Goldbach and Twin Prime Pairs: A Sieve Method to Connect the Two

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Abstract

This paper proposes, and demonstrates the efficacy of, a method for establishing a lower bound for cardinalities of selected sets of twin primes, and shows that the proof employed may be modified for selected sets of Goldbach pairs. Our sieve method is centred on the restrictive properties of intervals, specifically regarding divisibility distributions. We implicitly use the Chinese Remainder Theorem by way of the use of the midpoint in our intervals, and consider the sieve of Eratosthenes in such a way as to find a set of primes whose distribution is mirror-symmetrical about that midpoint. Bounds are established through the use of the formulae closely associated with the Prime Number theorem and the Mertens theorem. We show that the Goldbach conjecture is true if the Riemann hypothesis is true.

Keywords: Goldbach pairs, twin primes, prime pairs, Goldbach and Twin Primes equivalence, Euler totient, divisibility distributions, mirror symmetry, folded number scale.

Introduction

Throughout this paper, for any set S that is explicitly stated to be an 'interval' or is written $[x, y]$ for some real x and y, S is to be taken to be a nonempty set of integers; $\mathbb{N}$ will be the set of non-negative integers; p_n for $n = 1, 2, \ldots$ will be the sequence of primes; for any finite subset K of $\mathbb{N}$ and any integer i, $[K]^i$ will be the set of all subsets, M, of K for which $|M| = i$. Finally, $P(n)$ will be $\{p_1, p_2, \ldots, p_n\}$.

For any integer i, we begin with the interval $[1, i]$ and any k-element interval, I. We study the number of elements of our relevant set, T_I, of integers (i.e. the union of I and another set of integers, which serves to provide the conditions that we impose), that are not coprime to $\prod_{p \in J} p$, where J is any set of primes. It is straightforward to show, as we do by combining Lemmas 2 and 4, that $|T_{[1, I-1]}| - |T_I|$ is at most $|[J]^3|$. That is to say, in our scheme, each three-element subset of J can serve to increase the number of elements of our relevant set, that are not coprime to $\prod_{p \in J} p$, by at most one. For the purposes of this introduction may call any element of $[J]^3$ that yields in such a way, a *yielding triple*. We show, through combining Lemmas 2 and 3, that it is impossible for more than one yielding triple to exist in any four-element subset of J. This is the core of the proof of Theorem 1, which establishes an upper bound for the number of elements of I that are not coprime to $\prod_{p \in J} p$, and, using $J = P(n)$, will ultimately enable us to use the Euler totient in conjunction, tacitly, with the Chinese Remainder Theorem. Our tacit use of the Chinese Remainder theorem is the part of the method that amounts to our essential concept of the folding of the number scale. We use the Mertens theorem to develop this approach.

0.1. **Extended introduction.** Our first theorem is a prelude to the sieve method that is the focus of this paper. For Theorems 2 and 3, key to our method is the set $Z_{i,r}$, where r is even and i is as in our Introduction (above), and which is specified as the set of all sets, G, for which we have the following. For each $s \in \{0, r\}$ and $p \in P(\pi(\sqrt{2i}))$, precisely one element of $\{\{1 \leq m \leq i : p \mid m - s\} : s \in \{0, r\}\}$ is in G. We ultimately use $Z_{i,r}$ in the form $\bigcup(\bigcup Z_{i,r})$. For each $K \in Z_{i,r}$, we show through Lemma 8 that $|\bigcup K|$ has an upper bound equal to that for the number of elements of any i-element interval that are not coprime to $\prod_{k=1}^{n} p_k$, which is found by Theorem 1. We then show how our sieve method may be used to address the Twin Primes conjecture when $r = 2$ and the Goldbach conjecture when $r = 2i$. Such is explained in the section Method Outline 2.1.

Let $n = \pi(\sqrt{2i})$. Consider any element, E_p, of

$$\{\{1 \leq m \leq i : p \mid m - s\} : s \in \{0, r\}\}$$

to be called a p-*sieve* (this way, we employ the sieve of Eratosthenes, which justifies our use of p_n^2, as discussed below). Suppose we say that the p-sieve has a larger *magnitude* than the q-sieve, where $q \in P(n) \setminus \{p\}$, when $|E_p| > |E_q|$. Then we may formulate a value, w_n, using the upper bound, j_n, on $|\bigcup K|$ as found by Theorem 1. We treat w_n in a similar way to the treatment of each $p \in P(n)$ with respect to the Euler totient. Specifically, we create a quasi-sieve, which might be called a w_n-sieve, that may be shown to provide an upper bound on $|\bigcup(\bigcup Z_{i,r})|$, given by

$$i \left(1 - \frac{1}{2} \prod_{q \in (P(n)\setminus\{2\}) \cup \left\{ \frac{i \prod_{k=1}^n (1 - 1/p_k)}{j_n - i \left(1 - \prod_{k=1}^n (1 - 1/p_k) \right)} \right\}} \left(1 - \frac{2}{q} \right) \right). \tag{1}$$

(The numerator two, for $2/q$, where q is the bound variable cited as being any element of $P(n) \setminus \{2\}$, is attributable to the folding of the number scale.) In the above expression, w_n is the element in the set, beneath the product, for which there is a union with $P(n) \setminus \{2\}$. Such a use of w_n is justified by our forthcoming (58) (take it that T, in (58), is any element of $Z_{i,r}$ and K_T is any subset of T; take it also that $u_{I,n}$ as in (58) is equal to w_n), combined with the fact that the second term between the outer brackets on the right side of the first relation of (58) is equal to zero when $K_T = T$.

In an analogous way, we can now apply our idea of sieve magnitude to the quasi-sieve; for the purposes of this introduction we can use the phrase *quasi-sieve magnitude*. Indeed, from here, Theorems 2 and 3 follow by simple algebra combined with known bounds for the prime counting function, all combined, in the case of the Twin Primes conjecture where we take $r = 2$, with the Mertens theorem. To address the Goldbach conjecture, we use a work by Nicolas, through which a lower bound on the prime count, connected with the Euler-Mascheroni constant, can be deduced subject to the Riemann hypothesis being true.

1. Furnishing a Sieve Method

Theorem 1. *Let J be any set of primes for which $|J| \geq 4$. Let I be any interval. Then*

$$|\{m \in I : (m, \prod_{p \in J} p) \neq 1\}| - |\{m \in [0, |I| - 1] : (m, \prod_{p \in J} p) \neq 1\}|$$

$$\leq \frac{5|J|^2}{8}. \tag{2}$$

Lemma 1. *Let A and A' be any set of integers and T be any set of primes. For any $V \in \{A, A'\}$, let*

$$h_{V,T} = \sum_{m \in V} |\{p \in T : p \mid m\}| - \sum_{m \in V} |[\{p \in T : p \mid m\}]^2|$$

$$+ \sum_{\substack{m \in V \\ |\{p \in T : p \mid m\}| > 2}} (|[\{p \in T : p \mid m\}]^2| - |\{p \in T : p \mid m\}| + 1). \tag{3}$$

Then

$$|\{m \in A : (m, \prod_{p \in T} p) \neq 1\}| - |\{m \in A' : (m, \prod_{p \in T} p) \neq 1\}| = h_{A,T} - h_{A',T}. \tag{4}$$

2

Proof. We have

$$|\{m \in V : (m, \prod_{p \in T} p) \neq 1\}|$$

$$= \sum_{m \in A} |\{p \in T : p \mid m\}| - \sum_{\substack{m \in V \\ (m, \prod_{p \in T} p) \neq 1}} (|\{p \in T : p \mid m\}| - 1)$$

$$= \sum_{m \in V} |\{p \in T : p \mid m\}| - \sum_{m \in V} |[\{p \in T : p \mid m\}]^2|$$

$$+ \sum_{\substack{m \in V \\ |\{p \in T : p \mid m\}| > 2}} (|[\{p \in T : p \mid m\}]^2| - |\{p \in T : p \mid m\}| + 1), \quad (5)$$

implying (4). Here, the 'minus one' and, as in the case where the immediately preceding term is negative, 'one' terms in the expressions for the summands are found by the following. There is necessarily at least one element of T that divides the bound variable m cited in the $|\{p \in T : p \mid m\}| - 1$ that is the expression for the summands of the second term on the right side of the first relation of (5). Accordingly, $|\{m\} \cap \{s \in V : (s, \prod_{p \in T} p) \neq 1\}| = 1$ is a constant function of $|\{p \in T : p \mid m\}|$.

