

Constrained Multi-Slot Optimization for Ranking Recommendations

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ABSTRACT

Ranking items to be recommended to users is one of the main problems in large scale social media applications. This problem can be set up as a multi-objective optimization problem to allow for trading off multiple, potentially conflicting objectives (that are driven by those items) against each other. Most previous approaches to this problem optimize for a single slot without considering the interaction effect of these items on one another.

In this paper, we develop a constrained multi-slot optimization formulation, which allows for modeling interactions among the items on the different slots. We characterize the solution in terms of problem parameters and identify conditions under which an efficient solution is possible. The problem formulation results in a quadratically constrained quadratic program (QCQP). We provide an algorithm that gives us an efficient solution by relaxing the constraints of the QCQP minimally. Through simulated experiments, we show the benefits of modeling interactions in a multi-slot ranking context, and the speed and accuracy of our QCQP approximate solver against other state of the art methods.

CCS Concepts

•**Theory of computation** → **Mathematical optimization**; *Quadratic programming*; •**Information systems** → *Content ranking*;

Keywords

Large scale multi objective optimization, Multi-slot optimization, feed ranking, recommendation systems

1. INTRODUCTION

Ranking of items on a recommendation platform has become one of the most important problems in most internet and social media applications. Popular examples include the feed on most social media applications like Facebook, LinkedIn and Instagram (Figure 1). The People You May

Know (PYMK) application on LinkedIn which recommends professional connections to the user, email digests with various news articles that can be interesting to the user are other such examples (Figure 2). All of these involve showing a list of ranked items from a larger set of candidates to the user. As the scale of these applications becomes progressively larger with time, they have more data to optimize and provide a better user experience. In the case of feed ranking at LinkedIn [3, 4] for example, there can be multiple items of different types (articles shared by first degree connections, status updates, connection updates, job anniversaries to name a few) all of which need to be consolidated to generate a ranked list which should be the most engaging to the user. Due to the cost of optimization of most ranking objectives in machine learning, scaling ranking methods to very large data sets becomes difficult [12, 22]. In most real life applications, these ranked recommendations are generated by separately sorting each of these classes of items (via predicted estimates of their click-through rate (CTR) or some appropriate notion of engagement) and then mixing them together using an algorithm.

At the same time the businesses operating these applications also aim to grow by potentially trading off immediate user engagement (via CTR) with monetization metrics like revenue from native ads and/or user retention. This necessitates a formulation where multiple such objectives can be traded off efficiently while trying to provide a balanced user experience. There has been a plethora of research addressing this problem, commonly known as Multi-Objective Optimization (MOO) [1, 2, 5, 13, 18, 19], both in theory and practice.

However most of the practical, scalable approaches **consider generating scores for each candidate item for a single slot, rather than considering a global optimization for the entire list of candidate items**. As a result, they fail to take into account the interaction effect these items can have on the user while occurring in multiple slots. Consider the example of a user being shown two similar items, both of which are separately very relevant to her, but if we rank them consecutively at the top of her feed, the user is less likely to click on the second item and the user experience suffers. These ideas can also be extended to cover approaches like impression discounting of a group of items (which governs that a user should not see too many impressions of the same type on consecutive visits to the feed) as well as diversity of items (which governs that items of the same type should not occur together). The importance of context of an item gets magnified on the mobile screen due

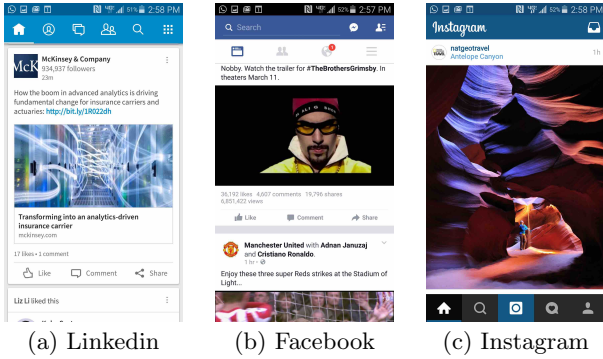


Figure 1: (News) Feed from various social networks. Each of them has recommendations ranked in some order over multiple slots

to the limited attention span and display space.

In this paper, we try to address this class of problems by coming up with a formulation which allows for this MOO trade off to be set up as a constrained optimization problem taking into account interactions between multiple slots along the lines of [2]. Our main contributions can be summarized as following:

- We provide an efficient solution for the MOO problem on the feed (without assuming multi-slot dependency) by setting it up as a QP and optimizing the dual of the QP. Although this method has been explored in [2], we can handle much more general classes of constraints and provide a trick of efficiently obtaining the primal optimum from the dual by evaluating a cheap projection. This enables us to come up with a deterministic serving plan for serving items on the feed in a sorted order to maximize some notion of user engagement while also keeping other business metrics above accepted thresholds.
- We formulate the problem of MOO in the multi-slot case, where we allow for interactions between the different items. We identify specific interaction models, mathematical conditions and assumptions which result in the problem being a QCQP. Without those, the problem is much more computationally intractable. We show that solving this QCQP results in a far superior feed than without modeling the multi-slot dependence.
- One of the big challenges is to solve a large scale QCQP in an efficient manner. We devise an algorithm which approximates the quadratic constraints by a set of linear constraints, thus converting the problem into a quadratic program (QP) whose solution is a pretty close approximation for the original optimum. We show results of convergence as well as experiments comparing our algorithm to existing QCQP solvers and how fast it can scale to large scale data. To the best of our knowledge, this technique is new and has not been previously explored in the optimization literature.

The rest of the paper is structured as follows. In Section 2, we introduce the problem of feed ranking as a multi-objective optimization problem. Section 3 gives an efficient

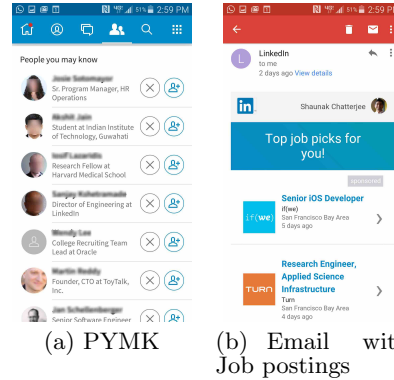


Figure 2: Other applications including People Recommendation and Emails with job recommendations, having a ranking of items over multiple slots.

solution of the posed problem. In Section 4, we describe the interaction model and the several characterizations which translate the original QP into a QCQP. We develop a new technique to solve the large scale QCQP in Section 5. Experimental results follow in Section 6 and we conclude with some discussion in Section 7.

2. FEED RANKING AS A MOO PROBLEM

We begin by introducing some notation which we use throughout the paper. Let $i = 1, \dots, n$ denote the i -th user, $j = 1, \dots, J$ denote the j -th item to be shown to the user and $k = 1, \dots, K$ denote the k -th slot on the feed. Let x_{ijk} be the serving probability of item j at slot k to user i and p_{ijk} denote the probability of clicking item j at slot k by user i conditional on the fact that j -th item was shown at position k to user i .

We allow both organic and sponsored items on the feed. Let c_{ijk} be the dollar value associated to clicking item j and \mathcal{J}_s denote the set of sponsored items. We will assume throughout that the revenue generated from a sponsored item does not depend on the position of the item, i.e. $c_{ijk} = c_j > 0$ for all i, k and $j \in \mathcal{J}_s$. We also assume that $c_{i,j,k} = 0$ for $j \notin \mathcal{J}_s$. We introduce the vector notation for $x_{ijk}, p_{ijk}, c_{ijk}$ as \mathbf{x}, \mathbf{p} and \mathbf{c} respectively. Further let $\mathbf{\$} = \mathbf{p} \cdot \mathbf{c}$ where \cdot denotes the element wise multiplication of the two vectors.

We set up the problem of feed ranking in the presence of sponsored items as a constrained optimization problem. This formulation has been familiar in the literature [1, 2] and targets mainly at maximizing the overall click-through rate (CTR) while simultaneously trying to maximize revenue by showing sponsored content on the feed. These aspects can often conflict with each other (since organic items on the feed can have higher CTR than sponsored items) which leads to the modified problem of maximizing the CTR under constraints imposed by business rules, in terms of maintaining the revenue beyond a certain threshold, showing more than a certain number of posts of a specific type (e.g. news articles) among others.

