Achievable Diversity-Rate Tradeoff of MIMO AF Relaying Systems with MMSE Transceivers

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Abstract—This paper investigates the diversity order of the minimum mean squared error (MMSE) based optimal transceivers in multiple-input multiple-output (MIMO) amplify-and-forward (AF) relaying systems. While the diversity-multiplexing tradeoff (DMT) analysis accurately predicts the behavior of the MMSE receiver for the positive multiplexing gain, it turned out that the performance is very unpredictable via DMT for the case of fixed rates, because MMSE strategies exhibit a complicated rate dependent behavior. In this paper, we establish the diversity-rate tradeoff performance of MIMO AF relaying systems with the MMSE transceivers as a closed-form for all fixed rates, thereby providing a complete characterization of the diversity order together with the earlier work on DMT.

I. INTRODUCTION

In multiple-input multiple-output (MIMO) systems, although suboptimal, the linear minimum mean squared error (MMSE) receivers have widely been adopted as a low complexity alternative to the optimal maximum likelihood (ML) receiver. This leads to a large amount of research on the performance of MMSE receivers [1]–[3], but their performance is not fully understood yet in MIMO relaying channels.

A fundamental criterion to evaluate the performance of a MIMO system is the *"diversity-multiplexing tradeoff" (DMT)*. Thus, many analyses have been conducted based on the DMT in MIMO relaying systems [4]–[6]. Under the MMSE strategy, however, the DMT is not sufficient to characterize the diversity order, because the DMT framework an asymptotic notion for the high signal-to-noise ratio (SNR), cannot distinguish between different spectral efficiencies that correspond to the same multiplexing gain which we denote by r.

In fact, it is known in point-to-point (P2P) MIMO channels that while the DMT analysis accurately predicts the behavior of the MMSE receiver for the positive multiplexing gain ($r > 0$), the extrapolation of the DMT to $r = 0$ is unable to predict the performance especially at low rates. This rate-dependent behavior of MMSE receivers has first been observed by Hedayat in [1] and comprehensively analyzed by Mehana in [3] by performing the *"diversity-rate tradeoff (DRT)"* analysis for all fixed rates. A similar phenomenon can be observed in MMSE-based MIMO AF relaying systems, but the analysis has not been made so far.

In this paper, we investigate the achievable DRT of the linear MMSE transceivers in MIMO amplify and forward (AF) relaying systems for all fixed data rates, where the relay

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transceiver and the destination receiver are jointly optimized with respect to the MMSE. The optimal MMSE transceiver designs have been proposed in [7] and [8] using different approaches. In this paper, we focus on the method in [8] based on the error covariance decomposition, which allows further analysis tractable. In fact, the DRT analysis does not impose any restriction on the number of antennas at each node, because a certain diversity gain is always achievable at arbitrarily low rates. Thus, we first provide a new result of the error covariance decomposition that can be applied to any kinds of antenna configurations, and then establish the DRT performance as a closed-form. Our analysis complements the earlier work on DMT [6] which is only valid for a positive multiplexing gain, and thus allows us to fully characterize the diversity order of the MMSE transceivers in MIMO AF relaying systems. Again, we note that the result of our DRT analysis is unpredictable via DMT analysis. Finally, simulations results will demonstrate the accuracy of the analysis.

Throughout this paper, normal letters represent scalar quantities, boldface letters indicate vectors and boldface uppercase letters designate matrices. We use $\mathbb{C}$ to denote a set of complex numbers. The superscript $(\cdot)^H$ stands for conjugate transpose. $\mathbf{I}_N$ is defined as an $N \times N$ identity matrix, and $E[\cdot]$ and $\lceil \cdot \rceil$ means the expectation and rounding up to the next higher interger, respectively. $[\mathbf{A}]_{k,k}$ and $\text{Tr}(\mathbf{A})$ denote the k-th diagonal element and trace function of a matrix $\mathbf{A}$, respectively. The k-th element of a vector $\mathbf{a}$ is denoted by a_k.

II. SYSTEM MODEL

Figure 1 shows the input-output system model for quasi-static flat fading MIMO AF relaying channels equipped with N_S, N_R, and N_D number of antennas at the source, the relay, and the destination, respectively. A single channel encoder supports all the data streams at the source so that coding is applied jointly across antennas. We assume no channel state information (CSI) at the source, while both the relay and the destination have perfect CSI of both links. Due to loop interference at the relay, it is assumed that each data transmission occurs in two separate phases (time or frequency). A direct link between the source and the destination is ignored due to large pathloss.

In the first phase, the source transmits the signal vector $\mathbf{x} = [x_1, x_2, \ldots, x_{N_S}]^T \in \mathbb{C}^{N_S \times 1}$ to the relay, and then the received signals at the relay, $\mathbf{y}_R \in \mathbb{C}^{N_R \times 1}$ is given by $\mathbf{y}_R = \mathbf{H}\mathbf{x} + \mathbf{n}_R$, where $\mathbf{H} \in \mathbb{C}^{N_R \times N_S}$ and $\mathbf{n}_R \in \mathbb{C}^{N_R \times 1}$ denote the source-to-relay channel matrix and the noise vector at the relay, respectively. Due to no CSI at the source, transmit
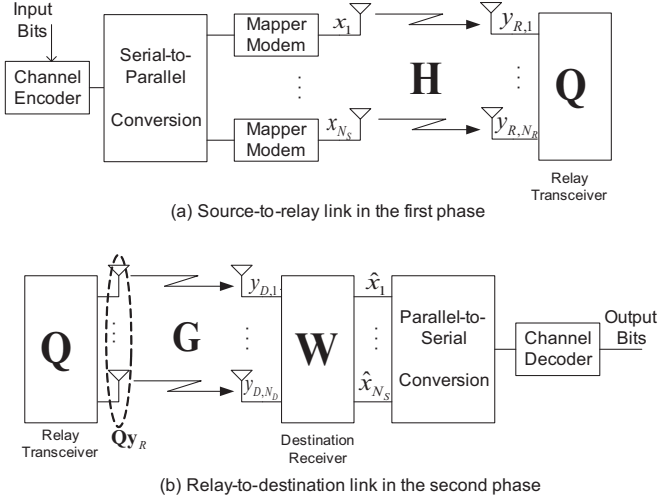
Fig. 1. Joint encoding/decoding structure for MIMO AF relaying systems with linear MMSE transceivers

antennas operate with equal power as $E[|x_k|^2] = \frac{P_S}{N_S} = \rho$ for all k, where P_S represents the total transmit power at the source.

