A new DEA ranking system based on interval Cross Efficiency and interval Analytic Hierarchy Process methods

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September 27, 2018

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Abstract

The aim of this paper is to present a novel approach for ranking of all DMUs using the interval Cross-Efficiency (ICE) and interval Analytic Hierarchy Process (IAHP) methods. The approach includes two basic stages. In the first stage using DEA models the interval cross-efficiency values of each DMU are specified. In the second stage, the interval pairwise comparison matrix generated in the first stage is utilized to rank the units (DMUs) via the one step process of interval AHP (i.e. the AHP model with interval decision maker judgements). The numerical example is presented in this paper and we compare our approach with some other approaches.

Keywords: Data envelopment analysis; Interval Analytical Hierarchy Process; Interval Cross efficiency; Ranking.

1 Introduction

Data envelopment analysis (DEA) is a tool for evaluation and measuring the efficiency of a set of decision making units that consume multiple inputs and produce multiple outputs, first introduced by Charnes, Cooper and Rhodes (CCR) (1978) and extended by Banker, Charnes and Cooper (BCC) (1984). One important issue in DEA which has been studied by many DEA researchers, is to discriminate between efficient DMUs. Several authors have proposed methods

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for ranking the best performers (Jin-Xiao Chen and Mingrong Deng (2011), Jahanshahloo et al. (2007) Khodabakhshi and Aryavash (2012), Hosseinzadeh Lotfi et al. (2011), Adler et al. (2000), Andersen and Petersen (1993) and Y.M. Wanga, Y. Luob and L. Liang (2009) among others). Perhaps the most widely known and applied ranking method is the super-efficiency DEA model (see, e.g., Andersen and Petersen (1993), Sueyoshi (1999) and Tone (2002)). The problem with super-efficiency DEA is inability to rank non-extreme DEA-efficient DMUs and that under certain conditions infeasibility occurs which limits the applicability of the technique (see Seiford et al. (1999) for detail). Some authors used the Analytic Hierarchy Process (AHP) technique to rank DMUs. Sinuany-Stern et al. (2000) proposed an two stage approach based on the relationship between DEA and AHP to rank DMUs. Their two-stage approach has some problems; such as illogic comparison two DMUs in one DEA model and it is not compatible with DEA ranking in the case of multiple inputs/outputs. Some other authors used the IAHP model for the evaluation and classification of efficient units, because it makes it possible to incorporate an appropriate level of uncertainty in the analysis that is typical for economic decision making problems (see Jablonsky (2007)). Another effective tool to rank the performance of DMUs is cross efficiency method, proposed by Sexton et al. (1986). Cross efficiency evaluates the performance of each DMU with the optimal input and output weights of all the DMUs instead of with its own weights only. The drawback of cross efficiency evaluation is the possible existence of multiple optimal weights that leads to different rankings of units. In order to remedy this drawback; Sexton et al. (1986) suggested to use secondary goals to choose the weights among the optimal solutions. The well-known secondary goals are benevolent and aggressive formulations (Sexton et al., 1986; Doyle and Green, 1994). The benevolent formulation selects the weights that maximizing the cross efficiencies of the other DMUs as much as possible while maintaining the self-evaluation efficiency score of the DMU under evaluation, whereas the aggressive formulation also minimizing the efficiency scores of all the DMUs while maintains the self-evaluation efficiency score. For more secondary goals, see Liang et al. (2008a), Wang et al. (2010a) and (2010b), for instance. Nevertheless, the benevolent or aggressive formulations still have some drawbacks; such as their inability to obtain an identical ranking and sometimes the weight sets induced by the aggressive or benevolent formulation are non-unique. The idea of the interval cross efficiency introduced by Liang et al. (2008b) for the first time. Yang et al. (2012) used interval cross efficiency to rank all DMUs. They considered all possible weight sets in weight space when computing the cross efficiency and each DMU is given an interval cross efficiency. They viewed the interval cross efficiency matrix as a stochastic multicriteria decision making (MCDM) problem. The aim of this paper is to propose a method for ranking all DMUs using interval Cross Efficiency and interval Analytic Hierarchy Process methods. We call this new method as ICE-IAHP method, hereafter. Our approach includes two stages. In the first stage, using DEA models, the (modified) interval cross-efficiency value of each DMU is specified. In the second stage, the interval pairwise comparison matrix generated in the first stage is utilized to rank the DMUs via the IAHP. In order to generate interval weights of the interval pairwise comparison matrix, the Arbel's preference programming method (Arbel (1989)) is used. Finally, a simple preference ranking method (Wang et al. (2005b)) is used to compare or rank the generated interval weights (and so, rank all DMUs). We show that the generated interval pairwise comparison matrix in the first stage is consistent and therefore, the ranking orders for alternatives (DMUs) are reliable. The motivation of this paper is that one can obtain *the degree of preference* of each DMU over other DMUs. The degree of preference is the degree of interval weight of one DMU being greater than another one (Wang et al. (2005a)) and therefore, the corresponding DMU having better performance than another one. None of the ranking methods have this property. This paper is organized as follows. Section 2 presents a brief introduction to the (interval) cross efficiency evaluation, (interval) AHP. The new ranking method for all DMUs are developed in Section 3. Numerical example is examined in Section 4. The paper is concluded in Section 5.