Further, for all m for which $|\{p \in T : p \mid m\}| = 2$, we have

$$|\{m\}| = |\{p \in T : p \mid m\}| - |[\{p \in T : p \mid m\}]^2|, \quad (6)$$

giving the term, in the second relation, $-\sum_{m \in V} |[\{p \in T : p \mid m\}]^2|$. Contrastingly, for all k for which $|\{p \in T : p \mid k\}| > 2$ we have

$$|\{k\}| \neq |\{p \in T : p \mid k\}| - |[\{p \in T : p \mid k\}]^2| \quad (7)$$

which, through the right side of the first relation, accounts for the final two terms of the final expression for the summands and completes the proof. $\square$

1.1. **Definition.** Let I be as in Theorem 1. For any $t \in \{0, 1\}$, any set L of primes and any integer r, let $Q_{I,L,t,r}$ and $Q'_{I,L,t,r}$ be any sets of integers for which I to III, below, are all true:

I. $\max\{|\{p \in L : p \mid m\}| : m \in Q_{I,L,t,r} \cup Q'_{I,L,t,r}\} \leq r$;

II. for each $s \in \{j \in \mathbb{N} : 1 \leq j \leq r\}$ and each $M \in [L]^s$

$$|\{m \in [t, |I| - 1] \cup Q_{I,L,t,r} : \prod_{p \in M} p \mid m\}| = |\{m \in I \cup Q'_{I,L,t,r} : \prod_{p \in M} p \mid m\}|; \quad (8)$$

III. $|[t, |I| - 1] \cup Q_{I,L,t,r}| = |I \cup Q'_{I,L,t,r}|$.

We note that (8) implies that

$$\sum_{m \in [t, |I| - 1] \cup Q_{I,L,t,r}} |[\{p \in L : p \mid m\}]^s| = \sum_{m \in I \cup Q'_{I,L,t,r}} |[\{p \in L : p \mid m\}]^s|. \quad (9)$$

Lemma 2. *Let I be as in Theorem 1. Then for any three-element set H of primes for which*

$$|\{m \in I : \prod_{p \in H} p \mid m\}| - |\{1 \leq m \leq |I| - 1 : \prod_{p \in H} p \mid m\}| = 1$$

we have

$$|\{m \in I \cup Q'_{I,H,1,2} : (m, \prod_{p \in H} p) \neq 1\}| - |\{m \in [1, |I| - 1] \cup Q_{I,H,1,2} : (m, \prod_{p \in H} p) \neq 1\}|$$

$$= 1.$$

Proof. Conditions I to III in Definition 1.1 require through Lemma 1 for $A = I \cup Q'_{I,H,1,2}$ and $A' = [1, |I| - 1] \cup Q_{I,H,1,2}$ and $T = H$, that

$$|\{m \in I \cup Q'_{I,H,1,2} : (m, \prod_{p \in H} p) \neq 1\}| - |\{m \in [1, |I| - 1] \cup Q_{I,H,1,2} : (m, \prod_{p \in H} p) \neq 1\}|$$

$$= \sum_{\substack{m \in I \cup Q'_{I,H,1,2} \\ |\{p \in H : p | m\}| > 2}} (|[\{p \in H : p \mid m\}]^2| - |\{p \in H : p \mid m\}| + 1)$$

$$- \sum_{\substack{m \in [1, |I| - 1] \cup Q_{I,H,1,2} \\ |\{p \in H : p | m\}| > 2}} (|[\{p \in H : p \mid m\}]^2| - |\{p \in H : p \mid m\}| + 1)$$

$$= 1 \tag{10}$$

which completes the proof. $\qquad \square$

Lemma 3. *Let I be as in Theorem 1. Let Z be any four-element set of primes for which*

$$|\{0 \leq m \leq |I| - 1 : \prod_{p \in Z} p \mid m\}| - |\{m \in I : \prod_{p \in Z} p \mid m\}| = 1. \tag{11}$$

Then

$$|\{m \in I \cup Q'_{I,Z,0,3} : (m, \prod_{p \in Z} p) \neq 1\}| - |\{m \in [0, |I| - 1] \cup Q_{I,Z,0,3} : (m, \prod_{p \in Z} p) \neq 1\}|$$

$$= 1. \tag{12}$$

Proof. By Lemma 1, for $A = I \cup Q'_{I,Z,0,3}$, $A' = [0, |I| - 1] \cup Q_{I,Z,0,3}$ and $T = Z$, the conditions on $Q_{I,Z,0,3}$ and $Q'_{I,Z,0,3}$ require that, for some $0 \leq m < |I| - 1$ such that $|\{p \in Z : p \mid m\}| = 4$,

$$|\{u \in I \cup Q'_{I,Z,0,3} : (u, \prod_{p \in Z} p) \neq 1\}| - |\{u \in [0, |I| - 1] \cup Q_{I,Z,0,3} : (u, \prod_{p \in Z} p) \neq 1\}|$$

$$= \sum_{\substack{u \in I \cup Q'_{I,Z,0,3} \\ |\{p \in Z : p | u\}| > 2}} (|[\{p \in Z : p \mid u\}]^2| - |\{p \in Z : p \mid u\}| + 1)$$

$$- \sum_{\substack{u \in [0, |I| - 1] \cup Q_{Z,J,0,3} \\ |\{p \in Z : p | u\}| > 2}} (|[\{p \in Z : p \mid u\}]^2|$$

$$- |\{p \in Z : p \mid u\}| + 1)$$

$$= |[\{p \in Z : p \mid m\}]^3| \left(\binom{3}{2} - 3 + 1 \right) - (|[\{p \in Z : p \mid m\}]^2|$$

$$- |\{p \in Z : p \mid m\}| + 1)$$

$$= 4 - (6 - 4 + 1)$$

$$= 1 \tag{13}$$

which completes the proof. $\qquad \square$

Lemma 4. *Let J be as in Theorem 1. Let A and A' be any sets of integers for which, for each $r \in \{1, 2, 3\}$,*

$$\sum_{m \in A} |[\{p \in J : p \mid m\}]^r| = \sum_{m \in A'} |[\{p \in J : p \mid m\}]^r| \tag{14}$$

and

$$\sum_{\substack{m \in A \\ |\{p \in J : p | m\}| > 3}} \frac{|[\{p \in J : p \mid m\}]^3|}{|\{u \in A : |\{p \in J : p \mid u\}| > 3\}|}$$

$$> \sum_{\substack{m \in A' \\ |\{p \in J : p | m\}| > 3}} \frac{|[\{p \in J : p \mid m\}]^3|}{|\{u \in A' : |\{p \in J : p \mid u\}| > 3\}|}.$$

Then

$$|\{m \in A : (m, \prod_{p \in J} p) \neq 1\}| < |\{m \in A' : (m, \prod_{p \in J} p) \neq 1\}|. \tag{15}$$

Proof. We have I and II, below.

I. For all $k \geq 3$,

$$\frac{\binom{k}{3} k}{\binom{k}{2}^2} = \frac{4(k-1)(k-2)}{6(k-1)^2}$$

$$= \frac{2(k-2)}{3(k-1)} \tag{16}$$

is an increasing function of k. We may substitute $k = |\{p \in J : p \mid m\}|$ for any $m \in [1, |I| - 1] \cup I$ such that $|\{p \in J : p \mid m\}| \geq 3$.

II. In (16), for $k = |\{p \in J : p \mid m\}|$, the quotients express

$$\frac{\binom{|\{p \in J : p | m\}|}{3}}{\binom{|\{p \in J : p | m\}|}{2}} \tag{17}$$

as a ratio of

$$\frac{\binom{|\{p \in J : p | m\}|}{2}}{|\{p \in J : p \mid m\}|}. \tag{18}$$

Combining I and II gives

$$\sum_{\substack{m \in A \\ |\{p \in J : p | m\}| \in \{1,2\}}} \frac{|[\{p \in J : p \mid m\}]^2|}{|\{p \in J : p \mid m\}|} > \sum_{\substack{m \in A' \\ |\{p \in J : p | m\}| \in \{1,2\}}} \frac{|[\{p \in J : p \mid m\}]^2|}{|\{p \in J : p \mid m\}|}.$$

Further,

III. (14) for $r = 1$ implies that

$$|\{m \in A' : (m, \prod_{p \in J} p) \neq 1\}| - |\{m \in A : (m, \prod_{p \in J} p) \neq 1\}|$$

$$= \sum_{m \in A} (|\{p \in J : p \mid m\}| - 1) - \sum_{m \in A'} (|\{p \in J : p \mid m\}| - 1) \tag{19}$$

with, for any $m \in A \cup A'$, $|\{p \in J : p \mid m\}| - 1 = 0$ when $|\{p \in J : p \mid m\}| = 1$.

For all $m \in A \cup A'$ for which $|\{p \in J : p \mid k\}| > 2$, and all k for which $|\{p \in J : p \mid k\}| = 2$ we have

$$\frac{|[\{p \in J : p \mid m\}]^2|}{|\{p \in J : p \mid m\}|} > \frac{|[\{p \in J : p \mid k\}]^2|}{|\{p \in J : p \mid k\}|}. \tag{20}$$

Therefore, combining III and (19) gives (15). $\qquad \square$

Lemma 5. *Let I and J be as in Theorem 1. Then*

$$|\{m \in I \cup Q'_{I,J,0,2} : (m, \prod_{p \in J} p) \neq 1\}| - |\{m \in [0, |I|-1] \cup Q_{I,J,0,2} : (m, \prod_{p \in J} p) \neq 1\}|$$

$$\leq \frac{|J|^2}{8}. \tag{21}$$

Proof. Let S_J be any subset of $[J]^3$ for which $|S_J|$ is an upper bound for the value on the left side of (21). We shall proceed to progressively justify choosing S_J so that $|S_J|$ is equal to the right side of (21).

