GLOBAL ROBUST H_∞ CONTROL FOR NON-MINIMUM-PHASE UNCERTAIN NONLINEAR SYSTEMS WITHOUT STRICT TRIANGULAR STRUCTURE [1]

Shengzhi Zhao *, **Jun Zhao** *, **Georgi M. Dimirovski** **

* *Key Laboratory of Process Industry Automation, Ministry of Education, Northeastern University, Shenyang, 110004, P. R. Chain*
** *Department of Computer Engineering Faculty of Engineering, Dogus University, Kadikoy, TR-34722, Istanbul Turkey*

Abstract: This paper deals with the robust H_∞ control problem for a class of multi-input non-minimum-phase nonlinear systems with parameter uncertainty. A system of this class is assumed to be in a special interlaced form, which includes a strict triangular form as a special case. By using an extension of backstepping, nonlinear static-state feedback controllers are designed such that the closed-loop system is input-to-state stable with respect to the disturbance input and has the prescribed L_2-gain from the disturbance input to the controlled output for all admissible parameter uncertainties.

Keywords: nonlinear systems, H_∞ control, L_2-gain, state feedback, backstepping.

1. INTRODUCTION

H_∞ control has become a powerful tool to solve the robust stabilization and disturbance attenuation problem and has been investigated heavily. There are some approaches which have been used to provide solutions to nonlinear H_∞ control problem. One is based on the dissipativity theory and differential games theory in (Basar and Bernhard, 1991; Ball and Helton, 1989). The other is based on the nonlinear version of classical bounded real lemma in (Isidori and Astolfi, 1992; Isidori, 1991; Van der Schaft, 2000). These results involve solving Hamilton-Jacobi-Issacs equations (HJIEs), whose nice feature is that they are parallel to the results of linear H_∞ control. However, for the aforementioned results, the lack of efficient numerical procedures for solving the HJIEs is a formidable difficulty. This motivates some attempts to look for methods which solve reduced-order HJIEs or need not to solve any HJIEs. By using "normal form" and backstepping technique, the H_∞ control problem has been investigated extensively. Results in (Isidori, 1996a; Marino, et al., 1994; Guo, et al., 2000) deal with minimum-phase systems and (Isidori, 1996b; Su, et al., 1999; Lin, et al., 1999) provide results for non-minimum-phase systems whose zero dynamics are divided into stable parts and unstable but stabilizable parts. Backstepping method has been extended to investigate the H_∞ control problem for systems with block-strict-triangular form (Xie and Su, 1997), subject to parameter uncertainty (Xie

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and Su, 1997; Su, et al., 1999) and with time-delay (Guo, et al., 2000).

Though backstepping play an important role in nonlinear H_∞ control problem, a significant restriction of the method is that it is only suitable to the systems with inherent strict triangular form.

In this paper, we will extend the backstepping technique to multi-input non-minimum-phase systems which are non-strict triangular form. Such a system, which includes the strict triangular form as a special case, consists of one autonomous subsystem (we call it zero dynamics of the system) and two single-input subsystems with special interconnections. We allow the existence of not only feedback but also feedforward interconnections among the subsystems. Also, our zero dynamics equation is allowed to contain the states of all subsystems, which is obviously much more general than those in papers mentioned above, where zero dynamics equation can only contain the states of the first subsystem. Such special interconnections in this paper make it possible to construct the robust H_∞ controllers using an extension of backstepping. The main design procedure is as follows. First, using backstepping technique, we design the robust H_∞ controller for the first subsystem. Then, by analyzing the resulting closed-loop subsystem through the interconnection to the next subsystem, another augmented subsystem is obtained, which is also in a strict triangular form. Finally, the second robust H_∞ controller can be designed by backstepping. The results of this paper can also be viewed as a generalization of the robust stabilization result in (Liu, et al., 1999), where no H_∞ control is considered.

