

# Extracting Frequent Gradual Patterns Using Constraints Modeling

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**Abstract.** In this paper, we propose a constraint-based modeling approach for the problem of discovering frequent gradual patterns in a numerical dataset. This SAT-based declarative approach offers an additional possibility to benefit from the recent progress in satisfiability testing and to exploit the efficiency of modern SAT solvers for enumerating all frequent gradual patterns in a numerical dataset. Our approach can easily be extended with extra constraints, such as temporal constraints in order to extract more specific patterns in a broad range of gradual patterns mining applications. We show the practical feasibility of our SAT model by running experiments on two real world datasets.

## 1 Introduction

Frequent pattern mining is a well-known and essential problem of data mining. Its goal is to efficiently discover in large volumes of data the hidden patterns having more occurrences than a pre-defined threshold. This problem started with the analysis of transactional data [1] (frequent itemsets), and quickly expanded to the analysis of data having more complex structures such as sequences, trees or graphs. A new pattern mining problem has then been introduced in transactional databases where attributes can have a numeric value: mining frequent gradual itemsets (also known as gradual patterns). Frequent gradual patterns mining problem consists to discover frequent co-variations between numerical attributes in Databases [3], such as: The higher the age, the higher the salary. This problem has been tackled since many years [28] and find the numerous applications for the numerical databases such that the biological and medical databases. Several algorithms have been proposed in the literature in order to address this problem. Most of these algorithms use very often the data mining algorithms for automatically extract the gradual patterns. [16,4,24,8,9,20].

In [16], for extracting such patterns, the authors apply a linear regression analysis between pairs of attributes and the validity of the gradual tendency between two attributes is evaluate from the quality of regression. This validity is measured by the normalized mean squared error  $R^2$ , together with the slope of the regression line: attribute pairs that are insufficiently correlated are rejected, as well as pairs for which one attribute remains almost constant while the other one increases, which can be detected by a low slope of the regression line. In [4], the authors use for the first time the data mining approaches from a adaptation of **Apriori** algorithm for extracting gradual patterns. They formulate the problem of extraction of such patterns as the problem of discovery classical itemset in a suitable set of transactions  $\Delta'$  obtained from the initial data set  $\Delta$ . Each pair of objects in the initial data is associated to a transaction in the derived data set  $\Delta'$ ; each item  $I$  of  $\Delta$  defines two items  $I^{\leq}$  and  $I^{\geq}$  in  $\Delta'$  instead one item. A transaction  $t$  in  $\Delta'$  then possesses an item  $I^*$  ( $* \in \{\geq, \leq\}$ ) if the pair of objects  $(x, x')$  of  $\Delta$  satisfies the constraint imposed by  $I^*$ , i.e.  $A(x) * A(x')$ . Formulate thus, a gradual pattern in  $\Delta$  is equivalent to a classical itemset extracted from  $\Delta'$ . The computing of support of gradual pattern makes this approach complex. In fact, the support is compute by considering all object couples and explicitly building the data set  $\Delta'$  to apply a classic frequent itemset mining algorithm have too high a computational cost.

Whereas the proposed algorithm in [4] are limited to six attributes, in [9] a first efficient algorithm for mining gradual itemsets and gradual rules capable of handling databases with hundreds of attributes was proposed. One of the major problem of the mining gradual pattern approaches is the exponential combination space to explore and the problem of handling the quantity of extracted patterns which can be also of exponential size. This combinatorial explosion is tackled in [20,3,10]. In fact, in [20], the authors propose an approach for extracting gradual patterns from large datasets which takes advantage of a binary representation of lattice structure. In [3,10], in order to reduce the quantity of patterns, it is proposed to mine only the closed frequent gradual patterns.

Our approach for extracting all frequent closed gradual patterns in a numerical dataset differs from all the previous specialized approaches. It follows the SAT-Based framework proposed in [18] for mining frequent closed itemsets. This new framework offers a declarative and flexible representation model. New constraints often require new implementations in specialized approaches, while they can be easily integrated in such Boolean satisfiability framework. It allows data mining problems to benefit from several generic and efficient SAT solving techniques. In [6], the authors show how some typical constraints (e.g. frequency, closure, monotonicity) used in itemset mining can be formulated for use in Boolean satisfiability. This first study leads to the first SAT approach for itemset mining displaying nice declarative opportunities without neglecting efficiency. Considering the promising results obtained from this framework, we propose to heavily exploit the declarative language Boolean satisfiability (SAT) and these associated efficient and generic solving techniques. First, we propose a new SAT model for the problem of mining frequent gradual pattern that in-

