Prediction-Correction Algorithms for Time-Varying Constrained Optimization

Andrea Simonetto and Emiliano Dall'Anese

Abstract—This paper develops online algorithms to track solutions of time-varying constrained optimization problems. Particularly, resembling workhorse Kalman filtering-based approaches for dynamical systems, the proposed methods involve prediction-correction steps to provably track the trajectory of the optimal solutions of time-varying convex problems. The merits of existing prediction-correction methods have been shown for unconstrained problems and for setups where computing the inverse of the Hessian of the cost function is computationally affordable. This paper addresses the limitations of existing methods by tackling constrained problems and by designing first-order prediction steps that rely on the Hessian of the cost function (and do not require the computation of its inverse). The proposed methods can track saddle-point solutions of time-varying minimax problems (e.g., optimizers of given double-smoothed Lagrangian functions), and are shown to improve the convergence speed of existing prediction-correction methods when applied to unconstrained problems. Numerical simulations corroborate the analytical results and showcase performance and benefits of the proposed algorithms.

Index Terms—Time-varying optimization, non-stationary optimization, parametric programming, prediction-correction methods, minimax problems.

I. INTRODUCTION

We consider constrained optimization problems that vary continuously in time. We outline the main ideas by first focusing on problems with a time-varying objective function. Consider then the following optimization problem:

$$\boldsymbol{x}^*(t) := \operatorname*{argmin}_{\boldsymbol{x} \in X} f(\boldsymbol{x};t), \quad \text{for } t \geqslant 0 , \qquad (1)$$

where $X \subseteq \mathbb{R}^n$ is a convex set; $t \in \mathbb{R}_+$ is non-negative, continuous, and it is used to index time; and, $f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is a *smooth strongly convex function*. The goal is to find (and track) the solution $\boldsymbol{x}^*(t)$ of (1) for each time t – hereafter referred to as the optimal solution *trajectory*.

Problem (1) is a generalization of traditional time-invariant (i.e., static) problems, and can naturally model: a) control problems where one seeks to generate a control action depending on a (parametric) varying optimization problem [1]–[3]; b) signal processing problems [4], where states of a dynamical process are estimated online based on time-varying observations – including time-varying compressive sensing settings [5]–[8]; and, c) inferential problems on dynamic networks [9]. Additional application domains include robotics [10]–[12], smart grids [13], [14], economics [15], and real-time magnetic resonance imaging (MRI) [16].

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The problem (1) might be solved in a centralized setting based on a continuous time platform [17]–[22]; however, motivated by the discrete nature of communication in networks (where our algorithms could be implemented), we focus on discrete time settings and we use sampling arguments to reinterpret (1) as a sequence of time-invariant problems. In particular, upon sampling the objective functions $f(\boldsymbol{x};t)$ at time instants t_k, $k = 0, 1, 2, \ldots$, where the sampling period $h := t_k - t_{k-1}$ can be chosen arbitrarily small, one can solve the sequence of time-invariant problems

$$\boldsymbol{x}^*(t_k) := \operatorname*{argmin}_{\boldsymbol{x} \in X} f(\boldsymbol{x};t_k), \quad k \in \mathbb{N}. \qquad (2)$$

By decreasing h, an arbitrary accuracy may be achieved when approximating problem (1) with (2). However, solving (2) for each sampling time t_k may not be computationally affordable in many application domains, even for moderate-size problems.

Similar arguments apply to the following minimax program:

$$\min_{\boldsymbol{x} \in X} \max_{\boldsymbol{\mu} \in V} K(\boldsymbol{x}, \boldsymbol{\mu};t), \quad \text{for } t \geqslant 0 , \qquad (3)$$

where $X \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^q$ are convex sets, and function $K : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}_+ \to \mathbb{R}$ is *smooth strongly convex - strongly concave function*. Function $K(\boldsymbol{x}, \boldsymbol{\mu};t)$ coincides with the regularized Lagrangian associated with a time-varying constrained optimization program (with time-varying inequality and equality constrains), with a double-smoothing regulatization [23], [24]. However, it is worth pointing out that problem (3) is relevant also in robust optimization tasks [25].

Focusing on unconstrained optimization problems, the works in [26], [27] developed a prediction-correction method to find and track the solution trajectory $\boldsymbol{x}^*(t)$ up to a bounded asymptotical error, starting from an arbitrary guess $\boldsymbol{x}_0$. This methodology arises from non-stationary optimization [28], [29], parametric programming [15], [30]–[32], and continuation methods in numerical mathematics [33]. It also resembles path-following methods in interior point solvers [34]. This paper significantly broadens the method of [26] by offering the following contributions.

(i) We develop prediction-correction methods to track the solutions of the time-varying *constrained* problems (1) and (3);

(ii) We develop first-order algorithms that do not involve the computation of the inverse of the Hessian of the cost function, as required in [26]; the proposed prediction-correction method is computationally lighter, as it requires only matrix-vector multiplications. Further, we offer a trade-off between tracking capabilities and computational effort; and,

(iii) We show that the proposed prediction-correction algorithm improves on the method in [26] when applied to unconstrained optimization problems; particularly, it exhibits enhanced local convergence properties by relying on Newton-like prediction steps.

The design and analysis of proposed prediction-correction methods are grounded on the theory of generalized equations and implicit function theorems [31].

Organization. In Section II, we describe the prediction-correction methodology for constrained time-varying optimization problems. The special case of unconstrained case is developed in Section III. Convergence analysis is discussed in Section IV, while the extension to minimax problems in Section V. Numerical examples are displayed in Section VI, and in Section VII, we draw our conclusions. The proofs of all the propositions and theorems are given in the appendices.

Notation. Vectors are written as $\boldsymbol{x} \in \mathbb{R}^n$ and matrices as $\boldsymbol{A} \in \mathbb{R}^{n \times n}$. We use $\|\cdot\|$ to denote the Euclidean norm in the vector space, and the respective induced norms for matrices and tensors. The gradient of the function $f(\boldsymbol{x}; t)$ with respect to $\boldsymbol{x}$ at the point $(\boldsymbol{x}, t)$ is denoted as $\nabla_{\boldsymbol{x}} f(\boldsymbol{x}; t) \in \mathbb{R}^n$, while the partial derivative of the same function with respect to (w.r.t.) t at $(\boldsymbol{x}, t)$ is written as $\nabla_t f(\boldsymbol{x}; t) \in \mathbb{R}$. Similarly, the notation $\nabla_{\boldsymbol{xx}} f(\boldsymbol{x}; t) \in \mathbb{R}^{n \times n}$ denotes the Hessian of $f(\boldsymbol{x}; t)$ w.r.t. $\boldsymbol{x}$ at $(\boldsymbol{x}, t)$, whereas $\nabla_{t\boldsymbol{x}} f(\boldsymbol{x}; t) \in \mathbb{R}^n$ denotes the partial derivative of the gradient of $f(\boldsymbol{x}; t)$ w.r.t. the time t at $(\boldsymbol{x}, t)$, i.e. the mixed first-order partial derivative vector of the objective. The tensor $\nabla_{\boldsymbol{xxx}} f(\boldsymbol{x}; t) \in \mathbb{R}^{n \times n \times n}$ indicates the third derivative of $f(\boldsymbol{x}; t)$ w.r.t. $\boldsymbol{x}$ at $(\boldsymbol{x}, t)$, the matrix $\nabla_{\boldsymbol{x}t\boldsymbol{x}} f(\boldsymbol{x}; t) = \nabla_{t\boldsymbol{xx}} f(\boldsymbol{x}; t) \in \mathbb{R}^{n \times n}$ indicates the time derivative of the Hessian of $f(\boldsymbol{x}; t)$ w.r.t. the time t at $(\boldsymbol{x}, t)$, and the vector $\nabla_{tt\boldsymbol{x}} f(\boldsymbol{x}; t) \in \mathbb{R}^n$ indicates the second derivative in time of the gradient of $f(\boldsymbol{x}; t)$ w.r.t. the time t at $(\boldsymbol{x}, t)$.

II. PREDICTION-CORRECTION STRATEGY

In this section, we first focus on problem (1) and design a prediction-correction algorithm to track the (unique) trajectory of the optimal solution. As explained in the introduction, consider sampling (1) at times t_k, $k \in \mathbb{N}$, and constructing a sequence of time-invariant problems (2). In lieu of solving (2) at each time step, the goal of the prediction-correction strategy is to determine an approximate optimizer for (1) at t_{k+1} in a computationally affordable way, given the current approximate optimizer at t_k.

A. Prediction

Suppose that $\boldsymbol{x}_k$ is an approximate solution of (2) at time t_k. Given $\boldsymbol{x}_k$, the prediction step seeks an approximate optimizer for (1) at t_{k+1}, given the only information available at time t_k. Let $\boldsymbol{x}_{k+1|k}$ denote the output of the prediction step, which is computed as explained next.

Notice first that solving the time-invariant problem (2) associated with time t_k is equivalent to solving the generalized equation

$$\nabla_{\boldsymbol{x}} f(\boldsymbol{x}^*(t_k); t_k) + N_X(\boldsymbol{x}^*(t_k)) \ni \boldsymbol{0} \quad (4)$$

where $N_X : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the normal cone operator, while $\boldsymbol{x}^*(t_k)$ is the optimizer of (2) at t_k. With (4) in place, the prediction step seeks the solution of the following perturbed generalized equation

$$
\begin{aligned}
\nabla_{\boldsymbol{x}} f(\boldsymbol{x}_{k+1|k}; t_{k+1}) + N_X(\boldsymbol{x}_{k+1|k}) \approx \\
\nabla_{\boldsymbol{x}} f(\boldsymbol{x}_k; t_k) + \nabla_{\boldsymbol{xx}} f(\boldsymbol{x}_k; t_k)(\boldsymbol{x}_{k+1|k} - \boldsymbol{x}_k) \\
+ h \nabla_{t\boldsymbol{x}} f(\boldsymbol{x}_k; t_k) + N_X(\boldsymbol{x}_{k+1|k}) \ni \boldsymbol{0}. \quad (5)
\end{aligned}
$$

That is, the prediction step produces a solution that is optimal w.r.t. a perturbed (first-order) version of the original generalized equation (4). We can now replace (5) with the following equivalent formulation

$$
\begin{aligned}
\boldsymbol{x}_{k+1|k} = \operatorname*{argmin}_{\boldsymbol{x} \in X} \frac{1}{2} \boldsymbol{x}^\mathsf{T} \nabla_{\boldsymbol{xx}} f(\boldsymbol{x}_k; t_k) \boldsymbol{x} \\
+ (\nabla_{\boldsymbol{x}} f(\boldsymbol{x}_k; t_k) + h \nabla_{t\boldsymbol{x}} f(\boldsymbol{x}_k; t_k) - \nabla_{\boldsymbol{xx}} f(\boldsymbol{x}_k; t_k) \boldsymbol{x}_k)^\mathsf{T} \boldsymbol{x}.
\end{aligned}
\quad (6)
$$