2. PRELIMINARIES

This paper considers a nonlinear uncertain system with two inputs, which is described by

$$\begin{aligned}
\dot{\eta} &= f_1(\eta,\xi_1,\theta) + p_0(\eta,\xi_1,\theta)w + f_2(\eta,\xi,u,\zeta_1,\theta)\zeta_1,\\
\dot{\xi}_1 &= \xi_2 + \mu_1(\eta,\xi_1,\theta) + p_1(\eta,\xi_1,\theta)w\\
&\quad + \varphi_1(\eta,\xi,u,\zeta_1,\theta)\zeta_1,\\
&\cdots\\
\dot{\xi}_{m-1} &= \xi_m + \mu_{m-1}(\eta,\xi_1,\ldots,\xi_{m-1},\theta)\\
&\quad + p_{m-1}(\eta,\xi_1,\ldots,\xi_{m-1},\theta)w\\
&\quad + \varphi_{m-1}(\eta,\xi,u,\zeta_1,\theta)\zeta_1,\\
\dot{\xi}_m &= u + \mu_m(\eta,\xi_1,\ldots,\xi_m,\theta)\\
&\quad + p_m(\eta,\xi_1,\ldots,\xi_m,\theta)w\\
&\quad + \varphi_m(\eta,\xi,u,\zeta_1,\theta)\zeta_1,\\
\dot{\zeta}_1 &= \zeta_2 + \phi_1(\eta,\xi,u,\zeta_1,\theta) + q_1(\eta,\xi,u,\zeta_1,\theta)w,\\
&\cdots\\
\dot{\zeta}_{n-1} &= \zeta_n + \phi_{n-1}(\eta,\xi,u,\zeta_1,\ldots,\zeta_{n-1},\theta)\\
&\quad + q_{n-1}(\eta,\xi,u,\zeta_1,\ldots,\zeta_{n-1},\theta)w,\\
\dot{\zeta}_n &= v + \phi_n(\eta,\xi,u,\zeta_1,\ldots,\zeta_n,\theta)\\
&\quad + q_n(\eta,\xi,u,\zeta_1,\ldots,\zeta_n,\theta)w,\\
y &= h(\eta,\xi_1,\theta) + d(\eta,\xi_1,\theta)w,
\end{aligned} \tag{1}$$

where $\eta \in R^l$, $\xi = [\xi_1,\ldots,\xi_m]^T \in R^m$, $\zeta = [\zeta_1,\ldots,\zeta_n]^T \in R^n$, $u, v \in R$ are the control inputs, θ is a uncertain parameter vector belonging to a known compact set Ω, $w \in R^r$ is the disturbance input, $y \in R^k$ is the controlled output. All vector fields are assumed to be smooth. We also assume $f_1(0,0,\theta) = 0$, $\mu_i(0,\ldots,0,\theta) = 0$, $\phi_i(0,\ldots,0,\theta) = 0$, $i = 1,\ldots,n$, and $h(0,0,\theta) = 0$ for any $\theta \in \Omega$.

Remark 1. In single-input case, system (1) with $\zeta_1 \equiv 0$ reduces to a strict triangular form, which has been studied extensively (see, for example, Isidori, 1996a; Isidori, 1996b; Su, et al., 1999). Obviously, system (1) is not in a strict feedback form and its zero dynamics (η-subsystem) contains not only the state of the first subsystem ξ but also the state of the second subsystem ζ. Therefore, system (1) covers much broader class of nonlinear systems. At the same time, the results aforementioned are no longer suitable for system (1).

Remark 2. Here we study two inputs case only for the sake of simplicity. All results can be extended to multi-input case without any difficulty.

The following assumptions will be needed in the sequel.

