

Sparse (group) learning with Lipschitz loss functions: a unified analysis

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Abstract

We study a family of sparse estimators defined as minimizers of some empirical Lipschitz loss function—which include hinge, logistic and quantile regression losses—with a convex, sparse or group-sparse regularization. In particular, we consider the L1-norm on the coefficients, its sorted Slope version, and the Group L1-L2 extension. First, we propose a theoretical framework which simultaneously derives new L2 estimation upper bounds for all three regularization schemes. For L1 and Slope regularizations, our bounds scale as $(k^*/n) \log(p/k^*)$ — $n \times p$ is the size of the design matrix and k^* the dimension of the theoretical loss minimizer β^* —matching the optimal minimax rate achieved for the least-squares case. For Group L1-L2 regularization, our bounds scale as $(s^*/n) \log(G/s^*) + m^*/n$ — G is the total number of groups and m^* the number of coefficients in the s^* groups which contain β^* —and improve over the least-squares case. We additionally show that when the signal is strongly group-sparse Group L1-L2 is superior to L1 and Slope. Our bounds are achieved both in probability and in expectation, under common assumptions in the literature. Second, we propose an accelerated proximal algorithm which computes the convex estimators studied when the number of variables is of the order of 100,000. We additionally compare their statistical performance of our estimators against standard baselines for settings where the signal is either sparse or group-sparse. Our experiments findings reveal (i) the good empirical performance of L1 and Slope regularizations for sparse binary classification problems, (ii) the superiority of Group L1-L2 regularization for group-sparse classification problems and (iii) the appealing properties of sparse quantile regression estimators for sparse regression problems with heteroscedastic noise.

1 Introduction

We consider a training data of independent samples $\{(\mathbf{x}_i, y_i)\}_{i=1}^n$, $(\mathbf{x}_i, y_i) \in \mathbb{R}^p \times \mathcal{Y}$ from a distribution $\mathbb{P}(\mathbf{X}, \mathbf{y})$. We fix a loss f and consider a theoretical minimizer β^* of the theoretical loss $\mathcal{L}(\beta) = \mathbb{E}(f(\langle \mathbf{x}, \beta \rangle; y))$:

$$\beta^* \in \operatorname{argmin}_{\beta \in \mathbb{R}^p} \mathbb{E}(f(\langle \mathbf{x}, \beta \rangle; y)). \quad (1)$$

In the rest of this paper, $f(\cdot, y)$ will be assumed to be Lipschitz and to admit a subgradient. We denote $k^* = \|\beta^*\|_0$ the number of non-zeros coefficients of the theoretical minimizer and $R = \|\beta^*\|_1$ its L1 norm. We consider the L1-constrained learning problem

$$\min_{\beta \in \mathbb{R}^p: \|\beta\|_1 \leq 2R} \frac{1}{n} \sum_{i=1}^n f(\langle \mathbf{x}_i, \beta \rangle; y_i) + \Omega(\beta), \quad (2)$$

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where $\Omega(\boldsymbol{\beta})$ is a regularization function. We study sparse estimators, i.e. with a small number of non-zeros. To this end, we restrict $\Omega(\cdot)$ to a class of the sparsity-inducing regularizations. We first consider the L1 regularization, which is well-known to encourage sparsity in the coefficients [31]. Problem (2) becomes:

$$\min_{\boldsymbol{\beta} \in \mathbb{R}^p: \|\boldsymbol{\beta}\|_1 \leq 2R} \frac{1}{n} \sum_{i=1}^n f(\langle \mathbf{x}_i, \boldsymbol{\beta} \rangle; y_i) + \lambda \|\boldsymbol{\beta}\|_1. \quad (3)$$