cludes different types of constraints. We propose the new constraints different from those proposed in [6]. The first one, encodes that a gradual itemset not must contain both a gradual item and its complementary gradual item. The second one, allows us for a given gradual itemset  $s$ , to place uniquely one transaction in each position of the longest sequence of transactions respecting  $s$ . The third constraint captures the fact that a transaction should not be placed in more than one position of the longest sequence of transactions respecting  $s$ . The others constraints allow to link the gradual item with the transactions in order to compare the transactions with respect to gradual item of a given gradual itemset. This link allows us to detect for a given gradual itemset  $s$ , all the sequences of transactions which respect  $s$ . Any gradual itemset, representing co-variations items has a symmetric gradual itemset where the items are the same and the variations are all reversed. If a gradual itemset is frequent, then its symmetric gradual itemset is also frequent and does not carry additional information. This symmetry of problem allows to generate only half of gradual itemsets for automatically deduce the other ones. We take into account this symmetry by forcing the SAT solver during the search process to affect always a positive polarity to the first variable selected by branching heuristic at each restart.

Finally, we provide a boolean formulation of closedness constraints, in order to search for frequent closed gradual patterns. This allows us to obtain a SAT-based model for enumerating frequent closed gradual patterns in a numerical dataset.

The paper is organized as follows: in Section 2, we present the problem of mining gradual itemsets from numerical databases and some efficient algorithms proposed in the data mining domain for the automatic extracting such patterns. We also recall the definition of Boolean satisfiability problem, called SAT. Section 3 describes the SAT-Based enumeration procedure to deal with the problem of enumerating all models of a CNF formula. In Section 4, we describe our SAT encoding of frequent gradual itemset mining problem and show through an example how it can be applied to find frequent gradual itemsets in a numerical dataset. Finally, section 5 presents detailed experiments carried out over real datasets, showing the applicability and the interest of our approach.

## 2 Preliminaries

In this section, we formally describe the problem of mining frequent gradual itemsets (patterns) and the mining closed gradual patterns in a numerical dataset. We then present some approaches of state of the art proposed to automatically extract such patterns. We also recall the boolean satisfiability problem, commonly called SAT and the definitions of gradual patterns given in [9].

### 2.1 Gradual patterns mining problem

The problem of mining gradual patterns consists in mining attribute co-variations in numerical dataset of the form "*The more/less X, . . . , the more/less Y*". We

assume here that we are given a dataset  $\Delta$  containing a set of objects  $\mathcal{T}$  that defined a relation on a attribute set  $\mathcal{I}$  with numerical values  $\mathcal{I}$  i.e. for  $t \in \mathcal{T}$ ,  $t[i]$  denotes the value of the attribute  $i$  over object  $t$ .

For instance, we consider the numerical dataset given in Table 1 which gives for each date, the quantity of each species presents in a given ecosystem. This table contains eight objects ( $\{t_1, \dots, t_8\}$ ) and three attributes represents by the scientific names of the different species (*Poaceae*, *Secale*, *Rumex*).

Dates	Poaceae (p)	Secale (s)	Rumex (r)
t1	4	3	13
t2	6	9	11
t3	8	1	9
t4	13	7	5
t5	4	5	10
t6	9	6	8
t7	10	6	12
t8	13	7	13

**Table 1.** Abundance of species in a Ecosystem

Each attribute will hereafter be considered twice: once to indicate its increasing, and once to indicate its decreasing, using the  $\leq$  and  $\geq$  operators. This leads to consider new kinds of items, called gradual items.

**Definition 1 (Gradual Item).** *Let  $\Delta$  be a dataset defined on a numerical attribute set  $\mathcal{I}$ , A gradual item is defined under the form  $i^*$ , where  $i$  is a attribute of  $\mathcal{I}$  and  $*$   $\in \{\geq, \leq\}$  be a comparison operator.*

If we consider the numerical dataset given in table 1, *Poaceae* $\geq$  (respectively *Poaceae* $\leq$ ) is a gradual item meaning that the values of attribute *Poaceae* is increasing (respectively decreasing).

A gradual itemset (gradual pattern) is then defined as follow:

**Definition 2 (Gradual Itemset).** *A gradual itemset  $s = (i_1^{*1}, \dots, i_k^{*k})$  is a non empty set of gradual items. A gradual  $k$ -itemset is an gradual itemset containing  $k$  gradual items.*

For example,  $\{Poaceae \geq, Rumex \leq\}$  is a gradual itemset meaning that "more the values of attribute *Poaceae* increase, more the values of attribute *Rumex* decrease".

A gradual itemset imposes a variation constraint on several attributes simultaneously. The length of a gradual itemset is equal to number of gradual item that it contains.  $(Poaceae \geq, Rumex \leq)$  is a gradual 2-itemsets.