These requirements can be met by setting up a multi-objective optimization (MOO) problem. Specifically, we wish to maximize the expected clicks on the feed while keeping the revenue above a certain level and the total number

of impressions at a certain level. Under the above notation, the expected number of clicks is given by

$$\mathbb{E}(\text{Clicks}) = \sum_{i,j,k} x_{ijk} p_{ijk}.$$

while the two constraints can be written as

$$\begin{aligned} \sum_{i,j,k} x_{ijk} p_{ijk} c_{ijk} &\geq R \\ \sum_{j \in \mathcal{J}_I} \sum_{i,k} x_{ijk} &\geq I \end{aligned}$$

where R and I refer to business specified thresholds while \mathcal{J}_I defines a subset of items whose impressions are required to be more than a specified amount. Further, we know that since x_{ijk} is a probability we have $0 \leq x_{ijk} \leq 1$. Moreover, we know that for each slot k we show an item. Hence, $\sum_{j=1}^J x_{ijk} = 1$ for all i, k . And if we sum across k we get the probability of showing the j -th item. Thus we have $0 \leq \sum_{k=1}^K x_{ijk} \leq 1$ for all i, j . Combining all of these together we can formulate the problem of maximizing the clicks on the multi-slot case as follows.

$$\begin{aligned} \text{Maximize}_x \quad & \sum_{i,j,k} x_{ijk} p_{ijk} \\ \text{subject to} \quad & \sum_{i,j,k} x_{ijk} p_{ijk} c_{ijk} \geq R \\ & \sum_{i,j,k} x_{ijk} d_{ijk} \geq I \\ & 0 \leq x_{ijk} \leq 1 \quad \forall i, j, k \\ & \sum_{j=1}^J x_{ijk} = 1, \quad \forall i, k \\ & 0 \leq \sum_{k=1}^K x_{ijk} \leq 1 \quad \forall i, j, \end{aligned} \quad (1)$$

where $d_{ijk} = 1$ for $j \in \mathcal{J}_I$ and zero otherwise.

Infact, for each user i the last three constraints create a local constraint and we rewrite the equations as $\mathbf{K}_i \mathbf{x}_i \leq \mathbf{b}^i$. Replacing the maximization to a minimization problem (to exploit the convexity of the objective) and introducing a strongly convex regularization term for ease of optimization (See [2] for details) we get the following optimization problem,

$$\begin{aligned} \text{Minimize}_x \quad & -\mathbf{x}^T \mathbf{p} + \frac{\gamma}{2} \mathbf{x}^T \mathbf{x} \\ \text{subject to} \quad & \mathbf{x}' \mathbf{\$} \geq R \\ & \mathbf{x}' \mathbf{d} \geq I \\ & \mathbf{x}_i \in \mathcal{K}_i \quad \forall i \end{aligned} \quad (2)$$

where $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$ and \mathcal{K}_i denotes the convex set created by the linear equations, $\mathbf{K}_i \mathbf{x}_i \leq \mathbf{b}^i$.

The main hurdle in optimizing (2) is the prohibitively expensive cost arising as the number of users n is in the order of millions for most real life web applications. Thus, instead of solving it directly we try to evaluate the dual variables corresponding to the first two global constraints as in [2]. In [5], the authors show that if we have the optimal dual variables then it possible to get the optimal primal solution in an online setting efficiently by solving a small quadratic programming problem.

3. AN EFFICIENT SOLUTION

Instead of solving the optimization problem (2) directly, we employ a two step procedure as in [2]. In the first stage we solve the dual problem of (2) to get optimal dual variables. Using the dual variables we obtain the primal solution through a neat conversion trick. Finally, we describe how to get a deterministic serving plan from the probabilistic optimal primal solution. Each step of this procedure is described below.

3.1 Dual Solution

The local constraints $\mathbf{x}_i \in \mathcal{K}_i$ for all $i = 1, \dots, n$ can be combined together to get a constraint like $\mathbf{K}\mathbf{x} \leq \mathbf{b}$, where $\mathbf{K} = \text{diag}(\mathbf{K}_i, i = 1, \dots, n)$. Thus with this notation we can transform the problem (2) as

$$\begin{aligned} \text{Minimize}_x \quad & -\mathbf{x}^T \mathbf{p} + \frac{\gamma}{2} \mathbf{x}^T \mathbf{x} \\ \text{subject to} \quad & \mathbf{x}^T \mathbf{\$} \geq R \\ & \mathbf{x}^T \mathbf{d} \geq I \\ & \mathbf{K}\mathbf{x} \leq \mathbf{b} \end{aligned} \quad (3)$$

Writing the Lagrangian of the above problem, we have

$$\begin{aligned} L(\mathbf{x}, \mu_0, \mu_1, \boldsymbol{\eta}) = & -\mathbf{x}^T \mathbf{p} + \frac{\gamma}{2} \mathbf{x}^T \mathbf{x} + \mu_0 (R - \mathbf{x}^T \mathbf{\$}) \\ & + \mu_1 (I - \mathbf{x}^T \mathbf{d}) + \boldsymbol{\eta}^T (\mathbf{K}\mathbf{x} - \mathbf{b}). \end{aligned}$$

Finding the minimum with respect to x we see get,

$$\hat{\mathbf{x}} = \frac{1}{\gamma} (\mu_0 \mathbf{\$} + \mu_1 \mathbf{d} + \mathbf{p} - \mathbf{K}^T \boldsymbol{\eta}).$$

Plugging this back into the Lagrangian we get,

$$\begin{aligned} L(\hat{\mathbf{x}}, \mu_0, \mu_1, \boldsymbol{\eta}) = & \frac{\gamma}{2} \hat{\mathbf{x}}^T \hat{\mathbf{x}} - \hat{\mathbf{x}}^T (\mu_0 \mathbf{\$} + \mu_1 \mathbf{d} + \mathbf{p} - \mathbf{K}^T \boldsymbol{\eta}) \\ & + \mu_0 R + \mu_1 I - \boldsymbol{\eta}^T \mathbf{b} \\ = & -\frac{\gamma}{2} \hat{\mathbf{x}}^T \hat{\mathbf{x}} + \mu_0 R + \mu_1 I - \boldsymbol{\eta}^T \mathbf{b}. \end{aligned}$$

Writing $\boldsymbol{\xi} = (R, I, -\mathbf{b}^T)^T$, $\mathbf{A} = (\mathbf{\$} : \mathbf{d} : -\mathbf{K}^T)$ and $\mathbf{y} = (\mu_0, \mu_1, \boldsymbol{\eta})$ the above problem simplifies to

$$\begin{aligned} \text{Maximize}_y \quad & \boldsymbol{\xi}^T \mathbf{y} - \frac{1}{2\gamma} (\mathbf{p} + \mathbf{A}\mathbf{y})^T (\mathbf{p} + \mathbf{A}\mathbf{y}) \\ \text{subject to} \quad & \mathbf{y} \geq 0 \end{aligned} \quad (4)$$

which can be further simplified to

$$\begin{aligned} \text{Minimize}_y \quad & \frac{1}{2} \mathbf{y}^T \mathbf{M} \mathbf{y} - \mathbf{y}^T (\boldsymbol{\xi} - \mathbf{A}^T \mathbf{p} / \gamma) \\ \text{subject to} \quad & \mathbf{y} \geq 0 \end{aligned} \quad (5)$$

where $\mathbf{M} = \mathbf{A}^T \mathbf{A} / \gamma$. We solve this by the operator splitting algorithm [21] to obtain μ_0 and μ_1 .

REMARK 1. *We cannot apply block splitting algorithm [21] to the dual problem since the objective function is no longer separable. We may apply it to the primal but obtaining the dual solution may be not be possible in all situations. For details, we refer to [5].*

The ability to solve this problem on a large scale completely depends on the sparsity of matrix \mathbf{M} .

DEFINITION 1. *Let \mathbf{A} be any $m \times n$ matrix. We define the sparsity ratio, $\psi(\mathbf{A})$ as k if the number of non-zero entries in the matrix is kmn .*

From previous experimentation we were able to solve the dual problem having n variables through operator splitting if $\psi(\mathbf{M})$ is of order $O(1/n)$. We shall show in this setup too, $\psi(\mathbf{M})$ is of the same order.

LEMMA 1. Consider the dual problem for the multi-slot optimization problem as given in (5). Let \mathcal{J}_s and \mathcal{J}_I denote the sets of sponsored and impression important items. If there are K slots, J total items and n users, then,

$$\psi(\mathbf{M}) = \frac{1 + n(J + \beta + K(3 + \beta) + 7JK)}{(1 + nJ + nK + nJK)^2}.$$

where $\beta = |\mathcal{J}_s| + |\mathcal{J}_I|$.

PROOF. Note that, using the previous notation we can write,

$$\mathbf{M} = \frac{1}{\gamma} \mathbf{A}^T \mathbf{A} = \frac{1}{\gamma} \begin{bmatrix} \mathbf{\$}^T \mathbf{\$} & \mathbf{\$}^T \mathbf{d} & -\mathbf{\$}^T \mathbf{K}^T \\ \mathbf{d}^T \mathbf{\$} & \mathbf{d}^T \mathbf{d} & -\mathbf{d}^T \mathbf{K}^T \\ -\mathbf{K} \mathbf{\$} & -\mathbf{K} \mathbf{d} & \mathbf{K} \mathbf{K}^T \end{bmatrix}.$$

Without loss of generality we assume that $\mathcal{J}_s \cap \mathcal{J}_I \neq \emptyset$. Then the top-left 2×2 principal sub matrix of M contains 4 non-zero terms. Otherwise only the diagonals are positive and we only lose a count of 2 to the total number of non-zero terms. Note that for each $i = 1, \dots, n$ we can write \mathbf{K}_i^T as

$$\mathbf{K}_i^T = [\mathbf{I}_{JK} \quad -\mathbf{I}_{JK} \quad \mathbf{B}^T \quad -\mathbf{B}^T \quad \mathbf{C}^T \quad -\mathbf{C}^T],$$

where,

$$\mathbf{B} = [\mathbf{I}_K \quad \mathbf{I}_K \quad \dots \quad \mathbf{I}_K]_{K \times JK}, \quad \mathbf{C} = \mathbf{I}_J \otimes \mathbf{1}_K^T,$$

and \mathbf{I}_n denotes the identity matrix of dimension n and \otimes denote the Kronecker product. Note that the matrices in \mathbf{K}_i denotes the constraints for the i -user of optimization problem (1). Let $\text{card}(\mathbf{A})$ denotes the number of non-zero entries in a matrix \mathbf{A} . Now it is easy to see that,

$$\text{card}(\mathbf{K}_i \mathbf{\$}_i) = 2(K + |\mathcal{J}_s| + K|\mathcal{J}_s|) \quad (6)$$

$$\text{card}(\mathbf{K}_i \mathbf{d}_i) = 2(K + |\mathcal{J}_I| + K|\mathcal{J}_I|). \quad (7)$$