In lieu of seeking an exact solution of (6) (which is a constrained optimization problem with quadratic cost), consider the less computational demanding task of finding an approximate solution of (6) by computing a number of projected gradient descent steps – the first key step towards a *first-order* prediction-correction method. Particularly, let $\widehat{\boldsymbol{x}}^0$ be a dummy variable initialized as $\widehat{\boldsymbol{x}}^0 = \boldsymbol{x}_k$; then, the following steps are performed:

$$
\begin{aligned}
\widehat{\boldsymbol{x}}^{p+1} = \mathbb{P}_X[\widehat{\boldsymbol{x}}^p - \alpha(\nabla_{\boldsymbol{xx}} f(\boldsymbol{x}_k; t_k)(\widehat{\boldsymbol{x}}^p - \boldsymbol{x}_k) \\
+ h \nabla_{t\boldsymbol{x}} f(\boldsymbol{x}_k; t_k) + \nabla_{\boldsymbol{x}} f(\boldsymbol{x}_k; t_k))], \quad (7)
\end{aligned}
$$

for $p = 0, 1, \ldots, P-1$, where P is a pre-determined number of gradient steps, $\alpha > 0$ is the stepsize, and $\mathbb{P}_X$ is the projection operator over the convex set X. Once P steps are performed, $\tilde{\boldsymbol{x}}_{k+1|k}$ is set to:

$$\tilde{\boldsymbol{x}}_{k+1|k} = \widehat{\boldsymbol{x}}^P. \quad (8)$$

B. Correction

Once the cost function $f(\cdot; t_{k+1})$ becomes available, the correction step is performed to refine the estimate of the optimal solution $\boldsymbol{x}^*(t_{k+1})$. To this end, a first-order projected gradient method is considered next. Particularly, let $\widehat{\boldsymbol{x}}^0 = \tilde{\boldsymbol{x}}_{k+1|k}$ be a dummy variable; then, consider the following projected gradient steps

$$\widehat{\boldsymbol{x}}^{c+1} = \mathbb{P}_X[\widehat{\boldsymbol{x}}^c - \beta(\nabla_{\boldsymbol{x}} f(\widehat{\boldsymbol{x}}^c; t_{k+1})], \quad (9)$$

for $c = 0, 1, \ldots, C-1$, where C is a predetermined number of gradient steps and $\beta > 0$ the stepsize. The estimate of the optimal solution $\boldsymbol{x}_{k+1}^*$ is then computed as $\boldsymbol{x}_{k+1} = \widehat{\boldsymbol{x}}^C$.

Notice that a Newton step could be implemented; however, to develop computationally light online schemes that naturally afford a distributed implementation, this paper considers first-order methods.

C. Complete Algorithm

The complete algorithm Constrained - First Order Prediction Correction (C-FOPC) is tabulated as Algorithm 1. Steps 4-7 are utilized to compute $\tilde{\boldsymbol{x}}_{k+1|k}$ based on the information available at t_k. Provided that the projection operator is easy

Algorithm 1 Constrained First-Order Prediction-Correction (C-FOPC)

Require: Initial variable $\boldsymbol{x}_0$. Initial objective function $f(\boldsymbol{x}; t_0)$, no. of prediction steps P and correction steps C

1: **for** $k = 0, 1, 2, \ldots$ **do**
2:      // time t_k
3:      Prediction: initialize $\widehat{\boldsymbol{x}}^0 = \boldsymbol{x}_k$
4:      **for** $p = 0 : P - 1$ **do**
5:         Predict the variable by the gradient step [cf (7)]

$$\widehat{\boldsymbol{x}}^{p+1} = \mathbb{P}_X[\widehat{\boldsymbol{x}}^p - \alpha(\nabla_{\boldsymbol{xx}} f(\boldsymbol{x}_k; t_k)(\widehat{\boldsymbol{x}}^p - \boldsymbol{x}_k) + \\ h\,\nabla_{t\boldsymbol{x}} f(\boldsymbol{x}_k; t_k) + \nabla_{\boldsymbol{x}} f(\boldsymbol{x}_k; t_k))]$$

6:      **end for**
7:      Set the predicted variable $\tilde{\boldsymbol{x}}_{k+1|k} = \widehat{\boldsymbol{x}}^P$
8:      // time t_{k+1}
9:      Acquire the updated function $f(\boldsymbol{x}; t_{k+1})$
10:     Initialize the sequence of corrected variables $\widehat{\boldsymbol{x}}^0 = \tilde{\boldsymbol{x}}_{k+1|k}$
11:     **for** $c = 0 : C - 1$ **do**
12:        Correct the variable by the gradient step [cf (9)]

$$\widehat{\boldsymbol{x}}^{c+1} = \mathbb{P}_X[\widehat{\boldsymbol{x}}^c - \beta(\nabla_{\boldsymbol{x}} f(\widehat{\boldsymbol{x}}^c; t_{k+1})]$$

13:     **end for**
14:     Set the corrected variable $\boldsymbol{x}_{k+1} = \widehat{\boldsymbol{x}}^C$
15: **end for**

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to carry out (set X is simple), and the Hessian is easy to evaluate, the computational complexity of these steps is $O(Pn^2)$, which is quadratic (due to matrix-vector multiplications) in the number of scalar decision variables. This is in contrast with the algorithms presented in [26], which involve the computation of the Hessian inverse. Steps 10-14 are utilized to compute $\boldsymbol{x}_{k+1}$, based on the information available at t_{k+1}. Provided that the projection operator is easy to performt (set X is simple) and the gradient is easy to evaluate, the computational complexity of these steps is $O(Cn)$, which is linear in the number of scalar decision variables.

Remark 1: [Distributed implementation] C-FOPC naturally affords a distributed implementation for networked time-varying problems, i.e., for problems for which the gradient and Hessian can be computed via one-hop communication with neighboring nodes/agents and the constraint set X is the cartesian product of local constraints (in fact, in these cases, C-FOPC involves only one round of communication among neighboring nodes to perform one step of prediction and one step of correction). This is in contrast with exact prediction schemes that necessitate the computation of an approximate inverse of the Hessian in a distributed fashion [27]. The exact characterization of how this aspect affect the asymptotical error and convergence speed is left as future research.

Remark 2: [Time derivative approximation] The time derivative of the gradient $\nabla_{t\boldsymbol{x}} f(\boldsymbol{x}; t)$ can be substituted with an approximate version, as explained in [26].

III. SPECIAL CASE: UNCONSTRAINED PROBLEMS

In this section, we focus on the special case of unconstrained problems (i.e., $X = \mathbb{R}^n$). Although [26] has given an extensive characterization of methods for unconstrained problems, we will see in this section that further important improvements can be achieved based on the prediction generalized equation (5).

For unconstrained problems, the prediction equation (5) can be rewritten as

$$\nabla_{\boldsymbol{x}} f(\boldsymbol{x}_{k+1|k}; t_{k+1}) \approx \nabla_{\boldsymbol{x}} f(\boldsymbol{x}_k; t_k) + \\ \nabla_{\boldsymbol{xx}} f(\boldsymbol{x}_k; t_k)(\boldsymbol{x}_{k+1|k} - \boldsymbol{x}_k) + h\,\nabla_{t\boldsymbol{x}} f(\boldsymbol{x}_k; t_k) = \boldsymbol{0}. \quad (10)$$

Since the suboptimality at time t_k can be easily characterized by the gradient, i.e., $\nabla_{\boldsymbol{x}} f(\boldsymbol{x}_k; t_k)$, we can also modify (10) to tune the prediction step. A way to do that is to require $\boldsymbol{x}_{k+1|k}$ to reduce the suboptimality by a factor of $1 - \gamma \in [0, 1]$, and pose the problem as

$$\nabla_{\boldsymbol{x}} f(\boldsymbol{x}_{k+1|k}; t_{k+1}) \approx \nabla_{\boldsymbol{x}} f(\boldsymbol{x}_k; t_k) + \nabla_{\boldsymbol{xx}} f(\boldsymbol{x}_k; t_k) \times \\ (\boldsymbol{x}_{k+1|k} - \boldsymbol{x}_k) + h\,\nabla_{t\boldsymbol{x}} f(\boldsymbol{x}_k; t_k) = (1 - \gamma)\nabla_{\boldsymbol{x}} f(\boldsymbol{x}_k; t_k). \quad (11)$$

When $\gamma = 1$, (11) boils down to (10) (when one seeks an "optimal prediction"); on the other hand, when $\gamma = 0$, (11) coincide with [26], where a prediction vector that maintains the same suboptimality (i.e., the same gradient) between successive time steps is sought. Notice that the possibility of tuning the algorithm via γ is possible only for unconstrained optimization problems, since we have access to a "suboptimality measure".

From (11) we obtain the prediction vector via the following update

$$\boldsymbol{x}_{k+1|k} = \boldsymbol{x}_k - \nabla_{\boldsymbol{xx}} f(\boldsymbol{x}_k; t_k)^{-1} \times \\ (h\,\nabla_{t\boldsymbol{x}} f(\boldsymbol{x}_k; t_k) + \gamma\,\nabla_{\boldsymbol{x}} f(\boldsymbol{x}_k; t_k)), \quad (12)$$

which combines a Newton-like step in the direction on the changing cost function, and a (damped) Newton's step towards the optimizer at time t_k. When $\gamma = 0$, (12) coincides with the prediction step in [26]. To obtain a first-order update, it is then easy to modify (7) as

$$\widehat{\boldsymbol{x}}^{p+1} = \widehat{\boldsymbol{x}}^p - \alpha(\nabla_{\boldsymbol{xx}} f(\boldsymbol{x}_k; t_k)(\widehat{\boldsymbol{x}}^p - \boldsymbol{x}_k) \\ + h\,\nabla_{t\boldsymbol{x}} f(\boldsymbol{x}_k; t_k) + \gamma\,\nabla_{\boldsymbol{x}} f(\boldsymbol{x}_k; t_k)), \quad (13)$$

where the notation is the same as in (7). On the other hand, the correction update becomes in this case

$$\widehat{\boldsymbol{x}}^{c+1} = \widehat{\boldsymbol{x}}^c - \beta(\nabla_{\boldsymbol{x}} f(\widehat{\boldsymbol{x}}^c; t_{k+1}). \quad (14)$$

The resultant Unconstrained - First Order Prediction Correction (U-FOPC) method is tabulated as Algorithm 2.

IV. CONVERGENCE ANALYSIS

In this section, we establish analytical results to bound the discrepancy between the optimal solution $\boldsymbol{x}^*(t)$ and the iterates $\boldsymbol{x}_k$ produced by the prediction-correction schemes developed in Section II. Particularly, we will show that $\boldsymbol{x}_k$ tracks $\boldsymbol{x}^*(t)$ up to an error term that depends on the discrete-time sampling period. To this end, some technical conditions are required as stated next

Assumption 1: The function $f(\boldsymbol{x}; t)$ is twice differentiable and m-strongly convex in $\boldsymbol{x} \in X$ and uniformly in t; that is, the Hessian of $f(\boldsymbol{x}; t)$ with respect to $\boldsymbol{x}$ is bounded below by m for each $\boldsymbol{x} \in X$ and uniformly in t,

$$\nabla_{\boldsymbol{xx}} f(\boldsymbol{x}; t) \succeq m\mathbf{I}, \quad \forall \boldsymbol{x} \in X, t.$$

Algorithm 2 Unconstrained First-Order Prediction-Correction (U-FOPC)

Require: Initial variable $\boldsymbol{x}_0$. Initial objective function $f(\boldsymbol{x}; t_0)$, no. of prediction steps $P + 1$ and correction steps $C + 1$, sub-optimality requirement $\gamma \in [0, 1]$
1: **for** $k = 0, 1, 2, \ldots$ **do**
2:     // time t_k
3:     Prediction: initialize $\widehat{\boldsymbol{x}}^0 = \boldsymbol{x}_k$
4:     **for** $p = 0 : P - 1$ **do**
5:        Predict the variable by the gradient step [cf (7)]

$$\widehat{\boldsymbol{x}}^{p+1} = \widehat{\boldsymbol{x}}^p - \alpha(\nabla_{\boldsymbol{xx}} f(\boldsymbol{x}_k; t_k)(\widehat{\boldsymbol{x}}^p - \boldsymbol{x}_k)$$
$$+ h \nabla_{t\boldsymbol{x}} f(\boldsymbol{x}_k; t_k) + \gamma \nabla_{\boldsymbol{x}} f(\boldsymbol{x}_k; t_k))$$