In the second phase, the relay signal $\mathbf{y}_R$ is amplified by the relay matrix $\mathbf{Q} \in \mathbb{C}^{N_R \times N_R}$ and transmitted to the destination. Then, the received signal at the destination is written by

$$\mathbf{y}_D = \mathbf{GQy}_R + \mathbf{n}_D = \mathbf{GQHx} + \mathbf{GQn}_R + \mathbf{n}_D, \qquad (1)$$

where $\mathbf{n}_D$ designates the noise vector at the destination. Note that the relay matrix $\mathbf{Q}$ must satisfy the relay power constraint P_R as $E[\|\mathbf{Qy}_R\|^2] = \mathrm{Tr}(\mathbf{Q}(\rho\mathbf{HH}^H + \mathbf{I}_{N_R})\mathbf{Q}^H) \leq P_R$. Finally, when a linear MMSE receiver $\mathbf{W} \in \mathbb{C}^{N_S \times N_D}$ is employed at the destination, the estimated signal waveform $\hat{\mathbf{x}} \in \mathbb{C}^{N_S \times 1}$ is expressed as $\hat{\mathbf{x}} = \mathbf{Wy}_D$.

Unlike the open-loop P2P MIMO systems, the diversity order may vary according to the forwarding scheme $\mathbf{Q}$ at the relay. In this paper, we examine the diversity order of the MMSE transceivers $\mathbf{Q}$ and $\mathbf{W}$ which are jointly optimized with respect to the MMSE [7] [8][1]. Throughout the paper, we assume that all channel matrices have random entries which are independent and identically distributed (i.i.d.) complex Gaussian $\sim \mathcal{CN}(0, 1)$, but remain constant over a codeword duration. All elements of the noise vectors $\mathbf{n}_R$ and $\mathbf{n}_D$ are also assumed to be i.i.d. $\sim \mathcal{CN}(0, 1)$.

III. OPTIMAL TRANSCEIVER DESIGN

We would like to mention that the MMSE transceiver design between the relay and the destination has first developed in [7]. However, it turns out that the approach in [7] which is based on the singular-value decomposition is cumbersome to be dealt with due to the complicated structure of a compound channel matrix and colored noise at the destination. In this section, we introduce an alternative design method based on

[1]The diversity order of other relaying strategies are currently under investigation for our future works.

the error covariance decomposition, which makes the analysis more tractable. This is an extension of the result in [8].

We define error vector $\mathbf{e} \triangleq \hat{\mathbf{x}} - \mathbf{x}$ and its covariance matrix $\mathbf{R}_e \triangleq E[\mathbf{ee}^H]$. Then, the joint MMSE optimization problem for $\mathbf{Q}$ and $\mathbf{W}$ is written by

$$\min_{\mathbf{Q}, \mathbf{W}} \mathrm{Tr}\left(\mathbf{R}_e\right) \quad s.t. \quad \mathrm{Tr}\left(\mathbf{Q}(\rho\mathbf{HH}^H + \mathbf{I}_{N_R})\mathbf{Q}^H\right) \leq P_R. \quad (2)$$

By the orthogonality principle $E[\mathbf{ey}_D^H] = \mathbf{0}$, it is easy to find that the optimal receiver at the destination is given by $\hat{\mathbf{W}} = \rho\mathbf{H}^H\mathbf{Q}^H\mathbf{G}^H(\rho\mathbf{GQHH}^H\mathbf{Q}^H\mathbf{G}^H + \mathbf{I}_{N_D})^{-1}$. Therefore, the remaining work is now to determine the relay transceiver $\mathbf{Q}$.

The following lemma [9, Lemma 1] shows that the optimal relay matrix $\mathbf{Q}$ can be expressed as a product of two matrices.

Lemma 1: Under the MMSE strategy, the optimal relay matrix $\mathbf{Q}$ consists of the relay precoder $\mathbf{B} \in \mathbb{C}^{N_R \times N_S}$ and the relay receiver $\mathbf{L} \in \mathbb{C}^{N_S \times N_R}$ as

$$\hat{\mathbf{Q}} = \mathbf{BL}, \qquad (3)$$

where $\mathbf{B}$ is an arbitrary matrix, while $\mathbf{L} = \rho\mathbf{H}^H(\rho\mathbf{HH}^H + \mathbf{I}_{N_R})^{-1}$ is an MMSE receiver for the first hop channel $\mathbf{H}$ with input signal $\mathbf{x}$.

Now, let us define $\mathbf{y} \in \mathbb{C}^{N_S \times 1}$ as the relay receiver output signal, i.e., $\mathbf{y} \triangleq \mathbf{Ly}_R$ and its covariance matrix $\mathbf{R}_y \triangleq E[\mathbf{yy}^H] \in \mathbb{C}^{N_S \times N_S}$ as

$$\mathbf{R}_y = \mathbf{L}(\rho\mathbf{HH}^H + \mathbf{I}_{N_R})\mathbf{L}^H. \qquad (4)$$

Then, the estimated signal vector $\hat{\mathbf{x}}$ and the relay power constraint in (2) are respectively rephrased as

$$\hat{\mathbf{x}} = \mathbf{W}(\mathbf{GBy} + \mathbf{n}_D) \quad \text{and} \quad \mathrm{Tr}(\mathbf{BR}_y\mathbf{B}^H) \leq P_R. \qquad (5)$$

Since the rank of $\mathbf{R}_y$ equals $M \triangleq \min(N_S, N_R)$, $\mathbf{R}_y$ becomes clearly non-invertible when $N_S > N_R$. This fact makes the problem more challenging, but has not been fully addressed in conventional literature. In the following, we revisit the previous works in [7] and [8], and provide a more generalized and insightful design strategy without restriction on the number of antennas at the source.