To complete the proof, we note that,

$$\text{card}(\mathbf{I}_{JK}) = \text{card}(\mathbf{B}) = \text{card}(\mathbf{C}) = JK, \quad (8)$$

$$\text{card}(\mathbf{B}\mathbf{B}^T) = K, \quad \text{card}(\mathbf{C}\mathbf{C}^T) = J, \quad \text{card}(\mathbf{B}\mathbf{C}^T) = JK. \quad (9)$$

Now, using the structure is $\mathbf{K}\mathbf{K}^T$, and equations (8) and (9), we get

$$\text{card}(\mathbf{K}\mathbf{K}^T) = 4n(J + K + 7JK). \quad (10)$$

The result follows from using (6), (7) and (10) and observing that the total number of entries in the matrix \mathbf{M} is $(2 + 2nK + 2nJ + 2nJK)^2$. \square

From Lemma 1, it is easy to see that $\psi(\mathbf{M}) = O(1/(nJK))$. Thus, for example if we are able to solve the single slot for 2 million users, having $K = 5$ and $J = 10$ we can solve this multi-slot problem at the same time and same accuracy for 40000 users.

3.2 Dual to Primal Trick

The main idea behind this technique is to exploit the KKT conditions of the problem, so that we can write the primal

solution as a function of the dual solution and the input parameters. To explain this technique, let us start by rewriting the problem given in (2) as

$$\begin{aligned} \underset{\mathbf{x}}{\text{Minimize}} \quad & -\mathbf{x}^T \mathbf{p} + \frac{\gamma}{2} \mathbf{x}^T \mathbf{x} + \mathbb{1}_{\mathcal{K}_1 \times \mathcal{K}_2 \times \dots \times \mathcal{K}_n}(\mathbf{x}) \\ \text{subject to} \quad & \mathbf{x}^T \mathbf{\$} \geq R, \\ & \mathbf{x}^T \mathbf{d} \geq I, \end{aligned} \quad (11)$$

where $\mathcal{K}_1 \times \mathcal{K}_2 \times \dots \times \mathcal{K}_n$ denotes the domain of \mathbf{x} and $\mathbb{1}_{\mathcal{C}}(\mathbf{x}) = 0$ if $\mathbf{x} \in \mathcal{C}$ and ∞ otherwise. Introducing the dual variables μ_0 and μ_1 for the two constraints we can write the Lagrangian as follows.

$$L(\mathbf{x}, \mu_0, \mu_1) = -\mathbf{x}^T \mathbf{p} + \frac{\gamma}{2} \mathbf{x}^T \mathbf{x} + \mu_0(R - \mathbf{x}^T \mathbf{\$}) + \mu_1(I - \mathbf{x}^T \mathbf{d}),$$

for $\mathbf{x} \in \mathcal{K}_1 \times \mathcal{K}_2 \times \dots \times \mathcal{K}_n$. From this it is easy to see that the optimal solution satisfies

$$\mathbf{x}^{opt} = \Pi_{\mathcal{K}_1 \times \mathcal{K}_2 \times \dots \times \mathcal{K}_n} \left(\frac{1}{\gamma} (\mu_0 \mathbf{\$} + \mu_1 \mathbf{d} + \mathbf{p}) \right), \quad (12)$$

where $\Pi_{\mathcal{C}}(\cdot)$ denotes the projection function onto \mathcal{C} . Note that, since we have a product domain, we can write this as

$$\mathbf{x}_i^{opt} = \Pi_{\mathcal{K}_i} \left(\frac{1}{\gamma} (\mu_0 \mathbf{\$} + \mu_1 \mathbf{d} + \mathbf{p}) \right) \quad \forall i = 1, \dots, n.$$

Thus once we know μ_0 and μ_1 by solving (5), x_i can be obtained simply by the projection onto \mathcal{K}_i which is a small quadratic problem and can be solved fast and efficiently.

3.3 Deterministic Serving Plan

Here we present the algorithm to generate the serving plan for user i using the optimal probabilistic solution x_{ijk} . We know for each slot k the items follow a multinomial distribution with probability x_{ijk} for $j = 1, \dots, J$. While serving we have the added criteria that no two items can be repeated. Thus for each slot k we sample from $Mult(\{x_{ijk}\}_{j=1}^J)$ and we resample if we get the same item for two slots. This guarantees that that serving plan obeys the optimal serving distribution. The detailed steps are written out in Algorithm 1.

Algorithm 1 Deterministic Serving Plan

- 1: Input : Optimal primal solution \mathbf{x}
 - 2: Output : Deterministic serving plan for each user
 - 3: Set S empty $n \times K$ matrix
 - 4: **for** $i = 1 : n$ **do**
 - 5: $Sample = \emptyset$
 - 6: **for** $k = 1 : K$ **do**
 - 7: Pick $j \sim Mult(\{x_{ijk}\}_{j=1}^J)$
 - 8: **if** $j \in Sample$ **then**
 - 9: Go to step 7.
 - 10: **else**
 - 11: Add j to $Sample$.
 - 12: **end if**
 - 13: **end for**
 - 14: $S[i, :] = Sample$
 - 15: **end for**
 - 16: **return** S
-

4. MODELING INTERACTION

The previous sections outline a scheme to display feed contents using a constrained optimization formulation assuming the items act independently of each other. However in most situations, the feed content has an interaction effect. In other words, the user's probability of clicking on a post might change depending on the type of items he has seen in his feed until now, particularly if he has been displayed items from the same content group. In order to avoid this problem, most feed algorithms use a mixer to mix content from different channels. In practice, for each channel or group we have a sorted content list based on a single slot optimization problem. The output of each channel is then mixed to create the final feed. However, doing this clearly results in a loss of the optimal ranking of the feed content and a sub-par feed experience.

One of the ways of tackling this problem is to set it up as a multi-slot optimization problem so that we can directly optimize for the best feed ranking under constraints while also respecting the dependency structure among multiple slots. Note that the same idea also comes in useful for tackling problems related to impression discounting or diversity on the feed, which influence the ranking

In this section, we develop a technique to solve the dependency problem as a multi-slot optimization problem instead of using ad-hoc mixing algorithms to individual channels. Following the notation in the previous sections, we consider the following generic optimization problem which encompasses the objectives described before.

$$\begin{aligned} \underset{\mathbf{x}}{\text{Minimize}} \quad & -\mathbf{x}^T \mathbf{p} + \frac{\gamma}{2} \mathbf{x}^T \mathbf{x} \\ \text{subject to} \quad & \mathbf{x}^T \mathbf{r} \leq P \\ & \mathbf{K} \mathbf{x} \leq \mathbf{b} \end{aligned} \quad (13)$$

Note that \mathbf{p} and \mathbf{r} are the only two parameters in the problem and we currently assume that they are independent of each other. (For the more generalized formulation, see Section 4.3). A lot of practical problems can be posed as above. Some examples include multi-slot feed where $-\mathbf{r}$ can be thought of as a substitute of $\mathbf{\$}$ from (2), multi-slot product email updates with \mathbf{p} denoting probability of clicks and \mathbf{r} denoting the probability of complaints on sending the email, the People-You-May-Know application which is a recommended list of people you can professionally connect with, where \mathbf{p} can denote the probability of clicks while \mathbf{r} can represent the probability of dismissing the recommendation.

We model the interaction effect as follows

$$\mathbf{p} = -\mathbf{Q}_p \mathbf{x} \quad \text{and} \quad \mathbf{r} = \mathbf{Q}_r \mathbf{x}. \quad (14)$$

where \mathbf{Q}_p and \mathbf{Q}_r are some positive definite matrices (exact choice of which is discussed in Section 4.1). Using (14) in (13) we re-write the problem as,

$$\begin{aligned} \underset{\mathbf{x}}{\text{Minimize}} \quad & \mathbf{x}' \left(\mathbf{Q}_p + \frac{\gamma}{2} \mathbf{I} \right) \mathbf{x} \\ \text{subject to} \quad & \mathbf{x}' \mathbf{Q}_r \mathbf{x} \leq P \\ & \mathbf{K} \mathbf{x} \leq \mathbf{b} \end{aligned} \quad (15)$$

Note that this makes the problem much more complicated since we transform a quadratic programming problem (QP) to a quadratically constrained quadratic program (QCQP) a general version of which is actually NP-hard [9]. Moreover the choice of \mathbf{Q}_p and \mathbf{Q}_r is extremely important since

convexity of the problem depends on $\mathbf{Q}_p, \mathbf{Q}_r$ being positive definite.