6:     **end for**
7:     Set the predicted variable $\tilde{\boldsymbol{x}}_{k+1|k} = \widehat{\boldsymbol{x}}^P$
8:     // time t_{k+1}
9:     Acquire the updated function $f(\boldsymbol{x}; t_{k+1})$
10:    Initialize the sequence of corrected variables $\widehat{\boldsymbol{x}}^0 = \tilde{\boldsymbol{x}}_{k+1|k}$
11:    **for** $c = 0 : C - 1$ **do**
12:       Correct the variable by the gradient step [cf (9)]

$$\widehat{\boldsymbol{x}}^{c+1} = \mathbb{P}_X[\widehat{\boldsymbol{x}}^c - \beta(\nabla_{\boldsymbol{x}} f(\widehat{\boldsymbol{x}}^c; t_{k+1})]$$

13:    **end for**
14:    Set the corrected variable $\boldsymbol{x}_{k+1} = \widehat{\boldsymbol{x}}^C$
15: **end for**

Assumption 2: The function $f(\boldsymbol{x}; t)$ is sufficiently smooth both in $\boldsymbol{x} \in X$ and in t; in particular, $f(\boldsymbol{x}; t)$ has bounded second and third order derivatives with respect to $\boldsymbol{x} \in X$ and t:

$$\|\nabla_{\boldsymbol{xx}} f(\boldsymbol{x}; t)\| \leqslant L, \|\nabla_{t\boldsymbol{x}} f(\boldsymbol{x}; t)\| \leqslant C_0, \|\nabla_{\boldsymbol{xxx}} f(x; t)\| \leqslant C_1,$$
$$\|\nabla_{\boldsymbol{x}t\boldsymbol{x}} f(\boldsymbol{x}; t)\| \leqslant C_2, \quad \|\nabla_{tt\boldsymbol{x}} f(\boldsymbol{x}; t)\| \leqslant C_3.$$

Assumption 1 guarantees that problem (1) is strongly convex and has a *unique* solution for each time instance. On course, uniqueness of the solution implies that the solution trajectory is also unique. This setting is common in the the time-varying optimization domain; see, for instance [4], [15], [26], [27], [29], [35]. Assumption 2 ensures that the Hessian is bounded from above; this property is equivalent to the Lipschitz continuity of the gradient; it also ensures that the third derivative tensor $\nabla_{\boldsymbol{xxx}} f(x; t)$ is bounded above (typically required for the analysis of Newton-type algorithms), as well as boundedness of the temporal variability of gradient and Hessian. These last properties ensure the possibility to build a prediction scheme based on the knowledge of (or an estimate of) how the function and its derivatives change over time. A similar assumption was required (albeit only locally) for the local convergence analysis in [15, Eq. (3.2)].

Assumptions 1 and 2 are sufficient to show that the solution mapping $t \mapsto \boldsymbol{x}^*(t)$ is single-valued and locally Lipschitz continuous in t; in particular, from [31, Theorem 2F.10] we have that

$$\|\boldsymbol{x}^*(t_{k+1}) - \boldsymbol{x}^*(t_k)\| \leqslant \frac{1}{m} \|\nabla_{t\boldsymbol{x}} f(\boldsymbol{x}; t)\|(t_{k+1} - t_k) \leqslant \frac{C_0 h}{m}, \tag{15}$$

for sufficiently small sampling periods h. This result established link between the sampling period h and the temporal variability of the optimal solutions; further, (15) will be utilized to substantiate convergence and tracking capabilities of the proposed prediction-correction methods.

We begin the convergence analysis by deriving an upper bound on the norm of the approximation error $\boldsymbol{\Delta}_k \in \mathbb{R}^n$ incurred by the Taylor expansion in (5). In particular, given the optimal solution $\boldsymbol{x}^*(t_k)$ at t_k, the objective is to characterize the error that one commits when $\boldsymbol{x}^*(t_{k+1})$ is replaced $\boldsymbol{x}^*_{k+1|k}$ (here we use the superscript $*$ to indicate that we start from $\boldsymbol{x}^*(t_k)$ and not any $\boldsymbol{x}_k$); that is,

$$\boldsymbol{\Delta}_k := \boldsymbol{x}^*_{k+1|k} - \boldsymbol{x}^*(t_{k+1}). \tag{16}$$

Deriving an upper bound on $\|\boldsymbol{\Delta}_k\|$ is key towards to establishing convergence and tracking results, since it substantiates the error introduced by the prediction step. An upper bound on $\|\boldsymbol{\Delta}_k\|$ is derived next.

Proposition 1: Under Assumptions 1-2, the error norm $\|\boldsymbol{\Delta}_k\|$ is upper bounded by

$$\|\boldsymbol{\Delta}_k\| \leqslant \frac{h^2}{2} \left[\frac{C_0^2 C_1}{m^3} + \frac{2 C_0 C_2}{m^2} + \frac{C_3}{m} \right] =: \Delta = O(h^2). \tag{17}$$

Proof: See Appendix A. ∎

Proposition 1 asserts that the norm of the discretization error $\|\boldsymbol{\Delta}_k\|$ is bounded above by a constant which is of order $O(h^2)$. Incidentally, the bound (17) is the same as the one in [26] for unconstrained optimization problems; however, the proof in Appendix A follows different steps.

A. Convergence analysis

We study the convergence properties of the sequence $\{\boldsymbol{x}_k\}_{k \in \mathbb{N}}$ generated by the algorithm C-FOPC, for different choices of the stepsize. In the following theorem, we show that the optimality gap $\|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\|$ converges exponentially to a given error bound.

Theorem 1: Consider the sequence $\{\boldsymbol{x}_k\}_{k \in \mathbb{N}}$ generated by C-FOPC, and let Assumptions 1-2 hold true. Define the following quantities

$$\varrho_{\text{P}} = \max\{|1 - \alpha m|, |1 - \alpha L|\}, \quad \varrho_{\text{C}} = \max\{|1 - \beta m|, |1 - \beta L|\}, \tag{18}$$

and let stepsizes α and β be such that

$$\alpha < 2/L, \quad \beta < 2/L. \tag{19}$$

Further, select $\tau \in (0, 1)$, the number of prediction steps P, and the number of correction steps C in a way that $\varrho_{\text{P}}^P \varrho_{\text{C}}^C < \tau$.

There exist an upper bound on the sampling period $\bar{h}$ and a convergence region $\bar{R}$, such that if the sampling period is chosen as $h \leqslant \bar{h}$ and the initial optimality gap satisfy $\|\boldsymbol{x}_0 - \boldsymbol{x}^*(t_0)\| \leqslant \bar{R}$, then the sequence $\{\|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\|\}_{k \in \mathbb{N}}$ converges linearly with rate τ to an asymptotical error bound, and

$$\limsup_{k \to \infty} \|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\| = O(h^2 \varrho_{\text{C}}^C) + O(h \varrho_{\text{P}}^P \varrho_{\text{C}}^C). \tag{20}$$

In addition, the bounds $\bar{h}$ and $\bar{R}$ are given as

$$\bar{h} = \frac{\tau - \varrho_{\text{C}}^C \varrho_{\text{P}}^P}{\varrho_{\text{C}}^C (\varrho_{\text{P}}^P + 1)} \left(\frac{C_1 C_0}{m^2} + \frac{C_2}{m} \right)^{-1}, \bar{R} = \frac{2m}{C_1} \left(\frac{C_1 C_0}{m^2} + \frac{C_2}{m} \right) (\bar{h} - h). \tag{21}$$

Proof: See Appendix B, where we also derive all the constants in the right-hand side of (20). ∎

Theorem 1 asserts that the sequence $\{\boldsymbol{x}_k\}_{k\in\mathbb{N}}$ generated by C-FOPC locks to a neighborhood of the optimal solution trajectory $\boldsymbol{x}^*(t)$. In particular, for a choice of prediction and correction steps P and C, there exist an upper bound on the sampling period and an attraction region, such that if the sampling period is smaller than the bound and the initial optimality gap is in the attraction region, then the sequence converge (at least) linearly to an asymptotic bound. The bound depends on the sampling period and on the selection of prediction and correction steps P and C. When one performs an optimal prediction (that is $P \to \infty$), then the bound goes as $O(h^2)$, which is similar to the bounds derived in [26]. When one performs the correction step exactly, i.e., $C \to \infty$, then the asymptotic bound goes to zero (in fact, in that case each the time-invariant problem is solved exactly).

The presence of an attraction region is due to Newton steps in the prediction stage (that is, the presence of the gradient $\nabla_{\boldsymbol{x}} f(\boldsymbol{x}_k; t_k)$ in the generalized equation (5)). When the function is quadratic, then $C_1 = 0$, and the convergence is global. One could then utilize a warm start procedure to initialize the C-FOPC sequence and this obtains global convergence. Details are omitted in the interest of space.

B. Unconstrained Algorithm Convergence

The results of Theorem 1 can be tuned to the case of unconstrained problems as shown in the following.

Theorem 2: Consider the sequence $\{\boldsymbol{x}_k\}_{k\in\mathbb{N}}$ generated by U-FOPC. Let Assumptions 1-2 hold true. Define,

$$\varrho_{\mathrm{P}} = \max\{|1-\alpha m|, |1-\alpha L|\}, \quad \varrho_{\mathrm{C}} = \max\{|1-\beta m|, |1-\beta L|\}, \tag{22}$$

and set the stepsizes α and β as

$$\alpha < 2/L, \quad \beta < 2/L. \tag{23}$$

Let $\tau \in (0,1)$, P, C be such that

$$(1-\gamma)\varrho_{\mathrm{C}}^C(1+\varrho_{\mathrm{P}}^P) + \varrho_{\mathrm{P}}^P \varrho_{\mathrm{C}}^C < \tau. \tag{24}$$

There exist an upper bound on the sampling period $\bar{h}$ and a convergence region $\bar{R}$, such that if the sampling period is chosen as $h \leqslant \bar{h}$ and the initial optimality gap satisfy $\|\boldsymbol{x}_0 - \boldsymbol{x}^*(t_0)\| \leqslant \bar{R}$, then the sequence $\{\|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\|\}_{k\in\mathbb{N}}$ converges linearly with rate τ to an asymptotical error bound, and

$$\limsup_{k\to\infty} \|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\| = O(h^2 \varrho_{\mathrm{C}}^C) + O(h\,\varrho_{\mathrm{P}}^P \varrho_{\mathrm{C}}^C). \tag{25}$$

In addition, the bounds $\bar{h}$ and $\bar{R}$ are given as

$$\bar{h} = \left(\frac{\tau - \varrho_{\mathrm{C}}^C \varrho_{\mathrm{P}}^P}{\varrho_{\mathrm{C}}^C(\varrho_{\mathrm{P}}^P + 1)} - 1 + \gamma\right)\left(\frac{C_1 C_0}{m^2} + \frac{C_2}{m}\right)^{-1}, \tag{26}$$

$$\bar{R} = \frac{2\,m}{\gamma C_1}\left(\frac{C_1 C_0}{m^2} + \frac{C_2}{m}\right)(\bar{h} - h). \tag{27}$$

Proof: See Appendix C, where we also derive all the constants in the right-hand side of (25). The proofs leverage generalized equation theory and the special unconstrained nature of the problem. ∎

Theorem 2 can be seen as a generalization of Theorem 1 in [26] and a special, yet more detailed, version of Theorem 1. In particular, by properly selecting the parameter γ, one

obtains the results of both Theorem 1 and [26]; when $\gamma = 1$, we obtain the results of Theorem 1 and this indicates that considering constrained problems does not add extra errors to the asymptotical bounds. On the other hand, if $\gamma = 0$, we are implicitly assuming that the prediction stage leads one to navigate into the iso-residual manifold; since $\bar{R} \to \infty$, we also obtain global convergence. Furthermore, one can use γ as a tuning mechanism to enlarge the convergence region.