2 Background

2.1 Interval Cross-Efficiency (ICE)

Consider a set of n DMUs which use m inputs to produce s outputs. Particularly, DMU_j ($j \in J = 1, ..., n$) consumes amount x_{ij} of input i and produces amount y_{rj} of output r. Let $X_j = (x_{1j}, ..., x_{mj})$ in which $X_j \geq 0$ & $X_j \neq 0$ and $Y_j = (y_{1j}, ..., y_{rj})$ in which $Y_j \geq 0$ & $Y_j \neq 0$.

The CCR model corresponding DMU_k can be written as (Charnes et al., 1978):

$$
\begin{aligned}
\theta^*_{kk} = \max \quad & \sum_{r=1}^{s} u_{rk} y_{rk} \\
s.t. \quad & \sum_{r=1}^{s} u_{rk} y_{rj} - \sum_{i=1}^{m} v_{ik} x_{ij} \leq 0, \quad j = 1, ...n \\
& \sum_{i=1}^{m} v_{ik} x_{ik} = 1, \\
& u_{rk} \geq 0, \qquad r = 1, ..., s \\
& v_{ik} \geq 0, \qquad i = 1, ..., m
\end{aligned}
\tag{1}
$$

Let $u^*_{rk}(r = 1, 2, ..., s)$ and $v^*_{ik}(i = 1, ..., m)$ be the optimal solution to the linear programming LP (1). Then,

$$
\theta^*_{pk} = \frac{\sum_{r=1}^{s} u^*_{rk} y_{rp}}{\sum_{i=1}^{m} v^*_{ik} x_{ip}}, \qquad p, k = 1, ..., n \tag{2}
$$

is referred to as a cross-efficiency value of DMU_p using the weight(s) of DMU_p and reflects the peer evaluation of DMU_p to DMU_k.

In general, the optimal weights obtained using classical DEA are not unique. Therefore, the values θ_{pk} will change depending on these values. To remedy this drawback, the aggressive model (3) and the benevolent model (4) were suggested by Doyle and Green (1994):

$$
\begin{aligned}
\min \quad & \sum_{r=1}^{s} u_{rp} \Big(\sum_{j=1, j \neq p}^{n} y_{rj} \Big) \\
s.t. \quad & \sum_{r=1}^{s} u_{rp} y_{rj} - \sum_{i=1}^{m} v_{ip} x_{ij} \leq 0, \qquad j = 1, ...n, \ j \neq p \\
& \sum_{i=1}^{m} v_{ip} \Big(\sum_{j=1, j \neq p}^{n} x_{ij} \Big) = 1, \\
& \sum_{r=1}^{s} u_{rp} y_{rp} - \theta^*_{pp} \sum_{i=1}^{m} v_{ip} x_{ip} = 0 \\
& u_{rp} \geq 0, \qquad r = 1, ..., s \\
& v_{ip} \geq 0, \qquad i = 1, ..., m
\end{aligned}
\tag{3}
$$