4.1 Choice of \mathbf{Q}_p and \mathbf{Q}_r

The choice of these matrices is a hard problem in practice since we do not know the exact dependency structure of \mathbf{p} and \mathbf{r} on \mathbf{x} . However, due to the presence of enormous empirical data for clicks as well as complaints and dismisses, estimation of $\mathbf{Q}_p, \mathbf{Q}_r$ is not a hard problem, if there exists a parametric form. Since \mathbf{Q}_p and \mathbf{Q}_r can have similar parametric form, without loss of generality we explain the form through \mathbf{Q}_p .

Throughout we assume $\mathbf{Q}_p = \text{Diag}(\mathbf{Q}_i, i = 1, \dots, n)$, i.e. the probabilities across the users are independent and each \mathbf{Q}_i is a $JK \times JK$ dimensional positive definite matrix which we parametrize below. We assume that all K slots are viewable by a user, and the chance of an event at any slot depends on all the rest. Moreover we hide the i in the subscript for notational simplicity.

Let us begin by introducing some more notation. For any $j \in \{1, \dots, J\}$, let $\mathbf{p}_j, \mathbf{x}_j$ denote K -dimensional vectors comprising of elements $\{p_{jk}\}_{k=1}^K$ and $\{x_{jk}\}_{k=1}^K$ respectively. Let $\mathbf{Q}_{jj'}$ denote the $K \times K$ dependence matrix between event j and observation j' . This leads to the following expression

$$\mathbf{p} = \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \vdots \\ \mathbf{p}_J \end{pmatrix} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} & \dots & \mathbf{Q}_{1J} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23} & \dots & \mathbf{Q}_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{Q}_{J1} & \mathbf{Q}_{J2} & \mathbf{Q}_{J3} & \dots & \mathbf{Q}_{JJ} \end{bmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_J \end{pmatrix}$$

Now we characterize each $\mathbf{Q}_{jj'}$. Note that for any j, k , p_{jk} is the probability of an event conditional of the fact that $x_{jk} = 1$. Thus, it depends on \mathbf{x}_j only through x_{jk} . Hence it is easy to see that for any $j \in \{1, \dots, J\}$,

$$\mathbf{Q}_{jj} = \tilde{p}_j \mathbf{I}_K \quad (16)$$

where \tilde{p}_j is the prior event probability conditional on seeing item j irrespective of its position, and \mathbf{I}_K is the $K \times K$ identity matrix. The values of \tilde{p}_j for $j = 1, \dots, J$ can be estimated from empirical data.

Now for any $j \neq j'$, we consider the dependency between the event corresponding to item j and the observation of j' . When we are considering p_{jk} , since we know that $x_{jk} = 1$, we must have that the contribution from $x_{j'k} = 0$ for $j' \neq j$. Moreover, that is the only coefficient which is zero, since we let p_{jk} depend on $x_{j'k'}$ for $k' \neq k$. Thus, we have,

$$\mathbf{Q}_{jj'} = \begin{bmatrix} 0 & a_{12} & a_{13} & \dots & a_{1K} \\ a_{21} & 0 & a_{23} & \dots & a_{2K} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{K1} & a_{K2} & a_{K3} & \dots & 0 \end{bmatrix}. \quad (17)$$

We estimate each of the $a_{\ell\ell'}$ using empirical data. Finally to bring in symmetry in the problem, we assume that

$$\mathbf{Q}_{j'j} = \mathbf{Q}_{jj'}^T \quad (18)$$

for all j and j' . This condition will often be true for similar items, or for dissimilar items which have similar effects on one another (whether mutually synergistic or antagonistic). However, we also acknowledge that there will be certain cases where this condition will not hold, and that will make the problem non-convex.

Combining the structure in equations (16), (17) and (18), we get the complete parametric form of the matrix \mathbf{Q} .

4.2 Practical considerations

If we assume the parametric form of \mathbf{Q} as given in Section 4.1, the postive-definiteness of \mathbf{Q} is a concern since we estimate each of the $a_{\ell\ell'}$ from the data. An easy fix to this problem, is to add $\eta\mathbf{I}_{JK}$ to \mathbf{Q} for some appropriate value of η . This enforces that \mathbf{Q} to be positive-definite by only slightly increasing the elements corresponding to \tilde{p}_j . Thus instead of using the estimated \mathbf{Q} directly we use,

$$\tilde{\mathbf{Q}} := \begin{cases} \mathbf{Q} + (-\lambda_1(\mathbf{Q}) + \epsilon)\mathbf{I}_{JK} & \text{if } \lambda_1(\mathbf{Q}) < 0 \\ \mathbf{Q} & \text{otherwise} \end{cases} \quad (19)$$

where $\lambda_1(\mathbf{Q})$ denotes the minimum eigenvalue of \mathbf{Q} and $\epsilon > 0$ is arbitrary.

An advantage of using this technique from a computational point of view is that since each \mathbf{Q} is of small dimension (JK being small), the minimum eigenvalue calculation can be done extremely fast and can be made highly parallel for different users i . Secondly, this guarantees that $\mathbf{Q}_p = \text{Diag}(\tilde{\mathbf{Q}}_i, i = 1, \dots, n)$ is also positive-definite.

4.3 Dependence of Parameters

Whenever we are modeling the interaction effect, we have assumed that vectors \mathbf{p} and \mathbf{r} are independent. However, there may arise problems where that is not the case. For example, if we consider our original problem (2), then $\mathbf{r} = \mathbf{S} \cdot \mathbf{p} \cdot \mathbf{c}$ and hence they are clearly not independent. Here, it is clear that if we construct $\mathbf{Q}_r = \mathbf{Q}_p \cdot \mathbf{c}$, where $\mathbf{Q}_p \cdot \mathbf{c}$ is a matrix whose ℓ -th row is the ℓ -th row of \mathbf{Q}_p multiplied by c_ℓ , then \mathbf{Q}_r loses its positive-definiteness since it is not even symmetric. For a general dependency structure, we have the following result.

THEOREM 1. *Suppose we have an optimization problem of the form (13). Moreover, assume that $\mathbf{r} = f(\mathbf{p})$ for some function f , that maps $\mathbb{R}^n \rightarrow \mathbb{R}^n$ and let $\mathbf{Q}_p = \text{Diag}(\tilde{\mathbf{Q}}_i, i = 1, \dots, n)$ where $\tilde{\mathbf{Q}}_i$'s are generated by (19). If there exists a positive-definite matrix \mathbf{Q}_r and a vector \mathbf{c}_r such that*

$$\mathbf{Q}_r \mathbf{x} - 2\mathbf{Q}_r \mathbf{c}_r = f(-\mathbf{Q}_p \mathbf{x}) \quad (20)$$

for all \mathbf{x} , then there exists a convex QCQP formulation of (13).

PROOF. Assume there exists a positive-definite matrix \mathbf{Q}_r and \mathbf{c}_r such that such that (20) holds. Thus, using the dependence structure and modeling the interaction effect, we can write,

$$\begin{aligned} \mathbf{x}^T \mathbf{r} &= \mathbf{x}^T f(\mathbf{p}) = \mathbf{x}^T f(-\mathbf{Q}_p \mathbf{x}) \\ &= \mathbf{x}^T \mathbf{Q}_r \mathbf{x} - 2\mathbf{x}^T \mathbf{Q}_r \mathbf{c}_r \\ &= (\mathbf{x} - \mathbf{c}_r)^T \mathbf{Q}_r (\mathbf{x} - \mathbf{c}_r) - \mathbf{c}_r^T \mathbf{Q}_r \mathbf{c}_r. \end{aligned}$$

Thus we have,

$$\mathbf{x}^T \mathbf{r} \leq P \iff (\mathbf{x} - \mathbf{c}_r)^T \mathbf{Q}_r (\mathbf{x} - \mathbf{c}_r) \leq P + \mathbf{c}_r^T \mathbf{Q}_r \mathbf{c}_r.$$

The result follows from observing the fact that since \mathbf{Q}_r is positive definite, $\mathbf{c}_r^T \mathbf{Q}_r \mathbf{c}_r > 0$. \square

REMARK 2. *Few simple functions f for which the convex reformulation Theorem holds include the linear shift opera-*

tors and the positive scaling operators. The exact characterization of the function class may be hard and is beyond the scope this paper.

5. SOLUTION TO THE QCQP

In this section we describe the technique we use to solve the following optimization problem.

$$\begin{aligned} \underset{\mathbf{x}}{\text{Minimize}} \quad & (\mathbf{x} - \mathbf{a})^T \mathbf{A} (\mathbf{x} - \mathbf{a}) \\ \text{subject to} \quad & (\mathbf{x} - \mathbf{b})^T \mathbf{B} (\mathbf{x} - \mathbf{b}) \leq \tilde{b} \\ & \mathbf{C}\mathbf{x} \leq \mathbf{c}, \end{aligned} \quad (21)$$

where \mathbf{A}, \mathbf{B} are positive definite matrices. Note that this is a QCQP in its general form having a single quadratic constraint. The problem as stated in (15) is a special case of this formulation. It is already known that solving a general QCQP is NP-hard [9]. Before we discuss our methodology, we discuss few of the common techniques used in literature.