Assumption 1. The η-subsystem of (1) with $\zeta_1 \equiv 0$ can be decomposed into the following two cascade-connected subsystems:

$$\begin{aligned}
\dot{\eta}_1 &= f_{01}(\eta_1,\eta_2,\xi_1,\theta) + p_{01}(\eta_1,\eta_2,\xi_1,\theta)w,\\
\dot{\eta}_2 &= f_{02}(\eta_2,\xi_1,\theta),
\end{aligned} \tag{2}$$

where $\eta_1 \in R^{m_1}, \eta_2 \in R^{m_2}$, $m_1 + m_2 = l$, $\eta = [\eta_1{}^T, \eta_2{}^T]^T$, and with $f_{01}(0,0,0,\theta) = 0$ and $f_{02}(0,0,\theta) = 0$ for any $\theta \in \Omega$.

Assumption 2.

(a) For the η_1-subsystem, there exists a real-valued function $W_1(\eta_1,\theta)$, which is smooth in η_1, and positive definite and proper for any $\theta \in \Omega$, such that

$$\begin{aligned}
&\frac{\partial W_1}{\partial \eta_1}[f_{01}(\eta_1,\eta_2,\xi_1,\theta) + p_{01}(\eta_1,\eta_2,\xi_1,\theta)w]\\
&\le -\alpha_1\|\eta_1\|^2 + \gamma_0{}^2\|w\|^2 + k_1(\eta_2,\xi_1)
\end{aligned} \tag{3}$$

for any $\theta \in \Omega$, some positive-definite function $k_1(\eta_2,\xi_1)$ and some positive constants α_1 and γ_0.

(b) For the η_2-subsystem, there exists a real-valued function $\mu(\eta_2)$ with $\mu(0) = 0$ and a real-valued function $W_2(\eta_2,\theta)$, which is smooth and positive definite in η_2 for any $\theta \in \Omega$, such that

$$\begin{aligned}
&\frac{\partial W_2}{\partial \eta_2} f_{02}(\eta_2,\mu(\eta_2),\theta) \le -\alpha_2 W_2(\eta_2,\theta),\\
&\alpha_3\|\eta_2\|^2 \le W_2(\eta_2,\theta)
\end{aligned} \tag{4}$$

for any $\theta \in \Omega$ and some positive constants α_2 and α_3.

Remark 3. Here we adopt the standard conditions similar to that in (Su, et al., 1999; Isidori, 1996b). Assumption 1 and Assumption 2 mean that when $\zeta \equiv 0$ the zero dynamics of (1) take two cascade-connected parts, one part is input-to-state stable, the other part may be unstable but stabilizable.

The following condition is crucial to the construction of a storage function of the system even in single-input strict triangular form, and is commonly applied in the literature (see, for example, Xie and Su, 1997; Su, et al. 1999).

Assumption 3. $d(\eta, \xi_1, \theta)$ is uniformly bounded, i.e., there exists a positive real number γ_d such that for any $\theta \in \Omega$,

$$\|d(\eta, \xi_1, \theta)\| \leq \gamma_d, \forall [\eta^T, \xi_1]^T \in R^{l+1}.$$

This paper deals with the following robust H_∞ control problem for system (1):

Given any scalar $\gamma > \gamma_d$, design feedback control laws $u = u(\eta, \xi)$ with $u(0,0) = 0$ and $v = v(\eta, \xi, \zeta)$ with $v(0,0,0) = 0$ such that for any $\theta \in \Omega$:

(a) the resulting closed-loop system is input-to-state stable with respect to the disturbance input w.

(b) the L_2-gain from disturbance input w to controlled output y of the closed-loop is not larger than γ for any $\theta \in \Omega$, i.e., there exists a function $\beta : R^l \times R^m \times R^n \times \Omega \to R$ with $\beta(0,0,0,\theta) = 0$, $\forall \theta \in \Omega$, such that for any initial condition (η^0, ξ^0, ζ^0) and all $w \in L_2[0, \infty)$, it holds that

$$\begin{aligned} &\int_0^\infty y^T(\tau) y(\tau) d\tau \leq \gamma^2 \int_0^\infty w^T(\tau) w(\tau) d\tau \\ &\qquad\qquad + \beta(\eta^0, \xi^0, \zeta^0, \theta), \\ &\forall \theta \in \Omega. \end{aligned} \tag{5}$$

Remark 4. Since global asymptotic stability with zero input does not imply stability when they are subjected to some non-zero inputs, and the notion input-to-state stability (ISS) in (Sontag and Wang, 1995) describes stronger and desirable stability property of systems with bounded inputs. Here, the internal stability with zero initial value adopted in general H_∞ control problem (Isidori, 1996a; Isidori, 1996b; Guo, et al. 2000; Lin, et al. 1999) is replaced by ISS.