For any distinct H and H' in $[J]^3$, the fact that zero divides $\prod_{p \in H \cup H'} p$ implies that

$$|\{1 \leq m \leq |I| : \prod_{p \in H} p \mid m\} \cap \{1 \leq m \leq |I| : \prod_{p \in H'} p \mid m\}|$$

$$\leq |\{m \in I : \prod_{p \in H} p \mid m\} \cap \{m \in I : \prod_{p \in H'} p \mid m\}|. \tag{22}$$

Further, for each $M \in [J]^3$ we have

$$|\{m \in [0, |I|-1] : \prod_{p \in M} p \mid m\}| \geq |\{m \in I : \prod_{p \in M} p \mid m\}|. \tag{23}$$

Therefore, combining all of Lemmas 2, 3 and 4 for $A = [0, |I|-1] \cup Q_{I,J,0,3}$ and $A' = I \cup Q'_{I,J,0,3}$, gives, for all subsets, N, of J,

$$|\{m \in I \cup Q'_{I,J,0,3} : (m, \prod_{p \in J} p) \neq 1\}| - |\{m \in [0, |I|-1] \cup Q_{I,J,0,3} : (m, \prod_{p \in N} p) \neq 1\}|$$

$$\leq |\{m \in I \cup Q'_{I,J,1,2} : (m, \prod_{p \in N} p) \neq 1\}|$$

$$- |\{m \in [1, |I|-1] \cup Q_{I,J,1,2} : (m, \prod_{p \in N} p) \neq 1\}| + 1$$

$$\leq |[N]^3| + 1. \tag{24}$$

Here, the first relation, when taken together with the second, is found by combining Lemma 2 and Lemma 4, itself combined with (22). The first term of the left side of the second relation is found by Lemma 2. The second term is found by Lemma 3 for its use of zero for the third parameter of $Q_{I,Z,0,3}$ and $Q'_{I,Z,0,3}$, where Z is as in Lemma 3 with $Z \in [J]^4$. We note that, in (24), for each $k \in \{0,1\}$ and $d \in \{2,3\}$, when the second parameter of $Q_{I,J,k,d}$ and $Q'_{I,J,k,d}$ is given as N instead of J, the result is unchanged. The second term on the right side of (24) is found by the fact that the left-hand endpoint, zero, of $[0, |I|-1]$ is not coprime to $\prod_{p \in N} p$.

For any $M \in [J]^3$ and $W \in [J]^4$ for which $M \subset W$, let $f_{S_J}(M, W)$ be equal to one when $M \in S_J$ and equal to zero when $M \notin S_J$.

By varying N, it follows through (24) that, for all $W \in [J]^4$ and any $D_W \in [W]^3$, the combination of Lemma 2 for $H = D_W$, and Lemma 3 for $Z = W$ justifies our choosing S_J so that $|[W]^3| - 1 = 3$ elements of $[W]^3$ are excluded from S_J. This is to say, the fact that the right sides of both (10) and (12) are equal to one implies that

I. for all $W \in [J]^4$ for which $f_{S_J}(D_W, W) = 1$, for each $M \in [W]^3 \setminus \{D_W\}$ our assumptions on S_J allow that $f_{S_J}(M, W) = 0$.

For any $H \in [J]^3$, any $F \in [J]^4$ for which $H \subset F$, and any $T \in [H]^2$, let $g_{S_J}(T, H, F) = f_{S_J}(H, F)$. Then for any distinct T and T' in $[J]^2$, if $g_{S_J}(T, H, F) = 1$ it follows by I that, for any $V \in [J]^4$ and $U \in [V]^3$ for which $T' \subset U$, we have

II. $g_{S_J}(T', U, V) = 1$ if and only if $T \cap T' = \emptyset$. Otherwise, when $T \cup T' \neq H$, for some distinct L and L' in $[T \cup T' \cup H]^3$ we would have $f_{S_J}(L, T \cup T' \cup H) = 1$ and $f_{S_J}(L', T \cup T' \cup H) = 1$, which is contrary to I, above.

For any subset K of $[J]^2$ for which $|K| = \lfloor |J|/2 \rfloor$, combining (24) and II (above) gives

$$|\{m \in I \cup Q'_{I,J,0,2} : (m, \prod_{p \in J} p) \neq 1\}| - |\{m \in [0, |I| - 1] \cup Q_{I,J,0,2} : (m, \prod_{p \in J} p) \neq 1\}|$$

$$\leq \binom{\lfloor |K|/2 \rfloor}{2} + 1$$

$$< \frac{|J|^2}{8} \tag{25}$$

where the final two parameters of $Q_{I,J,0,2}$ and $Q'_{I,J,0,2}$ are justified through the final relation of (24) (where we use $Q_{I,J,1,2}$ and $Q'_{I,J,1,2}$), combined with the fact that we have the second term on the right side of (24). We thereby have (21). $\qquad\square$

1.2. **Proof of Theorem 1.**

Proof. Let A and A' be as in Lemma 1. Let I and J be as in Theorem 1. Then, when A and A' are each intervals with $|A| = |A'|$,

$$\left| \sum_{m \in A} |[\{p \in J : p \mid m\}]^2| - \sum_{m \in A'} |[\{p \in J : p \mid m\}]^2| \right|$$

$$\leq |[J]^2|$$

$$= \frac{|J|(|J| - 1)}{2}. \tag{26}$$

Therefore, combining Lemma 5 and Lemma 1 for $A = I$ and $A' = [0, |I| - 1]$ gives

$$|\{m \in I : (m, \prod_{p \in J} p) \neq 1\}| - |\{m \in [0, |I| - 1] : (m, \prod_{p \in J} p) \neq 1\}|$$

$$\leq \frac{|J|^2}{8} + \frac{|J|(|J| - 1)}{2}$$

$$< \frac{|J|^2}{8} + \frac{|J|^2}{2}$$

$$= \frac{(2 + 8)|J|^2}{16}$$

$$= \frac{5|J|^2}{8}. \tag{27}$$

Since for each $p \in J$,

$$|\{m \in I : p \mid m\}| \leq |\{0 \leq m \leq |I| - 1 : p \mid m\}|,$$

in (27) we have tacitly substituted zero for the two that is the final parameter of $Q_{I,J,0,2}$ and $Q'_{I,J,0,2}$. Condition I in Definition 1.1 enables us to dispense with $Q_{I,J,0,0}$ and $Q'_{I,J,0,0}$, whence (2) follows. $\qquad\square$

2. The Folded the Number Scale

2.1. **Method outline.** Let y be any integer > 1. Our forthcoming exposition employs mirror symmetry in the context of the interval, $[1, y]$ and, for any integer n, the primes in $P(n)$ that divide y and finally the sieve of Eratosthenes, which in turn contextualises our 'fold' of the number scale. For the Goldbach conjecture our interest is in primes a and b in $[y, 2y]$ for which $a + b = 2y$. Here, one side, which we may call the lower side, of the

fold will be taken to be $[1, y]$; the other, the upper side, will be $[y, 2y]$. Indeed, the context of our use of y will imply a rephrasing of the Goldbach Conjecture, familiar as *every even number greater than two is the sum of two primes*, to the equivalent *every integer greater than three is the arithmetic mean of two primes*.

For i and r as in our Extended Introduction, consider the set we denoted by $Z_{i,r}$. For both the Goldbach and Twin primes conjectures, the fold of the number scale occurs at $r/2$. When we address the Goldbach conjecture we use $r = 2y$. For the Twin Primes conjecture we instead take $r = 2$ and $y = p_n^2$. Thus, in contrast to our treatment of the Goldbach conjecture, in which we only consider positive integers, by folding the number scale we tacitly map contiguous negative integers, descending from -1, onto positive integers up to p_n^2. Then, using the sieve of Eratosthenes to establish coprimality, when we reverse the signage of each of the negative integers the value $|\bigcup Z_{p_n^2, 2}|$ will be seen to be the number of primes, a, such that $p_n + 2 < a < p_n^2$, for which $a - 2$ is also prime.

In short, we shall consider the objects of interest in our folded number scale as being subdivided into two distributions. One such is the distribution of all integers that are co-prime to $\prod_{k=1}^{n} p_k$, where n is any integer. This distribution is the object that is folded, which is to say that when we apply functions to it, they are in the context of the mirror symmetry discussed above. The other is, in effect, an intermediate to the two sides of this folded distribution and one that tacitly invokes the Chinese Remainder theorem.

2.2. **Remark.** Let $n \geq 1$ and I be as in Theorem 1. Key to our method is the expression

$$\prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right) \prod_{k=2}^{n} \left(1 - \frac{1}{p_k - 1}\right). \tag{28}$$

We note that, for any $p \in P(n)$ and any r, when $p \mid r$ we have (by virtue of the mirror symmetry, about either r (as applies to our treatment of the Twin Primes conjecture) or $r/2$ (as applies to our treatment of the Goldbach conjecture), of the distribution of integer multiples of p),

$$\{m \in I : p \mid m\} = \{m \in I : p \mid m - r\}, \tag{29}$$

but when $p \nmid r$, (29) does not hold, bringing into play the expression

$$\prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right) \prod_{\substack{q \in P(n) \\ q \nmid r}} \left(1 - \frac{1}{q - 1}\right). \tag{30}$$

Hence the fact that, when $p = 2$,

$$\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p - 1}\right) = 0, \tag{31}$$

ultimately justifies our imposing the condition in our forthcoming exposition that r is even.

2.3. **Introduction to Theorems 2 and 3.** For i and r as in our Extended Introduction, recall that $Z_{i,r}$ is the set of all sets, G, for which, for each $s \in \{0, r\}$ and $p \in P(\pi(\sqrt{2i}))$, precisely one element of $\{\{1 \leq m \leq i : p \mid m - s\} : p \in P(\pi(\sqrt{2i})) \ \& \ s \in \{0, r\}\}$ is in G. Recall further that we noted that we shall ultimately use $Z_{i,r}$ in the form $\bigcup(\bigcup Z_{i,r})$. For any integer n, we shall show that the proof of the Goldbach conjecture resides fundamentally in taking $r = 2z_n$ where $p_n^2/2 < z_n < p_{n+1}^2/2$ and $i = z_n$, and the proof of the Twin Primes conjecture, on which we shall focus in this introduction (since the justification of Theorem 3 is essentially founded in the lemmas that prove Lemma 2), taking $r = 2$ and $i = p_n^2$. Theorem 1 enables us to use (1) as an upper bound for $|\bigcup(\bigcup Z_{i,r})|$.