The second problem we study is inspired by the sorted L1-penalty aka the Slope norm [7, 3], used in the context of least-squares problems for its statistical properties. We note \mathcal{S}_p the set of permutations of $\{1, \dots, p\}$ and consider a sequence $\lambda \in \mathbb{R}^p: \lambda_1 \geq \dots \geq \lambda_p > 0$. For $\eta > 0$, we define the L1-constrained Slope estimator as a solution of the convex minimization problem:

$$\min_{\boldsymbol{\beta} \in \mathbb{R}^p: \|\boldsymbol{\beta}\|_1 \leq 2R} \frac{1}{n} \sum_{i=1}^n f(\langle \mathbf{x}_i, \boldsymbol{\beta} \rangle; y_i) + \eta \|\boldsymbol{\beta}\|_S, \quad \text{where} \quad \|\boldsymbol{\beta}\|_S = \max_{\phi \in \mathcal{S}_p} \sum_{j=1}^p |\lambda_j| |\beta_{\phi(j)}| = \sum_{j=1}^p \lambda_j |\beta_{(j)}| \quad (4)$$

is the Slope regularization and $|\beta_{(1)}| \geq \dots \geq |\beta_{(p)}|$ is a non-increasing rearrangement of $\boldsymbol{\beta}$.

Finally, in several applications, sparsity is structured—the coefficient indices occur in groups a-priori known and it is desirable to select a whole group. In this context, group variants of the L1 norm are often used to improve the performance and interpretability [34, 16]. We consider the use of a Group L1-L2 regularization [1] and define the L1-constrained Group L1-L2 problem:

$$\min_{\boldsymbol{\beta} \in \mathbb{R}^p: \|\boldsymbol{\beta}\|_1 \leq 2R} \frac{1}{n} \sum_{i=1}^n f(\langle \mathbf{x}_i, \boldsymbol{\beta} \rangle; y_i) + \lambda \sum_{g=1}^G \|\boldsymbol{\beta}_g\|_2. \quad (5)$$

where, $g = 1, \dots, G$ denotes a group index (the groups are disjoint), $\boldsymbol{\beta}_g$ denotes the vector of coefficients belonging to group g , \mathcal{I}_g the corresponding set of indexes, $n_g = |\mathcal{I}_g|$ and $\boldsymbol{\beta} = (\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_G)$. In addition, we denote $g_* := \max_{g=1, \dots, G} n_g$, $\mathcal{J}^* \subset \{1, \dots, G\}$ the smallest subset of group indexes such that the support of $\boldsymbol{\beta}^*$ is included in the union of these groups, $s^* := |\mathcal{J}^*|$ the cardinality of \mathcal{J}^* , and m^* the sum of the sizes of these s^* groups.

What this paper is about: In this paper, we propose a unified statistical and computational analysis of a large class of estimators, defined as solutions of Problems (3), (4) and (5) when $f(\cdot, y)$ is a convex Lipschitz loss function which admits a subgradient (cf. Assumption 1, Section 2.2), e.g. when f is the hinge loss, the logistic loss or the quantile regression loss. In a first part, we propose a statistical study which derives new error bounds for the L2 norm of the difference between the empirical and theoretical minimizers $\|\hat{\boldsymbol{\beta}}(\lambda, R) - \boldsymbol{\beta}^*\|_2$ where $\hat{\boldsymbol{\beta}}(\lambda, R)$ ¹ is a solution of Problem (3), (4) or (5). Our bounds are reached under standard assumptions in the literature, and hold with high probability and in expectation. As a critical step, we derive stronger versions of existing cone conditions and restricted strong convexity conditions in the following Theorems 1 and 2. Our method draws inspiration from the least-squares approaches [5, 3, 16] and illustrates the distinction between regression and classification studies. Our framework is flexible enough to apply to coefficient-based and group-based regularizations, while enhancing the differences between these two classes of problems. For Problem (3) and (4), our bounds scale as $(k^*/n) \log(p/k^*)$. They improve over existing results for all three losses considered with L1 regularization [24, 27, 4], and match the best minimax rate achieved in the least-squares case [26]. For the Group L1-L2 Problem (5), our bounds appear to be the first existing results for all three losses and scale as $(s^*/n) \log(G/s^*) + m^*/n$. This rate is better than the existing ones for the least-squares problems [16] due to a stronger cone condition (cf. Theorem 1). Similarly to [16], we additionally show that when the signal is strongly group-sparse, Group L1-L2 regularization is