Before developing the main results, we review some results about robust H_∞ control for single-input strict triangular systems described as

$$\begin{aligned} &\dot{\eta} = f_1(\eta, \xi_1, \theta) + p_0(\eta, \xi_1, \theta) w, \\ &\dot{\xi}_1 = \xi_2 + \mu_1(\eta, \xi_1, \theta) + p_1(\eta, \xi_1, \theta) w, \\ &\quad \vdots \\ &\dot{\xi}_{m-1} = \xi_m + \mu_{m-1}(\eta, \xi_1, \ldots, \xi_{m-1}, \theta) \\ &\qquad + p_{m-1}(\eta, \xi_1, \ldots, \xi_{m-1}, \theta) w, \\ &\dot{\xi}_m = u + \mu_m(\eta, \xi_1, \ldots, \xi_m, \theta) \\ &\qquad + p_m(\eta, \xi_1, \ldots, \xi_m, \theta) w, \\ &y = h(\eta, \xi_1, \theta) + d(\eta, \xi_1, \theta) w. \end{aligned} \tag{6}$$

Lemma 1 (Su, 1999). The global robust H_∞ control problem for the uncertain nonlinear system (6) is solvable if Assumptions 1∼3 are satisfied.

According to (Su, 1999), we can easily have the following fact.

Proposition 1. If the uncertain nonlinear system (6) satisfies Assumptions 1∼3, then there exist functions $\sigma_1(\eta, \xi_1)$,..., $\sigma_m(\eta, \xi_1, \ldots, \xi_m)$ with $\sigma_j(0, \ldots, 0) = 0$,$j = 1, \ldots, m$, and storage function $S_m(\eta, \xi_1, \ldots, \xi_m, \theta)$, which is a positive definite Class K_∞ function, such that along the trajectory of (6) with $u = \sigma_m(\eta, \xi_1, \ldots, \xi_m)$, we have

$$\begin{aligned} &\dot{S}_m + \|y\|^2 - \varepsilon_m^2 \|w\|^2 \\ &\leq -l_m(\|\eta\|^2 + (\xi_1 - \sigma_0)^2 + (\xi_2 - \sigma_1)^2 \\ &\quad + \ldots + (\xi_m - \sigma_{m-1})^2) \end{aligned} \tag{7}$$

for some constants ε_m satisfying $\gamma > \varepsilon_m > \gamma_d$ and $l_m > 0$, where $\sigma_0(\eta) = \mu(\eta_2)$.

We recall the following result when (6) becomes the form

$$\begin{aligned} &\dot{x}_1 = f_1(x_1, \theta) + q(x_1, x_2, \theta) x_2 + p_1(x_1, \theta) w, \\ &\dot{x}_2 = u + f_2(x_1, x_2, \theta) + p_2(x_1, x_2, \theta) w, \\ &y = h_0(x_1, \theta) + d_0(x_1, \theta) w. \end{aligned} \tag{8}$$

Lemma 2 (Su, 1999). Suppose that for a given scalar $\tau_1 > 0$, there exists a storage function $V_1(x_1, \theta)$ for system (8) with $x_2 \equiv 0$ satisfying

$$\begin{aligned} &\dot{V}_1(x_1, \theta) + \|y\|^2 - \tau_1^2 \|w\|^2 \leq -c_1 \|x_1\|^2, \\ &\forall \theta \in \Omega \end{aligned} \tag{9}$$

for some positive real number c_1. Then for any scalar $\tau_2 > \tau_1$, there exists a control law $u = u(x_1, x_2)$, such that storage function
$V_2(x_1, x_2, \theta) = V_1(x_1, \theta) + \frac{1}{2} x_2^T x_2$
for system (8) satisfies

$$\begin{aligned} &\dot{V}_2(x_1, x_2, \theta) + \|y\|^2 - \tau_2^2 \|w\|^2 \\ &\leq -c_2(\|x_1\|^2 + \|x_2\|^2), \forall \theta \in \Omega \end{aligned} \tag{10}$$

for some positive real number c_2.

