A few of the results given above were also obtained by S. A. Goudsmit, and the author acknowledges with thanks communications from him containing unpublished formulas in both the two- and three-dimensional cases. Thanks are also due to D. G. Kendall, and to P. I. Richards, for making available their results (together with derivations).

* The first part of this paper appeared in the October 1964 issue of these PROCEEDINGS.

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*AVERAGES FOR POLYGONS FORMED BY RANDOM LINES**

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When a plane is uniformly covered with random straight lines, it is divided into an infinite set of interlocking convex polygons (cf. Fig. 1). This aggregate of random polygons was first studied by S. A. Goudsmit,[1] who obtained the mean number of sides, the mean perimeter, the mean area, and the mean area-squared. Recently, R. E. Miles[2] has summarized current knowledge of this problem, presenting a number of original results and generalizations. In this paper, I shall derive some relations that yield additional new averages by a method that generalizes Goudsmit's original approach.

Simple Results.—Among many particular results that can be obtained by this method, the following simple ones seem worth recording specifically. Let $E[\ldots]$ denote the mean of a quantity, with each polygon weighted equally. Then,

$$E[A^2R] = 16/k^5,\ E[AI] = 24/k^6,\ E[\phi] = 8/k^3. \tag{1}$$

Here, A is the area of a typical polygon, R is the mean separation of two random points within it, I is its moment of inertia about its center of gravity, and ϕ is its "Newtonian self-energy": $\phi = \iint |\mathbf{r} - \mathbf{r}'|^{-1}d\sigma\, d\sigma'$. The parameter k determines the density of the random lines; it is the mean number of lines crossing any straight segment of unit length. (Thus, $1/k$ is the "mean free length" of a line; in Miles' notation,[2] $\tau = \pi k/2$.)

The relations in equation (1) give some approximate information on the shapes of the polygons. Another relation of this type is the following. If a circle of radius b is centered over a random point inside a typical polygon, let A_b be the mean of the overlapping area (cf. Fig. 2). Then

$$E[A\cdot A_b] = (8/k^4)\ [1 - (1 + kb)\ e^{-kb}]. \tag{2}$$

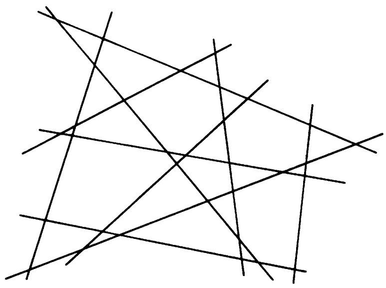

FIG. 1.—Examples of random lines forming random polygons.

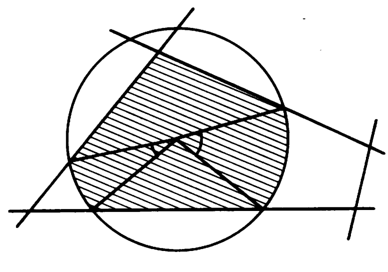

FIG. 2.—The center of the circle is a random point within the random polygon. The mean area of the shaded region is A_b; the mean of the sum of the marked angles is θ_b.

If we let the radius b approach zero, we find $E[A] = 4/(\pi k^2)$, while $b \to \infty$ yields $E[A^2] = 8/k^4$, in agreement with Goudsmit's original results.[1] A relation concerning the mean angular arc θ, cut from such a circle by the polygon, follows by differentiating equation (2) and dividing by b.

$$E[A\theta_b] = (8/k^2)\ e^{-kb}. \tag{3}$$

Additional simple results are

$$E[A^3R] = 2^{11}\pi/(21\ k^7) \tag{4}$$

and

$$E[LI_L] = (8 + 6\pi^2)/k^4, \tag{5}$$

where L is the perimeter of a typical polygon and I_L is the moment of inertia of the perimeter about its own center of gravity.

Basic Method.—Following Goudsmit, we consider points placed at random in the plane. To avoid dealing with infinite sets, we may employ Goudsmit's elegant device of mapping the problem onto a very large sphere of area D, where the random lines become great circles, uniformly distributed over the surface. Keeping the density k of these "lines" constant, we can later let the sphere increase in size and asymptotically approach the plane. Alternatively, we may confine the points to a large but finite domain of area D in the plane and verify that the edge effects will vanish in the limit, as $D \to \infty$.