The support (frequency) of a gradual itemset in a dataset amounts to the extent to which a gradual pattern is present in a given database. Several support

definitions have been proposed in the literature, showing that gradual itemsets can follow different semantics. The choice between them generally depends on the considered application.

A gradual itemset is said to be frequent if its frequency is greater than or equal to a user-defined threshold.

**Definition 3 (Frequent Gradual Itemsets Mining Problem).** *Let  $\Delta$  be a numerical dataset and  $minSupp$  a minimum support threshold. The problem of mining gradual itemsets is to find the set of all frequent gradual itemsets of  $\Delta$  with respect to  $minSupp$ .*

In the following section, we present the different semantics and algorithms that have been proposed to automatically extract gradual itemsets from numerical dataset.

## 2.2 Discovering frequent gradual patterns

Gradual patterns can be compared to fuzzy gradual rules that have first been used for command systems some years ago [11,12], for instance for braking systems: the closer the wall, the stronger the brake force. Whereas such fuzzy gradual rules are expressed in the same way as the gradual patterns, the main difference is that fuzzy gradual rules were not discovered automatically from data. They were designed by human experts and provided as input to expert systems.

Several works in the pattern mining field have shown that it was feasible to mine automatically such rules from raw data [4,16,9]. However, the quantity of mined patterns (and, consequently, the quantity of extracted rules) makes their exploitation difficult. So, as mentioned above, in [3,10], the authors propose to mine only closed gradual patterns in order to reduce the number of patterns extracted without loss of information. This preliminary work didn't exploit closure properties to improve the mining algorithm and reduce execution time. However, mining gradual patterns is a costly task in terms of computation time. It was proposed in [21] to exploit the parallel processing capabilities of multi-core architectures in order to reduce computation time.

The evaluation of the support of gradual patterns has been defined in different manners depending on the semantic and the application considered. In [16], it is based on regression, while [4] and [20] consider the number of transactions that are concordant and discordant, in the idea of exploiting the Kendalls tau ranking correlation coefficient [19]. This means that given a gradual itemset  $s$ , all pairs of transactions  $(t_i, t_j)$  will be compared according to the order induced by  $s$ , and the support will be based on the proportion of these pairs that satisfy all gradual items in  $s$ . The interest of this definition is that it makes possible to take into account the amplitude of the distortion for data that do not satisfy the gradual patterns.

In contrast, the definition of support proposed in [9] is based on the length of the longest sequence of transactions that can be ordered consecutively according to a gradual pattern  $s$ . The interest of this definition is that such transaction

sequences can then be easily presented to the analyst, allowing to isolate and reorder a part of data and to label it with a description in terms of co-variations (the gradual itemset being this description).

The main contribution of the present work is to provide a boolean satisfiability encoding of the problem of mining frequent gradual patterns by considering the gradual patterns definition used in the association rule formulation [9]. We then exploit the scalability of modern SAT solvers to discover frequent closed gradual patterns from the models of the obtained boolean formula.

In this paper, For a given attribute  $i$  in a dataset  $\Delta$ , we consider two gradual items  $i^{\leq}$  and  $i^{\geq}$ , as consider in the algorithms of extracting frequent gradual itemsets proposed in the data mining domain [8,9,20]. We use the variation semantic proposed by [9], which defined the support of a gradual itemset as being the maximum number of transactions (the size of the longest sequences of transactions) that can be ordered w.r.t. this gradual itemset. In order to explain this semantic, we present first the definition of the order induced by a gradual itemset [10].

**Definition 4 (Gradual itemset induced order).** *Let  $s = (i_1^{*1}, \dots, i_k^{*k})$  be a gradual itemset, and  $\Delta$  be a numerical dataset. Two objects  $t$  and  $t'$  of  $\Delta$  can be ordered w.r.t.  $s$  if all the values of the corresponding items from the gradual itemset can be ordered to respect all the variations of the gradual items of  $s$ : for every  $l \in [1, k]$ ,  $t[i_l] \leq t'[i_l]$  if  $*_l = \geq$  and  $t[i_l] \geq t'[i_l]$  if  $*_l = \leq$ . The fact that  $t$  precedes  $t'$  in the order induced by  $s$  is denoted  $t \triangleleft_s t'$ .*

Referring to the previous example from Table 1,  $t_1$  and  $t_2$  can be ordered with respect to gradual itemset  $s_1 = (Poaceae^{\geq}, Rumex^{\leq})$  as  $t_1[Poaceae] \leq t_2[Poaceae]$  and  $t_1[Rumex] \geq t_2[Rumex]$ : we have  $t_1 \triangleleft_{s_1} t_2$ .