From (Su, 1999), we know that the control law $u = u(x_1, x_2)$ in Lemma 2 satisfies $u(0,0) = 0$.

3. MAIN RESULT

In this section, we extend the backstepping technique to solve the robust H_∞ control problem for system (1). The main theorem is as follows.

Theorem 1. Assume Assumptions 1∼3 are satisfied. Then the global robust H_∞ control problem for system (1) is solvable.

Proof. We divide the proof into three steps.

Step 1. Consider the (η, ξ)-subsystem, i.e., system (6). Proposition 1 gives (7).

Step 2. Make the global change of coordinates

$$\begin{array}{l} \bar{\eta} = \eta, \\ \bar{\xi}_i = \xi_i - \sigma_{i-1}, i = 1, \ldots, m, \\ \bar{\zeta}_i = \zeta_i, i = 1, \ldots, n, \end{array} \tag{11}$$

and impose the feedback $u = \sigma_m$, where $\sigma_i, i = 0, 1, \ldots, m$, are shown in Proposition 1. Then, system (1) can be expressed as

$$\begin{aligned} \dot{z} &= F_1(z,\theta) + F_2(z,\zeta_1,\theta)\zeta_1 + P(z,\theta)w, \\ \dot{\zeta}_1 &= \zeta_2 + \phi_1(\eta,\xi,\sigma_m,\zeta_1,\theta) \\ &\quad + q_1(\eta,\xi,\sigma_m,\zeta_1,\theta)w, \\ &\vdots \\ \dot{\zeta}_{n-1} &= \zeta_n + \phi_{n-1}(\eta,\xi,\sigma_m,\zeta_1,\ldots,\zeta_{n-1},\theta) \\ &\quad + q_{n-1}(\eta,\xi,\sigma_m,\zeta_1,\ldots,\zeta_{n-1},\theta)w, \\ \dot{\zeta}_n &= v + \phi_n(\eta,\xi,\sigma_m,\zeta_1,\ldots,\zeta_n,\theta) \\ &\quad + q_n(\eta,\xi,\sigma_m,\zeta_1,\ldots,\zeta_n,\theta)w, \\ y &= h(\eta,\xi_1,\theta) + d(\eta,\xi_1,\theta)w, \end{aligned} \tag{12}$$

where, $z = [\eta^T, \bar{\xi}_1, \ldots, \bar{\xi}_m]^T$,

$$F_1(z,\theta) = \begin{pmatrix} f_1 \\ \xi_2 + \mu_1 - \frac{\partial\sigma_0}{\partial\eta} f_1 \\ \xi_3 + \mu_2 - \frac{\partial\sigma_1}{\partial\eta} f_1 - \frac{\partial\sigma_1}{\partial\xi_1}(\xi_2+\mu_1) \\ \vdots \\ \xi_m + \mu_{m-1} - \frac{\partial\sigma_{m-2}}{\partial\eta} f_1 - \frac{\partial\sigma_{m-2}}{\partial\xi_1}(\xi_2+\mu_1) \\ - \ldots - \frac{\partial\sigma_{m-2}}{\partial\xi_{m-2}}(\xi_{m-1}+\mu_{m-2}) \\ \sigma_m + \mu_m - \frac{\partial\sigma_{m-1}}{\partial\eta} f_1 - \frac{\partial\sigma_{m-1}}{\partial\xi_1}(\xi_2+\mu_1) \\ - \ldots - \frac{\partial\sigma_{m-1}}{\partial\xi_{m-1}}(\xi_m+\mu_{m-1}) \end{pmatrix},$$