In fact, when the relay matrix has the form of (3), the error covariance matrix $\mathbf{R}_e$ in (2) can be expressed as a sum of two individual covariance matrices, each of which represents the first hop and the second hop MIMO channels, respectively. This result has been proved in [8], but the proof was limited to the cases of $N_S \leq \min(N_R, N_D)$. For the sake of completeness, we give a new result of error decomposition that can be applied to any kind of antenna configurations.

Lemma 2: Define the eigenvalue decomposition $\mathbf{R}_y = \mathbf{U}_y\mathbf{\Lambda}_y\mathbf{U}_y^H$ where $\mathbf{U}_y \in \mathbb{C}^{N_S \times N_S}$ is a unitary matrix and $\mathbf{\Lambda}_y \in \mathbb{C}^{N_S \times N_S}$ represents a square diagonal matrix with eigenvalues $\lambda_{y,k}$ for $k = 1, \ldots, N_S$ arranged in descending order. Then, without loss of MMSE optimality, we have

$$\mathbf{R}_e = (\mathbf{H}^H\mathbf{H} + \rho^{-1}\mathbf{I}_{N_S})^{-1}$$
$$+ \widetilde{\mathbf{U}}_y(\widetilde{\mathbf{U}}_y^H\mathbf{B}^H\mathbf{G}^H\mathbf{GB}\widetilde{\mathbf{U}}_y + \widetilde{\mathbf{\Lambda}}_y^{-1})^{-1}\widetilde{\mathbf{U}}_y^H, \qquad (6)$$

where $\widetilde{\mathbf{U}}_y \in \mathbb{C}^{N_S \times M}$ is a matrix constructed by the first M columns of $\mathbf{U}_y$ and $\widetilde{\mathbf{\Lambda}}_y = \widetilde{\mathbf{U}}_y^H \mathbf{R}_y \widetilde{\mathbf{U}}_y$ indicates the $M \times M$ upper-left submatrix of $\mathbf{\Lambda}_y$.

Proof: As the relay receiver $\mathbf{L}$ follows the receive Wiener filter structure, its output signal $\mathbf{y}$ should satisfy the orthogonality principle [10], i.e., $E\left[(\mathbf{y} - \mathbf{x})\mathbf{y}^H\right] = \mathbf{0}$. Now, using $\mathbf{y}$, we can express the MSE as $E\left[\|\mathbf{e}\|^2\right] = E\left[\|\hat{\mathbf{x}} - \mathbf{y} + \mathbf{y} - \mathbf{x}\|^2\right]$. Then, due to the orthogonality principle above, it is true that the signal $\mathbf{y} - \mathbf{x}$ becomes orthogonal to $\hat{\mathbf{x}}$ as well as $\mathbf{y}$, since $\hat{\mathbf{x}} = \mathbf{W}\mathbf{y}_D = \mathbf{W}(\mathbf{GB}\mathbf{y} + \mathbf{n}_D)$ is also a function of $\mathbf{y}$ and independent noise $\mathbf{n}_D$. Therefore, it follows

$$E\left[\|\mathbf{e}\|^2\right] = \mathrm{MSE}_H + \mathrm{MSE}_G,$$

where $\mathrm{MSE}_H \triangleq E\left[\|\mathbf{y} - \mathbf{x}\|^2\right]$ and $\mathrm{MSE}_G \triangleq E\left[\|\mathbf{W}\mathbf{y}_D - \mathbf{y}\|^2\right]$. This result also illustrates that for a given structure of $\mathbf{Q} = \mathbf{BL}$, the optimal destination receiver $\mathbf{W}$ can be alternatively expressed as $\hat{\mathbf{W}} = \mathbf{R}_y\mathbf{B}^H\mathbf{G}^H(\mathbf{GB}\mathbf{R}_y\mathbf{B}^H\mathbf{G}^H + \mathbf{I}_{N_D})^{-1}$ which amounts to an MMSE receiver for the second hop channel $\mathbf{G}$ with input signal $\mathbf{y}$. In what follows, we will show that MSE_H and MSE_G in (7) can be expressed as the first and second term in (6), respectively.

Let us first have a look at MSE_G. Then, it follows

$$\mathrm{MSE}_G = E[\mathrm{Tr}\left((\mathbf{W}\mathbf{y}_D - \mathbf{y})(\mathbf{W}\mathbf{y}_D - \mathbf{y})^H\right)]$$
$$= \mathrm{Tr}\left(\mathbf{R}_y - \mathbf{R}_y\mathbf{B}^H\mathbf{G}^H(\mathbf{GB}\mathbf{R}_y\mathbf{B}^H\mathbf{G}^H + \mathbf{I}_{N_D})^{-1}\mathbf{G}^H\mathbf{B}^H\mathbf{R}_y\right).$$