In what follows, first we use the Euler totient. Second, we implicitly use, multiplicatively, the value $\prod_{p \in \{P(n) : p \nmid r\}} (p - 2)$, where n is any integer and r is even. This will enable future use, in the form of (28), of our resulting expression (by substituting $P(n) \setminus \{2\}$ for

the implicitly used $\{p \in P(n) : p \nmid r\}$ and using our forthcoming (33)). We note, incidentally, that since $\phi(p) - 1 = p - 2$ while $\phi(p) = p - 1$, (30) is equal to $(1/2) \prod_{k=2}^{n}(1 - 2/p_k)$ when no prime in $P(n) \setminus \{2\}$ divides r.

Key to our use of the bound variable m, for $\{m \in I : (m, \prod_{k=1}^{n} p_k) \neq 1\}$ and for $\{m \in I : (m - r, \prod_{k=1}^{n} p_k) \neq 1\}$, where I is any interval, is the value $m(m - r)$. Here, the primes in $P(n)$ that divide either m or $m - r$ are also those primes in $P(n)$ that divide $m(m - r)$. Substituting $r = 2$ will give our proof of the Twin Primes conjecture. Here,

$$\left\{ (m, m - 2) : p_n + 2 \leq m \leq \frac{p_n^2 + 1}{2} \ \& \ \left(m(m - 2), \prod_{k=1}^{n} p_k \right) = 1 \right\} \quad (32)$$

is a subset of the set of all $(p, p - 2)$ such that $p < p_n^2/2$, and p and $p - 2$ are together twin primes.

Theorem 2. *There are infinitely many pairs, (p, q), of primes such that $p + 2 = q$.*

Lemma 6. *Let s be even. Let J be any set of integers for which, for any $z \in J$, z is coprime to $\prod_{d \in J \setminus \{z\}} d$. Then*

$$\left| \left\{ 1 \leq m \leq \prod_{d \in J} d : \left(m(m - s), \prod_{d \in J} d \right) = 1 \right\} \right| = \phi\left(\prod_{d \in J} d\right) \prod_{\substack{d \in J \\ d \nmid s}} \left(1 - \frac{1}{d - 1} \right). \quad (33)$$

Proof. For each $q \in J$ for which $q \nmid s$ and all $i \in \mathbb{N}$,

$$|\{1 + i \leq m \leq q + i : (m(m - s), q) = 1\}| = |\{1 + i \leq m \leq q + i : (m, q) = 1\}| - 1. \quad (34)$$

Since the first term on the right side of (34) is equal to $\phi(q) = q - 1$ and the left side is equal to $\phi(q) - 1 = q - 2$, and since for any real x, $1 - 1/x = (x - 1)/x$, taking $x = q - 1$ for our final line we have

$$|\{1 \leq m \leq \prod_{d \in J} d : (m(m - s), \prod_{d \in J} d) = 1\}| = \phi\left(\prod_{d \in J} d\right) \prod_{\substack{d \in J \\ d \nmid s}} \frac{d - 2}{d - 1}$$

$$= \phi\left(\prod_{d \in J} d\right) \prod_{\substack{d \in J \\ d \nmid s}} \left(1 - \frac{1}{d - 1} \right) \quad (35)$$

which completes the proof. $\qquad \square$

Lemma 7. *Let s be even. Let J be as in Lemma 6 with the additional conditions that $2 \in J$ and more than one element of J divides s. Let t be any integer for which two is the sole element of J that divides t. Then*

$$|\{1 \leq m \leq \prod_{d \in J} d : (m(m - s), \prod_{d \in J} d) = 1\}|$$

$$= \prod_{d \in J} d \prod_{d \in J} \left(1 - \frac{1}{d} \right) \prod_{\substack{d \in J \setminus \{2\} \\ d \mid s}} \left(1 - \frac{1}{d - 1} \right) \quad (36)$$

$$> |\{1 \leq m \leq \prod_{d \in J} d : (m(m - t), \prod_{d \in J} d) = 1\}|$$

$$= \phi\left(\prod_{d \in J} d\right) \prod_{d \in J \setminus \{2\}} \left(1 - \frac{1}{d - 1} \right)$$

$$= \prod_{d \in J} d \prod_{d \in J} \left(1 - \frac{1}{d} \right) \prod_{d \in J \setminus \{2\}} \left(1 - \frac{1}{d - 1} \right). \quad (37)$$

Proof. We have $\{d \in J : d \nmid s\} \subset \{d \in J : d \nmid t\}$. Therefore

$$\prod_{\substack{d \in J \\ d \nmid s}} \left(1 - \frac{1}{d-1}\right) > \prod_{\substack{d \in J \\ d \nmid t}} \left(1 - \frac{1}{d-1}\right) \tag{38}$$

Since $\prod_{d \in J} d \prod_{d \in J}(1 - 1/d) = \phi(\prod_{d \in J} d)$, combining (38) and Lemma 6 for s as current gives (37). $\square$

2.4. **Definition.** For any even r, any $n \geq 1$, any set N of integers, any $p \in P(n)$, and any integer k define

$$V_{N,p,k} = \{h \in N : \ p \mid h - k\}. \tag{39}$$

For any set M of integers such that for any distinct elements p and q of $P(n)$ and any s and s' in $\{0, r\}$, $V_{M,p,s} \neq V_{M,q,s'}$, define

$$\begin{aligned}
R(M,&n,r) \\
&= \{F \subseteq \{V_{M,p,k} : p \in P(n) \ \& \ k \in \{0,r\}\} : \ \text{for each } q \in P(n) \text{ we have} \\
&\quad |\{V_{M,q,k} : k \in \{0,r\}\} \cap F| = 1\}. \tag{40}
\end{aligned}$$

Hence $R(M, n, r)$ is the set of all sets, G, such that each $T \in G$ satisfies $T = \{m \in M : q \mid m - j\}$ for some $q \in P(n)$ and $j \in \{0, r\}$, and for each $p \in P(n)$, precisely one element of $\{\{m \in M : p \mid m - s\} : s \in \{0, r\}\}$ is in G. We note that N satisfies all stated conditions on M when N is an interval with $|N| \geq 2p_n$. We note further that, for i, r and $Z_{i,r}$, as in our Extended Introduction, $Z_{i,r} = R([1,i], \pi(\sqrt{2i}), r)$. Finally,

$$\bigcup \left(\bigcup R(M,n,r)\right) = \ \{m \in M : (m(m-r), \prod_{k=1}^{n} p_k) \neq 1\}. \tag{41}$$

Lemma 8. *For any integer n, any even r, any j for which there exists $R([1,j], n, r)$, and any $T \in R([1,j], n, r)$ there exists an interval J_T for which $|J_T| = j$ and*

$$|\bigcup T| = |\{m \in J_T : (m, \prod_{k=1}^{n} p_k) \neq 1\}|. \tag{42}$$

Proof. For any $p \in P(n)$ and any $v > 0$, the v-th lowest element of $\bigcup(\{\{1 \leq m \leq j : p \mid m - s\} : s \in \{0, r\}\} \cap T)$ is equal to $(v-1)p + \min\{m \in J_T : p \mid m\} - \min J_T + 1$. The fact that we may substitute $v = 1$ implies that there exists an integer i_T for which $1 \leq i_T \leq \prod_{k=1}^{n} p_k$, such that

$$\begin{aligned}
\{\{(v - k)q + \min\{i_T \leq m \leq j + i_T - 1 : q \mid m\} \\
\quad - i_T + 1 : 1 \leq k \leq |\{i_T \leq m \leq j + i_T - 1 : q \mid m\}|\} : q \in P(n)\} \\
= T. \tag{43}
\end{aligned}$$

Thus $[i_T, j + i_T - 1]$ satisfies all stated conditions on J_T, giving (42). $\square$

Lemma 9. *For any n, let b be any positive integer that has no factors in $P(n)$. Then for any subset V of $P(n)$*

$$|\{1 \leq m \leq \prod_{q \in P(n) \cup \{b\}} q : m \in \mathbb{N} \, \& \, (m, \prod_{q \in V \cup \{b\}} q) \neq 1\}$$

$$\cap \, \{1 \leq m \leq \prod_{q \in P(n) \cup \{b\}} q : m \in \mathbb{N} \, \& \, (m, \prod_{q \in P(n) \backslash V} q) \neq 1\}|$$

$$= \left(1 - \prod_{q \in V \cup \{b\}} \left(1 - \frac{1}{q}\right)\right)$$

$$\times |\{1 \leq m \leq \prod_{q \in P(n) \cup \{b\}} q : m \in \mathbb{N} \, \& \, (m, \prod_{q \in P(n) \backslash V} q) \neq 1\}|. \quad (44)$$

Proof. For any distinct v and w in $[1, \prod_{q \in V \cup \{b\}} q]$ we have

$$\{v + m \prod_{q \in V \cup \{b\}} q : m \in \mathbb{N}\} \cap \{w + m \prod_{q \in V \cup \{b\}} q : m \in \mathbb{N}\} = \emptyset. \quad (45)$$