$$F_2(z,\zeta_1,\theta) = \begin{pmatrix} f_2 \\ \varphi_1 - \frac{\partial\sigma_0}{\partial\eta} f_2 \\ \varphi_2 - \frac{\partial\sigma_1}{\partial\eta} f_2 - \frac{\partial\sigma_1}{\partial\xi_1}\varphi_1 \\ \vdots \\ \varphi_{m-1} - \frac{\partial\sigma_{m-2}}{\partial\eta} f_2 - \frac{\partial\sigma_{m-2}}{\partial\xi_1}\varphi_1 \\ - \ldots - \frac{\partial\sigma_{m-2}}{\partial\xi_{m-2}}\varphi_{m-2} \\ \varphi_m - \frac{\partial\sigma_{m-1}}{\partial\eta} f_2 - \frac{\partial\sigma_{m-1}}{\partial\xi_1}\varphi_1 \\ - \ldots - \frac{\partial\sigma_{m-1}}{\partial\xi_{m-1}}\varphi_{m-1} \end{pmatrix},$$

$$P(z) = \begin{pmatrix} p_0 \\ p_1 - \frac{\partial\sigma_0}{\partial\eta} p_0 \\ p_2 - \frac{\partial\sigma_1}{\partial\eta} p_0 - \frac{\partial\sigma_1}{\partial\xi_1} p_1 \\ \vdots \\ p_{m-1} - \frac{\partial\sigma_{m-2}}{\partial\eta} p_0 - \frac{\partial\sigma_{m-2}}{\partial\xi_1} p_1 \\ - \ldots - \frac{\partial\sigma_{m-2}}{\partial\xi_{m-2}} p_{m-2} \\ p_m - \frac{\partial\sigma_{m-1}}{\partial\eta} p_0 - \frac{\partial\sigma_{m-1}}{\partial\xi_1} p_1 \\ - \ldots - \frac{\partial\sigma_{m-1}}{\partial\xi_{m-1}} p_{m-1} \end{pmatrix}.$$

Applying (7) to system (12) with $\zeta_1 \equiv 0$ results in

$$\frac{\partial\bar{S}_m}{\partial z}(F_1 + Pw) + \|y\|^2 - \varepsilon_m^2\|w\|^2 \leq -l_m\|z\|^2, \tag{13}$$

where $\bar{S}_m(z,\theta) = S_m(\eta,\xi_1,\ldots,\xi_m,\theta)$.

Step 3. Since the coordinate transformation (11) is a global diffeomorphism, the robust H_∞ control problem for system (1) is solvable if the robust H_∞ control problem for system (12) is solvable. It is obvious that system (12) is in strict triangular form. Therefore, we can realize the robust H_∞ control for system (12) by a recursive application of lemma 2. To this end, let constants $\varepsilon_{m+1},\ldots,\varepsilon_{m+n}$ satisfy $\varepsilon_m < \varepsilon_{m+1} < \ldots < \varepsilon_{m+n} = \gamma$, $\beta_0(z) = 0$ and $\zeta_{n+1} = v$ and consider the following system for $t = 1, \ldots, n$,

$\sum_t$:

$$
\begin{aligned}
&\dot{z} = F_1(z,\theta) + F_2(z,\zeta_1,\theta)\zeta_1 + P(z,\theta)w,\\
&\dot{\zeta}_1 = \zeta_2 + \phi_1(\eta,\xi,\sigma_m,\zeta_1,\theta)\\
&\qquad + q_1(\eta,\xi,\sigma_m,\zeta_1,\theta)w,\\
&\quad\vdots\\
&\dot{\zeta}_t = \zeta_{t+1} + \phi_t(\eta,\xi,\sigma_m,\zeta_1,\ldots,\zeta_t,\theta)\\
&\qquad + q_t(\eta,\xi,\sigma_m,\zeta_1,\ldots,\zeta_t,\theta)w,\\
&y = h(\eta,\xi_1,\theta) + d(\eta,\xi_1,\theta)w.
\end{aligned}
\tag{14}
$$