This order is only a partial order. For example consider  $t_2$  and  $t_5$  of Table 1: they can't be ordered according to  $s_1$ . In fact, the pattern  $s_1$  is not relevant to explain the variations between  $t_2$  and  $t_5$ , and more generally all transaction pairs that it can't order. Conversely, a gradual pattern is relevant to explain the variations occurring in the transactions that it can order. The support definition that we consider in this paper for our encoding SAT goes further and focuses on the size of the longest sequences of objects that can be ordered according to a gradual itemset. The intuition being that such patterns will be supported by long continuous variations in the data (long periods of co-evolution between paleoecological indicators in the case of paleoecological data given by table 1), such continuous variations being particularly desirable to extract in order to better understand the data.

**Definition 5 (Support of a Gradual Itemset).** *Let  $\Delta$  be a numerical dataset containing a set of objects  $\{t_1, \dots, t_n\}$  and  $L = \langle t_{i_1}, \dots, t_{i_s} \rangle$  be a sequence of objects from  $\Delta$ , with  $\forall k \in [1..s], i_k \in [1..n]$  and  $\forall k, k' \in [1..s], k \neq k' \Rightarrow i_k \neq i_{k'}$ . Let  $s$  be a gradual itemset.  $L$  respects  $s$  if  $\forall k \in [1..s-1]$  we have  $t_{i_k} \triangleleft_s t_{i_{k+1}}$ . Let  $L_s$  be the set of objects that respect  $s$ . The support of  $s$  is define by  $Support(s) = \frac{\max_{L \in L_s} (|L|)}{|\Delta|}$ . i.e. it is the size of the longest list of tuples that respects  $s$ .*

Note that the support of a gradual itemset containing a single gradual item is always 100% as it is always possible to order all the tuples by one column.

By considering the dataset of table 1 and the pattern  $s_1 = (Poaceae^{\geq}, Rumex^{\leq})$ , the set of all the lists of sequence of objects respecting  $s_1$  is  $L_{s_1} = \{\langle t_1, t_2, t_3, t_6, t_4 \rangle, \langle t_1, t_5, t_3, t_6, t_4 \rangle, \langle t_1, t_7, t_4 \rangle, \langle t_1, t_8, t_4 \rangle\}$ . Two lists from  $L_{s_1}$  have a maximal size, which is 5. Hence,  $support(s_1) = \frac{5}{8} = 0.625$ , meaning that 62.5% of the input objects can be ordered consecutively according to  $s_1$ .

**Definition 6 (Complementary Gradual Itemset).** *Let  $s = (i_1^{*1}, \dots, i_k^{*k})$  be a gradual itemset, and  $c$  be a function such that  $c(\geq) = \leq$  and  $c(\leq) = \geq$ . Then  $c(s) = (i_1^{*c}, \dots, i_k^{*c})$  is the complementary (symmetric) gradual itemset of  $s$  and is defined as  $\forall j \in [1..k], *j^c = c(*j)$ .*

The complementary gradual itemset (symmetric gradual itemset) of  $s_1$  is denoted  $c(Poaceae^{\geq}, Rumex^{\leq}) = (Poaceae^{\leq}, Rumex^{\geq})$ .

**Proposition 1.**  $Support(s) = Support(c(s))$ .

The proposition 1 given in [9] avoids unnecessary computations, as generating only half of the gradual itemsets is sufficient to automatically deduce the other ones. This means that for each gradual itemset there is a symmetric gradual itemset having the same support.

### 2.3 Closed gradual patterns

In data mining, closed patterns are key to obtain a condensed representation of the patterns without loss of information [26]. A pattern  $I$  is said closed if there is no pattern  $I'$  such that  $I \subset I'$  and  $support(I') = support(I)$ . This notion of closure has been introduced for the first time in the gradual patterns in [3] where the author propose an pair of functions  $(f, g)$  defining a Galois closure operator for gradual patterns.

Given a set of sequence of transactions  $\mathcal{L}$  of a dataset, the function  $f$  returns the gradual pattern  $s$  (all the attributes (items) associated with their respective variations) respecting all transaction sequences in  $\mathcal{L}$ . While the function  $g$  returns for a given gradual pattern  $s$  the set of the maximal sequences of transactions  $\mathcal{L}$  which respects the variations of all gradual attribute in  $s$ .

Provided these definitions, a gradual pattern  $s$  is said to be closed if  $f(g(s)) = s$ . In [3], the authors use these definitions rather as a post-processing step. In [10], these definitions are included in the mining process and allow to benefit from the runtime and memory reduction.