There are two main relaxation techniques that are used to solve a QCQP, namely, semi-definite programming (SDP) and reformulation-linearization technique (RLT) [9]. However both of them introduce a new variable $\mathbf{X} = \mathbf{x}\mathbf{x}^T$ so that the problem becomes linear in \mathbf{X} . They relax the condition $\mathbf{X} = \mathbf{x}\mathbf{x}^T$ using different means. However, in doing so they increase the number of variables from n to $O(n^2)$. This makes these methods prohibitively expensive for most large scale methods.

However, based on the Operator Splitting method from [5] we are capable of solving a large enough QP that scales to web applications. This motivated us to try to convert the QCQP into an approximate QP by linearizing the constraints, which we can then solve efficiently. To the best of our knowledge this technique is new and has not been previously explored in the literature. The rest of this section is devoted to explain the linearization of the quadratic constraint. We also give several results showing convergence guarantees to the original problem.

5.1 QCQP to QP Approximation

Here we describe the linearization technique to convert the quadratic constraint into a set of N linear constraints. The main idea behind this approximation, is the fact that given any convex set in the Euclidean plane, there exists a convex polytope that covers the set.

Let us begin with a few notations. Let \mathcal{P} denote the optimization problem (21). Let $\mathbf{x} \in \mathbb{R}^s$. Define,

$$\mathcal{S} := \{\mathbf{x} \in \mathbb{R}^s : (\mathbf{x} - \mathbf{b})^T \mathbf{B} (\mathbf{x} - \mathbf{b}) \leq \tilde{b}\}. \quad (22)$$

Let $\partial\mathcal{S}$ denote the boundary of the ellipsoid \mathcal{S} .

To generate the N linear constraints for this one quadratic constraint, we generate a set of N points, $\mathbf{x}_1, \dots, \mathbf{x}_N \in \partial\mathcal{S}$. The sampling technique to select these points are given in Section 5.2. Corresponding to these N points we get the following set of N linear constraints,

$$(\mathbf{x} - \mathbf{b})^T \mathbf{B} (\mathbf{x}_j - \mathbf{b}) \leq \tilde{b} \quad \text{for } j = 1, \dots, N. \quad (23)$$

If we look at it geometrically, it is not hard to see that each of these linear constraints are just tangent planes to \mathcal{S} at \mathbf{x}_j for $j = 1, \dots, N$. Figure 3 shows a set of six linear constraints for an ellipsoidal feasible set in two dimensions. Thus, using these N linear constraints we can write the approximate optimization problem, $\mathcal{P}(N)$, as follows.

$$\begin{aligned}
& \underset{\mathbf{x}}{\text{Minimize}} && (\mathbf{x} - \mathbf{a})^T \mathbf{A} (\mathbf{x} - \mathbf{a}) \\
& \text{subject to} && (\mathbf{x} - \mathbf{b})^T \mathbf{B} (\mathbf{x}_j - \mathbf{b}) \leq \tilde{b} \quad \text{for } j = 1, \dots, N \\
& && \mathbf{C} \mathbf{x} \leq \mathbf{c}.
\end{aligned} \tag{24}$$

Thus, instead of solving \mathcal{P} , we solve $\mathcal{P}(N)$ for a large enough value of N . Note that as we sample more points, our approximation gets more and more accurate to the original solution.

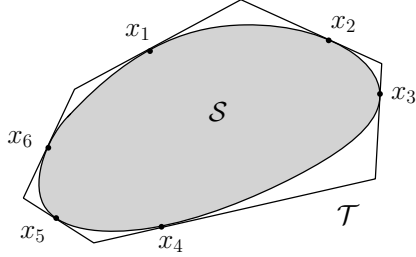


Figure 3: Converting a quadratic constraint into linear constraints. The tangent planes through the 6 points $\mathbf{x}_1, \dots, \mathbf{x}_6$ create the approximation to \mathcal{S} .

5.2 Sampling Scheme

Note that the accuracy of the solution of $\mathcal{P}(N)$ completely depends on the choice of the N points. The tangent planes to \mathcal{S} at those N points create a cover of \mathcal{S} . Throughout this section, we are going to use the notion of a bounded cover, which we define as follows.

DEFINITION 2. Let \mathcal{T} be the convex polytope generated by the tangent planes to \mathcal{S} at the points $\mathbf{x}_1, \dots, \mathbf{x}_n \in \partial\mathcal{S}$. \mathcal{T} is said to be a bounded cover of \mathcal{S} if,

$$\sup_{\mathbf{t} \in \mathcal{T}} d(\mathbf{t}, \mathcal{S}) < \infty,$$

where $d(\mathbf{t}, \mathcal{S}) = \inf_{\mathbf{x} \in \mathcal{S}} \|\mathbf{t} - \mathbf{x}\|$ and $\|\cdot\|$ denotes the Euclidean distance.

The first result shows that we need a certain minimum number of points to get a bounded cover.

LEMMA 2. Let \mathcal{S} be an s dimensional ellipsoid as defined in (22). Then we need at least $s + 1$ points on $\partial\mathcal{S}$ to be get a bounded cover.

PROOF. Note that since \mathcal{S} is a compact convex body in \mathbb{R}^s , there exists a s -dimensional simplex T such that $\mathcal{S} \subseteq T$. We can always shrink T such that each edge touches \mathcal{S} tangentially. Since there are $s + 1$ faces, we will get $s + 1$ points whose tangent surface creates a bounded cover.

To complete the proof we need to show that we cannot create a bounded cover using only s or fewer points. Consider any set of s points, $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_s$ on $\partial\mathcal{S}$. The equation of their tangent planes are,

$$(\mathbf{x} - \mathbf{b})^T \mathbf{B} (\mathbf{x}_j - \mathbf{b}) = \tilde{b}$$

for $j = 1, \dots, s$. Note that we can rewrite this as $\mathbf{A} \mathbf{x} = \gamma$, where each row $\mathbf{a}_j = (\mathbf{x}_j - \mathbf{b})^T \mathbf{B}$ and $\gamma_j = \tilde{b} + \mathbf{b}^T \mathbf{B} (\mathbf{x}_j - \mathbf{b})$ for $j = 1, \dots, s$.

Without loss of generality we can assume that $\gamma \in \mathcal{C}(\mathbf{A})$, otherwise the system of linear equations is inconsistent and this case, it is easy to see that the tangent planes do not create a compact polytope. Now if \mathbf{A} is not of full rank, there exists a continuum of solutions to $\mathbf{A} \mathbf{x} = \gamma$. Hence, the polytope is not bounded. Finally if \mathbf{A} is invertible, then there exists a unique solution to the system of equations, call it \mathbf{x}_0 . Since all the planes intersect at a single point, the polytope is divergent. Thus, it is not possible to construct a bounded cover with only s points. Similar proof holds for fewer than s points. This completes the proof of the lemma. \square

Lemma 2 states that we need a minimum of $s + 1$ points to create the bounding cover \mathcal{T} , but it does not guarantee the its existence. Moreover, we wish to pick our points in a way such that the constructed \mathcal{T} is close as possible to \mathcal{S} , thus having a better approximation. Formally, we introduce the notion of optimal bounded cover.

DEFINITION 3. $\mathcal{T}^* = \mathcal{T}(\mathbf{x}_1^*, \dots, \mathbf{x}_n^*)$ is said to be an optimal bounded cover, if

$$\sup_{\mathbf{t} \in \mathcal{T}^*} d(\mathbf{t}, \mathcal{S}) \leq \sup_{\mathbf{t} \in \mathcal{T}} d(\mathbf{t}, \mathcal{S})$$

for any bounded cover \mathcal{T} generated by any other n -point sets. Moreover, $\{\mathbf{x}_1^*, \dots, \mathbf{x}_n^*\}$ are defined to be the optimal n -point set.

Note that we can think of the optimal n -point set as that set of n points which minimizes the maximum distance between \mathcal{T}^* and \mathcal{S} . It is not hard to see that the optimal n -point set on the unit circle are the n th roots of unity, unique upto rotation.

5.2.1 Riesz Energy and Equidistribution

It has been shown that the n th roots of unity minimize the discrete Riesz energy for the unit circle [15]. Riesz energy of a point set $A_n = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is defined as

$$E_s(A_n) := \sum_{i \neq j=1}^n \|\mathbf{x}_i - \mathbf{x}_j\|^{-s} \tag{25}$$

for positive real parameter s . There is a vast literature on Riesz energy and its association with “good” configuration of points. In fact, we can associate the optimal n -point set to the set of n points that minimize the Riesz energy on \mathcal{S} . It is well known that the measures associated to the optimal point set that minimizes the Riesz energy on \mathcal{S} converges to the normalized surface measure of \mathcal{S} . For sake of compactness we do not go deeper into this. For more details see [16, 17] and the references therein.

Thus, to pick the optimal n -point set, we try to choose a point set, which has a very good equidistribution property that is lacking in random uniform sampling. One such point set in $[0, 1]^s$ is called the (t, m, s) -net. To describe this point set we begin with a few definitions. Throughout these definitions $b \geq 2$ is an integer base, $s \geq 1$ is an integer dimension and $\mathbb{Z}_b = \{0, 1, \dots, b - 1\}$.