Now, we write the relay precoder $\mathbf{B}$ in a more general form as $\mathbf{B} = \check{\mathbf{B}}\mathbf{U}_y^H$ where $\check{\mathbf{B}} = [\mathbf{B}_1 \ \mathbf{B}_2]$ with $\mathbf{B}_1 \in \mathbb{C}^{N_R \times M}$ and $\mathbf{B}_2 \in \mathbb{C}^{N_R \times (N_S - M)}$. Since $\mathbf{R}_y$ is a rank M matrix, setting $\mathbf{B}_2 = \mathbf{0}$ has no impact on both the MSE and the relay power consumption, i.e., $\mathrm{Tr}(\mathbf{B}\mathbf{R}_y\mathbf{B}^H)$. Therefore, without loss of generality, MSE_G is further rephrased as

$$\mathrm{MSE}_G = \mathrm{Tr}\big(\widetilde{\mathbf{U}}_y(\widetilde{\mathbf{\Lambda}}_y - \widetilde{\mathbf{\Lambda}}_y^H\mathbf{B}_1^H\mathbf{G}^H$$
$$\times (\mathbf{GB}_1\widetilde{\mathbf{\Lambda}}_y\mathbf{B}_1^H\mathbf{G}^H + \mathbf{I}_{N_D})^{-1}\mathbf{G}^H\mathbf{B}_1^H\widetilde{\mathbf{\Lambda}}_y)\widetilde{\mathbf{U}}_y^H\big)$$
$$= \widetilde{\mathbf{U}}_y\big(\mathbf{B}_1^H\mathbf{G}^H\mathbf{GB}_1\widetilde{\mathbf{U}}_y + \widetilde{\mathbf{\Lambda}}_y^{-1}\big)^{-1}\widetilde{\mathbf{U}}_y^H$$
$$= \widetilde{\mathbf{U}}_y\big(\widetilde{\mathbf{U}}_y^H\mathbf{B}^H\mathbf{G}^H\mathbf{GB}\widetilde{\mathbf{U}}_y + \widetilde{\mathbf{\Lambda}}_y^{-1}\big)^{-1}\widetilde{\mathbf{U}}_y^H,$$

where the last equality follows from $\mathbf{B}_1 = \mathbf{B}\widetilde{\mathbf{U}}_y$.

Meanwhile, the case of MSE_H is equivalent to a situation of P2P MIMO channels with the input signal vector $\mathbf{x}$. Thus, the proof simply follows the previous results in [10], and thus omitted. ∎

When $N_S \leq N_R$, Lemma 2 is equivalent to one in [8]; thus is more general. Now, the result of Lemma 2 illustrates that the original joint optimization problem in (2) can be reduced to optimizing $\mathbf{B}$, since the first term of $\mathbf{R}_e$ consists of known parameters. Define eigenvalue decomposition $\mathbf{G}^H\mathbf{G} = \mathbf{V}_g\mathbf{\Lambda}_g\mathbf{V}_g^H$ where $\mathbf{\Lambda}_g$ designates a square diagonal matrix with eigenvalues $\lambda_{g,k}$ for $k = 1, \ldots, N_d$ arranged in descending order. Then, we can show that the optimal relay precoder $\mathbf{B}$ can be generally written by $\hat{\mathbf{B}} = \widetilde{\mathbf{V}}_g\mathbf{\Phi}\widetilde{\mathbf{U}}_y^H$ where

$\widetilde{\mathbf{V}}_g \in \mathbb{C}^{N_R \times M}$ denotes a matrix constructed by the first M columns of $\mathbf{V}_g$ and $\mathbf{\Phi} \in \mathbb{C}^{M \times M}$ is an arbitrary matrix [7].

Now, substituting $\hat{\mathbf{B}}$ into (6), the modified problem determines the optimal $\mathbf{\Phi}$:

$$\hat{\mathbf{\Phi}} = \arg\min_{\mathbf{\Phi}} \ \left(\mathbf{\Phi}\widetilde{\mathbf{\Lambda}}_g\mathbf{\Phi}^H + \widetilde{\mathbf{\Lambda}}_y^{-1}\right)^{-1} \ s.t. \ \mathrm{Tr}(\mathbf{\Phi}\widetilde{\mathbf{\Lambda}}_y\mathbf{\Phi}^H) \leq P_R.$$

Here $\widetilde{\mathbf{\Lambda}}_g$ represents the $M \times M$ upper-left submatrix of $\mathbf{\Lambda}_g$. Since we have $\mathrm{Tr}(\mathbf{A}^{-1}) \geq \sum_{i=1}^{M} ([\mathbf{A}]_{k,k})^{-1}$ for a positive definite matrix $\mathbf{A}$ [11], it is easy to check that the minimum MSE is achieved when $\mathbf{\Phi}$ is a diagonal matrix, which leads to a simple convex problem. The remaining procedure simply follows from previous works in [7], [8], and [12]. Finally, in combination with the relay receiver $\mathbf{L}$ in (3), we have

$$\hat{\mathbf{Q}} = \hat{\mathbf{B}}\mathbf{L} = \widetilde{\mathbf{V}}_g\hat{\mathbf{\Phi}}\widetilde{\mathbf{U}}_y^H\mathbf{L}, \tag{7}$$

where the k-th diagonal element of $\hat{\mathbf{\Phi}}$ is determined by $|\hat{\phi}_k|^2 = \frac{1}{\lambda_{y,k}\lambda_{g,k}}\left(\sqrt{\frac{\lambda_{y,k}\lambda_{g,k}}{\nu}} - 1\right)^+$ for $k = 1, 2, \ldots, M$ with $(\cdot)^+ = \max(\cdot, 0)$ and ν being chosen to satisfy the relay power constraint in (5). Note that if $\lambda_{g,k} = 0$, we have $|\hat{\phi}_k|^2 = 0$.