Let us consider the dataset of the Table 1. Thus, we have for example :  $g(\{Poaceae^{\geq}, Rumex^{\leq}\}) = \{\langle t_1, t_2, t_3, t_6, t_4 \rangle, \langle t_1, t_5, t_3, t_6, t_4 \rangle\}$  and  $f(\{\langle t_1, t_2, t_3, t_6, t_4 \rangle, \langle t_1, t_5, t_3, t_6, t_4 \rangle\}) = \{Poaceae^{\geq}, Rumex^{\leq}\}$ . Therefore,  $\{Poaceae^{\geq}, Rumex^{\leq}\}$  is a closed gradual pattern.

Compared to the context of classical items, the main issue here is to manage the fact that the function  $g$  does not return a set of transactions but it returns a set of sequences of transactions. We propose below a new SAT-based approach for

discovering frequent closed gradual patterns in the numerical dataset. Our proposed approach allows to extract all the frequent gradual patterns with respect to the minimum support threshold by benefiting from the impressive progress in boolean satisfiability checking [5] and from the scalability of modern SAT solvers.

## 2.4 Boolean satisfiability

In this section, we introduce the Boolean satisfiability problem, called SAT. It corresponds to the problem of deciding if a formula of propositional classical logic is consistent or not. It is one of the most studied NP-complete decision problem. In this work, we consider the associated problem of boolean model enumeration.

We consider the conjunctive normal form (*CNF*) representation for the propositional formulas. A (*CNF*) formula  $\mathcal{F}$  is a conjunction of clauses, where a *clause* is a disjunction of literals. A *literal* is a positive ( $l$ ) or negated ( $\neg l$ ) propositional variable. The two literals ( $l$ ) and ( $\neg l$ ) are called complementary. We note by  $\bar{l}$  the complementary literal of  $l$ . For a set of literals  $L$ ,  $\bar{L}$  is defined as  $\{\bar{l} \mid l \in L\}$ .

Let us recall that any propositional formula can be translated to (*CNF*) using linear Tseitin’s encoding [29]. The set of variables occurring in  $\mathcal{F}$  is noted  $Var(\mathcal{F})$ .

An *interpretation*  $\rho$  of a boolean formula  $\mathcal{F}$  is a function which associates a value  $\rho(l) \in \{0, 1\}$  (0 correspond to false and 1 to true) to the variables  $x \in Var(\mathcal{F})$ . A model of a formula is an *interpretation*  $\rho$  that satisfies the formula. SAT problem consists in deciding if a given formula admits a model or not.

## 3 SAT-Based enumeration procedure

In this, we describe the SAT-Based enumeration procedure to deal with the problem of enumerating all models of a CNF formula. SAT is a decision problem. When the answer is positive, the current SAT solvers provide a model satisfying the formula. In the sequel, we briefly describe the basic components of moderns SAT solvers so call CDCL SAT solvers [25,14] designed to enumerate all the models of a given CNF formula. To be exhaustive, these solvers incorporate unit propagation (enhanced by efficient and lazy data structures), variable activity-based heuristic, literal polarity phase, clause learning, restarts and a learnt clauses database reduction policy.

Algorithm 1 depicts the general scheme of CDCL SAT solver extended for model enumeration. A SAT solver is a tree-based backtrack search procedure; at each node of the search tree, the assigned literals (decision literal and the propagated ones) are labeled with the same decision level starting from 1 and increased at each decision (or branching).

Typically, this solver performs a tree-based backtrack search procedure. Each branch of the binary search tree can be seen as a sequence of decision and



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**Algorithm 1:** CDCL Based Enumeration solver

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**Input:** a CNF formula  $\mathcal{F}$   
**Output:** All models of  $\mathcal{F}$

```
1  $\rho = \emptyset$  ; /* interpretation */
2  $\delta = \emptyset$  ; /* learnt clauses database */
3  $dl = 0$  ; /* decision level */
4 while (true) do
5   Prop ;
6    $\gamma = \text{unitPropagation}(\mathcal{F}, \mathcal{I})$  ;
7   if  $\gamma \neq \text{null}$  then
8      $\beta = \text{conflictAnalysis}(\mathcal{F}, \mathcal{I}, \gamma)$  ;
9      $btl = \text{computeBackjumpLevel}(\beta, \mathcal{I})$  ;
10    if  $btl == 0$  then
11      return UNSAT ;
12     $\delta = \delta \cup \{\beta\}$  ;
13    if restart() then
14       $btl = 0$  ;
15    backjump( $btl$ ) ;
16     $dl = btl$  ;
17  else
18    if  $\rho \models \mathcal{F}$  then
19      extractPatternFromModel( $\rho$ ) ;
20      addBlockedClause( $\rho$ ) ;
21      backjumpUntil(0) ;
22      goto Prop ;
23    if (timeToReduce()) then
24      reduceDB( $\delta$ ) ;
25       $l = \text{selectDecisionVariable}(\mathcal{F})$  ;
26       $dl = dl + 1$  ;
27       $\rho = \rho \cup \{\text{selectPhase}(l)\}$  ;
```