Finally, for $\gamma = 0$ and $P \to \infty$, then the results of theorem (25) boil down to [26]. In particular, from (26), it can be seen that the sampling period must satisfy the following relationship

$$\varrho_{\mathrm{C}}^C\left[1 + h\left(\frac{C_1 C_0}{m^2} + \frac{C_2}{m}\right)\right] < \tau < 1, \tag{28}$$

which is the same requirement of [26, Theorem 1].

V. MINIMAX PROBLEMS

This section focuses on constrained time-varying minimax problems, and develops a prediction-correction algorithm to track the optimizer of the minimax problem over time. This setting is particularly relevant when one seeks the optimizer of the Lagrangian associated with a time-varying constrained optimization program (with time-varying inequality and equality constrains), with a double-smoothing regularization [23], [24]. However, minimax problems appear in many research areas, from game theory, machine learning, and signal processing; for example, robust estimation over a worst case scenario is a minimax problem [25]. Consider the following problem [cf. (3)]

$$\min_{\boldsymbol{x}\in X} \max_{\boldsymbol{\mu}\in V} K(\boldsymbol{x}, \boldsymbol{\mu}; t), \quad \text{for } t \geqslant 0. \tag{29}$$

where $X \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^q$ are convex sets, $t \in \mathbb{R}_+$ is a non-negative continuous variable indexing time, and $K : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}_+ \to \mathbb{R}$ is a *smooth strongly convex - strongly concave function*. The goal is to determine the unique saddle-point solution $(\boldsymbol{x}^*(t), \boldsymbol{\mu}^*(t))$ of (29) for each time t – hereafter referred to as the saddle-point *trajectory*.

In par with the technical approach of Section II, we first sample the minimax problem at sampling times $\{t_k\}_{k\in\mathbb{N}}$. The prediction-correction scheme is then design as follows.

A. Prediction

Consider writing the optimality conditions of a sampled version of (29) in terms of generalized equations. Upon introducing a perturbation via Taylor expansion, we arrive at the following set of conditions:

$$\nabla_{\boldsymbol{x}} K(\boldsymbol{x}_k, \boldsymbol{\mu}_k; t_k) + \nabla_{\boldsymbol{xx}} K(\boldsymbol{x}_k, \boldsymbol{\mu}_k; t_k)(\boldsymbol{x}_{k+1|k} - \boldsymbol{x}_k)+$$
$$\nabla_{\boldsymbol{\mu x}} K(\boldsymbol{x}_k, \boldsymbol{\mu}_k; t_k)(\boldsymbol{\mu}_{k+1|k} - \boldsymbol{\mu}_k)+$$
$$h\,\nabla_{t\boldsymbol{x}} K(\boldsymbol{x}_k, \boldsymbol{\mu}_k; t_k) + N_X(\boldsymbol{x}_{k+1|k}) \ni \boldsymbol{0} \tag{30a}$$

$$-\nabla_{\boldsymbol{\mu}} K(\boldsymbol{x}_k, \boldsymbol{\mu}_k; t_k) - \nabla_{\boldsymbol{x\mu}} K(\boldsymbol{x}_k, \boldsymbol{\mu}_k; t_k)(\boldsymbol{x}_{k+1|k} - \boldsymbol{x}_k)-$$
$$\nabla_{\boldsymbol{\mu\mu}} K(\boldsymbol{x}_k, \boldsymbol{\mu}_k; t_k)(\boldsymbol{\mu}_{k+1|k} - \boldsymbol{\mu}_k)-$$
$$h\,\nabla_{t\boldsymbol{\mu}} K(\boldsymbol{x}_k, \boldsymbol{\mu}_k; t_k) + N_V(\boldsymbol{\mu}_{k+1|k}) \ni \boldsymbol{0}. \tag{30b}$$

The saddle-point $(\boldsymbol{x}_{k+1|k}, \boldsymbol{\mu}_{k+1|k})$ could in principle be obtained by solving a quadratic program. However, to design computationally affordable prediction strategies, gradient-based approaches are utilized next to find an approximate solution to (30). To this end, set $\hat{\boldsymbol{x}}^0 = \boldsymbol{x}_k$, and $\hat{\boldsymbol{\mu}}^0 = \boldsymbol{\mu}_k$, and perform P times the following steps:

$$\hat{\boldsymbol{x}}^{p+1} = \mathbb{P}_X[\hat{\boldsymbol{x}}^p - \alpha(\nabla_{\boldsymbol{xx}}K(\boldsymbol{x}_k, \boldsymbol{\mu}_k; t_k)(\hat{\boldsymbol{x}}^p - \boldsymbol{x}_k) +$$
$$\nabla_{\boldsymbol{\mu x}}K(\boldsymbol{x}_k, \boldsymbol{\mu}_k; t_k)(\hat{\boldsymbol{\mu}}^p - \boldsymbol{\mu}_k) + h\,\nabla_{t\boldsymbol{x}}K(\boldsymbol{x}_k, \boldsymbol{\mu}_k; t_k) +$$
$$K_{\boldsymbol{x}}(\boldsymbol{x}_k, \boldsymbol{\mu}_k; t_k))] \quad (31a)$$

$$\hat{\boldsymbol{\mu}}^{p+1} = \mathbb{P}_V[\hat{\boldsymbol{\mu}}^p + \alpha(\nabla_{\boldsymbol{\mu\mu}}K(\boldsymbol{x}_k, \boldsymbol{\mu}_k; t_k)(\hat{\boldsymbol{\mu}}^p - \boldsymbol{\mu}_k) +$$
$$\nabla_{\boldsymbol{x\mu}}K(\boldsymbol{x}_k, \boldsymbol{\mu}_k; t_k)(\hat{\boldsymbol{x}}^p - \boldsymbol{x}_k) + h\,\nabla_{t\boldsymbol{\mu}}K(\boldsymbol{x}_k, \boldsymbol{\mu}_k; t_k) +$$
$$K_{\boldsymbol{\mu}}(\boldsymbol{x}_k, \boldsymbol{\mu}_k; t_k))], \quad (31b)$$

where P is the number of prediction steps and $\alpha > 0$ is the stepsize; then, $\tilde{\boldsymbol{x}}_{k+1|k} = \hat{\boldsymbol{x}}^P$ and $\tilde{\boldsymbol{\mu}}_{k+1|k} = \hat{\boldsymbol{\mu}}^P$.

B. Correction

At time t_{k+1}, first-order projected saddle-point gradient methods are utilized to refine the estimate of $\boldsymbol{x}^*(t_{k+1}), \boldsymbol{\mu}^*(t_{k+1}))$. Define the auxiliary variables $\hat{\boldsymbol{x}}^0 = \tilde{\boldsymbol{x}}_{k+1|k}$ and $\hat{\boldsymbol{\mu}}^0 = \tilde{\boldsymbol{\mu}}_{k+1|k}$, and apply the projected saddle-point gradient method as

$$\hat{\boldsymbol{x}}^{c+1} = \mathbb{P}_X[\hat{\boldsymbol{x}}^c - \beta(\nabla_{\boldsymbol{x}}K(\hat{\boldsymbol{x}}^c, \hat{\boldsymbol{\mu}}^c; t_{k+1}))], \quad (32a)$$
$$\hat{\boldsymbol{\mu}}^{c+1} = \mathbb{P}_V[\hat{\boldsymbol{\mu}}^c + \beta(\nabla_{\boldsymbol{\mu}}K(\hat{\boldsymbol{x}}^c, \hat{\boldsymbol{\mu}}^c; t_{k+1}))], \quad (32b)$$

for $c = 0, 1, \ldots, C-1$, where C is the total number of gradient steps and $\beta > 0$ the stepsize. Once C steps are performed, the variables $\boldsymbol{x}_{k+1}$ and $\boldsymbol{\mu}_{k+1}$ are updated as $\boldsymbol{x}_{k+1} = \hat{\boldsymbol{x}}^C$, $\boldsymbol{\mu}_{k+1} = \hat{\boldsymbol{\mu}}^C$.

C. Complete Algorithm

The steps of the Constrained - First Order Prediction Correction - Minimax (C-FOPC-MM) method are tabulated as Algorithm 3. Steps 4-7 predict $\boldsymbol{x}^*(t_{k+1})$ and $\boldsymbol{\mu}^*(t_{k+1})$ based on the information available at t_k. Provided that the projection operator is easy to carry out (sets X and V are simple) and the Hessian matrices are easy to evaluate, the computational complexity of these steps is $O(Pn^2)$. The correction steps 10-14 are utilized to fine-tune the estimates of $\boldsymbol{x}^*(t_{k+1})$ and $\boldsymbol{\mu}^*(t_{k+1})$, based on the information available at t_{k+1}. Provided that the projection operator is easy to carry out (sets X and V are simple), and the gradient is easy to evaluate, the computational complexity of these steps is $O(Cn)$, which is linear in the number of scalar decision variables.

D. Convergence Analysis

In this section, we establish convergence results for Algorithm 3. Compared to rage results of Section IV, the requirements on the stepsize β will be more stringent; this is because we now focus on saddle-point operators and not subdifferential operators (see e.g., [36]).

Let $\boldsymbol{z} := [\boldsymbol{x}^\mathsf{T}, \boldsymbol{\mu}^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{n+q}$, and define the set $Z = X \times V$. Define further the functions

$$\boldsymbol{\Phi}(\boldsymbol{z}; t) : \mathbb{R}^{n+q} \times \mathbb{R}_+ \to \mathbb{R}^{n+q}, \quad (\boldsymbol{z}; t) \mapsto \begin{bmatrix} \nabla_{\boldsymbol{x}}K(\boldsymbol{x}; t) \\ -\nabla_{\boldsymbol{\mu}}K(\boldsymbol{\mu}; t) \end{bmatrix}. \quad (33)$$

Algorithm 3 Constrained First-Order Prediction-Correction Minimax (C-FOPC-MM)

Require: Initial variables $\boldsymbol{x}_0, \boldsymbol{\mu}_0$. Initial objective function $K(\boldsymbol{x}, \boldsymbol{\mu}; t_0)$, no. of prediction steps P and correction steps C
1: **for** $k = 0, 1, 2, \ldots$ **do**
2:    // time t_k
3:    Prediction: initialize $\hat{\boldsymbol{x}}^0 = \boldsymbol{x}_k$, $\hat{\boldsymbol{\mu}}^0 = \boldsymbol{\mu}_k$
4:    **for** $p = 0 : P - 1$ **do**
5:       Predict the variables by the saddle-point gradient step [cf (31)]
6:    **end for**
7:    Set the predicted variables $\tilde{\boldsymbol{x}}_{k+1|k} = \hat{\boldsymbol{x}}^P$, $\tilde{\boldsymbol{\mu}}_{k+1|k} = \hat{\boldsymbol{\mu}}^P$
8:    // time t_{k+1}
9:    Acquire the updated function $K(\boldsymbol{x}, \boldsymbol{\mu}; t_{k+1})$
10:   Initialize the sequence of corrected variables $\hat{\boldsymbol{x}}^0 = \tilde{\boldsymbol{x}}_{k+1|k}$, $\hat{\boldsymbol{\mu}}^0 = \tilde{\boldsymbol{\mu}}_{k+1|k}$
11:   **for** $c = 0 : C - 1$ **do**
12:      Correct the variable by the saddle-point gradient step [cf (32)]
13:   **end for**
14:   Set the corrected variable $\boldsymbol{x}_{k+1} = \hat{\boldsymbol{x}}^C$, $\boldsymbol{\mu}_{k+1} = \hat{\boldsymbol{\mu}}^C$
15: **end for**

With this notation, the exact prediction (30) can be rewritten as

$$\boldsymbol{\Phi}(\boldsymbol{z}_k; t_k) + \nabla_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z}_k; t)(\boldsymbol{z}_{k+1|k} - \boldsymbol{z}_k) +$$
$$h\,\nabla_t\boldsymbol{\Phi}(\boldsymbol{z}_k; t) + N_Z(\boldsymbol{z}_{k+1|k}) \ni \boldsymbol{0}, \quad (34)$$

and, in particular, $\nabla_{\boldsymbol{z}}\boldsymbol{\Phi}$ is a positive definite matrix; in fact, for any vector $\boldsymbol{z} \in Z$: $\boldsymbol{z}^\mathsf{T}\nabla_{\boldsymbol{z}}\boldsymbol{\Phi}\boldsymbol{z} = \boldsymbol{x}^\mathsf{T}\nabla_{\boldsymbol{xx}}K(\boldsymbol{x}, \boldsymbol{\mu}; t)\boldsymbol{x} - \boldsymbol{\mu}^\mathsf{T}\nabla_{\boldsymbol{\mu\mu}}K(\boldsymbol{x}, \boldsymbol{\mu}; t)\boldsymbol{\mu} > 0$.