DEFINITION 4. For $k_j \in \mathbb{N}_0$ and $c_j \in \mathbb{Z}_b^{k_j}$ for $j = 1, \dots, s$, the set

$$\prod_{j=1}^s \left[\frac{c_j}{b^{k_j}}, \frac{c_j + 1}{b^{k_j}} \right)$$

is a b -adic box of dimension s .

DEFINITION 5. For integers $m \geq t \geq 0$, the points $\mathbf{x}_1, \dots, \mathbf{x}_{b^m} \in [0, 1]^s$ are a (t, m, s) -net in base b if every b -adic box of dimension s with volume b^{t-m} contains precisely b^t of the \mathbf{x}_i .

The nets have good equidistribution (low discrepancy) because boxes $[0, a]$ can be efficiently approximated by unions of b -adic boxes. Digital nets can attain a discrepancy of $O((\log(n))^{s-1}/n)$. There is vast literature on easy construction of these point sets. For more details on digital nets we refer to [14, 20].

5.2.2 Area preserving map to $\partial\mathcal{S}$

Now once we have a point set on $[0, 1]^s$ we try to map it to $\partial\mathcal{S}$ using a measure preserving transformation so that the equidistribution property remains. We describe the mapping in two steps. First we map the point set from $[0, 1]^s$ to the hyper-sphere $\mathbb{S}^s = \{\mathbf{x} \in \mathbb{R}^{s+1} : \mathbf{x}^T \mathbf{x} = 1\}$. Then we map it to $\partial\mathcal{S}$. The mapping from $[0, 1]^s$ to \mathbb{S}^s is based on [10].

The cylindrical coordinates of the d -sphere, can be written as

$$\begin{aligned} \mathbf{x} = \mathbf{x}_s &= (\sqrt{1 - t_s^2} \mathbf{x}_{s-1}, t_s) \\ &\dots \\ \mathbf{x}_2 &= (\sqrt{1 - t_2^2} \mathbf{x}_1, t_2) \\ \mathbf{x}_1 &= (\cos \phi, \sin \phi) \end{aligned}$$

where $0 \leq \phi \leq 2\pi$, $-1 \leq t_d \leq 1$, $\mathbf{x}_d \in \mathbb{S}^d$ and $d = (1, \dots, s)$. Thus, an arbitrary point $\mathbf{x} \in \mathbb{S}^s$ can be represented through angle ϕ and heights t_2, \dots, t_s as,

$$\mathbf{x} = \mathbf{x}(\phi, t_2, \dots, t_s), \quad 0 \leq \phi \leq 2\pi, -1 \leq t_2, \dots, t_s \leq 1.$$

We map a point $\mathbf{y} = (y_1, \dots, y_s) \in [0, 1]^s$ to $\mathbf{x} \in \mathbb{S}^s$ using

$$\varphi_1(y_1) = 2\pi y_1, \quad \varphi_d(y_d) = 1 - 2y_d \quad (d = 2, \dots, s)$$

and cylindrical coordinates

$$\mathbf{x} = \Phi_s(\mathbf{y}) = \mathbf{x}(\varphi_1(y_1), \varphi_2(y_2), \dots, \varphi_s(y_s)).$$

LEMMA 3. Under the above notation, $\Phi_s : [0, 1]^s \rightarrow \mathbb{S}^s$ is an area preserving map.

PROOF. See [10]. \square

REMARK 3. Instead of using (t, m, s) -nets and mapping to \mathbb{S}^s , we could have also used spherical t -designs, the existence of which was proved in [8]. However, construction of such sets is still a hard problem in large dimensions. We refer to [11] for more details.

We consider the map from $\psi : \mathbb{S}^{s-1} \rightarrow \partial\mathcal{S}$ defined as follows.

$$\psi(\mathbf{x}) = \sqrt{\tilde{b}} B^{-1/2} \mathbf{x} + b. \quad (26)$$

The next result shows that this also an area-preserving map, in the sense of normalized surface measures.

LEMMA 4. Let ψ be a mapping from $\mathbb{S}^{s-1} \rightarrow \partial\mathcal{S}$ as defined in (26). Then for any set $A \subseteq \partial\mathcal{S}$,

$$\sigma_s(A) = \lambda_s(\psi^{-1}(A))$$

where, σ_s, λ_s are the normalized surface measure of $\partial\mathcal{S}$ and \mathbb{S}^{s-1} respectively

PROOF. Pick any $A \subseteq \partial\mathcal{S}$. Then we can write,

$$\psi^{-1}(A) = \left\{ \frac{1}{\sqrt{\tilde{b}}} B^{1/2} (\mathbf{x} - b) : \mathbf{x} \in A \right\}.$$

Now since the linear shift does not change the surface area, we have,

$$\begin{aligned} \lambda_s(\psi^{-1}(A)) &= \lambda_s \left(\left\{ \frac{1}{\sqrt{\tilde{b}}} B^{1/2} (\mathbf{x} - b) : \mathbf{x} \in A \right\} \right) \\ &= \lambda_s \left(\left\{ \frac{1}{\sqrt{\tilde{b}}} B^{1/2} \mathbf{x} : \mathbf{x} \in A \right\} \right) = \sigma_s(A), \end{aligned}$$

where the last equality follows from the definition of normalized surface measures. This completes the proof. \square

Following Lemmas 3 and 4 we see that the map

$$\psi \circ \Phi_{s-1} : [0, 1]^s \rightarrow \partial\mathcal{S},$$

is a measure preserving map. Using this map and the $(t, m, s-1)$ net in base b , we derive the optimal b^m -point set on $\partial\mathcal{S}$. The procedure is detailed as Algorithm 2.

Algorithm 2 Point Simulation on $\partial\mathcal{S}$

- 1: Input : B, b, \tilde{b} to specify \mathcal{S} and $N = k^m$ points
 - 2: Output : $\mathbf{x}_1, \dots, \mathbf{x}_N \in \partial\mathcal{S}$
 - 3: Generate $\mathbf{y}_1, \dots, \mathbf{y}_N$ as a $(t, m, s-1)$ -net in base k .
 - 4: **for** $i \in 1, \dots, N$ **do**
 - 5: $\mathbf{x}_i = \psi \circ \Phi_{s-1}(\mathbf{y}_i)$
 - 6: **end for**
 - 7: **return** $\mathbf{x}_1, \dots, \mathbf{x}_N$
-

Figure 4 shows how we transform a $(0, 7, 2)$ -net in base 2 to a sphere and then to an ellipsoid. For more general geometric constructions we refer to [6, 7].

5.3 Convergence guarantees

Consider the problem $\mathcal{P}(N)$ as stated in (24). We shall show that asymptotically as $N \rightarrow \infty$, we get back the original problem \mathcal{P} as stated in (21). We shall also prove some finite sample results to give some error bounds on the solution to $\mathcal{P}(N)$. We begin with a few notation. Let $\mathbf{x}^*, \mathbf{x}^*(N)$ denote the solution to \mathcal{P} and $\mathcal{P}(N)$ respectively. Let $f(\cdot)$ denote the objective function.

THEOREM 2. Let \mathcal{P} and $\mathcal{P}(N)$ be the optimization problems defined in (21) and (24) respectively. Then, $\mathcal{P}(N) \rightarrow \mathcal{P}$ as $N \rightarrow \infty$ in the sense that

$$\lim_{N \rightarrow \infty} \|\mathbf{x}^*(N) - \mathbf{x}^*\| = 0.$$

PROOF. Fix any N . Let \mathcal{T}_N denote the optimal bounded cover constructed with N points on $\partial\mathcal{S}$. Note that to prove the result, it is enough to show that $\mathcal{T}_N \rightarrow \mathcal{S}$ as $N \rightarrow \infty$. This guarantees that linear constraints of $\mathcal{P}(N)$ converge to the quadratic constraint of \mathcal{P} , and hence the two problems match. Now since $\mathcal{S} \subseteq \mathcal{T}_N$ for all N , it is easy to see that $\mathcal{S} \subseteq \lim_{N \rightarrow \infty} \mathcal{T}_N$.

To prove the converse, let $t_0 \in \lim_{N \rightarrow \infty} \mathcal{T}_N$ but $t_0 \notin \mathcal{S}$. Thus, $d(t_0, \mathcal{S}) > 0$. Let t_1 denote the projection of t_0 onto \mathcal{S} . Thus, $t_0 \neq t_1 \in \partial\mathcal{S}$. Choose ϵ to be arbitrarily small and consider any region A_ϵ on $\partial\mathcal{S}$ with diameter less than ϵ .