IV. Diversity-Rate Tradeoff Analysis

We now investigate the diversity order of the MMSE optimal transceiving scheme in MIMO AF relaying systems studied in the previous section, where data streams are jointly encoded across the antennas at the source (vertical encoding). The diversity analysis may be conducted by either outage probability or pairwise error probability (PEP) [3]. In this paper, we focus on the outage probability of mutual information (MI) assuming infinite length Gaussian codewords. For simplicity, we assume that $P_R = P_T = \rho N_t$, but the result can be easily extended to more general cases. We say that two functions $f(\rho)$ and $g(\rho)$ are exponentially equal when

$$\lim_{\rho \to \infty} \frac{\log f(\rho)}{\log \rho} = \lim_{\rho \to \infty} \frac{\log g(\rho)}{\log \rho},$$

and denoted by $f(\rho) \doteq g(\rho)$.

When the coding is applied across antennas with MMSE receivers, the MI is defined as [1]

$$\mathcal{I} = \frac{1}{2}\sum_{k=1}^{N_S} \log\left(1 + \gamma_k\right),$$

where $\gamma_k = \rho/[\mathbf{R}_e]_{k,k} - 1$. Then, we obtain

$$\mathcal{I} \overset{(a)}{\geq} -\frac{N_S}{2}\log\left(\frac{1}{\rho N_S}\mathrm{Tr}(\mathbf{R}_e)\right)$$
$$\overset{(b)}{=} -\frac{N_S}{2}\log\Big(\frac{1}{\rho N_S}\big(\mathrm{Tr}\big(\mathbf{H}^H\mathbf{H} + \rho^{-1}\mathbf{I}_{N_S}\big)^{-1}$$
$$+ \mathrm{Tr}\big(\hat{\mathbf{\Phi}}^H\widetilde{\mathbf{\Lambda}}_g\hat{\mathbf{\Phi}} + \widetilde{\mathbf{\Lambda}}_y^{-1}\big)^{-1}\big)\Big)$$
$$\overset{(c)}{\geq} -\frac{N_S}{2}\log\Big(\frac{1}{N_S}\big(\mathrm{Tr}(\rho\mathbf{\Lambda}_h + \mathbf{I}_{N_S})^{-1}$$
$$+ \mathrm{Tr}\big(\eta\rho\widetilde{\mathbf{\Lambda}}_g + \rho\widetilde{\mathbf{\Lambda}}_y^{-1}\big)^{-1}\big)\Big), \tag{8}$$

where (a) follows from the Jensen's inequality, (b) is due to the optimal relay precoder $\hat{\mathbf{B}}$ described in (7), and (c) holds by setting $\hat{\boldsymbol{\Phi}} = \sqrt{\eta}\mathbf{I}_{N_M}$, where η can be chosen to be 1 to satisfy the relay power constraint in (5) (see Appendix A). Let us define the outage probability as $P_{\text{out}} \triangleq (\mathcal{I} \leq R)$. Then, using the MI bound in (8) and setting the target data rate as R, we obtain the outage probability upperbound as $P_{\text{out}} \leq P_{\text{out}}^U$, where

$$P_{\text{out}}^U \triangleq P\Big(\sum_{k=1}^{M} \frac{1}{1 + \rho\lambda_{h,k}} + \sum_{k=1}^{M} \frac{1}{\rho\lambda_{g,k} + \rho\lambda_{y,k}^{-1}} \geq m \Big), \quad (9)$$

with $m \triangleq N_S 2^{-\frac{2R}{N_S}} - (N_S - M)$.

First, let us first set the target data rate as $R = r\log\rho$. Then, the resulting outage exponent leads to the DMT performance which captures the tradeoff between the multiplexing gain r and block error probability at high SNR ($\rho \to \infty$).

Theorem 1: For MIMO AF relaying systems with positive multiplexing gain $r > 0$, the achievable DMT of the MMSE transceivers is given by

$$d(r) = \begin{cases} (N_R - N_S + 1)\Big(1 - \frac{2r}{N_S}\Big)^+ & \text{if } N_S \leq \min(N_R, N_D) \\ 0 & \text{otherwise} \end{cases}$$

Proof: The proof is simply obtained from [6] by assuming that the direct link between the source and the destination can be ignored. Details are omitted for brevity. ∎

As described in Theorem 1, the DMT analysis accurately predicts the diversity order of the MMSE transceivers when the multiplexing gain is positive ($r > 0$). However, when the target rate R is fixed with respect to ρ, i.e., $r = 0$ and sufficiently low, it is observed that the performance is in stark contrast to one predicted by the DMT analysis. In the following, we will analyze the fixed rate diversity of the MMSE transceivers as a function of rate R and the number of antennas at each node.

Theorem 2: For MIMO AF relaying systems with fixed rate R ($r = 0$), the achievable DRT of the MMSE transceivers is

$$d(R) = \min\Big(\overline{m}(N_R + N_S - 2M + \overline{m}),$$
$$(N_R - M + \overline{m})(N_D - M + \overline{m})^+\Big),$$

where $\overline{m} \triangleq \Big\lceil \big(N_S 2^{-\frac{2R}{N_S}} + M - N_S\big)^+ \Big\rceil$.

Proof: We begin by defining $\alpha_k \triangleq -\log\lambda_{h,k}/\log\rho$ and $\beta_k \triangleq -\log\lambda_{g,k}/\log\rho$ for $k = 1, \ldots, M$. Then, P_{out}^U in (9) is alternatively expressed as

$$P_{\text{out}}^U$$
$$\overset{(a)}{=} P\Big(\sum_{k=1}^{M} \frac{1}{1 + \rho^{1-\alpha_k}} + \sum_{k=1}^{M} \frac{1}{1 + \rho^{1-\beta_k} + \rho^{-(1-\alpha_k)}} \geq m \Big)$$

$$\overset{(b)}{=} P\Big(\sum_{k=1}^{M} \frac{1}{1 + \rho^{1-\alpha_k}} + \sum_{k=1}^{M} \frac{1}{1 + \rho^{1-\beta_k} + \rho^{-(1-\alpha_k)}} \geq \overline{m} \Big) \quad (10)$$