In par with Assumptions 1-2, the following is presupposed for the minimax problem.

Assumption 3: The function $K(\boldsymbol{x}, \boldsymbol{\mu}; t)$ is twice differentiable and m-strongly convex-concave in $\boldsymbol{x} \in X$ and $\boldsymbol{\mu} \in V$, respectively, and uniformly in t; that is, $\forall \boldsymbol{x} \in X, \boldsymbol{\mu} \in V, t$

$$\nabla_{\boldsymbol{xx}}K(\boldsymbol{x}, \boldsymbol{\mu}; t) \succeq m\mathbf{I}, \quad -\nabla_{\boldsymbol{\mu\mu}}K(\boldsymbol{x}, \boldsymbol{\mu}; t) \succeq m\mathbf{I}.$$

This implies that the matrix $(\nabla_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z}; t) - m\mathbf{I})$ is positive semidefinite.

Assumption 4: The function $\boldsymbol{\Phi}(\boldsymbol{z}; t)$ is sufficiently smooth in both $\boldsymbol{z} \in Z$ and t; and, in particular,

$$\|\nabla_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z}; t)\| \leqslant L, \|\nabla_{t\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z}; t)\| \leqslant C_0, \|\nabla_{\boldsymbol{zzz}}\boldsymbol{\Phi}(\boldsymbol{z}; t)\| \leqslant C_1,$$
$$\|\nabla_{t\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z}; t)\| \leqslant C_2, \quad \|\nabla_{tt\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z}; t)\| \leqslant C_3.$$

Based on Assumptions 3-4, the following convergence result holds.

Theorem 3: Consider the sequence $\{\boldsymbol{z}_k\}_{k\in\mathbb{N}}$ generated by the C-FOPC-MM algorithms, and let Assumptions 3-4 hold. Define ϱ_P as

$$\varrho_\mathrm{P} = \sqrt{1 - 2m\alpha + \alpha^2L^2}, \quad \varrho_\mathrm{C} = \sqrt{1 - 2m\beta + \beta^2L^2}, \quad (35)$$

and let the stepsizes α and β satisfy the following conditions

$$\alpha < 2m/L^2, \quad \beta < 2m/L^2. \quad (36)$$

Finally, select $\tau \in (0, 1)$, P, and C such that $\varrho_\mathrm{P}^P \varrho_\mathrm{C}^C < \tau$.

There exist an upper bound on the sampling period $\bar{h}$ and a convergence region $\bar{R}$, such that if the sampling period is chosen as $h \leqslant \bar{h}$ and the initial optimality gap satisfy $\|\boldsymbol{z}_0 -$
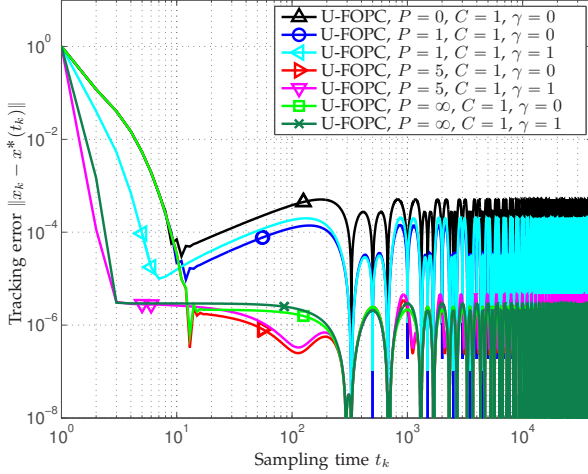
Fig. 1. Error with respect to the sampling time t_k for different algorithms applied to the scalar problem (39), with $h = 0.1$.
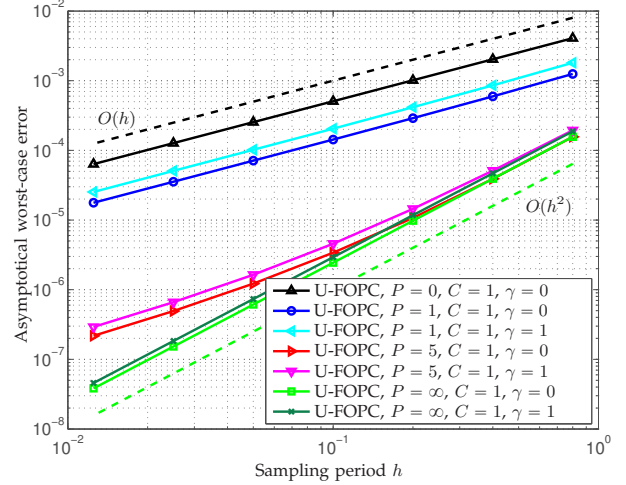

Fig. 2. Asymptotical worst case error floor with respect to the sampling time interval h for different algorithms applied to the scalar problem (39).

$\boldsymbol{z}^*(t_0)\| \leqslant \bar{R}$, then the sequence $\{\|\boldsymbol{z}_k - \boldsymbol{z}^*(t_k)\|\}_{k\in\mathbb{N}}$ converges linearly with rate τ to an asymptotical error bound, and

$$\limsup_{k\to\infty} \|\boldsymbol{z}_k - \boldsymbol{z}^*(t_k)\| = O(h^2 \varrho_{\mathrm{C}}^C) + O(h \varrho_{\mathrm{P}}^P \varrho_{\mathrm{C}}^C). \quad (37)$$

In addition, the bounds $\bar{h}$ and $\bar{R}$ are given by

$$\bar{h} = \frac{\tau - \varrho_{\mathrm{C}}^C \varrho_{\mathrm{P}}^P}{\varrho_{\mathrm{C}}^C(\varrho_{\mathrm{P}}^P + 1)}\left(\frac{C_1 C_0}{m^2} + \frac{C_2}{m}\right)^{-1}, \bar{R} = \frac{2\,m}{C_1}\left(\frac{C_1 C_0}{m^2} + \frac{C_2}{m}\right)(\bar{h} - h). \quad (38)$$

Proof: See Appendix D, where we also derive all the constants on the right-hand side of (37). ∎

VI. NUMERICAL EXPERIMENTS

In this section, we report two numerical example to showcase the performance of the proposed algorithms. First, we analyze the unconstrained case (Algorithm 2), then we switch our attention to the constrained one (Algorithm 1). In the interest of space, the evaluation of the minimax case is not included.

A. Unconstrained example

We use the same scalar example of [26, Section IV.A], and in particular we consider a time-varying cost function of the form

$$\min_{x\in\mathbb{R}} f(x; t) := \frac{1}{2}\left(x - \cos(\omega t)\right)^2 + \kappa \log[1 + \exp(\mu x)]. \quad (39)$$

The function in (39) represents, for instance, the goal of staying close to a periodically varying trajectory plus a logistic term that penalizes large values of x. The terms ω, κ, and μ are arbitrary nonnegative scalar parameters. In our experiments these parameters are set to $\omega = 0.02\,\pi$, $\kappa = 2$, and $\mu = 1.75$. The function $f(x; t)$ satisfies all the conditions in Assumptions 1 and 2. In particular, $m = 1$ and $L = 2.53$.

We choose the constant stepsizes as $\alpha = \beta = 0.56 < 2/L$ in the gradient methods stated in Algorithm 2 and initialize $x_0 = 0$ for all the algorithms. According to Theorem 2, for $\gamma = 0$, $P = 1$ and $C = 1$: we can set $\tau > 0.83$, the sampling period

needs to be chosen as $h \leqslant \bar{h} = 4.2$, and the convergence region is the whole $\mathbb{R}$, while for $\gamma = 1$, $P = 1$ and $C = 1$: we can set $\tau > 0.19$, $\bar{h} = 19.6$, and $\bar{R} = 2.46$ (notice that for greater P or C the requirements are less stringent).

In Figure 1, we plot the error $\|x_k - x^*(t_k)\|$ versus the discrete time t_k for a sampling period of $h = 0.1$, for different schemes (we fix the correction steps at $C = 1$ for all the algorithms and we study the behavior for different prediction steps P and γ values). Observe that the running gradient method [29], that is U-FOPC with no prediction and one correction step performs the worst, while U-FOPC with exact prediction (that is $P = \infty$) performs the best. Furthermore, we can notice that increasing P, one obtains better and better asymptotical error, with the drawback of an increased computational burden. Lastly, we notice how $\gamma = 1$, that is having a Newton-like prediction step helps in achieving a faster convergence, and yet it appears that the asymptotical error is slightly greater than using $\gamma = 0$ (i.e., tangential prediction).

The differences in performance can be also appreciated by varying h and observing the worst case error floor size which is defined as $\max_{k>\bar{k}}\{\|x_k - x^*(t_k)\|\}$, where $\bar{k} = 10^4$ in the simulations. Figure 2 illustrates the error as a function of h. The performance differences between the proposed methods that may be observed here corroborate the differences seen in Figure 1. In particular, the U-FOPC method with $P = 0, C = 1$ achieves the largest worst case error bound, while exact prediction with $\gamma = 0$ attains the best case error bound. Notice also the dashed lines displaying the theoretical performance of $O(h)$, $O(h^2)$, and the fact that for any finite P there is a trade-off between computational complexity and asymptotical error.

B. Constrained example

We consider here a mid-size optimization problem consisting of $n = 1000$ scalar variables. The cost function we study

has the form

$$f(\boldsymbol{x};t) = \frac{1}{2}\|\boldsymbol{x}+\mathbf{1}_n\|_{\mathbf{Q}}^2 + \sum_{i=1}^{n} \kappa_i \sin^2(\omega t + \varphi_i)\, \exp(\mu(x_{(i)}-2)^2), \tag{40}$$

where we have defined $\mathbf{1}_n$ as the column vector of all ones of dimension n, while $x_{(i)}$ is the i-th component of $\boldsymbol{x} \in \mathbb{R}^n$. In addition, the matrix $\mathbf{Q}$ is chosen as $\mathbf{Q} = \mathbf{I}_n + \boldsymbol{\mu}\boldsymbol{\mu}^{\mathsf{T}}/n$ with $\boldsymbol{\mu}$ being a vector randomly generated by a normal distribution of mean 0 and variance 1, $\kappa_i \sim \mathcal{U}_{[0,1]}$, $\omega = 0.1\pi$, $\varphi \sim \mathcal{N}(0,\pi)$, and $\mu = 0.25$.