and

$$
\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_{rp} \Big(\sum_{j=1, j \neq p}^{n} y_{rj} \Big) \\
s.t. \quad & \text{the same constraints as in (3).}
\end{aligned}
\tag{4}
$$

The interval cross efficiency is computed by solving the models (5) and (6):

$$
\begin{aligned}
\overline{\theta}_{pk} = \max \quad & \sum_{r=1}^{s} u_{rk} y_{rp} \\
s.t. \quad & \sum_{r=1}^{s} u_{rk} y_{rj} - \sum_{i=1}^{m} v_{ik} x_{ij} \leq 0, & j = 1, ...n \\
& \sum_{r=1}^{s} u_{rk} y_{rk} - \theta^*_{kk} \sum_{i=1}^{m} v_{ik} x_{ik} = 0, \\
& \sum_{i=1}^{m} v_{ik} x_{ip} = 1, \\
& u_{rk} \geq 0, & r = 1, ..., s \\
& v_{ik} \geq 0, & i = 1, ..., m
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
\underline{\theta}_{pk} = \min \quad & \sum_{r=1}^{s} u_{rk} y_{rp} \\
s.t. \quad & \text{the same constraints as in } (5).
\end{aligned}
\tag{6}
$$

where θ^*_{kk} is defined as above. In fact, the models (5) and (6) give the maximum and minimum cross efficiency values of DMU_p using the weight(s) of DMU_k, respectively. The following properties are hold for models (5) and (6):

Property 1. $0 < \underline{\theta}_{pk} \leq \overline{\theta}_{pk} \leq 1$.

Property 2. If model (1) has unique optimal solution then $\overline{\theta}_{pk} = \underline{\theta}_{pk}$.

Property 3. For each optimal solution (u^*_k, v^*_k) of the model (1); we have, $\theta_{pk} \in [\underline{\theta}_{pk}, \ \ \overline{\theta}_{pk}]$.

2.2 Interval Analytical Hierarchy Process (IAHP)

The AHP with interval decision maker's judgment is called interval AHP (IAHP) (Saaty and Vargas (1987)). An interval comparison matrix can be represented by:

$$
A = (a_{ij})_{n \times n} = \begin{bmatrix}
1 & [l_{12}, u_{12}] & \cdots & [l_{1n}, u_{1n}] \\
[l_{21}, u_{21}] & 1 & \cdots & [l_{2n}, u_{2n}] \\
. & . & . & . \\
. & . & . & . \\
. & . & . & . \\
[l_{n1}, u_{n1}] & [l_{n2}, u_{n2}] & \cdots & 1
\end{bmatrix}
$$

where l_{ij} and u_{ij} are the lower and upper bounds for preference relation between the i^{th} and j^{th} element, respectively. Also, $l_{ij} = 1/u_{ji}$, $u_{ij} = 1/l_{ji}$ and $l_{ij} \leq a_{ij} \leq u_{ij}$.

The following definition and theorem are about the above interval comparison matrix:

Definition 1.(Wang et al. (2005a)) *Let $A = (a_{ij})_{n\times n}$ is an interval comparison matrix defined by (12) with $l_{ij} \leq a_{ij} \leq u_{ij}$ and $a_{ii} = l_{ii} = u_{ii} = 1$ for $i, j = 1, ..., n$. If the convex feasible region $S_w = \{w = (w_1, ..., w_n) | l_{ij} \leq w_i/w_j \leq u_{ij}, \sum_{i=1}^{n} w_i = 1, w_i > 0, j = i, ..., n\}$ is nonempty, then A is said to be a consistent interval comparison matrix.*

Theorem 1: *$A = (a_{ij})_{n\times n}$ is a consistent interval comparison matrix if and only if it satisfies the following inequality constraints:*

$$\max_k (l_{ik} l_{kj}) \leq \min_k (u_{ik} u_{kj}) \textit{ for all i,j,k=1, ..., n.}$$

Proof. See Wang et al. (2005a).