¹When no confusion can be made, we drop the dependence upon the parameters λ, R .

superior to L1 and Slope. In a second part, we propose a computational study of our family of estimators. We design a proximal gradient algorithm to solve the fully tractable problems presented herein—our method uses Nesterov smoothing [22] in the case where f is a non-smooth loss—and we compare the estimators derived with standard non-sparse baselines through a variety of computational experiments. Our numerical findings enhance the numerical performance of our estimators of study for classification and regression settings where the signal is sparse and group-sparse.

Organization of paper: The rest of this paper is organized as follows. Section 2 builds our framework of study and presents our new theorems: our main statistical results appear in Theorem 3 and Corollary 1. Section 3 proposes a first order algorithm to solve Problems (3), (4) and (5) and presents a range of synthetic experiments which reveals the computational advantage of the estimators studied herein.

2 Statistical analysis

In this section, we study the statistical properties of the estimators defined as solutions of Problems (3), (4) and (5) and derive new upper bounds for L2 estimation.

2.1 Existing work on statistical performance

Statistical performance and L2 consistency for high-dimensional linear regression have been widely studied [11, 5, 10, 3, 19]. One important statistical performance measure is the L2 estimation error defined as $\|\hat{\beta} - \beta^*\|_2$ where β^* is the k^* -sparse vector used in generating the true model and $\hat{\beta}$ is an estimator. For regression problems with least-squares loss, [10] and [26] established a $(k^*/n) \log(p/k^*)$ lower bound for estimating the L2 norm of a sparse vector, regardless of the input matrix and estimation procedure. This optimal minimax rate is known to be achieved by a global minimizer of a L0 regularized estimator [9]. This minimizer is sparse and adapts to unknown sparsity—the degree k^* does not have to be specified—however, it is intractable in practice. Recently, [3] reached this optimal minimax bound for a Lasso estimator with knowledge of the sparsity k^* , and proved that a recently introduced and polynomial-time Slope estimator [6] achieves the optimal rate while adapting to unknown sparsity. In a related work, [33] reached a near-optimal $(k^*/n) \log(p)$ rate for L1 regularized least-angle deviation loss. [4] extended this bound for L1 regularized quantile regression. Finally, in the regime where sparsity is structured, [16] proved a $(s^*/n) \log(G) + m^*/n$ L2 estimation upper bound for a Group L1-L2 estimator—where, similarly to our notations, G is the number of groups, s^* the number of relevant groups and m^* their aggregated size—and showed that their Group L1-L2 estimator is superior to standard Lasso when the signal is strongly group-sparse, i.e. m^*/k^* is low and the signal is efficiently covered by the groups. [19] similarly showed that, in the multitask setting, a Group L1-L2 estimator is superior to Lasso.

Little work has been done on deriving estimation error bounds on high-dimensional classification problems. Existing work has focused on the analysis of generalization error and risk bounds [30, 35]. Unlike the regression case, for classification problems k^* is the sparsity of the theoretical minimizer to estimate. Recently, [24] proved a $(k^*/n) \log(p)$ upper-bound for L2 coefficients estimation of a L1 regularized Support Vector Machines (SVM). The authors recovered the rate proposed by [32], which considered a weighted L1 norm for linear models. [27] obtained a similar bound for a L1-regularized logistic regression estimator in a binary Ising graph. However, this rate of $(k^*/n) \log(p)$ is not the best known for a classification estimator: [25] proved a $k^* \log(p/k^*)$ error bound for estimating a single vector through sparse models—including 1-bit compressed sensing and logistic regression—over a bounded set of vectors. Contrary to this work, our approach does not assume a generative vector and applies to a larger class of losses (hinge, quantile regression) and regularizations (Slope, Group L1-L2). We are not aware of any existing result for group regularizations in classification settings.

2.2 Framework of study

We design herein our theoretical framework of study, using common assumptions in the literature.