We study the time-varying problem

$$\underset{\boldsymbol{x}\in[0,0.4]^n}{\text{minimize}} f(\boldsymbol{x};t). \tag{41}$$

We notice that the cost function f verifies the Assumptions 1-2 on on $[0,0.4]^n$, which is our optimization set (even though it would not satisfy them on the whole $\mathbb{R}^n$). In particular, $m = 1$ and $L = 6.07$.

One could run a similar analysis as the unconstrained example, however here we focus on realistic run-time constraints. Every time a new function is available, a number of correction steps are performed. The number depends on how fast we need the corrected variable to be available and the computational time necessary to compute the gradient and perform the correction step. We fix at $r_1 h$, with $r_1 < 1$ the time allocated for the correction steps, while t_{C} is the time to perform one correction step. For the above considerations, we can afford to run

$$C = \lfloor r_1 h/t_{\mathrm{C}}\rfloor, \tag{42}$$

correction steps. After the corrected variable is available, one can use it for the decision making process (which may require extra time to be performed). For the time-varying algorithm perspective, one can use the variable to either run P gradient prediction, or C' extra correction steps (to improve the corrected variable for having a better starting point when a new function becomes available). Fix at $r_2 h$, with $r_2 < 1$ the time allocated for the prediction (or extra correction) steps. The affordable number of prediction steps can be determined considering that P prediction steps require a time equal to $\bar{t} + Pt_{\mathrm{P}}$, where $\bar{t}$ is the time required to evaluate the Hessian, gradient, and time derivative of the gradient, while t_{P} is the time to perform one prediction calculation. Thus,

$$P = \lfloor (r_2 h - \bar{t})/t_{\mathrm{P}}\rfloor. \tag{43}$$

The affordable extra correction steps C' can be computed as in (42), substituting r_1 with r_2.

In the simulation example, we choose $r_1 = r_2 = 0.5$, while by running the experiments on a 1.8 GHz Intel Core i5, we empirically fix $t_{\mathrm{C}} = .76$ ms, $\bar{t} = 10$ ms, $t_{\mathrm{P}} = .62$ ms. Note that the time that would be needed to solve the prediction step exactly (by solving a quadratic program) is 190 ms, which is not affordable in the considered sampling period range.

In addition, we consider the situation in which one can use the whole sampling period to do correction, that is $r_1 = 1$, while $r_2 = 0$, and we call this case *total correction*. This situation is particularly interesting when one has to make a choice whether to stop the correction steps to perform prediction, or to continue to do correction steps till a new function
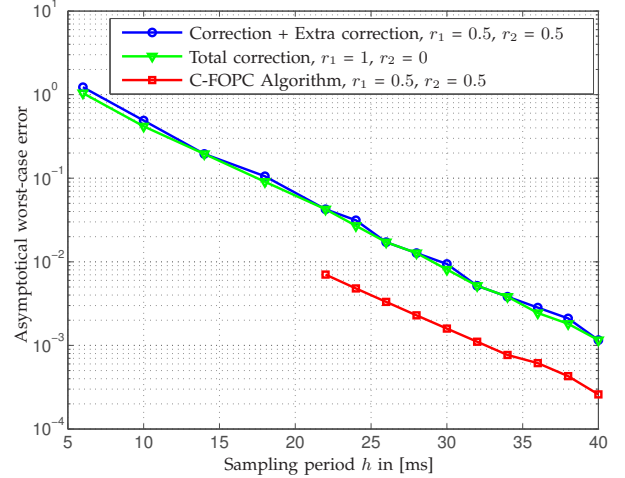

Fig. 3. Asymptotical worst case error floor with respect to the sampling time interval h for different algorithms applied to (41)

evaluation becomes available. Note that the correction+extra correction strategy is different from the total correction one, since the error is computed with the corrected variable (which is used for the decision making process), that is after $r_1 h$.

In Figure 3, we report the asymptotical worst-case error w.r.t the sampling period for the three considered cases (correction+extra correction, total correction, and prediction-correction, i.e. the C-FOPC algorithm), while the number of prediction steps and correction steps are optimized via the available resources as in Eq.s (42)-(43). With the simulation parameters, for $h = 6$ ms, we can perform $C = C' = 3$ steps of correction and extra correction, or $C = 7$ steps of (total) correction. For $h = 40$ ms, these values are $C = C' = 26$ and $C = 52$, respectively. For the prediction-correction strategy, for $h = 22$ ms, then $C = 14$ and $P = 1$, while for $h = 40$ ms, $C = 26$ and $P = 16$.

For sampling times below 22 ms, prediction cannot be performed due to time constraints. For sampling periods greater or equal than 22 ms, prediction can be performed and for $h = [22, 40]$ ms, then $\bar{h} = [90, 370]$ ms, and $\bar{R} = [.13, .68]$. We see clearly that, in this simulation example, if prediction is affordable, the prediction-correction strategy, that is our C-FOPC algorithm is to be preferred to traditional correction-only schemes, since it achieves a lower asymptotical worst-case error. We notice that this error is lower by an half order of magnitude, while the error of the correction-extra correction and the total correction strategy are practically the same. For completeness, we report that $\boldsymbol{x}_0$ is chosen to be zero, while the initial optimality gap is .30, which indicates that our bounds are somewhat conservative.

The result is quite remarkable, telling that performing Newton-like prediction steps on a fixed (Hessian, gradient, time derivative) triple can be computationally much more interesting that performing correction steps on a varying (i.e., re-updated) gradient.

In Figure 4, we report the time trajectories of a number of variables for the three strategies to appreciate how the constraints are in fact active.
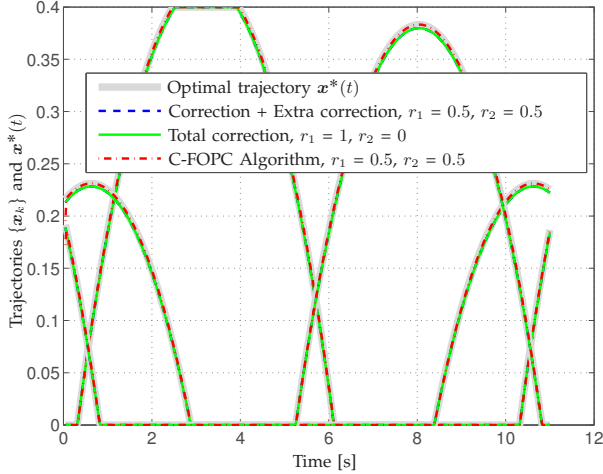
Fig. 4. Trajectories of $\{\boldsymbol{x}_k\}$ and optimal trajectory $\boldsymbol{x}^*(t)$ for different algorithms applied to (41), with $h = 22$ ms.

VII. CONCLUSIONS

We have proposed first-order algorithms to find and track the solution trajectory of strongly convex, strongly smooth constrained time-varying optimization problems. These algorithms rely on a discrete-time prediction-correction strategy, by which at each sampling time, the decision variables are corrected through one or multiple projected gradient steps, and then used to predict the next decision variables via successive projected gradient steps on a suitably defined quadratic program. The proposed algorithms exhibit better asymptotical accuracy than state-of-the-art correction-only schemes, even when computational complexity issues are taken into account.

APPENDIX A
PROOF OF PROPOSITION 1

Let us start by simplifying the notation. Define

$$\nabla_{\boldsymbol{x}} f_i = \nabla_{\boldsymbol{x}} f(\boldsymbol{x}^*(t_{k+i}); t_{k+i}), \quad \mathbf{Q}_i = \nabla_{\boldsymbol{x}\boldsymbol{x}} f(\boldsymbol{x}^*(t_{k+i}); t_{k+i}) \tag{44}$$

$$\boldsymbol{c}_i = \nabla_{t\boldsymbol{x}} f(\boldsymbol{x}^*(t_{k+i}); t_{k+i}), \quad \boldsymbol{x}_i = \boldsymbol{x}^*(t_{k+i}), \quad \boldsymbol{x} = \boldsymbol{x}^*_{k+1|k}. \tag{45}$$

With this notation in place, $\boldsymbol{\Delta}_k = \boldsymbol{x} - \boldsymbol{x}_1$ [Cf. (16)]. In addition, $\boldsymbol{x}$ is computed by the generalized equation (5),

$$\nabla_{\boldsymbol{x}} f_0 + \mathbf{Q}_0(\boldsymbol{x} - \boldsymbol{x}_0) + h \boldsymbol{c}_0 + N_X(\boldsymbol{x}) \ni \mathbf{0}, \tag{46}$$

while $\boldsymbol{x}_1$ is the solution of

$$\nabla_{\boldsymbol{x}} f_1 + N_X(\boldsymbol{x}_1) \ni \mathbf{0}. \tag{47}$$

Define the supporting functions,

$$G(\boldsymbol{y}) = \nabla_{\boldsymbol{x}} f_1 + \mathbf{Q}_1(\boldsymbol{y} - \boldsymbol{x}_1) + N_X(\boldsymbol{y}) \tag{48}$$

$$g(\boldsymbol{y}) = \nabla_{\boldsymbol{x}} f_0 + \mathbf{Q}_0(\boldsymbol{y} - \boldsymbol{x}_0) + h \boldsymbol{c}_0 + \\ - [\nabla_{\boldsymbol{x}} f_1 + \mathbf{Q}_1(\boldsymbol{y} - \boldsymbol{x}_1)]. \tag{49}$$

These two functions allows one to rewrite (46) as

$$(g + G)(\boldsymbol{x}) \ni \mathbf{0}. \tag{50}$$

It is also true that $(g + G)(\boldsymbol{x}_1) \ni g(\boldsymbol{x}_1)$, since for optimality $G(\boldsymbol{x}_1) \ni \mathbf{0}$. Define $F(\boldsymbol{y}) = (g + G)(\boldsymbol{y})$, and consider the parametric generalized equation $F(\boldsymbol{y}) + \boldsymbol{p} \ni \mathbf{0}$. Under the Assumptions 1-2, as for [31, Theorem 2F.9], we have that the solution mapping $\boldsymbol{p} \mapsto \boldsymbol{y}(\boldsymbol{p})$ for the generalized equation $F(\boldsymbol{y}) + \boldsymbol{p} \ni \mathbf{0}$ is every-where single valued and Lipschitz continuous with constant m^{-1}, that is

$$\|\boldsymbol{y}(\boldsymbol{p}) - \boldsymbol{y}(\boldsymbol{p}')\| \leqslant \frac{1}{m} \|\boldsymbol{p} - \boldsymbol{p}'\|. \tag{51}$$

We can set $\boldsymbol{p} = \mathbf{0}$ and $\boldsymbol{p}' = -g(\boldsymbol{x}_1)$, which leads to

$$\|\boldsymbol{x} - \boldsymbol{x}_1\| = \|\boldsymbol{\Delta}_k\| \leqslant \frac{1}{m} \|g(\boldsymbol{x}_1)\|. \tag{52}$$

We proceed now to bound $\|g(\boldsymbol{x}_1)\|$. We can write $g(\boldsymbol{x}_1)$ as

$$g(\boldsymbol{x}_1) = \nabla_{\boldsymbol{x}} f_0 + \mathbf{Q}_0(\boldsymbol{x}_1 - \boldsymbol{x}_0) + h \boldsymbol{c}_0 - \nabla_{\boldsymbol{x}} f_1, \tag{53}$$

which is nothing else but the error of the truncated Taylor expansion of $\nabla_{\boldsymbol{x}} f_1$, which due to Assumptions 1-2, can be upper bounded as

$$\|g(\boldsymbol{x}_1)\| \leqslant \frac{1}{2} C_1 \|\boldsymbol{x}_1 - \boldsymbol{x}_0\|^2 + h C_2 \|\boldsymbol{x}_1 - \boldsymbol{x}_0\| + \frac{1}{2} h^2 C_3. \tag{54}$$

By using the bound (15) on the variability of the optimizers $\boldsymbol{x}_1$ and $\boldsymbol{x}_0$, then

$$\|g(\boldsymbol{x}_1)\| \leqslant h^2 C_1 \frac{C_0^2}{2m^2} + h^2 C_2 \frac{C_0}{m} + \frac{1}{2} h^2 C_3, \tag{55}$$

and by combining this bound with (52), the claim (17) follows. ∎

APPENDIX B
PROOF OF THEOREM 1

We divide the proof in different steps. Step 1: we bound the prediction error in Propositions 2 and 3; Step 2: we bound the correction error; Step 3: we put the previous steps together and derive the convergence requirements and results.