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unit propagated literals. At each node, a decision variable is chosen (ligne 23), and assigned to the *true* or *false* polarity (*selectPhase*( $l$ ) - line 25). Then unit propagation is performed in line 6. All these literals (decision and propagated ones) assigned at a given node are labelled with the same level  $dl$ . If all literals are assigned without contadiction, then  $\rho$  is a model of  $\mathcal{F}$  and the formula is answered to be satisfiable (line 16). As our boolean formula represents the encoding of the closed frequent gradual itemset mining problem, each time a model is found, an gradual itemset is extracted from  $\rho$  (line 17). For model enumeration, the search continue by adding a blocked clause to avoid enumerating again the same models (line 18). Search restart at level 0, to search for the next models (lines 19-20). The other case, is reached when unit propagation (lines 8-14) leads to a conflict ( $\gamma$  is the conflict clause), a new asserting clause  $\beta$  is derived by conflict analysis (line 8), mostly following the First-UIP scheme ('Unique Implication Point' [31]) A backtrack level ( $btl$ ) is derived from the asserting clause (line 9). If  $btl$  is null, then the formula is answered unsatisfiable (line 10), otherwise  $\beta$  is added to

the learnt clauses database (line 11) and the algorithm backjump to the level *btl* (line 13). Regularly, the CDCL solver performs restarts, by backtracking to level 0 (line 12) using one of the various restart strategies ([15]). Such restarts define the frequency used by the solver to restart the search. Finally, another component concern the learnt clauses management policy. To maintain a learnt clauses database of reasonable size, a reduction is performed (line 22) using one of the various strategies proposed in the literature [2,13,22,17].

## 4 SAT-based encoding for the problem of discovering frequent gradual patterns

In this section, we show how the problem of mining all the frequent gradual itemset in a numerical dataset with respect to a minimum support threshold *minSupp* describe in section 2 can be encoded as a boolean formula in *CNF*. Our SAT encoding is inspired on the encodings proposed in [18].

In order to formally describe our encoding, we consider a numerical dataset  $\Delta = \mathcal{T} \times \mathcal{A}$  where  $\mathcal{A} = \{a_1, \dots, a_m\}$  is a set of attributes,  $\mathcal{T} = \{t_1, \dots, t_m\}$  a set of transactions and a minimum support threshold *minSupp*. In the follow, we denote by a parameter *k* the minimum support threshold. For reasons of clarity, the comparison operator " $\leq$ " (respectively " $\geq$ ") will be denoted "+<sup>-</sup>" (respectively "-<sup>-</sup>"). We denote by  $\mathcal{A}^*$  the set of attributes variations:  $\mathcal{A}^* = \{a_1^+, a_1^-, \dots, a_m^+, a_m^-\}$ . The SAT encoding of frequent gradual itemset mining that we propose is based on the use of propositional variables representing the items and the transaction identifiers in  $\Delta$

Let  $\mathcal{L} = \langle t_{i_1}, t_{i_2} \dots t_{i_k} \rangle$  the sequence ordering of the *k* first transactions as should be appear in the longest sequence of transactions required for a frequent gradual itemset. We denote by  $y_{ij}$  the fact that the transaction  $t_i$  is set on the *j*th position of  $\mathcal{L}$ .

First of all, we associate with each gradual attribute *a* two boolean variables respectively  $x_{a^+}$  and  $x_{a^-}$ .

The first constraint (1) allows to not consider gradual itemset involving both  $a^+$  and  $a^-$  of each attribute *a*.

$$\bigwedge_{a \in a_1 \dots a_n} (\neg x_{a^+} \vee \neg x_{a^-}) \quad (1)$$

*This first constraint solves the problem encountered with the specialized algorithm of frequent gradual itemsets mining GLCM [10] which often returns the gradual itemsets containing both the gradual items and their corresponding complementary gradual items.*

The second constraint (2) allows us to place uniquely one transaction  $t_i$  in the *j*th position of a gradual itemset *s*. To this end, a new boolean variable  $y_{ij}$  is added to indicate that the transaction  $t_i$  is putted in the position *j*.

$$\bigwedge_{1 \leq j \leq k} \left( \sum_{i=1}^n y_{ij} = 1 \right) \quad (2)$$

Constraint (3) is introduced to not allow a transaction to be placed in more than one position in  $s$ .