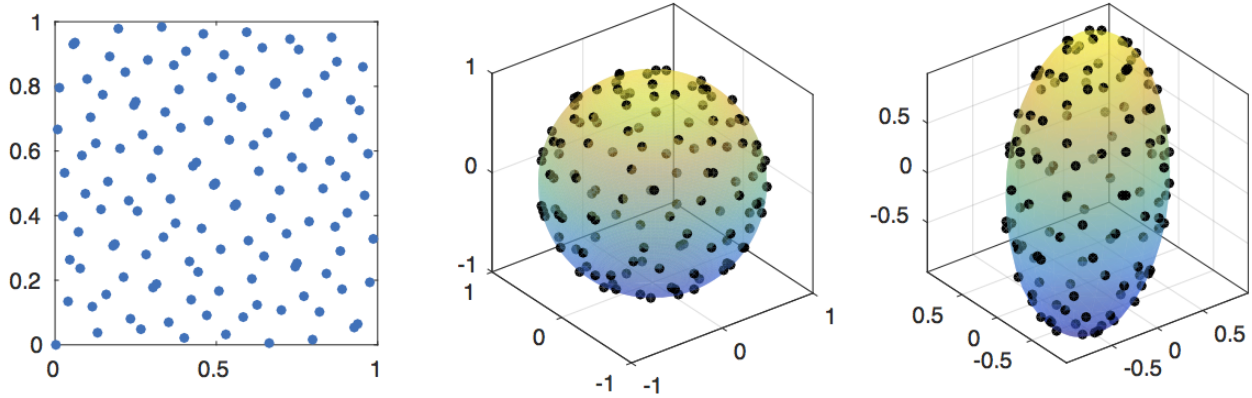


Figure 4: The left panel shows a $(0,7,2)$ -net in base 2 which is mapped to a sphere in 3 dimensions (middle panel) and then mapped to the ellipsoid as seen in the right panel.

Since, we are working with a limiting case, there exists infinitely many points in A_ϵ . Thus there exists a point $t^* \in A_\epsilon$, the tangent plane through which cuts the line joining t_0 and t_1 . Thus, $t_0 \notin \lim_{N \rightarrow \infty} \mathcal{T}_N$. Hence, we get a contradiction and the result is proved. \square

Note that Theorem 2 shows that $\lim_{N \rightarrow \infty} \|\mathbf{x}^*(N) - \mathbf{x}^*\| = 0$ and hence, $\lim_{N \rightarrow \infty} |f(\mathbf{x}^*(N)) - f(\mathbf{x}^*)| = 0$. Now we prove state some finite sample results.

THEOREM 3. *Let $g : \mathbb{N} \rightarrow \mathbb{R}$ such that $\lim_{n \rightarrow \infty} g(n) = 0$. Further assume that $\|\mathbf{x}^*(N) - \mathbf{x}^*\| \leq C_1 g(N)$ for some constant $C_1 > 0$. Then,*

$$|f(\mathbf{x}^*(N)) - f(\mathbf{x}^*)| \leq C_2 g(N)$$

where $C_2 > 0$ is a constant.

PROOF. Note that $f(\mathbf{x}) = (\mathbf{x} - \mathbf{a})^T A(\mathbf{x} - \mathbf{a})$ and $\nabla f(\mathbf{x}) = 2A(\mathbf{x} - \mathbf{a})$. Now, note that we can write

$$\begin{aligned} f(\mathbf{x}) &= f(\mathbf{x}^*) + \int_0^1 \langle \nabla f(\mathbf{x}^* + t(\mathbf{x} - \mathbf{x}^*)), \mathbf{x} - \mathbf{x}^* \rangle dt \\ &= f(\mathbf{x}^*) + \langle \nabla f(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^* \rangle \\ &\quad + \int_0^1 \langle \nabla f(\mathbf{x}^* + t(\mathbf{x} - \mathbf{x}^*)) - \nabla f(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^* \rangle dt \\ &= I_1 + I_2 + I_3 \text{ (say) .} \end{aligned}$$

Now, we can bound the last term as follows. Observe that using Cauchy-Schwarz inequality,

$$\begin{aligned} |I_3| &\leq \int_0^1 |\langle \nabla f(\mathbf{x}^* + t(\mathbf{x} - \mathbf{x}^*)) - \nabla f(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^* \rangle| dt \\ &\leq \int_0^1 \|\nabla f(\mathbf{x}^* + t(\mathbf{x} - \mathbf{x}^*)) - \nabla f(\mathbf{x}^*)\| \|\mathbf{x} - \mathbf{x}^*\| dt \\ &\leq 2\sigma_{\max}(A) \int_0^1 \|t(\mathbf{x} - \mathbf{x}^*)\| \|\mathbf{x} - \mathbf{x}^*\| dt \\ &= \sigma_{\max}(A) \|\mathbf{x} - \mathbf{x}^*\|^2, \end{aligned}$$

where $\sigma_{\max}(A)$ denotes the highest singular value of A . Thus,

we have

$$f(\mathbf{x}) = f(\mathbf{x}^*) + \langle \nabla f(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^* \rangle + \tilde{C} \|\mathbf{x} - \mathbf{x}^*\|^2 \quad (27)$$

where $|\tilde{C}| \leq \sigma_{\max}(A)$. Furthermore,

$$\begin{aligned} |\langle \nabla f(\mathbf{x}^*), \mathbf{x}^*(N) - \mathbf{x}^* \rangle| &= |\langle 2A(\mathbf{x}^* - \mathbf{a}), \mathbf{x}^*(N) - \mathbf{x}^* \rangle| \\ &\leq 2\sigma_{\max}(A) (\|\mathbf{x}^*\| + \|\mathbf{a}\|) \|\mathbf{x}^*(N) - \mathbf{x}^*\| \\ &\leq 2C_1 \sigma_{\max}(A) (s + \|\mathbf{a}\|) g(N) \end{aligned} \quad (28)$$

Combining (27) and (28) we have,

$$|f(\mathbf{x}^*(N)) - f(\mathbf{x}^*)| \leq C_2 g(N)$$

for some constant $C_2 > 0$. Thus, the result follows. \square

Note that the function g gives us an idea about how fast $\mathbf{x}^*(N)$ converges \mathbf{x}^* . To help, identify the function g we state the following results.

LEMMA 5. *If $f(\mathbf{x}^*) = f(\mathbf{x}^*(N))$, then $\mathbf{x}^* = \mathbf{x}^*(N)$. Furthermore, if $f(\mathbf{x}^*) \geq f(\mathbf{x}^*(N))$, then $\mathbf{x}^* \in \partial \mathcal{U}$ and $\mathbf{x}^*(N) \notin \mathcal{U}$, where $\mathcal{U} = \mathcal{S} \cap \{\mathbf{x} : C\mathbf{x} \leq c\}$ is the feasible set for (21)*

PROOF. Let $\mathcal{V} = \mathcal{T}_N \cap \{\mathbf{x} : C\mathbf{x} \leq c\}$. It is easy to see that $\mathcal{U} \subseteq \mathcal{V}$. Assume $f(\mathbf{x}^*) = f(\mathbf{x}^*(N))$, but $\mathbf{x}^* \neq \mathbf{x}^*(N)$. Note that $\mathbf{x}^*, \mathbf{x}^*(N) \in \mathcal{V}$. Since \mathcal{V} is convex, consider a line joining \mathbf{x}^* and $\mathbf{x}^*(N)$. For any point $\lambda_t = t\mathbf{x}^* + (1-t)\mathbf{x}^*(N)$,

$$f(\lambda_t) \leq tf(\mathbf{x}^*) + (1-t)f(\mathbf{x}^*(N)) = f(\mathbf{x}^*(N)).$$

Thus, f is constant on the line joining \mathbf{x}^* and $\mathbf{x}^*(N)$. But, it is known that f is strongly convex since A is positive definite. Thus, there exists only one unique minimum. Thus, we have a contradiction, which proves $\mathbf{x}^* = \mathbf{x}^*(N)$

Now let us assume that $f(\mathbf{x}^*) \geq f(\mathbf{x}^*(N))$. Clearly, $\mathbf{x}^*(N) \notin \mathcal{U}$. Suppose $\mathbf{x}^* \in \mathring{\mathcal{U}}$. Let $\tilde{\mathbf{x}} \in \partial \mathcal{U}$ denote the point on the line joining \mathbf{x}^* and $\mathbf{x}^*(N)$. Clearly, $\tilde{\mathbf{x}} = t\mathbf{x}^* + (1-t)\mathbf{x}^*(N)$ for some $t > 0$. Thus,

$$f(\tilde{\mathbf{x}}) < tf(\mathbf{x}^*) + (1-t)f(\mathbf{x}^*(N)) \leq f(\mathbf{x}^*)$$

But \mathbf{x}^* is the minimizer over \mathcal{U} . Thus, we have a contradiction, which gives $\mathbf{x}^* \in \partial \mathcal{U}$. This completes the proof. \square

LEMMA 6. Following the notation of Lemma 5, if $\mathbf{x}^*(N) \notin \mathcal{U}$, then \mathbf{x}^* lies on $\partial\mathcal{U}$ within the conic cap of \mathcal{U} generated from $\mathbf{x}^*(N)$.

PROOF. Since the gradient of f is linear, the result is easy to see from the proof of the second assertion in Lemma 5. \square

Now we can identify the function g by considering the maximum distance of the points lying on the intersection of the cone and \mathcal{S} . This is highly dependent on the shape of \mathcal{S} and on the cover \mathcal{T}_N . Explicit calculation can give us explicit rates of convergence, which we leave as future work.

6. EXPERIMENTAL RESULTS

A detailed study of solving the original optimization problem (2) via the efficient solution approach of Section 3 has been done in [5]. For compactness, we only report the results regarding the experimentation with modeling interactions.