Prediction error. The optimal prediction error, i.e., the distance between the optimal predicted variable $\boldsymbol{x}_{k+1|k}$ and the optimizer at time step t_{k+1}, $\boldsymbol{x}^*(t_{k+1})$ can be bounded as the following proposition.

Proposition 2: Under the same assumptions and notation of Theorem 1, we have that

$$\|\boldsymbol{x}_{k+1|k} - \boldsymbol{x}^*(t_{k+1})\| \leqslant \frac{C_1}{2m} \|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\|^2 + \\ h \left(\frac{C_1 C_0}{m^2} + \frac{C_2}{m} \right) \|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\| + \Delta. \tag{56}$$

Proof: We proceed as in the proof of Proposition 1. We use similar simplifications of (44), as

$$\nabla_{\boldsymbol{x}} f_k = \nabla_{\boldsymbol{x}} f(\boldsymbol{x}_k; t_k), \quad \mathbf{Q}_k = \nabla_{\boldsymbol{x}\boldsymbol{x}} f\boldsymbol{x}_k; t_k) \tag{57a}$$

$$\boldsymbol{c}_k = \nabla_{t\boldsymbol{x}} f(\boldsymbol{x}_k; t_k), \quad \boldsymbol{x} = \boldsymbol{x}_{k+1|k}. \tag{57b}$$

while $\nabla_{\boldsymbol{x}} f_1$ and $\boldsymbol{x}_1$ are defined just as in (44). The error $\|\boldsymbol{x}_{k+1|k} - \boldsymbol{x}^*(t_{k+1})\|$ is now $\boldsymbol{x} - \boldsymbol{x}_1$.

The vector $\boldsymbol{x}$ is computed by the generalized equation (5),

$$\nabla_{\boldsymbol{x}} f_k + \mathbf{Q}_k(\boldsymbol{x} - \boldsymbol{x}_k) + h \boldsymbol{c}_k + N_X(\boldsymbol{x}) \ni \mathbf{0}, \tag{58}$$

while $\boldsymbol{x}_1$ is the solution of (47).

Define the supporting functions,

$$G(\boldsymbol{y}) = \nabla_{\boldsymbol{x}} f_1 + \mathbf{Q}_1(\boldsymbol{y} - \boldsymbol{x}_1) + N_X(\boldsymbol{y}) \qquad (59)$$

$$g(\boldsymbol{y}) = \nabla_{\boldsymbol{x}} f_k + \mathbf{Q}_k(\boldsymbol{y} - \boldsymbol{x}_k) + h\,\boldsymbol{c}_k + \\ - [\nabla_{\boldsymbol{x}} f_1 + \mathbf{Q}_1(\boldsymbol{y} - \boldsymbol{x}_1)]. \qquad (60)$$

These two functions allows one to rewrite (58) as

$$(g + G)(\boldsymbol{x}) \ni \mathbf{0}. \qquad (61)$$

It is also true that $(g + G)(\boldsymbol{x}_1) \ni g(\boldsymbol{x}_1)$, since for optimality $G(\boldsymbol{x}_1) \ni \mathbf{0}$. Define $F(\boldsymbol{y}) = (g + G)(\boldsymbol{y})$, and consider the parametric generalized equation $F(\boldsymbol{y}) + \boldsymbol{p} \ni \mathbf{0}$. Due to Assumptions 1-2, and due to [31, Theorem 2F.9], we have that the solution mapping $\boldsymbol{p} \mapsto \boldsymbol{y}(\boldsymbol{p})$ of the generalized equation $F(\boldsymbol{y}) + \boldsymbol{p} \ni \mathbf{0}$ is every-where single valued and Lipschitz continuous as

$$\|\boldsymbol{y}(\boldsymbol{p}) - \boldsymbol{y}(\boldsymbol{p}')\| \leqslant \frac{1}{m}\|\boldsymbol{p} - \boldsymbol{p}'\|. \qquad (62)$$

Set $\boldsymbol{p} = \mathbf{0}$ and $\boldsymbol{p}' = -g(\boldsymbol{x}_1)$, then,

$$\|\boldsymbol{x} - \boldsymbol{x}_1\| \leqslant \frac{1}{m}\|g(\boldsymbol{x}_1)\|. \qquad (63)$$

We proceed now to bound $\|g(\boldsymbol{x}_1)\|$. We can write $g(\boldsymbol{x}_1)$ as

$$g(\boldsymbol{x}_1) = \nabla_{\boldsymbol{x}} f_k + \mathbf{Q}_k(\boldsymbol{x}_1 - \boldsymbol{x}_k) + h\,\boldsymbol{c}_k - \nabla_{\boldsymbol{x}} f_1, \qquad (64)$$

which is nothing else but the error of the truncated Taylor expansion of $\nabla_{\boldsymbol{x}} f_1$, which due to Assumptions 1-2, can be upper bounded as

$$\|g(\boldsymbol{x}_1)\| \leqslant \frac{1}{2}\,C_1\|\boldsymbol{x}_1 - \boldsymbol{x}_k\|^2 + h\,C_2\|\boldsymbol{x}_1 - \boldsymbol{x}_k\| + \frac{1}{2}h^2\,C_3. \qquad (65)$$

Since $\|\boldsymbol{x}_1 - \boldsymbol{x}_k\| \leqslant \|\boldsymbol{x}_1 - \boldsymbol{x}^*(t_k)\| + \|\boldsymbol{x}^*(t_k) - \boldsymbol{x}_k\|$, and we can bound the first term of the right-hand side by using (15), then

$$\|g(\boldsymbol{x}_1)\| \leqslant h^2\,C_1\frac{C_0^2}{2m^2} + h^2\,C_2\frac{C_0}{2m} + \frac{1}{2}h^2\,C_3 + \\ \frac{1}{2}\,C_1\|\boldsymbol{x}^*(t_k) - \boldsymbol{x}_k\|^2 + h\left(\frac{C_1 C_0}{m} + C_2\right)\|\boldsymbol{x}^*(t_k) - \boldsymbol{x}_k\| \qquad (66)$$

and by combining this bound with (63), the claim (56) follows. $\blacksquare$

On the other hand, the distance between the approximate prediction $\tilde{\boldsymbol{x}}_{k+1|k}$ and the optimal prediction $\boldsymbol{x}_{k+1|k}$ can be bounded by using standard results for the projected gradient method, for the proof see for instance [36].

Proposition 3: Under the same assumptions and notation of Theorem 1, we have that

$$\|\tilde{\boldsymbol{x}}_{k+1|k} - \boldsymbol{x}_{k+1|k}\| \leqslant \varrho_{\mathrm{P}}^{P}\|\boldsymbol{x}_k - \boldsymbol{x}_{k+1|k}\|, \qquad (67)$$

with $\varrho_{\mathrm{P}} = \max\{|1 - \alpha m|, |1 - \alpha L|\}$.

By putting together Propositions 2 and 3 and (15) , we obtain for the total error after prediction as

$$\|\tilde{\boldsymbol{x}}_{k+1|k} - \boldsymbol{x}^*(t_{k+1})\| \leqslant \|\tilde{\boldsymbol{x}}_{k+1|k} - \boldsymbol{x}_{k+1|k}\| + \|\boldsymbol{x}_{k+1|k} - \boldsymbol{x}^*(t_{k+1})\| \qquad (68a)$$

$$\leqslant \varrho_{\mathrm{P}}^{P}\|\boldsymbol{x}_k - \boldsymbol{x}_{k+1|k}\| + \|\boldsymbol{x}_{k+1|k} - \boldsymbol{x}^*(t_{k+1})\| \qquad (68b)$$

$$\leqslant \varrho_{\mathrm{P}}^{P}\left(\|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\| + \|\boldsymbol{x}^*(t_k) - \boldsymbol{x}^*(t_{k+1})\| + \\ \|\boldsymbol{x}^*(t_{k+1}) - \boldsymbol{x}_{k+1|k}\|\right) + \|\boldsymbol{x}_{k+1|k} - \boldsymbol{x}^*(t_{k+1})\| \qquad (68c)$$

$$\leqslant \varrho_{\mathrm{P}}^{P}\|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\| + \left(\varrho_{\mathrm{P}}^{P} + 1\right)\|\boldsymbol{x}_{k+1|k} - \boldsymbol{x}^*(t_{k+1})\| + \\ \varrho_{\mathrm{P}}^{P}\|\boldsymbol{x}^*(t_{k+1}) - \boldsymbol{x}^*(t_k)\| \qquad (68d)$$

$$\leqslant \eta_0\|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\|^2 + \eta_1\|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\| + \eta_2, \qquad (68e)$$

where we have defined

$$\eta_0 = (\varrho_{\mathrm{P}}^{P} + 1)\frac{C_1}{2\,m}, \qquad (69)$$

$$\eta_1 = \varrho_{\mathrm{P}}^{P} + h\,(\varrho_{\mathrm{P}}^{P} + 1)\left(\frac{C_1 C_0}{m^2} + \frac{C_2}{m}\right), \qquad (70)$$

$$\eta_2 = \varrho_{\mathrm{P}}^{P}\left(h\,\frac{C_0}{m} + \Delta\right) + \Delta, \qquad (71)$$

where Δ is defined as in Proposition 1.

Correction error. We look now at the correction step, which by using standard results for the projected gradient method, we have

$$\|\boldsymbol{x}_{k+1} - \boldsymbol{x}^*(t_{k+1})\| \leqslant \varrho_{\mathrm{C}}^{C}\,\|\tilde{\boldsymbol{x}}_{k+1|k} - \boldsymbol{x}^*(t_{k+1})\|, \qquad (72)$$

with $\varrho_{\mathrm{C}} = \max\{|1 - \alpha m|, |1 - \alpha L|\}$. And by putting together the result (68e) with (72), we obtain the recursive error bound,

$$\|\boldsymbol{x}_{k+1} - \boldsymbol{x}^*(t_{k+1})\| \leqslant \varrho_{\mathrm{C}}^{C}\Big(\eta_0\|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\|^2 + \\ \eta_1\|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\| + \eta_2\Big). \qquad (73)$$

Global error and convergence. Call for simplicity $\bar{\eta}_i = \varrho_{\mathrm{C}}^{C}\eta_i$, for $i = 0, 1, 2$. Then convergence is achieved if

1) Each iteration does not increase the error, so that

$$\bar{\eta}_0\|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\|^2 + \bar{\eta}_1\|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\| + \bar{\eta}_2 \leqslant \\ \tau\|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\| + \bar{\eta}_2 \qquad (74)$$

for a $\tau < 1$;

2) One can find a $\tau < 1$ such that (74) holds.