Therefore, the set whose cardinality is left side of (44) is otherwise written

$$\{v + m \prod_{q \in V \cup \{b\}} q : 1 \leq v \leq \prod_{q \in V \cup \{b\}} q \, \& \, m \in \mathbb{N} \, \& \, (v + m \prod_{q \in V \cup \{b\}} q, \prod_{q \in V \cup \{b\}} q) \neq 1\}$$

$$\cap \, \{1 \leq k \leq \prod_{q \in P(n) \cup \{b\}} q : k \in \mathbb{N} \, \& \, (k, \prod_{q \in P(n) \backslash V} q) \neq 1\}. \quad (46)$$

Here, we note that the bound variable $m \in \mathbb{N}$ can be replaced with $m \in [1, \prod_{p \in P(n) \backslash V} p]$ and the above set will be unchanged. Therefore, (45) combined with the fact that

I. for precisely one $1 - \prod_{q \in V \cup \{b\}}(1 - 1/q)$-th of all integers, i, that satisfy all stated conditions on v, we have $(i, \prod_{q \in V \cup \{b\}} q) \neq 1$;

all combined with the fact that

II. for

$$S_v = \{v + m \prod_{q \in V \cup \{b\}} q : m \in [1, \prod_{q \in P(n) \backslash V} q] \, \& \, (v + m \prod_{q \in V \cup \{b\}} q, \prod_{q \in P(n) \backslash V} q) \neq 1\},$$

we have $|S_v| = |S_w|$,

gives (44). $\qquad \square$

2.5. Definition. For any $n \geq 4$ and any set N of integers for which $R(N, n, t)$ exists, let $u_{N,n}$ be any rational number such that, for each $T \in R(N, n, t)$,

$$|\bigcup T| \leq |N| \left(1 - (1 - u_{N,n}) \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right)\right).$$

It follows through Lemma 8 for n as current, combined with Theorem 1 for $I = N$ and $|I| = j$ and $J = P(n)$ that, for each $c \in \{1, 2\}$, our assumptions on $u_{I,n}$ allow us that, when $j \geq 17$,

$$u_{I,n} = \frac{j \left(-\frac{1}{\log cj} + \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right)\right) + n + \frac{5n^2}{8}}{j \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right)}. \quad (47)$$

The term n on the numerator is attributable to the fact that the first n primes are not coprime to $\prod_{k=1}^{n} p_k$. Also, $x/\log x$, and thereby $j/\log cj$, is a lower bound for the prime counting function for all $x \geq 17$.[2]

2.6. **Remark.** In the ensuing exposition we occasionally introduce sets of a fixed cardinality. This is because of the self-explanatory nature of the written set.

Lemma 10. *Let n be any positive integer. Let r be even. Let I be any interval for which there exists $R(I, n, r)$. Let j be any element of $P(n) \setminus \{2\}$ for which $j \nmid r$. Let $X \in R(I, n, r)$. Let s be the element of $\{0, r\}$ for which $\{m \in I : j \mid m - s\} \in X$. Let $s' \in \{0, r\} \setminus \{s\}$. Finally, let*

$$f(X, j, n, r) = (X \setminus \{m \in I : j \mid m - s\}) \cup \{m \in I : j \mid m - s'\}.$$

Then

$$\left| \bigcup (X \cup f(X, j, n, r)) \right| \leq |I| \left(1 - (1 - 2u_{I,n}) \prod_{k=1}^{n} \left(1 - \frac{1}{p_k} \right) \right). \qquad (48)$$

Proof. First, we note that $f(X, j, n, r) \in R(I, n, r)$. Also,

$$\frac{\max\{|\bigcup Y| : Y \in R(I, n, r)\} - |I| \left(1 - \prod_{k=1}^{n} \left(1 - \frac{1}{p_k} \right) \right)}{|I| \prod_{k=1}^{n} \left(1 - \frac{1}{p_k} \right)}, \qquad (49)$$

satisfies all stated conditions on $u_{I,n}$. Here,

$$\left| \bigcup X \right| \leq |I| \left(1 - \prod_{q \in P(n) \cup \{1/u_{I,n}\}} \left(1 - \frac{1}{q} \right) \right). \qquad (50)$$

when $1/u_{I,n} \notin P(n)$.

Let $U_{X,j,n,r}$ be the set that satisfies either i or ii, below.

i. When $|\bigcup X| \geq \lfloor |I|(1 - \prod_{k=1}^{n}(1 - 1/p_k)) \rfloor$, then $U_{X,j,n,r}$ is the set of all $\lfloor |I|(1 - \prod_{k=1}^{n}(1 - 1/p_k)) \rfloor$-element subsets, M, of $\bigcup X$ and of $\bigcup f(X, j, n, r)$ for which

$$|M \setminus \{m \in I : \{p \in P(n) : p \mid m(m - r)\} = \{j\}\}|$$

$$= \max \left\{ |L \setminus \{m \in I : \{p \in P(n) : p \mid m(m - r)\} = \{j\}\}| : \right.$$

$$\left. L \subseteq \bigcup X \text{ or } L \subseteq \bigcup f(X, j, n, r) \And |L| = \left\lfloor |I| \left(1 - \prod_{k=1}^{n} \left(1 - \frac{1}{p_k} \right) \right) \right\rfloor \right\}. \qquad (51)$$

ii. When $|\bigcup X| < \lfloor |I|(1 - \prod_{k=1}^{n}(1 - 1/p_k)) \rfloor$, then $U_{X,j,n,r} = \{\bigcup X\}$.

Then we have two possible cases, Case A and Case B, depending on whether there exists $S \in U_{X,j,n,r}$ that satisfies (52), below.

Case A: We assume here that for some $S \in U_{X,j,n,r}$,

$$\left(\bigcup (X \cup \{\{m \in I : j \mid m - z\} : z \in \{0, r\}\}) \right) \setminus S$$

$$\subseteq \{m \in I : \{p \in P(n) : p \mid m(m - r)\} = \{j\}\}. \qquad (52)$$

Then

I. (48) follows through the fact that (49) satisfies all stated conditions on $u_{I,n}$. Here, the coefficient 'two' preceding $u_{I,n}$ on the right side of (48), is attributable to the fact that

12

$|\{\{m \in I : j \mid m-z\} : z \in \{0,r\}\}| = 2$ while $\bigcap \{\{m \in I : j \mid m-z\} : z \in \{0,r\}\} = \emptyset$.

Case B: We assume here that for all $S_j' \in U_{X,j,n,r}$,

$$\left(\bigcup (X \cup \{\{m \in I : j \mid m-z\} : z \in \{0,r\}\})\right) \setminus S_j'$$
$$\not\subseteq \bigcup \{\{m \in I : j \mid m-z\} : z \in \{0,r\}\}.$$

We have II, below.

II. For any $k \in P(n) \setminus \{j\}$ and any z and z' in $\{0,r\}$, the set $\{\{m \in I : k \mid m-z\}, \{m \in I : j \mid m-z'\}\}$ is a subset of some element of $R(I,n,r)$. Contrastingly, the assumption that $j \nmid r$ implies that, for all $M \in R(I,n,r)$, the set $\{\{m \in I : j \mid m-d\} : d \in \{0,r\}\}$ is not a subset of M.

Let $K = \max\{|\bigcup X|, |\bigcup f(X,j,n,r)|\}$. Let $y_X = s$ when $K = X$ and $y_X = s'$ when $K = f(X,j,n,r)$. Denote

$$d_{S_j'} = 2 - |(\bigcup K) \setminus (S_j' \cup \{m \in I : j \mid m - y_X\})|.$$

Since $\{m \in I : j \mid m - y_X\}$ is the sole element of K that is not a member of the sole element of $\{X, f(X,j,n,r)\} \setminus K$, we have the following. Combining all of I, II, and the fact that the right side of (49) satisfies all stated conditions on $u_{I,n}$, gives

$$|\bigcup (X \cup f(X,j,n,r))| \leq |I| \left(1 - (1 - d_{S_j'} u_{I,n}) \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right)\right). \qquad (53)$$

Combining I for Case A, and (53) for Case B, gives (48). $\qquad \square$

Lemma 11. *Let n, V and b all be as in Lemma 9. Let a be any positive integer less than b. Then for any a-element subset S of $[1,b]$,*

$$|(\{1 \leq m \leq \prod_{q \in P(n) \cup \{b\}} q : m \in \mathbb{N} \,\&\, (m, \prod_{q \in V \cup \{b\}} q) \neq 1\} \cup \{kwb : k \in \mathbb{N} \,\&\, w \in S\})$$
$$\cap \{1 \leq m \leq \prod_{q \in P(n) \cup \{b\}} q : m \in \mathbb{N} \,\&\, (m, \prod_{q \in P(n) \setminus V} q) = 1\}|$$
$$= \left(1 - \left(1 - \frac{a}{b}\right)\right.$$
$$\left. \times \prod_{q \in V} \left(1 - \frac{1}{q}\right)\right) |\{1 \leq m \leq \prod_{q \in P(n) \cup \{b\}} q : (m, \prod_{q \in P(n) \setminus V} q) = 1\}|. \qquad (54)$$