Introduce a global change of coordinates

$$
\bar{\zeta}_j = \zeta_j - \beta_{j-1}(z,\zeta_1,\ldots,\zeta_{j-1})\ ,\quad j = 1,\ldots,t,
$$

and let

$$
z_{t-1} = [z^T, \bar{\zeta}_1, \ldots, \bar{\zeta}_{t-1}]^T.
$$

System $\sum_t$ can be rewritten in the form of (8) with the state $(z_{t-1}, \bar{\zeta}_t)$ and a control input ζ_{t+1}, and (9) is satisfied. Using lemma 2, we know that there exist a control law

$$
\zeta_{t+1} = \beta_t(z,\zeta_1,\ldots,\zeta_t)
$$

with $\beta_t(0,0,\ldots,0) = 0$ and a storage function

$$
\begin{aligned}
&S_{m+t}(z,\zeta_1,\ldots,\zeta_t,\theta)\\
&= S_m + \tfrac{1}{2}(\zeta_1-\beta_0)^2 + \ldots + \tfrac{1}{2}(\zeta_t-\beta_{t-1})^2,
\end{aligned}
$$

such that for $\sum_t$ it holds that

$$
\begin{aligned}
&\dot{S}_{m+t} + \|y\|^2 - \varepsilon_{m+t}^2\|w\|^2\\
&\le -l_{m+t}(\|z\|^2 + (\zeta_1-\beta_0)^2 + \ldots + (\zeta_t-\beta_{t-1})^2)
\end{aligned}
$$

for some constant $l_{m+t} > 0$.

When $t = n$ we have

$$
\begin{aligned}
&\dot{S}_{m+n} + \|y\|^2 - \gamma^2\|w\|^2\\
&\le -l_{m+n}(\|z\|^2 + (\zeta_1-\beta_0)^2\\
&\quad + \ldots + (\zeta_n-\beta_{n-1})^2),
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
&S_{m+n}(z,\zeta_1,\ldots,\zeta_n,\theta)\\
&= S_m + \frac{1}{2}(\zeta_1-\beta_0)^2 + \ldots + \frac{1}{2}(\zeta_n-\beta_{n-1})^2.
\end{aligned}
\tag{16}
$$

From (15), we have

$$
\begin{aligned}
\dot{S}_{m+n} &\le -l_{m+n}(\|z\|^2 + (\zeta_1-\beta_0)^2\\
&\quad + \ldots + (\zeta_n-\beta_{n-1})^2) + \gamma^2\|w\|^2.
\end{aligned}
\tag{17}
$$

From (15)$\sim$ (17), it is known that $S_{m+n}(z,\zeta_1,\ldots,\zeta_n,\theta)$ is a Class K_∞ function for $\forall\theta \in \Omega$ and system (12) is input-to-state stable with respect to the disturbance input w, and we have

$$
\begin{aligned}
\int_0^\infty y^T(\tau)y(\tau)d\tau &\le \gamma^2\int_0^\infty w^T(\tau)w(\tau)d\tau\\
&\quad + \beta(z^0,\zeta_1^0,\ldots,\zeta_n^0,\theta),
\end{aligned}
\tag{18}
$$

$\forall\theta \in \Omega$,

where

$$
\beta(z^0,\zeta_1^0,\ldots,\zeta_n^0,\theta) = S_{m+n}(z^0,\zeta_1^0,\ldots,\zeta_n^0,\theta)
$$

with initial value $(z^0,\zeta_1^0,\ldots,\zeta_n^0)$. So the robust H_∞ control problem for system (12) is solvable. It is easily seen that the controllers

$$
u = \sigma_m(\eta,\xi_1,\ldots,\xi_m),
$$

and

$$
v = \beta_n(z,\zeta_1,\ldots,\zeta_n),
$$

solve the robust control problem for system (1). Thus we complete the proof of theorem 1. □