where (a) follows from $\rho\lambda_{y,k}^{-1} = 1 + \rho^{-(1-\alpha_k)}$ (see the definition of $\mathbf{R}_y$ in (4) and (16)) and (b) is due to the fact

that if $m < 0$, the outage always occurs. Now, at high SNR, we can write the exponential equality as

$$\frac{1}{1 + \rho^{1-\alpha_k}} \doteq \begin{cases} 0 & \text{if } \alpha_k < 1 \\ 1 & \text{if } \alpha_k > 1 \end{cases} \quad (11)$$

$$\frac{1}{1 + \rho^{1-\beta_k} + \rho^{-(1-\alpha_k)}} \doteq \begin{cases} 0 & \text{if } \alpha_k > 1 \text{ or } \beta_k < 1 \\ 1 & \text{if } \alpha_k < 1 \text{ and } \beta_k > 1 \end{cases}, \quad (12)$$

for all $k = 1, \ldots, M$. Note that in an asymptotic sense $\rho \to \infty$, the cases where the eigenvalues take on values that are comparable with $1/\rho$ can be ignored [2].

We first see from these results that in order for the outage to occur, at least $\overline{m}$ number of terms should be 1 among $2M$ summation terms in (10). The above results also reveal that two terms in (11) and (12) cannot be simultaneously 1 at a certain k. As will be clear later in this proof, this property allows us to obtain the full diversity order as $\overline{m}$ tends to be large. Remind that all eigenvalues are in descending order, which means that $\{\alpha_i\}$ and $\{\beta_i\}$ are ordered according to $\alpha_1 \leq \cdots \leq \alpha_M$ and $\beta_1 \leq \cdots \leq \beta_M$. Thus, if $\alpha_1 > 1$, the term in (12) converge to zero for all k, regardless of β.

For all $i = 1, \ldots, M$, let us define all possible events in which i number of terms in (10) equal 1 as

$$\mathcal{E}_{h,i} \triangleq \big\{\alpha_{M-i+1} > 1 > \alpha_{M-i}\big\} \quad \text{and}$$
$$\mathcal{E}_{g,i,j} \triangleq \big\{\beta_{M-i+1} > 1 > \beta_{M-i}\big\} \cap \mathcal{E}_{h,j} \text{ for } j = 0, 1, \ldots, i-1,$$

Then, it follows from the union bound that

$$P_{\text{out}}^U = P\Big(\bigcup_{i=\overline{m}}^{M} \Big[\mathcal{E}_{h,i} \cup \Big(\bigcup_{j=0}^{i-1} \mathcal{E}_{g,i,j}\Big)\Big]\Big)$$

$$\leq \sum_{i=\overline{m}}^{M} \Big(P(\mathcal{E}_{h,i}) + \sum_{j=0}^{i-1} P(\mathcal{E}_{g,i,j})\Big), \quad (13)$$

First, we define $P(\mathcal{E}_{h,i}) \doteq \rho^{-d_{h,i}(R)}$, $i = 1, \ldots, M$. Then, applying Varadhan's lemma as in [2] by using the pdf[2] of the random vector $\mathbf{a} = [\alpha_1, \ldots, \alpha_M]$ as

$$f(\mathbf{a}) \doteq \Big[\prod_{l=1}^{M} \rho^{-(N_S+N_R-2l+1)\alpha_l} \Big] \exp\Big(- \sum_{l=1}^{M} \rho^{-\alpha_l}\Big),$$

we obtain

$$d_{h,i}(R) = \inf_{\mathbf{a} \in \mathcal{E}_{h,i}, \forall \alpha_l > 0} \sum_{l=1}^{M}(N_S + N_R - 2l + 1)\alpha_l$$

$$= \sum_{l=1}^{M-i}(N_S + N_R - 2l + 1) \times 0$$

$$+ \sum_{l=M-i+1}^{M}(N_S + N_R - 2l + 1) \times 1$$

$$= i(N_R + N_S - 2M + i). \quad (14)$$

Now, let us examine the probability of the event $\mathcal{E}_{g,i,j}$, i.e., $P(\mathcal{E}_{g,i,j}) \doteq \rho^{-d_{g,i,j}(R)}$. Defining $L \triangleq \min(N_R, N_D)$, the pdf

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[2] The pdf is slightly different from [2], since the eigenvalue ordering is reversed.

of the random vector $\mathbf{b} = [\beta_1, \ldots, \beta_L]$ is given by

$$f(\mathbf{b}) \doteq \Big[\prod_{l=1}^{L} \rho^{-(N_R + N_D - 2l + 1)\beta_i} \Big] \exp\Big(-\sum_{l=1}^{L} \rho^{-\beta_l} \Big).$$

Then, the probability of the event $\mathcal{E}_{g,i,j}$ is

$$P(\mathcal{E}_{g,i,j}) = \int_{\mathcal{E}_{g,i,j}} f(\mathbf{a}, \mathbf{b}) d\mathbf{a} d\mathbf{b}$$

$$\doteq \int_{\mathcal{E}_{g,i,j}} \Big[\rho^{- \sum_{l=1}^{M}(N_S + N_R - 2l + 1)\alpha_l - \sum_{l=1}^{L}(N_R + N_D - 2l + 1)\beta_l} \Big]$$

$$\times \exp\Big(-\sum_{l=1}^{M} \rho^{-\alpha_l} - \sum_{l=1}^{L} \rho^{-\beta_l} \Big) d\mathbf{a} d\mathbf{b},$$

due to the independence of $\mathbf{a}$ and $\mathbf{b}$, and applying Varadhan's lemma again, we have

$$d_{g,i,j}(R) = \inf_{\substack{(\mathbf{a}, \mathbf{b}) \in \mathcal{E}_{g,i,j}, \\ \forall \alpha_l, \forall \beta_l > 0}} \sum_{l=1}^{M} (N_S + N_R - 2l + 1)\alpha_l$$