By the Theorem 1 one can judges whether or not an interval comparison matrix is consistent without solving any mathematical programming model.

2.3 Interval weight generation method for interval pairwise comparison matrices

In this paper the Arbel's preference programming method (Arbel (1989)) is used to derive the consistent interval weights due to the consistency of the interval pairwise comparison matrix (see Theorem 2) and its simplicity and effectiveness. The method is as follows:
The following pairs of linear programming (LP) models (7) is solved to find interval weights.

$$\begin{aligned} \max / \min \quad & w_i \\ s.t. \quad & W \in S_w, \end{aligned} \tag{7}$$

where $W = (w_1, ..., w_n)^t$ and $S_w = \{w = (w_1, ..., w_n) | l_{ij} \leq w_i/w_j \leq u_{ij}, \sum_{i=1}^{n} w_i = 1, w_i > 0, j = i, ..., n\}$. The solutions to the above pairs of LP models form the weight intervals denoted by $[w_i^L, w_i^U]$ $(i = 1, ..., n)$.

3 Proposed method

In this section we provide a two-stage method for ranking all DMUs using ICE approach and IAHP technique. In the first stage, for each pair of DMU_k and DMU_p $(k \neq p)$ we compute the interval cross-efficiency value of DMU_p i.e. $[\underline{\theta}_{pk}, \quad \overline{\theta}_{pk}]$ by using models (5) and (6). In order to use of the interval AHP technique and to construct a consistent interval pairwise comparison matrix, we consider the *modified* interval cross-efficiency as follows:

$$[\underline{a}_{pk}, \overline{a}_{pk}] = [\frac{\theta_{pp} + \underline{\theta}_{pk}}{\theta_{kk} + \overline{\theta}_{kp}}, \frac{\theta_{pp} + \overline{\theta}_{pk}}{\theta_{kk} + \underline{\theta}_{kp}}], \theta_{pp} = \theta_{pp}^*, \theta_{kk} = \theta_{kk}^* \tag{8}$$

The nominator of the first fraction of the above interval is the sum of the efficiency score of DMU_p and the minimum cross efficiency values of DMU_p using the weight(s) of DMU_k. The denominator is the sum of the efficiency score of DMU_k and the maximum cross efficiency values of DMU_k using the weight(s) of DMU_p. The second fraction of the above interval can be interpreted similarly. By property 3, for each DMUs DMU_p and DMU_k we have:

$$a_{pk} = \frac{\theta_{pp} + \theta_{pk}}{\theta_{kk} + \theta_{kp}} \in [\frac{\theta_{pp} + \underline{\theta}_{pk}}{\theta_{kk} + \overline{\theta}_{kp}}, \frac{\theta_{pp} + \overline{\theta}_{pk}}{\theta_{kk} + \underline{\theta}_{kp}}]$$

in which θ_{ij} is cross efficiency values of DMU_i using the weight(s) of DMU_j, $i \neq j$ and $i, j = p, k$. The expression $a_{pk} = \frac{\theta_{pp}+\theta_{pk}}{\theta_{kk}+\theta_{kp}}$ can be interpreted as the preference relation between the DMU_p and DMU_k with respect to some optimal weights of model (2) corresponding to DMU_p and DMU_k, say, (u_p^*, v_p^*) and (u_k^*, v_k^*). In fact by a_{pk} we compare the performance of DMU_p and DMU_k with together. If $a_{pk} < 1$, it means that unit p is evaluated less than unit k with respect to (u_p^*, v_p^*) and (u_k^*, v_k^*). Also $\underline{a}_{pk} = \frac{\theta_{pp}+\underline{\theta}_{pk}}{\theta_{kk}+\overline{\theta}_{kp}}$ is the lower bound and $\overline{a}_{pk} = \frac{\theta_{pp}+\overline{\theta}_{pk}}{\theta_{kk}+\underline{\theta}_{kp}}$ is the upper bound for a_{pk}.