Proof. Combining Lemma 9 and (45) for $v, w \in S$ gives

$$|(\{i + m \prod_{q \in V \cup \{b\}} q : 1 \leq i \leq \prod_{q \in V \cup \{b\}} q \,\&\, m \in \mathbb{N} \,\&\, (i + m \prod_{q \in V \cup \{b\}} q, \prod_{q \in V \cup \{b\}} q) \neq 1\}$$
$$\cup \{kwb : k \in \mathbb{N} \,\&\, w \in S\})$$
$$\cap \{1 \leq m \leq \prod_{q \in P(n) \cup \{b\}} q : m \in \mathbb{N} \,\&\, (m, \prod_{q \in P(n) \setminus V} q) \neq 1\}|$$
$$= \left(1 - \left(1 - \frac{a}{b}\right) \prod_{q \in V} \left(1 - \frac{1}{q}\right)\right) |\{1 \leq m \leq \prod_{q \in P(n) \cup \{b\}} q : m \in \mathbb{N} \,\&\, (m, \prod_{q \in P(n) \setminus V} q) \neq 1\}|.$$
$$(55)$$

Therefore, the fact that $\phi(b \prod_{q \in V} q) = \prod_{q \in V \cup \{b\}} q \prod_{q \in V \cup \{b\}} (1 - 1/q)$ implies (54). $\qquad \square$

13

Lemma 12. *Let n, I and r be as in Lemma 10. Let d be any integer that has no factors in $P(n)$. Then for any integer d' less than $2d$ for which $u_{I,n} = d'/d$,*

$$\left| \left\{ m \in I : \left(m(m-r), \prod_{k=1}^{n} p_k \right) = 1 \right\} \right|$$

$$\geq |I|(1 - 2u_{I,n}) \prod_{k=1}^{n} \left(1 - \frac{1}{p_k} \right) \prod_{\substack{p \in P(n) \\ p|r}} \left(1 - \frac{1}{p-1} \right). \tag{56}$$

Proof. Through our forthcoming (59), we shall use Lemmas 7, 10 and 11 to show that

$$\left| \bigcup \left(\bigcup R(I,n,r) \right) \right| \leq |I| \left(1 - (1 - 2u_{I,n}) \prod_{k=1}^{n} \left(1 - \frac{1}{p_k} \right) \prod_{\substack{p \in P(n) \\ p|r}} \left(1 - \frac{1}{p-1} \right) \right), \tag{57}$$

from which (56) immediately follows. We begin by noting I, below.

I. For any $T \in R(I,n,r)$, let K_T be any subset of T. Then for each $M \in \{K_T, T\}$,

$$\left| \bigcup M \right| \leq |I| \left(\left(1 - (1 - u_{I,n}) \prod_{\substack{q \in P(n)| \ \{m \in I: \\ q|m-s\} \in K_T \\ \text{for some } s \in \{0,r\}}} \left(1 - \frac{1}{q} \right) \right) \right.$$

$$+ \left(1 - \prod_{\substack{q \in P(n)| \ \{m \in I: \\ q|m-s\} \in T \backslash K_T \\ \text{for some } s \in \{0,r\}}} \left(1 - \frac{1}{q} \right) \right) (1 - u_{I,n}) \prod_{\substack{q \in P(n)| \ \{m \in I: \\ q|m-s\} \in K_T \\ \text{for some } s \in \{0,r\}}} \left(1 - \frac{1}{q} \right) \right)$$

$$= |I| \left(1 - (1 - u_{I,n}) \prod_{k=1}^{n} \left(1 - \frac{1}{p_k} \right) \right). \tag{58}$$

We see here that, on specifying T, the right side of the first relation is a constant function of K_T.

Each term between the outer brackets is found through Lemma 11 for

$$V = \{q \in P(n) : \{m \in I : q \mid m - s\} \in K_T \text{ for some } s \in \{0, r\}\}$$

and $a = d'$ and $b = d$, combined with the fact that, for each $H \in \{V \cup \{b\}, P(n) \setminus V\}$, we have $\phi(\prod_{q \in H} q) / \prod_{q \in H} q = \prod_{q \in H} (1 - 1/q)$.
   Denote

$$g_{K_T} = \left(1 - \prod_{\substack{q \in P(n)| \ \{m \in I: \\ q|m-s\} \in T \backslash K_T \\ \text{for some } s \in \{0,r\}}} \left(1 - \frac{1}{q} \right) \right) (1 - u_{I,n}) \prod_{\substack{q \in P(n)| \ \{m \in I: \\ q|m-s\} \in K_T \\ \text{for some } s \in \{0,r\}}} \left(1 - \frac{1}{q} \right),$$

14

whence g_{K_T} is the second term between the outer brackets on the right side of the first relation of (58). Then, for each $p \in P(n)$,

$$g_{T \setminus \{\{m \in I : p \mid m-s\} : \atop s \in \{0,r\}\}} = \frac{1 - u_{I,n}}{p} \prod_{q \in P(n) \setminus \{p\}} \left(1 - \frac{1}{q}\right).$$

Further, $g_T = 0$.

When $p \nmid r$ we have $\{m \in I : q \mid m\} \cap \{m \in I : q \mid m-r\} = \emptyset$. We may show now that combining Lemma 10 for $j = p$ and I (above) gives (57). Here, for the element j of $P(n)$ as in Lemma 10 and all $M \in R(I, n, r)$, the set $H = \{\{m \in I : j \mid m - s\} : s \in \{0, r\}\}$ is not a subset of M, since $j \nmid r$. However, some element of H is in M. Hence, for our above reference to Lemma 10, our assumptions on $u_{I,n}$ allow us to assume that, for f as in Lemma 10, some $X \in R(I, n, r)$ and some $p \in P(n)$,

$$u_{I,n} \geq \frac{\max\{|\bigcup X|, |\bigcup f(X, p, n, r)|\} - |I| \left(1 - \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right)\right)}{|I| \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right)}.$$

Our assumptions on $u_{I,n}$ allow us, also, to assume that $u_{I,n} = 0$ when, for all $M \in R(I, n, r)$, we have $|\bigcup M| \leq |I|(1 - \prod_{k=1}^{n}(1 - 1/p_k))$. Thus combining Lemma 11 (for $a = 2d'$ and $b = d$ and $V = P(n) \setminus \{p\}$ and subsequently $V = \{p\}$) and (58) for each $T \in \{X, f(X, p, n, r)\}$, itself combined with the fact that, as already noted, $g_T = 0$, and finally with Lemma 7 for $J = P(n) \cup \{d\}$ (whence we find the products over k in (57)), gives (57). With respect to Lemma 7, we note here that, for any real x,

$$\left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{x-1}\right) = \frac{(x-1)\frac{x-2}{x-1}}{x}$$

$$= 1 - \frac{2}{x}.$$

We substitute first $x = d$ and subsequently, when combining 11 with Lemma 7, $x = d'/d$, and finally $a = 2d'$ and $b = d$ to reach (57).

Since

$$\left|\bigcup\left(\bigcup R(I, n, r)\right)\right| = |\{m \in I : (m(m - r), \prod_{k=1}^{n} p_k) \neq 1\}|, \tag{59}$$

we may replace $(m(m - r), \prod_{k=1}^{n} p_k) \neq 1$ with $(m(m - r), \prod_{k=1}^{n} p_k) = 1$ whence (57) implies (56). $\qquad\square$

2.7. Remark. Let $n > 0$. Since, for all m such that $0 < m < p_n^2$ and all $k \geq 1$, we have $\{p \in P(n) : p \mid m\} \neq \{p_{n+k}\}$, there is no composite in $\{1 \leq j \leq p_n^2 : (j, \prod_{k=1}^{n} p_k) = 1\}$ (hence our respecting the sieve of Eratosthenes, by employing $p_n^2/2$ in our specifying the cardinality of intervals with which we are working). Therefore, for all integers, z_n, such that $p_n^2/2 < z_n < p_{n+1}/2$, for $s(m) = 1$ and $s'(m) = m - 2z_n$, and for each $y \in \{s, s'\}$ and each $d \in \{1, 2\}$,

$$\left\{1 \leq m \leq dz_n : \left(my(m), \prod_{k=1}^{n} p_k\right) = 1\right\}$$

is a subset of the set of all the primes, p, less than or equal to dz_n.

2.8. Definition. For any $x > 1$, let

$$\mathrm{Hi}(x) = \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x}\right).$$

It is a result of Dusart [1] that $\pi(x) < \mathrm{Hi}(x)$ for all $x \geq 355,991$. We find $\pi(355991) = 30,456$. Also, for $x \geq 355,991$, $\mathrm{Hi}(x)\log(x)/x$ is decreasing to one.

15

2.9. **Definitions.** For any real r, let q_r be the highest x such that $\mathrm{Hi}(x) = \mathrm{Hi}(r) + 1$.
We have

$$\mathrm{Hi}(355991) = \frac{355991}{\log(355991)} \left(1 + \frac{1}{\ln(355991)} + \frac{2.51}{\log^2(355991)} \right)$$
$$\approx 30456.026 \tag{60}$$

and

$$q_{355991} \approx 356003.80$$
$$q_{356003.80} \approx 356016.58 \tag{61}$$

with $q_r - q_{r-1}$ being an increasing function of r, through the fact that $\log r$ is increasing.

For any positive integer k, let $v(k)$ be the real number such that $(v(1), v(2), v(3), \ldots)$ is the sequence of real numbers for which I to III, below, are all true:

I. $\{j : j = v(u) \text{ for some } 1 \le u \le 30456\} = P(30456)$;

II. for all $c \ge 30458$ we have $v(c) = q_{v(c-1)}$;

III. for t such that $\mathrm{Hi}(t) = 30456$, we have $v(30457) = q_t$, whence $t \approx 355990.667$.
Then through the previously cited result of Dusart, $v(c) < p_c$. Also, $v(c+1) - v(c)$ is an increasing function of c. We note that $p_{30456} = 355969 < q_{p_{30455}} \approx 355979.783$. Contrastingly $p_{30457} = 356023 > 356003.456 \approx q_{355990.667} \approx v(30457)$.