By simple algebra, convergence is achieved if $\bar{\eta}_1 < \tau < 1$, that is if

$$\varrho_{\mathrm{P}}^{P}\varrho_{\mathrm{C}}^{C} < \tau, \quad h < \frac{\tau - \varrho_{\mathrm{C}}^{C}\varrho_{\mathrm{P}}^{P}}{\varrho_{\mathrm{C}}^{C}(\varrho_{\mathrm{P}}^{P} + 1)}\left(\frac{C_1 C_0}{m^2} + \frac{C_2}{m}\right)^{-1} = \bar{h}, \quad (75)$$

which sets the bounds on the number of prediction and correction steps as well as the sampling period, and if the initial optimality gap is chosen as

$$\|\boldsymbol{x}_0 - \boldsymbol{x}^*(t_0)\| \leqslant \frac{\tau - \bar{\eta}_1}{\bar{\eta}_0} = \bar{R}. \qquad (76)$$

The convergence region depends on the sampling period and on the prediction and correction steps. When $h \to 0$, then

$$\lim_{h \to 0} \bar{R} = \frac{2\,m}{C_1}\frac{\tau - \varrho_{\mathrm{P}}^{P}\varrho_{\mathrm{C}}^{C}}{\varrho_{\mathrm{C}}^{C}(\varrho_{\mathrm{P}}^{P} + 1)}. \qquad (77)$$

As for the convergence asymptotical error, by using (73) in combination with (74), we can show that

$$\|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\| \leqslant \tau^k \|\boldsymbol{x}_0 - \boldsymbol{x}^*(t_0)\| + \bar{\eta}_2 \frac{1 - \tau^k}{1 - \tau}, \quad (78)$$

from which, by letting $k \to \infty$, the result (20) follows. ∎

APPENDIX C
PROOF OF THEOREM 2

We proceed similarly to Appendix B. One of the main difference is a new Proposition 2 suited for the situation at hand.

Proposition 4: Under the same assumptions and notation of Theorem 2, we have that

$$\|\boldsymbol{x}_{k+1|k} - \boldsymbol{x}^*(t_{k+1})\| \leqslant \gamma \frac{C_1}{2\,m} \|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\|^2 +$$
$$\left[1 - \gamma + h\left(\frac{C_1 C_0}{m^2} + \frac{C_2}{m}\right)\right] \|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\| + \Delta. \quad (79)$$

Proof: Start by noticing that Proposition 1 holds true even when $\gamma < 1$ (in the unconstrained case), since $\nabla_{\boldsymbol{x}} f(\boldsymbol{x}^*(t_k); t_k) = \nabla_{\boldsymbol{x}} f_0 = \boldsymbol{0}$. In fact, Eq. (46) should read

$$\nabla_{\boldsymbol{x}} f_0 + \mathbf{Q}_0 (\boldsymbol{x} - \boldsymbol{x}_0) + h\,\boldsymbol{c}_0 = (1 - \gamma) \nabla_{\boldsymbol{x}} f_0, \quad (80)$$

but this is in fact equivalent to the original (46), since $\nabla_{\boldsymbol{x}} f_0 = \boldsymbol{0}$, and therefore the whole proposition is still valid, and in particular $\|\boldsymbol{x}^*_{k+1|k} - \boldsymbol{x}^*(t_{k+1})\| \leqslant \Delta$.

Call now $\delta \boldsymbol{x}_k = \boldsymbol{x}_{k+1|k} - \boldsymbol{x}_k$, and $\delta \boldsymbol{x}^*_k = \boldsymbol{x}^*_{k+1|k} - \boldsymbol{x}^*(t_k)$. Then,

$$\|\boldsymbol{x}_{k+1|k} - \boldsymbol{x}^*(t_{k+1})\| =$$
$$\|\boldsymbol{x}_k + \delta \boldsymbol{x}_k - (\boldsymbol{x}^*(t_k) + \delta \boldsymbol{x}^*_k) + (\boldsymbol{x}^*_{k+1|k} - \boldsymbol{x}^*(t_{k+1}))\|, \quad (81)$$

which given Proposition 1 and by using the Triangle inequality can be upper bounded as

$$\|\boldsymbol{x}_{k+1|k} - \boldsymbol{x}^*(t_{k+1})\| \leqslant (1-\gamma)\|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\| + \|\delta \check{\boldsymbol{x}}_k - \delta \boldsymbol{x}^*_k\| + \Delta, \quad (82)$$

where we have set $\delta \check{\boldsymbol{x}}_k = \delta \boldsymbol{x}_k + \gamma(\boldsymbol{x}_k - \boldsymbol{x}^*(t_k))$. Note that this decomposition may seem cumbersome, yet it is the cornerstone of the proof of this proposition.

For $\delta \check{\boldsymbol{x}}_k$ and $\delta \boldsymbol{x}^*_k$, it holds that

$$\gamma \nabla_{\boldsymbol{x}} f_k + \gamma \mathbf{Q}_k (\boldsymbol{x}_0 - \boldsymbol{x}_k) + \mathbf{Q}_k \delta \check{\boldsymbol{x}}_k + h\,\boldsymbol{c}_k = \boldsymbol{0}, \quad (83)$$
$$\gamma \nabla_{\boldsymbol{x}} f_0 + \mathbf{Q}_0 \delta \boldsymbol{x}^*_k + h\,\boldsymbol{c}_0 = \boldsymbol{0}, \quad (84)$$

where as in (57) we have used the simplifications

$$\nabla_{\boldsymbol{x}} f_k = \nabla_{\boldsymbol{x}} f(\boldsymbol{x}_k; t_k), \quad \mathbf{Q}_k = \nabla_{\boldsymbol{x}\boldsymbol{x}} f(\boldsymbol{x}_k; t_k) \quad (85a)$$
$$\boldsymbol{c}_k = \nabla_{t\boldsymbol{x}} f(\boldsymbol{x}_k; t_k), \quad \boldsymbol{x} = \boldsymbol{x}_{k+1|k}. \quad (85b)$$

while $\nabla_{\boldsymbol{x}} f_0$, $\mathbf{Q}_0$, $\boldsymbol{c}_0$, and $\boldsymbol{x}_0$ are defined just as in (44)-(45), for $i = 0$ (i.e., $\boldsymbol{x}_0 = \boldsymbol{x}^*(t_k)$, and so on). Define

$$g(\delta \boldsymbol{x}) = \gamma \left(\nabla_{\boldsymbol{x}} f_k - \nabla_{\boldsymbol{x}} f_0 + \mathbf{Q}_k (\boldsymbol{x}_0 - \boldsymbol{x}_k)\right) +$$
$$(\mathbf{Q}_k - \mathbf{Q}_0)\delta \boldsymbol{x} + h\,(\boldsymbol{c}_k - \boldsymbol{c}_0), \quad (86)$$
$$G(\delta \boldsymbol{x}) = \gamma \nabla_{\boldsymbol{x}} f_0 + \mathbf{Q}_0 \delta \boldsymbol{x} + h\,\boldsymbol{c}_0, \quad (87)$$

and notice that $(g + G)(\delta \check{\boldsymbol{x}}_k) = \boldsymbol{0}$, while $(g + G)(\delta \boldsymbol{x}^*_k) = g(\delta \boldsymbol{x}^*_k)$. With a similar argument as the one of the proof of Proposition 2, then

$$\|\delta \check{\boldsymbol{x}}_k - \delta \boldsymbol{x}^*_k\| \leqslant \frac{1}{m} \|g(\delta \boldsymbol{x}^*_k)\|. \quad (88)$$

Let us now bound $\|g(\delta \boldsymbol{x}^*_k)\|$. It is relatively straightforward to see that due to Assumptions 1-2,

$$\|g(\delta \boldsymbol{x}^*_k)\| \leqslant \gamma \frac{C_1}{2} \|\boldsymbol{x}_k - \boldsymbol{x}_0\|^2 + C_1 \|\boldsymbol{x}_k - \boldsymbol{x}_0\| \|\delta \boldsymbol{x}^*_k\| + h\,C_2 \|\boldsymbol{x}_k - \boldsymbol{x}_0\|. \quad (89)$$

The next step of the proof is to upper bound the term $\|\delta \boldsymbol{x}^*_k\| = \|\boldsymbol{x}^*_{k+1|k} - \boldsymbol{x}^*(t_k)\|$. We know that

$$\gamma \nabla_{\boldsymbol{x}} f_0 + \mathbf{Q}_0 \delta \boldsymbol{x}^*_k + h\boldsymbol{c}_0 = \boldsymbol{0}, \quad (90)$$

and since $\nabla_{\boldsymbol{x}} f_0 = \boldsymbol{0}$, then $\delta \boldsymbol{x}^*_k = -h\,\mathbf{Q}_0^{-1} \boldsymbol{c}_0$. Which yields

$$\|\delta \boldsymbol{x}^*_k\| \leqslant h \frac{C_0}{m}. \quad (91)$$

By putting together the bounds (82), (88), (89), and (91), the claim (79) follows. ∎

By using Proposition 4 along with the same arguments as the one in Eq.s (67) till (73), we arrive at the error recursion

$$\|\boldsymbol{x}_{k+1} - \boldsymbol{x}^*(t_{k+1})\| \leqslant \bar{\eta}'_0 \|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\|^2 + \bar{\eta}'_1 \|\boldsymbol{x}_k - \boldsymbol{x}^*(t_k)\| + \bar{\eta}'_2, \quad (92)$$

where,

$$\bar{\eta}'_0 = \gamma \varrho^C_{\mathsf{C}} (\varrho^P_{\mathsf{P}} + 1) \frac{C_1}{2\,m}, \quad (93)$$
$$\bar{\eta}'_1 = \varrho^C_{\mathsf{C}} \left[\varrho^P_{\mathsf{P}} + (\varrho^P_{\mathsf{P}} + 1)\left(1 - \gamma + h\left(\frac{C_1 C_0}{m^2} + \frac{C_2}{m}\right)\right)\right], \quad (94)$$
$$\bar{\eta}'_2 = \varrho^C_{\mathsf{C}} \left[\varrho^P_{\mathsf{P}} \left(h \frac{C_0}{m} + \Delta\right) + \Delta\right]. \quad (95)$$

By using now the same reasoning as in Eq.s (74) till Eq.s (78), the claim (25) can be proven. ∎

APPENDIX D
PROOF OF THEOREM 3

The proof follows closely the proof of Theorem 1 and we report here only the main differences. Considering the differences between the prediction (31)-(34) and (7), one can see that Proposition 2 holds as it is, with the substitution of $\boldsymbol{x}$ with $\boldsymbol{z}$. Proposition 3 does not hold anymore, since we are using a saddle-point projected gradient, and not a regular projected gradient. By using standard results, e.g. [23], [36] we can prove that

$$\|\tilde{\boldsymbol{z}}_{k+1|k} - \boldsymbol{z}_{k+1|k}\| \leqslant \varrho^P_{\mathsf{P}} \|\boldsymbol{z}_k - \boldsymbol{z}_{k+1|k}\|, \quad (96)$$

with $\varrho_{\mathsf{P}} = \sqrt{1 - 2\alpha m + \alpha^2 L^2}$. And a similar result holds also for the correction step (32). Putting these facts together, in a similar fashion as in the proof of Theorem 1, the claim (37) can be proven. ∎

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