$$\bigwedge_{1 \leq i \leq n} \left( \sum_{j=1}^k y_{ij} \leq 1 \right) \quad (3)$$

Constraint (4) aims to express given a gradual item  $a$ , the set of transactions that can be set in position  $j$  (respectively) those cannot be set constraint (5)).

$$\bigwedge_{a^\circ \in \mathcal{A}^*} \bigwedge_{1 \leq i \leq n} \bigwedge_{1 \leq j \leq k} (x_{a^\circ} \wedge y_{ij} \rightarrow \bigvee_{t_k(a) \diamond t_i(a)} y_{k(j+1)}) \quad (4)$$

Note that such constraint can be expressed differently by considering only those that are not allowed as stated in Constraint (5). In contrast to (4), constraint (5) allows to add only ternary clauses. However, their number is higher than those of (4).

$$\bigwedge_{a^\circ \in \mathcal{A}^*} \bigwedge_{1 \leq i \leq n} \bigwedge_{1 \leq j \leq k} (x_{a^\circ} \wedge y_{ij} \rightarrow \bigwedge_{t_k(a) \bar{\diamond} t_i(a)} \neg y_{k(j+1)}) \quad (5)$$

Finally, in order to eliminate symmetrical gradual itemsets, we add the following constraint:

$$\bigwedge_{a_i \in a_1 \dots a_n} (\neg x_{a_i^+} \vee \bigvee_{1 \leq j < i} \neg x_{a_i^-}) \quad (6)$$

In fact, each  $\sigma = (a_1^+, a_1^-) \dots (a_n^+, a_n^-)$  is a symmetry of the our encoding. Consequently, one can break such symmetry by adding the Symmetry Breaking Predicates [7].

Note that the equation (4) or (5) can be simplified to  $(\neg x_{a^\circ} \vee \neg y_{ij})$  when it is not possible to attribute transactions to positions between  $j + 1$  to  $k$  with transactions allowing to maintain the relation  $(\diamond)$  between positions  $j + 1$  and  $k$ . This is the case if  $(j - 1) < |\{l \mid t_i(a) \diamond t_l(a)\}|$  or  $(k - j) < |\{l \mid t_l(a) \diamond t_i(a)\}|$ . Note that the computation of  $|\{l \mid t_l(a) \diamond t_i(a)\}|$  can be done by double traversal of the transaction database  $\Delta$ . Such processing allows to reduce the number of added clauses if constraint (5) is used while it allows to reduce the size of added clauses if (4) is used. Note that  $(\sum_{i=1}^n y_{ij} = 1)$  (respectively  $(\sum_{j=1}^k y_{ij} \leq 1)$ ) represent linear equality (respectively inequality) commonly called exact-One (respectively atMostOne Constraint). Such constraint can be encoding in respectively  $O(n)$  (respectively  $O(k)$ ) clauses using  $O(n)$  (respectively  $O(k)$ ) additional variables as indicated in constraint (7) [30,27]. In fact,  $\sum_{i=1}^n x_i \leq 1$  can be encoded as follows using auxiliary variables  $\{p_1, \dots, p_{n-1}\}$ .

$$(\neg x_1 \vee p_1) \wedge (\neg x_n \vee \neg p_{n-1}) \wedge \bigwedge_{1 < i < n} (\neg x_i \vee p_i) \wedge (\neg p_{i-1} \vee p_i) \wedge (\neg x_i \vee \neg p_{i-1}) \quad (7)$$

## 4.1 Adding multiple constraints

The constraint (6) allows to avoid computing all gradual patterns and their corresponding symmetric gradual pattern. However, this constraint will add a certain number of variables and clauses to the final boolean formula. We propose another direction to take into account this symmetrical without add the constraint (6) but by adding two blocking clauses in the NCF formula each time a model is found. One clause to avoid finding the same model and another to avoid finding a model corresponding to the symmetric pattern.

Several other constraints over the pattern itself can be captured by the variable selection heuristic. In many application fields, interesting gradual patterns can be distinguished from irrelevant ones by specifying semantic constraints on the gradual pattern itself. For example, the authors of [23] designed an algorithm to mine temporal gradual patterns which are gradual patterns whose the longest sequence of transactions respect the temporal order. These kinds of gradual patterns are particularly interesting in the paleoecological domain where the experts search from their paleoecological numerical data the patterns which capture the simultaneously frequent co-evolutions between attributes. As the transactions are encoded in our CNF formula as boolean variables, the temporal constraint can be captured by selecting in the temporal order the propositional variables  $y_{ij}$  representing the transaction identifiers of the numerical dataset.