2.10. **Remark.** Let n be any integer. In the ensuing lemma, the use of $2((n+1)^2 - n^2)$ as a denominator is designed to invoke in a more congenial form the $5n^2/8$ that is found through Theorem 1 for $I = [1 + i, \lfloor p_n^2 \rfloor + i]$ for some integer i and $J = P(n)$.

Lemma 13. *Let* $n \ge 30457$. *Then*

$$\frac{\frac{v(n+1)^2}{\log(v(n+1)^2)} - \frac{v(n)^2}{\log v(n)^2}}{2((n+1)^2 - n^2)} = \log v(n) + O\left(\frac{\log^2 v(n)}{v(n)} \right).$$

Proof. We have

$$\frac{\frac{v(n+1)^2}{\log(v(n+1)^2)} - \frac{v(n)^2}{\log v(n)^2}}{2((n+1)^2 - n^2)} = \frac{\frac{v(n+1)^2}{\log(v(n+1)^2)} - \frac{v(n)^2}{\log v(n)^2}}{4n+2}$$

$$= \frac{\frac{v(n+1)^2}{\log(v(n+1)^2)} - \frac{v(n)^2}{\log v(n)^2}}{\frac{v(n)}{\log v(n)} + O\left(\frac{v(n)}{\log v(n)}\right)}$$

$$= \frac{v(n+1)^2 - v(n)^2}{v(n) + O\left(\frac{v(n)}{\log v(n)}\right)}$$

$$= \frac{2v(n)\log(v(n)) + \log^2 v(n)}{v(n) + O\left(\frac{v(n)}{\log v(n)}\right)}$$

$$= \log v(n) + \frac{\log^2 v(n)}{v(n)} + O\left(\frac{\log^2 v(n)}{v(n)}\right)$$

$$= \log v(n) + O\left(\frac{\log^2 v(n)^2}{v(n)}\right).$$

The second relation follows through the Prime Number theorem, whereby for real x, $\pi(x) \sim x/\log x$. In the fourth relation, the fact that $\mathrm{Hi}(x)\log(x)/x$ is decreasing to one implies that $v(n+1) - v(n) \sim \log v(n)$. More precisely, the fact that $\mathrm{Hi}(x)\log(x)/x$ is strictly decreasing to one implies that $(m_x - x)/(q_x - x)$ is strictly decreasing to one, where m_x is the highest j such that $j/\log j = 1 + x/\log x$. Thus we substitute

$$(v(n) + \log v(n))^2 - v(n)^2 = 2v(n)\log(v(n)) + \log^2 v(n) \tag{62}$$

for $v(n+1)^2 - v(n)^2$. This completes the proof. $\qquad\square$

2.11. **Proof of Theorem 2.**

Proof. In our proof we tacitly use i and r as in our Introduction to Theorems 2 and 3, with $i = \pi(\sqrt{2i})$ and $r = 2$. More precisely, for any integer n, we use Lemma 12 in conjunction with the fact that, for any $m \leq p_n^2$, when m and $m - 2$, and thereby $m(m-2)$, are each coprime to $\prod_{k=1}^n p_k$, we see that m and $m - 2$ are together a prime pair.

The fact that, for any m, $(m(m-2), \prod_{k=1}^n p_k) = 1$ if and only if both $(m, \prod_{k=1}^n p_k) = 1$ and $(m - 2, \prod_{k=1}^n p_k) = 1$, implies through Remark 2.7 first that

$$\left\{ (m, m-2) : 1 < m < p_n^2 \ \& \ \left(m(m-2), \prod_{k=1}^n p_k \right) = 1 \right\}$$
$$= \left\{ (p, p-2) : p_n + 2 < p < p_n^2 \ \& \ p \text{ prime} \ \& \ p - 2 \text{ prime} \right\} \tag{63}$$

and, thereby, second that

$$\left| \left\{ 1 < m \leq p_n^2 : \left(m(m-2), \prod_{k=2}^n p_k \right) = 1 \right\} \right|$$
$$= \left| \left\{ (p, p-2) : p_n + 2 < p \leq p_n^2 \ \& \ p \text{ prime} \ \& \ p - 2 \text{ prime} \right\} \right|. \tag{64}$$

The Mertens theorem [3] is given by

$$\lim_{n \to \infty} \log(p_n) \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right) = e^{-\gamma},$$

where e is the Euler number and γ is the Euler-Mascheroni constant. Since $\log p_n^2 = 2 \log p_n$, it follows through the Prime Number theorem that

$$\lim_{n \to \infty} \left(\sum_{k=1}^{\pi(p_n^2)} \frac{(\log(p_k) - \log p_{k-1})}{\pi(p_n^2)}\right) \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right) = 2e^{-\gamma}$$

$$\approx 1.12292, \qquad (65)$$

approximating upwards, so we may impose the assumption on $u_{[1,p_n^2],n}$ that $u_{[1,p_n^2],n} \sim 1 - 1/1.12292 < 1/2$. Combining Lemma 13, noting that $v(n) < p_n$ for $n \geq 30457$, and the Mertens theorem, all combined in turn with Lemma 12 for $I = [1 + i, p_n^2 + i]$, where i is any integer, themselves combined with (47) for $j = p_n^2$, $c = 2$ and n as current, thereby gives

$$|\{(p, p-2) : p_n + 2 < p \leq p_n^2 \ \& \ p \text{ prime } \& \ p - 2 \text{ prime}\}| \to \infty. \qquad (66)$$

Since the set whose cardinality is on the left side of (66) is a subset of all pairs, $(p, p-2)$, such that p is prime and less than p_n^2 and for which $p - 2$ is also prime, through Remark 2.7 for $d = 2$, the proof is complete. $\qquad \square$

Theorem 3. *Let $n > 4$. Then for any integer z_n for which $p_n^2/2 < z_n < p_{n+1}^2/2$ and any s_n for which*

$$s_n \geq -\frac{\pi(z_n)}{z_n} + \prod_{k=1}^{n} \left(1 - \frac{1}{p_k}\right),$$

and

$$z_n - \left(1 - \frac{2\left(z_n s_n + n + \frac{5n^2}{8}\right)}{z_n \prod_{k=1}^{n}\left(1 - \frac{1}{p_k}\right)}\right) i \prod_{k=1}^{n}\left(1 - \frac{1}{p_k}\right) \prod_{k=1}^{n}\left(1 - \frac{1}{p_k - 1}\right) \geq 1, \qquad (67)$$

there exist primes p and q that satisfy the Goldbach equation $p + q = 2z_n$.

Proof. We introduce our proof with I, below.

I. Suppose that w is an integer greater than $p_{30456}^2/2$. Then the number of ways of writing w as the arithmetic mean of two primes is greater than or equal to the cardinality of

$$L = \left\{1 \leq m \leq w : \left(m(2w - m), \prod_{k=1}^{\pi(\sqrt{2w})} p_k\right) = 1\right\}.$$

Here, for any two positive integers p and q for which $p < q$ and w is the arithmetic mean of p and q, and pq is coprime to $\prod_{k=1}^{\pi(\sqrt{2w})} p_k$, using the sieve of Eratosthenes we see that p and q are both prime; also, p is in L with $2w - p = q$; and for any two primes, a and b, the average of which is w, $a + b = 2w$ satisfies the Goldbach equation. We note here that $p(2w - p)$ is coprime to $\prod_{k=1}^{n} p_k$ if and only if both p and $2w - p$ are coprime to $\prod_{k=1}^{n} p_k$ and we substitute $p = m$ and $q = 2w - p$ where m is the bound variable used for L.

Substituting $w = z_n$, Theorem 3 follows by I combined with the fact that, for any even r,

$$\left| \left\{ 1 \le m \le z_n : \left(m(m-r), \prod_{k=1}^{n} p_k \right) = 1 \right\} \right|$$

$$\ge z_n - \frac{2 \left(z_n s_n + n + \frac{5n^2}{8} \right)}{\prod_{k=1}^{n} \left(1 - \frac{1}{p_k} \right)} \prod_{k=1}^{n} \left(1 - \frac{1}{p_k} \right) \prod_{k=2}^{n} \left(1 - \frac{1}{p_k - 1} \right). \quad (68)$$

The right side of (68) is found by Lemma 8 for $j = z_n$ combined with Lemma 12 for $I = [1+c, z_n+c]$ where c is any integer, by which we have (47), and we substitute

$$u_{I,n} = \frac{z_n s_n + n + \frac{5n^2}{8}}{z_n \prod_{k=1}^{n} \left(1 - \frac{1}{p_n} \right)}. \quad (69)$$

Here, the $5n^2/8$ term on the numerator is found through Theorem 1 for I as current and $J = P(n)$. The n term is found by the fact that the first n primes are not co-prime to $\prod_{k=1}^{n} p_k$ combined with the fact that $\pi(v(n)^2/2) \le v(n)^2/(2 \log v(n)^2/2) < z_n / \log v(n)^2$. The products over k in (68) are found through Lemma 7, specifically for the inequality in (37), combined with Lemma 12 for $r = 2z_n$, wherein we may take it that two is the sole element of $P(n)$ that divides r.