The intervals $[\underline{a}_{pk}, \overline{a}_{pk}]$ can be interpreted as follows:

If $1 \leq \underline{a}_{pk} < \overline{a}_{pk}$ it means that DMU_p is evaluated more than DMU_k with respect to all weights of DMU_k. If $\underline{a}_{pk} < \overline{a}_{pk} \leq 1$ it means that DMU_p is evaluated less than DMU_k with respect to all weights of DMU_k and if $\underline{a}_{pk} < 1 < \overline{a}_{pk}$ it means that in the interval $[\underline{a}_{pk}, 1]$, DMU_p is evaluated less than DMU_k and in the interval $[1, \overline{a}_{pk}]$, DMU_p is evaluated more than DMU_k with respect to some weights of DMU_k.

Let us define the interval pairwise comparison matrix A as follows:

$$A = \begin{bmatrix} 1 & [\underline{a}_{12}, \overline{a}_{12}] & \cdots & [\underline{a}_{1n}, \overline{a}_{1n}] \\ [\underline{a}_{21}, \overline{a}_{21}] & 1 & \cdots & [\underline{a}_{2n}, \overline{a}_{2n}] \\ \vdots & \vdots & \vdots & \vdots \\ [\underline{a}_{n1}, \overline{a}_{n1}] & [\underline{a}_{n2}, \overline{a}_{n2}] & \cdots & 1 \end{bmatrix}$$

The square matrix A of order n is a generalized pairwise comparison matrix in AHP; where n is the number of DMUs. Unlike the traditional one, all the values in pairwise comparison matrix A are not real but interval numbers. It is also found that the components in the diagonal are the special cases that $\underline{a}_{jj} = \overline{a}_{jj} = 1$ for any $j = 1, 2, ..., n$. It can be shown that $\underline{a}_{ij} = \dfrac{1}{\overline{a}_{ji}}$, $\overline{a}_{ij} = \dfrac{1}{\underline{a}_{ji}}$ and $\underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij}$ as done by IAHP.

The interval pairwise comparison matrix A is consistent by the following Theorem:

Theorem 2: *The interval pairwise comparison matrix A is consistent.*

Proof. By Theorem 1, proof is straightforward. □

Remark 1. In view of Theorem 2, the Arbel's preference programming method is used to derive the consistent interval weights of the interval pairwise comparison matrix A.

The following Theorems will be needed in the sequel:

Theorem 3: *Every tournament*[1] *contains a Hamiltonian path*[2]*.*

Proof. See Bondy et al. (1976) □

[1]If the edges of a complete graph are each given an orientation, the resulting directed graph is called a tournament.

[2]A Hamiltonian path in a digraph is a path containing all of the nodes.

We state the following Theorem without proof:

Theorem 4: *the Hamiltonian path of each tournament G is unique if and only if for each edges $(i,j),(j,k)\in G$ we have $(i,k)\in G$.*

In the second stage, using Arbel's method; studied in subsection 2.3, the interval weights vector of the interval pairwise comparison matrix A is computed, as shown in Table 1.