## 4.2 Solving the formula encoding gradual pattern mining problem

We solve our SAT boolean formula using *MiniSAT2.2* CDCL SAT solver [13]. Each model of our SAT formula (i.e., each solution) is a frequent gradual pattern of the input database with respect to a minimum support threshold. As a result, outputting all the frequent gradual pattern can be done by enumerating all the models that satisfy the CNF formula encoding the frequent gradual pattern mining problem.

The main procedure of our approach is given in algorithm 2. This procedure compute and output all frequent gradual patterns with respect to the minimum support threshold *minSupp*.

The procedure *findAllModel* corresponds to the algorithm 1 modified by adding to the CNF formula two blocking clauses instead of one blocking clause at each time that a model is found during the resolution process. One blocking clause to avoid finding the same model and another to avoid finding a model corresponding to the symmetric pattern of the extracted gradual pattern. More precisely, let  $s = (a_{i1}^{i1}, a_{i2}^{i2}, \dots, a_{ik}^{ik})$ , a frequent gradual pattern extracted from the current model, we add to the original formula the blocking clauses:  $c_1 = (\neg x_{a_{i1}^{i1}} \vee \neg x_{a_{i2}^{i2}} \vee \dots \vee \neg x_{a_{ik}^{ik}})$  and  $c_2 = (\neg x_{a_{j1}^{j1}} \vee \neg x_{a_{j2}^{j2}} \vee \dots \vee \neg x_{a_{jk}^{jk}})$ , with

$a_{j_1}^{i_1} = c(a_{i_1}^{j_1}), a_{j_2}^{i_2} = c(a_{i_2}^{j_2}), \dots, a_{j_k}^{i_k} = c(a_{i_k}^{j_k})$ . The clause  $c_2$  allows to discard from the set of patterns the complementary gradual pattern of  $s$ .

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**Algorithm 2:** SAT Based Gradual Patterns Enumeration

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**Input:** a numerical database  $\mathcal{DS}$ , a minimum support  $minSupp$

**Output:** Set of all frequent gradual patterns

1  $\mathcal{F} \leftarrow SATEncoding(\mathcal{DS}, minSupp)$  ;

2  $findAllModel(\mathcal{F})$  ;

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## 5 Experiments

In this section, we carried out an experimental evaluation of the performance of our proposed approach. we ran experiments on the paleoecological datasets. The paleoecological dataset are constituted of a set of numerical attributes whose the values correspond to the quantity of each paleoecological indicator contained in a sediment record taken, by coring operations, in a lake ecosystem. The sedimentary sequence obtained is then dated, sampled, and for each sample, at a given depth, a date is calculated. The abundance of each indicator is then recorded for each sample. The objects in this database correspond to the different dates obtained on the considered sedimentary record, and the columns to the different paleoecological recorded. We consider the paleoecological dataset constituted of the indicators of paleoecological anthropization (pollen grains). It contains 111 objects corresponding to different dates identified on the considered Lacustrine recording, and 117 attributes corresponding to different indicators of paleoecological anthropization (pollen grains). All the experiments were done on Intel Xeon quad-core machines with 32GB of RAM running at 2.66 Ghz. First, we present in the table 2 the size of the CNF formula (number of variables and clauses) encoding the frequent gradual patterns with respect to a minimum support.

In this table, we mention the formula encoding the whole problem in terms of number of variables ( $\#vars$ ) and clauses ( $\#clauses$ ) with respect to a minimum support threshold ( $\#minSupp$ ) given by the first column. The last column gives in seconds the cpu time need for encoding.

## 6 Conclusion

In this paper, we proposed SAT encoding to address the problem of mining frequent gradual patterns. This declarative approach offers an additional possibility to benefit from the recent progress in satisfiability testing. Several satisfiability based approach have been proposed for the classical patterns mining problem such that mining frequent itemsets in transactional data, mining frequent sequence in a data-sequence. However no satisfiability based approach has yet been proposed for the frequent gradual pattern mining problem. The problem of mining frequent gradual patterns differs from the classical cases related to simple itemsets. In fact, in this last case, for each line of the database, it is possible to

**Table 2.** CHARACTERISTICS OF THE INSTANCES & ENCODING TIME

#minSupp	#vars	#clauses	#encodingTime
5%	2 115	133 521	0.22s
10%	3 775	266 706	0.43s
20%	7 427	559 713	0.86s
30%	11 079	852 720	1.31s
40%	14 731	1 145 727	1.74s
50%	18 383	1 438 734	2.25s
60%	22 035	1 731 741	2.69s
70%	25 687	2 024 748	3.12s
80%	29 339	2 317 755	3.54s
90%	32 991	2 610 762	4.03s

say whether it supports the given itemset or not. In the gradual case, the entire database is needed for each count. This makes the problem of mining frequent gradual patterns more complex.

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