality (3.4) holds. Hence, $\mathcal{B}(x, p, l)$ is an APBC for $\mathbb{A}(\Sigma)$, which completes the proof. \square

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