Table 1 is a generalized interval AHP matrix for n DMUs. The last column is the vector interval weights of the DMUs. Finally, in order to rank all DMUs, we compare and rank interval weights. For this aim, using Wang's method (Wang et al. (2005b)) we construct a tournament graph, say graph G, in which each DMU is as a node of tournament G. Theorem 3, transitivity property[3] and Theorem 4 ensure that tournament G has a unique Hamiltonian path. This Hamiltonian path is used to rank all interval weights and therefore ranking all DMUs. The ranking process is outlined below:

Suppose $a=[a_1,a_2]=[w_p^L,w_p^U]$ and $b=[b_1,b_2]=[w_q^L,w_q^U]$ are the interval weights of DMU_p(=node p) and DMU_q(=node q); respectively. If $P(a>b)\geq P(b>a)$, then the edge (p,q) is added to the graph G; in which

$$P(a>b)=\frac{\max(0,a_2-b_1)-\max(0,a_1-b_2)}{(a_2-a_1)+(b_2-b_1)}$$

is the degree of preference of a over b (or $a>b$).

This means that DMU_p is preferred over DMU_q to a degree of $100P(a>b)\%$, denoted by $DMU_p \succ^{100P(a>b)\%} DMU_q$. If $P(a>b)=P(b>a)=0.5$, then, a is said to be indifferent to b, denoted by $a\sim b$. This means that DMU_p and DMU_q is not preferred over each other, denoted by $DMU_p\sim DMU_q$. In a similar manner we compare each pair of DMUs DMU_i and DMU_j and therefore, the edge (i,j) or (j,i) is added to the graph G. The resulting graph G will be a tournament. Therefore, by transitivity property and Theorems 3 and 4, the graph G is contain a unique Hamiltonian path. This unique Hamiltonian path is used to complete ranking of all DMUs. Moreover, each pair of DMUs can be compared with together and the degree preference of each DMU over the other one can be determined by the proposed method. It is the foremost advantage of the proposed method with respect to the other existing methods.

[3]Transitivity: If $P(a>b)\geqslant 0.5$ and $P(b>c)\geqslant 0.5$, then, $P(a>c)\geqslant 0.5$. (Wang et al. (2005b))

Table 1: A generalized interval AHP matrix for n DMUs.

	DMU_1	DMU_2	$\ldots$	$\ldots$	DMU_n	Interval weight
DMU_1	1	$[\underline{a}_{12}, \overline{a}_{12}]$	$\ldots$	$\ldots$	$[\underline{a}_{1n}, \overline{a}_{1n}]$	$[w_1^L, w_1^U]$
DMU_2	$[\underline{a}_{21}, \overline{a}_{21}]$	1	$\ldots$	$\ldots$	$[\underline{a}_{2n}, \overline{a}_{2n}]$	$[w_2^L, w_2^U]$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$	$\vdots$
DMU_n	$[\underline{a}_{n1}, \overline{a}_{n1}]$	$[\underline{a}_{n2}, \overline{a}_{n2}]$	$\ldots$	$\ldots$	1	$[w_n^L, w_n^U]$

4 A Numerical example

In this section, a numerical example is examined to illustrate the proposed method. Comparisons with other existing procedure (Aggressive-Benevolent (Doyle and Green (1994)) and Yang's methods (2012)) will also be made. The example is taken from Wong and Beasley (1990) and is about efficiency evaluation of seven academic departments in a university. The input and output variables are as follows:

Input variables:

x_1: number of academic staff;

x_2: academic staff salaries in thousands of pounds;

x_3: support staff salaries in thousands of pounds.

Output variables:

y_1: number of undergraduate students;

y_2: number of postgraduate students;

y_3: number of research papers.

Table 2 shows the input and output data of the seven departments together with their CCR efficiencies. Running the DEA model (1) will result six efficient units as 1, 2, 3, 5, 6 and 7.

By models (5) and (6), the interval pairwise comparison matrix of the seven DMUs (academic departments) is obtained and reported in Table 3.