$$+ \sum_{l=1}^{L} (N_R + N_D - 2l + 1)\beta_l$$

$$= \sum_{l=M-j+1}^{M} (N_S + N_R - 2l + 1)$$

$$+ \sum_{l=M-i+1}^{L} (N_R + N_D - 2l + 1)$$

$$= j(N_S + N_R - 2M + j)$$

$$+ (N_R + N_D - L - M + i)(L - M + i)^+$$

$$= j(N_S + N_R - 2M + j)$$

$$+ (N_R - M + i)(N_D - M + i)^+. \quad (15)$$

From the results in (13-15), we eventually conclude that

$$P_{\text{out}} \dot{\leq} P(\mathcal{E}_{h,\overline{m}}) + P(\mathcal{E}_{g,\overline{m},0}) \doteq \rho^{- \min(d_{h,\overline{m}}(R), d_{g,\overline{m},0}(R))},$$

because all other events causing the outage in (13) yield higher outage exponents than $\mathcal{E}_{h,\overline{m}}$ or $\mathcal{E}_{g,\overline{m},0}$; thus can be ignored, and the proof is concluded. ∎

Our result in Theorem 2 confirms and complements the earlier work on DMT in Theorem 1. We first see that when the rate is high, i.e., $R \geq \frac{N_S}{2} \log N_S$ (or $\overline{m} = 1$), both Theorem 1 and 2 yield the same diversity order. At high rate, therefore, the diversity order of the MMSE transceivers may be predictable by DMT analysis with setting $r = 0$, and thus very suboptimal compared to the optimal (ML) diversity [4] [13]. However, as the rate becomes lower, it is shown from Theorem 2 that higher diversity order is actually achievable than one predicted by the DMT analysis. In particular, when $R < \frac{N_S}{2} \log \frac{N_S}{N_S - 1}$ (or $\overline{m} = M$), the MMSE transceivers even exhibit the full diversity order $d = N_R \min(N_S, N_D)$, thereby achieving an ML-like performance. It is also interesting to observe that when the rate is sufficiently small, a certain diversity gain is still achievable even when $N_S > \min(N_R, N_D)$, which is often overlooked in MMSE-based relaying systems.
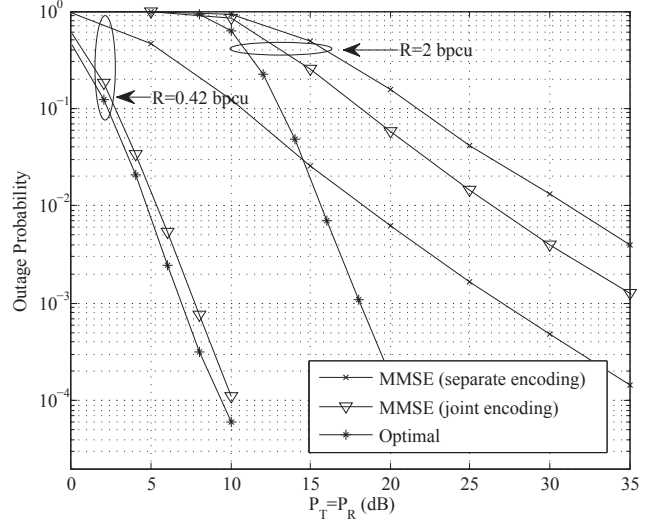

Fig. 2. Outage probabilities of $2 \times 2 \times 2$ MIMO AF relaying systems with R=0.42 and 2 bpcu

V. NUMERICAL RESULTS

In this section, we demonstrate the accuracy of our analysis using numerical simulations for the quasi-static i.i.d. Rayleigh fading model. Target rate R is measured in bits per channel use (bpcu). The notation $N_S \times N_R \times N_D$ is used to denote a system with N_S source, N_R relay and N_D destination antennas. Figure 2 shows the case of $2 \times 2 \times 2$ MIMO AF relaying systems with $R = 0.42$ ($\overline{m} = 2$) and 2 ($\overline{m} = 1$) bpcu, which leads to diversity order 4 and 1, respectively. Here, *"Optimal"* indicates the capacity achieving relaying scheme with the optimal receiver (ML) at the destination [13]. As predicted by our analysis, it is shown that as the rate becomes smaller, the MMSE transceiver with a joint encoding/decoding structure as in Figure 1 denoted by *"MMSE (joint encoding)"* exhibits near optimal performance with full diversity order, while the separate encoding gives a constant diversity for all rates. In Figure 3, simulation results for $2 \times 2 \times 1$ systems are given. This result illustrates that even when $N_S > \min(N_R, N_D)$, the MMSE scheme is still able to achieve a certain diversity gain at a low rate. This observation is flatly conflict with the assumption $N_S \leq \min(N_R, N_D)$ commonly adopted in designs of MMSE-based MIMO AF relaying systems. Remind that our design method in Section III can be applied to any kinds of antenna configuration without hurting the MMSE optimality.