6.1 Need for modeling interaction

We first show that if we ignore the interaction effect, we will get an increasingly worse solution as the dimension of the problem increases. We consider the following simple optimization problem for our purposes.

$$\begin{aligned} \underset{\mathbf{x}}{\text{Minimize}} \quad & -\mathbf{x}^T \mathbf{p} + \frac{\gamma}{2} \mathbf{x}^T \mathbf{x} \\ \text{subject to} \quad & \mathbf{x}' \mathbf{r} \leq P \\ & 0 \leq \mathbf{x} \leq 1 \end{aligned} \quad (29)$$

where the truth is $\mathbf{p} = -\mathbf{Q}_p \mathbf{x}$ and $\mathbf{r} = \mathbf{Q}_r \mathbf{x}$ for positive definite matrices $\mathbf{Q}_p, \mathbf{Q}_r$, as defined in Section 4.1. We solve the true optimization problem,

$$\begin{aligned} \underset{\mathbf{x}}{\text{Minimize}} \quad & \mathbf{x}^T \left(\mathbf{Q}_p + \frac{\gamma}{2} \mathbf{I} \right) \mathbf{x} \\ \text{subject to} \quad & \mathbf{x}' \mathbf{Q}_r \mathbf{x} \leq P \\ & 0 \leq \mathbf{x} \leq 1 \end{aligned} \quad (30)$$

to get \mathbf{x}^* . Now, ignoring the dependency structure we use $\mathbf{p} = \text{diag}(\mathbf{Q}_p)$ and $\mathbf{r} = \text{diag}(\mathbf{Q}_r)$ to solve the problem (29) to get $\hat{\mathbf{x}}$. Note that the diagonals are just the single slot estimates.

We simulate two random instances of the single slot estimate and create $\text{diag}(\mathbf{Q}_p), \text{diag}(\mathbf{Q}_r)$ as a decreasing function of the slot position. We then simulate the rest of the matrix \mathbf{Q}_p and \mathbf{Q}_r according to the structure in Section 4.1. For different values of sample size n , we calculate the relative error as

$$\text{err}(n) = \frac{\hat{\mathbf{x}}^T \left(\mathbf{Q}_p + \frac{\gamma}{2} \mathbf{I} \right) \hat{\mathbf{x}} - (\mathbf{x}^*)^T \left(\mathbf{Q}_p + \frac{\gamma}{2} \mathbf{I} \right) \mathbf{x}^*}{(\mathbf{x}^*)^T \left(\mathbf{Q}_p + \frac{\gamma}{2} \mathbf{I} \right) \mathbf{x}^*}$$

Figure 5 shows the log of two different objective values. It can be seen that we start getting increasing worse solutions as the dimension increases. The fluctuations are because of the randomness in the simulations.

Figure 6 show the log of the relative error as a function of the dimension. It can be seen from our simulation, that we get an average relative error of about 6×10^8 which is extremely high of a cost to pay for ignoring the interaction effect.

6.2 Comparative Study of QCQP Solutions

We compare our proposed technique to the current state-of-the-art solvers of QCQP. Specifically we compare it to

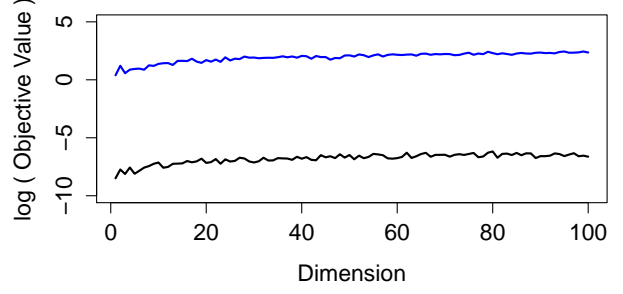


Figure 5: The black line shows the true objective value obtained from solving problem (30). The blue line shows the objective value using the solution from the ignored dependency problem.

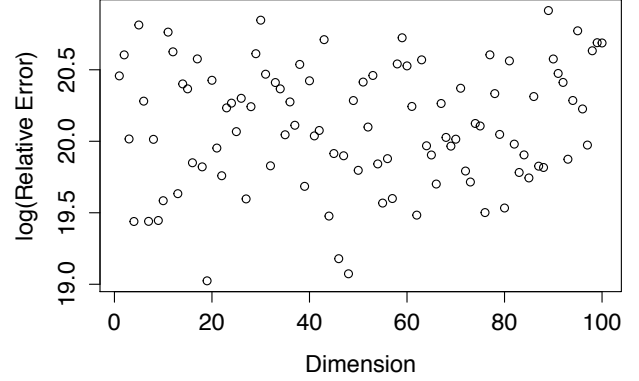


Figure 6: Plotting the log(relative error) against the dimension of the problem

the SDP and RLT relaxation procedures. For small enough problems, we also compare our method to the exact solution by interior point methods. Furthermore, we provide empirical evidence to show that our sampling technique is better than other simpler sampling procedures such as uniform sampling on the unit square or on the unit sphere and then mapping it subsequently to our domain as in Algorithm 2. We begin by considering a very simple QCQP for the form

$$\begin{aligned} \underset{\mathbf{x}}{\text{Minimize}} \quad & \mathbf{x}^T \mathbf{A} \mathbf{x} \\ \text{subject to} \quad & (\mathbf{x} - \mathbf{x}_0)^T \mathbf{B} (\mathbf{x} - \mathbf{x}_0) \leq \tilde{b}, \\ & \mathbf{l} \leq \mathbf{x} \leq \mathbf{u} \end{aligned} \quad (31)$$

We randomly sample $\mathbf{A}, \mathbf{B}, \mathbf{x}_0$ and \tilde{b} . The lower bound, \mathbf{l} and upper bounds \mathbf{u} are chosen in a way such that they intersect the ellipsoid. We vary the dimension n of the problem and tabulate the final objective value as well as the time taken for the different procedures to converge in Table 1. Throughout our simulations we have chosen $\eta = 2$ and the number of optimal points as $N = \max(1024, 2^m)$, where m is the smallest integer such that $2^m \geq 10n$.

Note that even though the SDP and the interior point methods converge very efficiently for small values of n , they

Table 1: The optimal objective value and convergence time

n	Our method	Sampling on $[0, 1]^n$	Sampling on \mathbb{S}^n	SDP relaxation	RLT relaxation	Exact
5	3.00 (4.61s)	2.99 (4.74s)	2.95 (6.11s)	3.07 (0.52s)	3.08 (0.51s)	3.07 (0.49)
10	206.85 (5.04s)	205.21 (5.65s)	206.5 (5.26s)	252.88 (0.53s)	252.88 (0.51s)	252.88 (0.51)
20	6291.4 (6.56s)	4507.8 (6.28s)	5052.2 (6.69s)	6841.6 (2.05s)	6841.6 (1.86s)	6841.6 (0.54)
50	99668 (15.55s)	15122 (18.98s)	26239 (17.32s)	1.11×10^5 (4.31s)	1.08×10^5 (2.96s)	1.11×10^5 (0.64)
100	1.40×10^6 (58.41s)	69746 (1.03m)	1.24×10^6 (54.69s)	1.62×10^6 (30.41s)	1.52×10^6 (15.36s)	1.62×10^6 (2.30s)
1000	2.24×10^7 (14.87m)	8.34×10^6 (15.63m)	9.02×10^6 (15.32m)	NA	NA	NA
10^5	3.10×10^8 (25.82m)	7.12×10^7 (24.59m)	8.39×10^7 (27.23m)	NA	NA	NA
10^6	3.91×10^9 (38.30m)	2.69×10^8 (39.15m)	7.53×10^8 (37.21m)	NA	NA	NA

Table 2: The Relative Error : $\|\mathbf{x}^*(N) - \mathbf{x}^*\|/\|\mathbf{x}^*\|$

n	Our method	Sampling on $[0, 1]^n$	Sampling on \mathbb{S}^n
5	0.0615	0.0828	0.0897
10	0.0714	0.1530	0.1229
20	0.0895	0.2455	0.2368
50	0.3352	3.8189	1.0472
100	0.8768	13.3709	2.0849

cannot scale to values of $n \geq 1000$, which is where the strength of our method becomes evident. From Table 1 we observe that the relaxation procedures SDP and RLT fail to converge within an hour of computation time for $n \geq 1000$, whereas all the approximation procedures can easily scale up to $n = 10^6$ variables. We can further notice that SDP performs slightly better than RLT for higher dimensions.

Furthermore, our procedure gives the best approximation result when compared to the remaining two sampling schemes. Lemma 5 shows that if the both the objective values are the same then we indeed get the exact solution. To see how much the approximation deviates from the truth, we also tabulate the relative error, i.e. $\|\mathbf{x}^*(N) - \mathbf{x}^*\|/\|\mathbf{x}^*\|$ for each of the sampling procedures in Table 2. We omit SDP and RLT results in Table 2 since both of them produce a solution very close to the exact minimum for small n . From the results in Table 2 it is clear that our procedure gets the smallest relative error compared to the other sampling schemes that we tried.

7. DISCUSSION

In this paper, we look at the problem of trading off multiple objectives while ranking recommendations over multiple slots. We give a deterministic serving plan under general constraints for a single slot and then give a formulation for the multi-slot setting assuming dependence in interaction of the items in the different slots. We characterize the conditions under which it is possible to efficiently solve

the problem and give an approximation algorithm for the QCQP, which involves relaxing the constraints via a non-trivial sampling scheme. This method can scale up to very large problem sizes while generating solutions which have good theoretical properties of convergence.

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