Table 2: Numerical example. Data set of seven academic departments in a university.

	input			output			CCR efficiency
Department (DMU)	x_1	x_2	x_3	y_1	y_2	y_3	
1	12	400	20	60	35	17	1
2	19	750	70	139	41	40	1
3	42	1500	70	225	68	75	1
4	15	600	100	90	12	17	0.820
5	45	2000	250	253	145	130	1
6	19	730	50	132	45	45	1
7	41	2350	600	305	159	97	1

Table 3: Numerical example. Interval pairwise comparison matrix of seven DMUs ($[\underline{a}_{pk}, \overline{a}_{pk}]$)

Dept.	DMU_k							Interval weight
DMU_p	1	2	3	4	5	6	7	$w_i = [w_i^L, w_i^U]$
1	1	[0.848, 1.451]	[0.896, 1.317]	[1.122, 1.899]	[0.745, 1,502]	[0.822, 1.321]	[0.817, 1.737]	[0.1233, 0.1865]
2	[0.689, 1.179]	1	[0.829, 1.071]	[1.2195, 1.328]	[0.884, 1.185]	[0.847, 1.025]	[0.778, 1.246]	[0.1284, 0.1563]
3	[0.759, 1.115]	[0.933, 1.205]	1	[1.3426, 1.7868]	[0.907, 1.390]	[0.874, 1.098]	[1.095, 1.539]	[0.1436, 0.1782]
4	[0.633, 0.998]	[0.843, 0.91]	[0.6634, 0.8472]	1	[0.7073, 0.7194]	[0.6222, 0.9473]	[0.603, 0.91]	[0.0983, 0.1203]
5	[0.665, 1.342]	[0.843, 1.130]	[0.719, 1.101]	[1.183, 1.421]	1	[0.739, 1.123]	[0.878, 1.311]	[0.1207, 0.1647]
6	[0.757, 1.215]	[0.975, 1.179]	[0.649, 0.913]	[1.2255, 1.8868]	[0.89, 1.353]	1	[0.805, 1.605]	[0.1372, 0.1750]
7	[0.575, 1.224]	[0.802, 1.285]	[0.910, 1.143]	[1.2195, 1.9493]	[0.762, 1.139]	[0,623, 1.242]	1	[0.1229, 0.1567]

By Theorem 2, Arbel's preference programming method can be used to derive the weight intervals. The results are summarized in the last column of Table 3. To give a complete ranking order for the seven interval weights, Table 4 records their degrees of preference. The corresponding directed diagram (tournament) is depicted in Fig. 1, from which it is clear that DMU5∼DMU2 to a degree of 50.42 percent, DMU3 is preferred over DMU1, DMU2, DMU4, DMU5, DMU6 and DMU7 to a degree of 56.18%, 79.55%, 100%, 73.06%, 56.55% and 80.73%, respectively; DMU2 over DMU4 and DMU7 to a degree of 100% and 54.29%, respectively; DMU5 over DMU2, DMU4 and DMU7 to a degree of 50.42%, 100% and 53.79%; DMU6 over DMU1, DMU2, DMU4, DMU5 and DMU7 to a degree of 51.19%, 70.82%, 100%, 66.30% and 72.80%, respectively and DMU1 is preferred over DMU2, DMU4, DMU5 and DMU7 to a degree of 63.73%, 100%, 61.23% and 65.57%, respectively; DMU7 is preferred over DMU4 to a degree of 100%. By Theorems 3 and 4 a unique hamiltonian path from node 3 (DMU3) to node 4 (DMU4) is found (i.e. $3 \to 6 \to 1 \to 5 \to 2 \to 7 \to 4$). So, the final ranking order should be $DMU3 \succ^{56.55\%} DMU6 \succ^{51.19\%} DMU1 \succ^{61.23\%} DMU5 \succ^{50.42\%} DMU2 \succ^{54.29\%} DMU7 \succ^{100\%} DMU4$. Moreover, all DMUs can be compared two by two, for example $DMU1 \succ^{63.73\%} DMU2$, $DMU2 \succ^{54.29\%} DMU7$, $DMU1 \succ^{65.57\%} DMU7$ and so on.