VI. CONCLUSION

In this paper, we investigated the DRT performance of the linear MMSE transceivers in MIMO AF relaying systems for all fixed rates and for any number of source, relay, and destination antennas. First, we generalized the previous error covariance decomposition lemma so that it can be applied to any kind of antenna configurations. Then, we derived the achievable DRT as a closed-form, which precisely characterizes the rate dependent behavior of the MMSE transceivers.
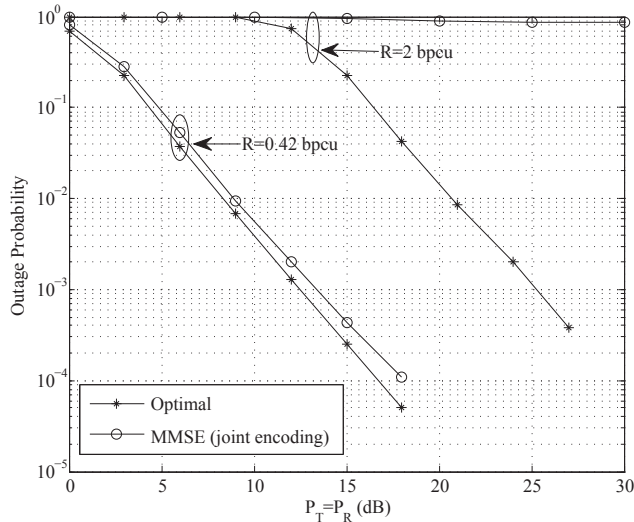
Fig. 3. Outage probabilities of $2 \times 2 \times 1$ MIMO AF relaying systems with R=0.42 and 2 bpcu

Our analysis allows us to completely characterize the diversity order of the MMSE transceivers together with the DMT which is only valid for a positive multiplexing gain. Finally, the analysis was confirmed by numerical simulations.

APPENDIX A
CHOOSING η IN (8)

From the definition of $\mathbf{R}_y$ in (5), we obtain

$$
\begin{aligned}
\mathbf{R}_y &= \rho \mathbf{H}^H \mathbf{H} (\mathbf{H}^H \mathbf{H} + \rho^{-1} \mathbf{I}_{N_S})^{-1} \\
&= \rho (\mathbf{H}^H \mathbf{H} + \rho^{-1} \mathbf{I}_{N_S} - \rho^{-1} \mathbf{I}_{N_S})(\mathbf{H}^H \mathbf{H} + \rho^{-1} \mathbf{I}_{N_S})^{-1} \\
&= \rho \mathbf{I}_{N_S} - (\mathbf{H}^H \mathbf{H} + \rho^{-1} \mathbf{I}_{N_S})^{-1}.
\end{aligned}
\tag{16}
$$

Then, it is obvious that $\mathbf{R}_y \prec \rho \mathbf{I}_{N_S}$ where $\prec$ or $\succ$ represent generalized inequality defined on the positive definite cone. Since we have $\mathrm{Tr}(\mathbf{X}) < \mathrm{Tr}(\mathbf{Y})$ for positive definite matrices $\mathbf{X} \prec \mathbf{Y}$, it follows that $\mathrm{Tr}(\mathbf{B}\mathbf{R}_y\mathbf{B}^H) < \mathrm{Tr}(\rho \mathbf{B}\mathbf{B}^H) = \mathrm{Tr}(\rho \mathbf{\Phi}\mathbf{\Phi}^H)$; thus setting $\mathbf{\Phi} = \mathbf{I}_M$ in (8) satisfies the relay power constraint (5).

REFERENCES

[1] A. Hedayat and A. Nosratinia, "Outage and Diversity of Linear Receivers in Flat-fading MIMO Channels," *IEEE Transactions on Signal Processing*, vol. 3, pp. 5868–5873, December 2007.

[2] K. R. Kumar, G. Caire, and A. L. Moustakas, "Asymptotic Performance of Linear Receivers in MIMO Fading Channels," *IEEE Transactions on Information Theory*, vol. 55, pp. 4398–4418, October 2009.

[3] A. H. Mehana and A. Nosratinia, "Diversity of MMSE MIMO Receivers," *IEEE Transactions on Information Theory*, vol. 58, pp. 6788–6805, November 2012.

[4] D. Gündüz, M. A. Khojastepour, A. Goldsmith, and V. Poor, "Multi-hop MIMO Relay Networks: Diversity-Multiplexing Trade-off Analysis," *IEEE Transactions on Wireless Communications*, vol. 9, pp. 1738–1747, May 2010.

[5] O. Lévêque, C. Vignat, and M. Yüksel, "Diversity-Multiplexing Trade-off for the MIMO Static Half-Duplex Relay," *IEEE Transactions on Information Theory*, vol. 56, pp. 3356–3368, July 2010.

[6] C. Song, K.-J. Lee, and I. Lee, "MMSE-Based MIMO Cooperative Relaying Systems: Closed-Form Designs and Outage Behavior," *IEEE Journal on Selected Areas in Communications*, vol. 30, pp. 1390–1401, September 2012.

[7] W. Guan and H. Luo, "Joint MMSE Transceiver Design in Non-Regenerative MIMO Relay Systems," *IEEE Communications Letters*, vol. 12, pp. 517–519, July 2008.

[8] C. Song, K.-J. Lee, and I. Lee, "MMSE Based Transceiver Designs in Closed-Loop Non-Regenerative MIMO Relaying Systems," *IEEE Transactions on Wireless Communications*, vol. 9, pp. 2310–2319, July 2010.

[9] S. Jang, J. Yang, and D. K. Kim, "Minimum MSE Design for Multiuser MIMO Relay," *IEEE Communications Letters*, vol. 14, pp. 812–814, September 2010.

[10] M. Joham, W. Utschick, and J. A. Nossek, "Linear Transmit Processing in MIMO Communications Systems," *IEEE Transactions on Signal Processing*, vol. 53, pp. 2700–2712, August 2005.

[11] N. Komaroff, "Bounds on Eigenvalues of Matrix Products with an Application to the Algebraic Riccati Equation," *IEEE Transactions on Automatic Control*, vol. 35, pp. 348–350, March 1990.

[12] D. P. Palomar, J. M. Cioffi, and M. A. Lagunas, "Joint Tx-Rx Beamforming Design for Multicarrier MIMO Channels: A Unified Framework for Convex Optimization," *IEEE Transactions on Signal Processing*, vol. 51, pp. 2381–2401, September 2003.

[13] X. Tang and Y. Hua, "Optimal Design of Non-Regenerative MIMO Wireless Relays," *IEEE Transactions on Wireless Communications*, vol. 6, pp. 1398–1407, April 2007.