II. The sieve of Eratosthenes justifies, through Remark 2.7 for $d = 1$, our assuming that, for I (above) the bound variable m appearing on the left side of (68) may be taken to be equal to p, with $q = 2w - p$ and $r = 2w$. Thus

$$\left\{ (m, 2z_n - m) : 1 < m < z_n^2 \ \& \ \left(m(2z_n - m), \prod_{k=1}^{n} p_k \right) = 1 \right\}$$

$$= \{ (p,q) : p_n < p \le z_n \le q < 2z_n \ \& \ p \text{ prime} \ \& \ q \text{ prime} \ \& \ p + q = 2z_n \}. \quad (70)$$

Also

$$\left| \left\{ 1 \le m \le z_n : \left(m(m - 2z_n), \prod_{k=2}^{n} p_k \right) = 1 \right\} \right|$$

$$= |\{ (p,q) : p_n < p \le z_n < q < 2z_n \ \& \ p \text{ prime} \ \& \ q \text{ prime} \ \& \ p + q = 2z_n \}|. \quad (71)$$

In (70), for the bound variable m, we have changed $(m(m - 2z_n), \prod_{k=1}^{n} p_k) = 1$, as in the preceding exposition for $r = 2z_n$, to $(m(2z_j - m), \prod_{k=1}^{j} p_k) = 1$. The set whose cardinality is the right side of (71) is a subset of the set of all pairs, (p, q), of primes such that $p_n < p \le z_n$ for which $z_j = (p + q)/2$.

Combining (68) and (71) gives Theorem 3. $\qquad \square$

Theorem 4. *If the Riemann hypothesis is true, the Goldbach conjecture is true.*

2.12. **Definition.** For any x, let $\theta(x) = \sum_{j=1}^{\pi(x)} \log p_j$.

Lemma 14. *For all $s > 30457$,*

$$\frac{\log p_s}{\log \theta p_m} < 1.007662 \quad (72)$$

Proof. For any positive integer k, let $j(k)$ be the highest y such that $y/\log y = k$. Then $j(s) > p_s$. Recall that the lowest k for which $v(k) \ne p_k$ is 30457, and that $v(s) < p_s$. We have I, below.

I. For all $n \geq 30457$, the ratios $v(n)/j(n)$, $(v(n+1) - v(n))/(j(n+1) - j(n))$ and $\log j(n+1)/(j(n+1) - j(n))$ are all strictly increasing to one.

We have

$$
\begin{aligned}
1.084175 &\approx \frac{j(30458) - j(30457)}{\log v(30458)} \\
&< \frac{j(30457)}{\theta(p_{30456}) + \log v(30457)} \\
&\approx 1.102878.
\end{aligned}
\tag{73}
$$

Since, for each $t \in \{j, v\}$, $t(s) = \sum_{k=1}^{s}(t(k) - t(k-1))$, combining I and (73) gives

$$
\frac{p_s}{\theta(p_s)} < \frac{p_s}{\sum_{k=1}^{s} \log v(k)}
\tag{74}
$$

$$
< \frac{j(s)}{\sum_{k=1}^{s} \log v(k)}
\tag{75}
$$

$$
< \frac{j(30457)}{\theta(p_{30456}) + \log v(30457)}.
\tag{76}
$$

Here, $j(30457) \approx 392277.800878$, $j(30458) \approx 392291.764798$, $v(30457) \approx 356003.455995$ and $\theta(p_{30456}) \approx 355685.674752$. Therefore,

$$
\begin{aligned}
\frac{\log p_s}{\log \theta(p_m)} &< \frac{\log j(30457)}{\log(\theta(p_{30456}) + \log v(30457))} \\
&\approx 1.007662
\end{aligned}
\tag{77}
$$

approximating upwards, whence we have (72). $\qquad\square$

2.13. **Proof of Theorem 4.**

Proof. Our proof may take I in the proof of Theorem 3 for its introduction, with the following added. It is a result of Nicolas [4] that if, for all $k \geq 2$,

$$
\frac{N_k}{\phi(N_k) \log \log N_k} > e^{\gamma}
\tag{78}
$$

where $N_k = \prod_{j=1}^{k} p_j$ and γ is the Euler-mascheroni constant, the Riemann hypothesis is true. Therefore, if the Riemann hypothesis is true, we have, for all n,

$$
\frac{1}{\log\left(\log \prod_{j=1}^{n} p_j\right) \prod_{j=1}^{n}\left(1 - \frac{1}{p_j}\right)} = \frac{1}{\log\left(\sum_{j=1}^{n} \log p_j\right) \prod_{j=1}^{n}\left(1 - \frac{1}{p_j}\right)}
$$
$$
> e^{\gamma}.
\tag{79}
$$

Let t_x be any real number for which for all $y > x$ we have $t_x > \log(\sqrt{y})/\log \theta(\sqrt{y})$. Then since (as mentioned earlier), for all $x \geq 17$, $\pi(x) < x/\log x$, it follows by (79) that

$$
\pi(y) > \frac{ye^{\gamma}}{2t_x} \prod_{j=1}^{\pi(y)}\left(1 - \frac{1}{p_j}\right).
\tag{80}
$$

Here, the coefficient 'two' in the denominator of the first quotient, is found by the fact that $\log y = 2 \log \sqrt{y}$. Hence (80) is found by a known lower bound on the prime count in the way that, through the Mertens theorem combined with the Prime Number theorem, whereby $\theta x \sim x$ and $\pi(x) \sim x/\log x$, we also have

$$
\lim_{x \to \infty} \frac{\pi(x)}{\frac{xe^{\gamma}}{2} \prod_{j=1}^{\pi(x)}\left(1 - \frac{1}{p_j}\right)} = 1.
$$

For n such that $p_n \geq x$, the fact that $v(n) \leq p_n$ thereby implies that

$$\pi(v(n)^2) > \frac{v(n)^2 e^\gamma}{2t_x} \prod_{j=1}^{n} \left(1 - \frac{1}{v(j)}\right). \tag{81}$$

Using, for convenience, $v(n)^2/(2\log v(n)^2)$ as a lower bound on $\pi(v(n)^2/2)$, we thereby have

$$\frac{v(n)^2 \prod_{j=1}^{n} \left(1 - \frac{1}{v(j)}\right)}{2} - \pi\left(\frac{v(n)^2}{2}\right) < \frac{1}{2}v(n)^2 \left(1 - \frac{e^\gamma}{2t_x}\right) \prod_{j=1}^{n} \left(1 - \frac{1}{v(j)}\right) - n \tag{82}$$

where the $-n$ term on the right side is found by the fact that the first n primes are not coprime to $\prod_{j=1}^{n} p_j$. Through Lemma 14, we now substitute $t_x = 1.007662$, with $x = v(30457)^2$, thence to combine (82) and Theorem 1 for $I = [1 + i, \lfloor v(n)^2/2 \rfloor + i]$ where i is any integer, and $J = P(n)$ to give the following. First, for Lemma 12, we may now substitute

$$u_{[1,\lfloor v(n)^2/2 \rfloor],n} = \frac{\frac{1}{2}v(n)^2 \left(1 - \frac{e^\gamma}{2 \times 1.007662}\right) + \frac{5n^2}{8}}{\frac{1}{2}v(n)^2 \prod_{j=1}^{30457} \left(1 - \frac{1}{v(j)}\right)}. \tag{83}$$

Here, the $-n$ term that appears in (82) becomes superfluous to consideration, since $u_{[1,\lfloor v(n)^2/2 \rfloor],n}$ is an upper bound on the number of integers that are not coprime to $\prod_{j=1}^{n} p_k$. The $5n^2/8$ is found through Theorem 1. Second, on the above substitution we have, for r as in Lemma 12,

$$\frac{v(30457)^2}{2} \left(1 - 2u_{[1,\lfloor v(30457)^2/2 \rfloor],n}\right) \prod_{j=1}^{30457} \left(1 - \frac{1}{v(j)}\right) \prod_{\substack{1 \leq j \leq n \\ p_j \nmid r}}^{30457} \left(1 - \frac{1}{v(j) - 1}\right)$$

$$\geq \frac{v(30457)^2}{2} \left(1 - 2u_{[1,\lfloor v(30457)^2/2 \rfloor],n}\right) \prod_{j=1}^{30457} \left(1 - \frac{1}{v(j)}\right) \prod_{j=2}^{30457} \left(1 - \frac{1}{v(j) - 1}\right)$$

$$\approx 56,611,211.95. \tag{84}$$

The first relation follows through Lemma 7, specifically for the inequality in (37), for $P(30457) = J$. Then, for z_n as in the Proof of Theorem 3 for n as current, the facts that we may increase z_n by increments of one and that $z_n > p_n > v(n)$ and that $(1 - 1/v(n))(1 - 1/(v(n) - 1))$ is increasing, all combined with Lemma 13, implies the following. Combining (84) with Theorem 3 for $s_n = 1 - e^\gamma/(2 \times 1.007662) - n/z_n$, gives I, below, for $i = 2z_n$.

I. If the Riemann hypothesis is true, then all integers, i, such that $\pi(\sqrt{i}) \geq 30457$ are the sum of two primes.

All even numbers, up to values greater than $p_{30457}^2 = 126,752,376,529$, which is less than 10^{12}, have been shown to be the sum of two primes by, for example, Richstein [5], who verified the Goldbach conjecture for all even numbers up to 4×10^{14}. Since Theorem 4 is, subject to the Riemann hypothesis being true, a statement of the Goldbach conjecture, it follows by I that if the Riemann hypothesis is true, the Goldbach conjecture is also true, which completes the proof. $\qquad\square$

2.14. **Conclusion.** We have shown through a single basic method that the Twin Primes is true, and the Goldbach conjecture is true subject to the Riemann hypothesis being true.

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