On the sphere or in the domain of area D, we place many independent pairs of random points. Let $f(r)$ be an arbitrary function, and consider the mean of $f(|\mathbf{r}\text{-}\mathbf{r}'|)$ taken over those pairs of points that happen to fall inside a single polygon. Since e^{-kx} is the probability that no line crosses a straight segment of length x, the mean will be (asymptotically for large D):

$$\int_0^\infty f(x)\ e^{-kx}\ 2\pi x\ dx/D.$$

On the other hand, this average can be evaluated in another way. The probability that two random points will both fall inside a polygon of area A is $(A/D)^2$, and the contribution of such a polygon to the foregoing mean will be

$$(A/D)^2 \iint f(|\mathbf{r}\text{-}\mathbf{r}'|)\ (d\sigma/A)\ (d\sigma'/A),$$

where $d\sigma$ represents a differential area within the polygon. Multiply this expression by the mean number of polygons in D with a given area and shape, namely, $N_D\ P(,.A\ ..)\ dA...$, where N_D is the total number of polygons in D, and $P(A,...)$ denotes the probability distribution for polygons with specified properties. Integrate this product over all sizes and shapes to obtain another expression for the foregoing mean. When we observe that $D/N_D = E[A] = 4/(\pi k^2)$, we find the relation,

$$E[\iint f(|\mathbf{r}-\mathbf{r}'|)\ d\sigma d\sigma'] = (8/k^2) \int_0^\infty x\ f(x)\ e^{-kx} dx, \tag{6}$$

where the double integral on the left extends over the interior of a typical polygon

The mean values displayed in equation (1) are obtained by choosing $f(r)$ equal to $1/r$, r, and r^2 in turn. The results given in equations (2) and (3) emerge when $f(r)$ is chosen equal to unity for $r < b$ and zero otherwise.

Cubic Averages.—Similarly, we may consider placing three random points in a large domain D and averaging some function of their locations. Now, the probability that no line separates one of the three points from the others is equal to $e^{-kp/2}$ where p is the perimeter of the triangle they form (compare Miles,[2] Theorem 3).

$$p = |\mathbf{r}_1 - \mathbf{r}_2| + |\mathbf{r}_2 - \mathbf{r}_3| + |\mathbf{r}_3 - \mathbf{r}_1|. \tag{7}$$

This suggests that we average an arbitrary function of this perimeter over those triples of points that happen to fall within a common polygon. This requires that we first evaluate the probability of obtaining a specified perimeter p when three points are dropped at random in a large domain D.

For a given value of p and specified locations of $\mathbf{r}_1$ and $\mathbf{r}_2$ (with $|\mathbf{r}_1 - \mathbf{r}_2| < p$), the locus of points $\mathbf{r}_3$ yielding the given p will lie on an ellipse with foci at $\mathbf{r}_1$ and $\mathbf{r}_2$. In this way, a tedious but straightforward calculation will show that three random points in a large area D have perimeter $p \pm dp/2$ with probability

$$P(p)\ dp = (2\pi^2/21)\ p^3\ dp/D^2$$

(asymptotically for large D, where edge effects may be neglected).

The remaining arguments are analogous to those for two random points, and the final result is

$$E[\iiint f(p)\ d\sigma_1 d\sigma_2 d\sigma_3] = (8\pi/21\ k^2) \int_0^\infty x^3 f(x)\ e^{-kx/2} dx, \tag{8}$$

where p is defined by equation (7) and the triple integral extends over the interior of a typical polygon. The choice $f(p) = 1$ yields $E[A^3]$, in agreement with unpublished results of D. G. Kendall and S. A. Goudsmit as reported by Miles.[2] The choice $f(p) = p$ yields the value of $E[A^3R]$ displayed in equation (4).

Line-Integral Averages.—The following relation requires a rather detailed proof.

$$E[\oint \oint f\ (|\mathbf{r} - \mathbf{r}'|)\ ds\ ds'] = (2/k) \int_0^\infty [4 + \pi^2 kx]\ f(x)\ e^{-kx}\ dx. \tag{9}$$

Here, $\mathbf{r}$ and $\mathbf{r}'$ are two points on the boundary of a typical polygon while ds and ds' denote differential lengths along the boundary. Choosing $f(r) = 1$ verifies $E[L^2]$ as first reported by Miles,[2] and choosing $f(r) = r^2$ yields the value of $E[L\ I_L]$ given in equation (5).

The remainder of this section is devoted to sketching a proof of equation (9). The basic method is again to consider pairs of points $(\mathbf{r},\mathbf{r}')$ in a large region D. This time, we require not only that the points fall inside a common polygon but also that each point lie within a small distance w of its boundary. We then average $f(|\mathbf{r} - \mathbf{r}'|)$ over such pairs and subsequently let w approach zero.

The new feature of this proof, then, is to determine the probability that two random points are not separated by a random line while each one lies within a distance w of such a line.

Given two points, let x be their separation, and divide the random lines of interest into four disjoint sets:

S: "separating lines" that pass between the points;
C: "common lines" that pass within w of both points at once but not between them;
E_1 and E_2: "end lines" that pass within w of one point but do not pass within w of the other and do not separate them.