Table 5 shows the different ultimate ranking results obtained by aggressive and benevolent methods (Doyle and Green (1994)) and interval cross efficiency (Yang et al. (2012)) method and our ICE-IAHP method. In Table 5, DMU4 is ranked in the last place based on any cross efficiency and interval cross efficiency methods and our method. On the other hand, DMU3 is ranked in the first place by our method but, DMU6 is ranked in the first place based on any cross efficiency and interval cross efficiency methods. The ranking results for all DMUs are almost consistent except for DMU3 and DMU5. Also by our method all CCR-efficient DMUs are 100% better than CCR-inefficient DMU4.

5 Conclusion

In this paper we propose a two-stage method for ranking all DMUs based on interval cross efficiency and interval AHP technique. In the first stage, interval cross efficiency of each DMU is calculated. In the second stage, the interval pairwise comparison matrix generated in the first step is utilized to rank the units via the IAHP. We show that the defined interval pairwise

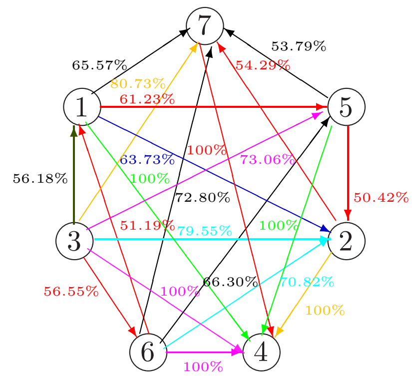

Figure 1: Preference relations in Numerical example.

Table 4: Numerical example. The matrix of degrees of preference for interval weights

p_{ij}	w_1	w_2	w_3	w_4	w_5	w_6	w_7
w_1	-	0.6373	0.4382	1.0000	0.6123	0.4881	0.6557
w_2	0.3627	-	0.2045	1.0000	0.4958	0.2918	0.5429
w_3	0.5618	0.7955	-	1.0000	0.7306	0.5655	0.8073
w_4	0.0000	0.0000	0.0000	-	0.0000	0.0000	0.0000
w_5	0.3877	0.5042	0.2694	1.0000	-	0.3370	0.5379
w_6	0.5119	0.7082	0.4345	1.0000	0.6630	-	0.7280
w_7	0.3443	0.4571	0.1927	1.0000	0.4621	0.2720	-

comparison matrix is always consistent and so, results in reliable weights and ranking orders for alternatives (DMUs). Also, in order to compare or rank interval weights (and so rank all DMUs) we use preference ranking method of interval numbers proposed by Wang et al. (2005b). There are some advantage of our approach over the other existing approaches. First, one can finds the degree of preference of each DMU over the other DMUs. Second, our method considers all the possible weight sets in weight space when cross efficiency scores are computed and so, we don't need to use any secondary goal when the non-uniqueness of cross efficiency scores are exist. Third, to rank all DMUs in the interval cross efficiency matrix, the interval Analytic Hierarchy Process, that is a powerful tool for analysis of complex decision problems, is used. So, we expect our ranking provides more reliable ranking. Fourth, the efficient and inefficient DMUs are ranked simultaneously by our method. In this paper we rank DMUs in constant return to

Table 5: Numerical example. The results of ranking

	Aggressive		Benevolent		Interval cross efficiency	ICE-IAHP method
	Efficiency	Ranking	Efficiency	Ranking	Ranking	Ranking
DMU_1	0.705	2	0.946	3	3	3
DMU_2	0.688	3	0.973	2	2	5
DMU_3	0.672	4	0.883	6	6	1
DMU_4	0.341	7	0.723	7	7	7
DMU_5	0.615	5	0.913	4	5	4
DMU_6	0.804	1	0.993	1	1	2
DMU_7	0.525	6	0.899	5	4	6

scale; in a similar way one can also ranks DMUs in variable return to scale. Initial studies had shown that our approach also can be applied with DEA models with ordinal, bounded and ratio-bounded data. We suggest as future works a deeper analysis in this subject.

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