4. EXAMPLE

As an illustration of the above design method, consider a simple nonlinear system of the form

$$
\begin{aligned}
&\dot{\eta} = -\eta + \eta\xi^2 + \eta\zeta,\\
&\dot{\xi} = u + \xi\sin(\theta(\eta+\xi)) + 0.25w + \zeta,\\
&\dot{\zeta} = v + w,\\
&y = \eta + 0.5w,
\end{aligned}
\tag{19}
$$

where $\eta,\xi,\zeta \in R$, and the constant unknown parameter $\theta \in [-50, 50]$. Note that the η-subsystem contains not only ξ but also ζ, and the system is not in the strict triangular structure. We will design a nonlinear state feedback controller for system (19) such that the closed-loop system is input-to-state stable and the L_2 -gain from w to y is not larger than $\sqrt{2}$.

System (19) is in the form of Eq. (1). It just contains the η_2-subsystem. Thus, $W_1 = 0$, $\alpha_1 = 0$, $\gamma_0 = 0$, $k_1 = 0$. Choose $\mu(\eta_2) \equiv 0$, $W_2 = 0.5\eta_2^2$. It is easily verified that Assumptions 1$\sim$3 are satisfied and theorem 1 holds. By using the procedure adopted in theorem 1, we obtain storage function

$$
S_3(\eta,\zeta,\zeta,\theta) = \eta^2 + \tfrac{1}{2}\xi^2 + \tfrac{1}{2}\zeta^2,
$$

and controller

$$
\begin{aligned}
&u = -2\eta^2\xi - 2\xi,\\
&v = -2\eta^2 - \xi - 2\zeta.
\end{aligned}
\tag{20}
$$

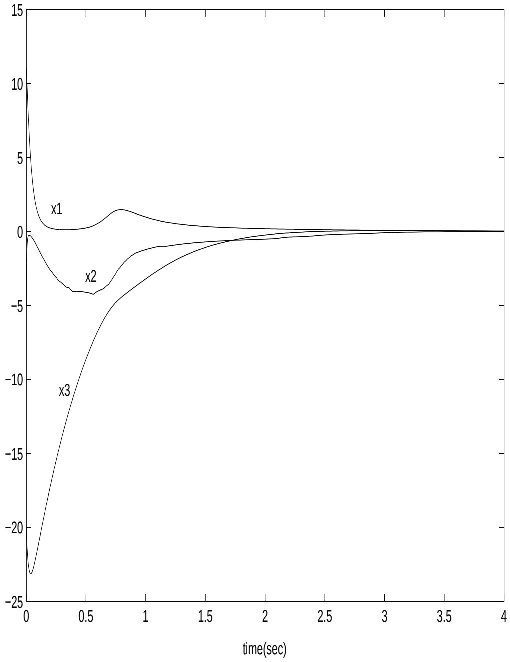

Fig. 1. state response of the resulting closed-loop system with w$\equiv$ 0 and $\theta = 50$, $\eta = x1$, $\xi = x2$, $\zeta = x3$.

For the closed-loop system (19),(20), it holds that

$$\dot{S}_3 + \|y\|^2 - 2\|w\|^2 \leq -\tfrac{3}{4}\eta^2 - \tfrac{3}{4}\xi^2 - \zeta^2$$

for $\forall\theta \in [-50, 50]$.

Thus, controller (20) solves the robust H_∞ control problem for system (19). Fig. 1 shows the state response of the resulting closed-loop system with zero disturbance and $\theta = 50$.

5. CONCLUSIONS

We have discussed the robust H_∞ control problem for a class of multi-input non-minimum-phase nonlinear systems with parameter uncertainty. A system of this class consists of several subsystems with both special feedback and feedforward interconnections and it may not be in strict triangular form. A robust H_∞ controller, which ensures that the closed-loop system is input-to-state stable with respect to the disturbance input and has a prescribed L_2-gain for all admissible parameter uncertainties, is obtained by using an extension of backstepping.

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