Since the random lines form a Poisson process, the probability that n lines actually occur in any set is $(M^n/n!)e^{-M}$ where M is the mean number of lines in the set. For disjoint line-sets, these probabilities may be multiplied, because the corresponding lines are independent. Our goal, then, is to evaluate the mean number of random lines falling in sets C and E_i; we already know that the mean number in S is

$$M_S = kx. \tag{10}$$

The mean number of lines that cross any convex region in the plane is the product, $k/2$ times the perimeter of the region.[3, 4] Now, the union of all four of our line-sets is precisely the set of lines that pass within a distance w of the line-segment x, between the two given points. The region of points within distance w of this segment has perimeter $2x + 2\pi w$, and thus

$$M_S + M_C + 2M_E = k(x + \pi w), \tag{11}$$

because the means for E_1 and E_2 are equal by symmetry.

The union of the line-sets S, C, and E_1 consists of lines that pass within w of one chosen point and/or cross the line-segment x. Such lines must cross the convex figure bounded by a disk (radius w) about the chosen point and a wedge with its vertex at the other point and its sides tangent to the disk. The perimeter of this figure is readily calculated, and we obtain

$$M_S + M_C + M_E = k[(\pi/2)w + w \sin^{-1}(w/x) + (x^2 - w^2)^{1/2}], \tag{12}$$

provided $w < x$. (Otherwise the figure degenerates to a disk, and the right-hand side should be replaced by πkw.)

Equations (10), (11), and (12) can now be solved for M_C and M_E. Since we shall later let w approach zero, we need only retain terms to order w^2.

$$M \;= kw^2/x + \ldots, \quad M_E = (\pi/2)kw - kw^2/2x + \ldots$$

These results now enable us to calculate the probability that two points with separation x lie inside a common polygon, each point being at a distance w or less from the boundary.

This probability is a sum of probabilities for two mutually exclusive events.

If we let n_1 denote the actual number of lines in the line-set E_1, and similarly for n_2, n_S, and n_C, then these two events are:

$$\text{(a)} \quad n_S = 0,\ n_C = 0,\ n_1 > 0,\ n_2 > 0$$

$$\text{(b)} \quad n_S = 0,\ n_C > 0 \text{ (and any } n_1, n_2).$$

From the Poisson formulas, we then find that the sum of the probabilities for these events is

$$(kw)^2[(kx)^{-1} + (\pi/2)^2]e^{-kx} + 0(w^3).$$

Since the probability that two points, chosen at random in a large domain of area D, have separation $x \pm dx/2$ is just $2\pi x\, dx/D$ neglecting edge effects, the average of $f(x)$ for points satisfying our criterion is

$$(kw)^2(2\pi/D)\int_0^\infty [k^{-1} + (\pi/2)^2 x]f(x)e^{-kx}dx + 0(w^3).$$

The remaining arguments are then analogous to those used in proving equations (6) and (8), and the final result is equation (9).

Other Relations.—The results displayed in equations (10), (11), and (12) enable us to derive several other general formulas. None of these seem to lead to particularly simple averages, but the following ones seem worth recording. With one point in the interior of a typical polygon and one point on its boundary,

$$E[\oint ds_1 \int d\sigma_2 f(|\mathbf{r}_1 - \mathbf{r}_2|)] = (4\pi/k)\int_0^\infty x f(x)e^{-kx}dx.$$

With $f = 1$, this verifies $E[AL]$ as first reported by Miles.[2] For one point on a corner of a polygon and one point in its interior,

$$E[\Sigma_i \csc(\phi_i) \int f(|\mathbf{r} - \mathbf{c}_i|)d\sigma] = \pi^2 \int_0^\infty x f(x)e^{-kx}dx,$$

where ϕ_i is the interior angle at the corner $\mathbf{c}_i$. For one point on a corner and the other on the boundary,

$$E[\Sigma_i \csc(\phi_i) \oint f(|\mathbf{r} - \mathbf{c}_i|)ds] = (\pi/2)\int_0^\infty [8 + \pi^2 kx]f(x)e^{-kx}dx.$$

If C denotes the sum $\Sigma_i \csc(\phi_i)$, these relations (with $f = 1$) yield $E[AC] = \pi^2/k^2$ and $E[LC] = (\pi/2k)(\pi^2 + 8)$, and it is easy to show directly that $E[C] = 2\pi$. These results might seem to suggest that C is a fundamental quantity for random polygons, but it is not difficult to show that $E[C^2]$ diverges.

I am indebted to Dr. S. A. Goudsmit for calling my attention to this problem and making available to me his unpublished results. My search for further results might well have ground to a halt unfruitfully without his encouragement and counsel. I am also indebted to Dr. R. E. Miles for preprints of his summary papers on this problem.

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