2.2.1 Lipschitz loss and existence of a subgradient

Our first assumption requires $f(\cdot, y)$ to be L -Lipschitz and to admit a subgradient $\partial f(\cdot, y)$. We list three main examples that fall into this framework.

Assumption 1 *The loss $f(\cdot, y)$ is non-negative, convex and Lipschitz continuous with constant L , that is, $|f(t_1, y) - f(t_2, y)| \leq L|t_1 - t_2|$, $\forall t_1, t_2$. In addition, there exists $\partial f(\cdot, y)$ such that $f(t_2, y) - f(t_1, y) \geq \partial f(t_1, y)(t_2 - t_1)$, $\forall t_1, t_2$.*

Support vectors machines (SVM) For $\mathcal{Y} = \{-1, 1\}$, the SVM problem learns a classification rule of the form $\text{sign}(\langle \mathbf{x}, \boldsymbol{\beta} \rangle)$ by solving Problem (2) with the hinge loss $f(t; y) = \max(0, 1 - yt)$. The loss admits as a subgradient $\partial f(t, y) = \mathbf{1}(1 - yt \geq 0)yt$ and satisfies Assumption 1 for $L = 1$.

Logistic regression We assume $\log(\mathbb{P}(y_i = 1 | \mathbf{X} = \mathbf{x}_i)) - \log(\mathbb{P}(y_i = -1 | \mathbf{X} = \mathbf{x}_i)) = \langle \mathbf{x}_i, \boldsymbol{\beta} \rangle$, $\forall i$. The maximum likelihood estimator solves Problem (2) for the logistic loss $f(t; y) = \log(1 + \exp(-yt))$. The loss satisfies Assumption 1 for $L = 1$ since $|\partial_t f(t, y)| = |1/(1 + e^{yt})| \leq 1$.

Quantile regression We consider $\mathcal{Y} = \mathbb{R}$ and fix $\theta \in (0, 1)$. Following [8], we assume the θ th conditional quantile of y given \mathbf{X} to be $Q_\theta(y | \mathbf{X} = \mathbf{x}) = \langle \mathbf{x}, \boldsymbol{\beta}_\theta \rangle$. We define the quantile loss² $\rho_\theta(t) = (\theta - \mathbf{1}(t \leq 0))t$. ρ_θ satisfies Assumption 1 for $L = \max(1 - \theta, \theta)$. In addition, it is known [17] that $\boldsymbol{\beta}_\theta \in \text{argmin}_{\boldsymbol{\beta} \in \mathbb{R}^p} \mathbb{E}[\rho_\theta(y - \langle \mathbf{x}, \boldsymbol{\beta} \rangle)]$. For $\theta = 1/2$, the quantile regression loss is proportional to the least-angle deviation loss: $\rho_\theta(y_i - \langle \mathbf{x}_i, \boldsymbol{\beta} \rangle) = \frac{1}{2}|y_i - \langle \mathbf{x}_i, \boldsymbol{\beta} \rangle|$, studied with L1 regularization in [33].

2.2.2 Differentiability of the theoretical loss

We additionally assume the unicity of $\boldsymbol{\beta}^*$ and the twice differentiability of the theoretical loss \mathcal{L} . [18] studied specific conditions under which Assumption 2 holds for the hinge loss (the result extends to the quantile regression loss). Assumption 2 is guaranteed for the logistic loss.

Assumption 2 *The theoretical minimizer is unique. In addition, the theoretical loss is twice-differentiable: we note its gradient $\nabla \mathcal{L}(\cdot)$ and its Hessian matrix $\nabla^2 \mathcal{L}(\cdot)$. It finally holds: $\nabla \mathcal{L}(\cdot) = \mathbb{E}(\partial f(\langle \mathbf{x}, \cdot \rangle; y) \mathbf{x})$.*

2.2.3 Sub-Gaussian entries

Our next assumption controls the entries of the design matrix. Let us first recall the definition of a sub-Gaussian random variable [28]:

Definition 1 *A random variable Z is said to be sub-Gaussian with variance $\sigma^2 > 0$ if $\mathbb{E}(Z) = 0$ and $\mathbb{P}(|Z| > t) \leq 2 \exp\left(-\frac{t^2}{2\sigma^2}\right)$, $\forall t > 0$.*

This variable will be noted $Z \sim \text{subG}(\sigma^2)$. Assumption 3 is then defined as follows:

Assumption 3 • *There exists $M > 0$ such that $\partial f(\langle \mathbf{x}_i, \boldsymbol{\beta}^* \rangle, y_i) x_{ij} \sim \text{subG}(L^2 M^2)$, $\forall i, j$ for $M > 0$.*

• *For Group L1-L2 regularization, we additionally assume that $\forall g, \forall \mathbf{u} \in \mathbb{R}^{n_g}$:*

$$\mathbb{P}\left(|\partial f(\langle \mathbf{x}_i, \boldsymbol{\beta}^* \rangle, y_i) (\mathbf{x}_i)_g^T \mathbf{u}_g| > t\right) \leq 2 \exp\left(-\frac{t^2}{2L^2 M^2 \|\mathbf{u}_g\|_2^2}\right), \forall t > 0, \forall i.$$

Under Assumptions 1 and 2, it holds $\mathbb{E}[\partial f(\langle \mathbf{x}_i, \boldsymbol{\beta}^* \rangle, y_i) x_{ij}] = 0$, $\forall i, j$ since $\boldsymbol{\beta}^*$ minimizes the theoretical loss. Hence, under Assumption 3 the random variables $\partial f(\langle \mathbf{x}_i, \boldsymbol{\beta}^* \rangle, y_i) x_{ij}$ $\forall i, j$ are sub-Gaussian with variance $L^2 M^2$.

²Note that the hinge loss is a translation of the quantile loss for $\theta = 0$.

2.2.4 Restricted eigenvalue conditions

The next assumption draws inspiration from the restricted eigenvalue conditions defined for all three L1, Slope and Group L1-L2 regularizations in the regression settings [5, 3, 19]. For an integer k , Assumption 4.1 ensures that the quadratic form associated with $n^{-1} \mathbf{X}^T \mathbf{X}$ is upper-bounded on the cone of k sparse vectors. Similarly, Assumptions 4.2, 4.3 and 4.4 ensure that the quadratic form associated with the Hessian matrix $\nabla^2 \mathcal{L}(\beta^*)$ is lower-bounded on a family of cones of \mathbb{R}^p —specific to the regularization used.

Assumption 4 • Let $k \in \{1, \dots, p\}$. Assumption 4.1(k) is satisfied if there exists a non-negative constant $\mu(k)$ such that almost surely:

$$\mu(k) \geq \sup_{\mathbf{z} \in \mathbb{R}^p: \|\mathbf{z}\|_0 \leq k} \frac{\sqrt{k} \|\mathbf{X} \mathbf{z}\|_1}{\sqrt{n} \|\mathbf{z}\|_1}.$$

• Let $\gamma_1, \gamma_2 > 0$. Assumption 4.2(k, γ) holds if there exists $\kappa(k, \gamma_1, \gamma_2)$ which almost surely satisfies:

$$0 < \kappa(k, \gamma_1, \gamma_2) \leq \inf_{|S| \leq k} \inf_{\mathbf{z} \in \Lambda(S, \gamma_1, \gamma_2)} \frac{\mathbf{z}^T \nabla^2 \mathcal{L}(\beta^*) \mathbf{z}}{\|\mathbf{z}\|_2^2},$$

where $\gamma = (\gamma_1, \gamma_2)$ and for every subset $S \subset \{1, \dots, p\}$, the cone $\Lambda(S, \gamma_1, \gamma_2) \subset \mathbb{R}^p$ is defined as:

$$\Lambda(S, \gamma_1, \gamma_2) = \{\mathbf{z} \in \mathbb{R}^p : \|\mathbf{z}_{S^c}\|_1 \leq \gamma_1 \|\mathbf{z}_S\|_1 + \gamma_2 \|\mathbf{z}_S\|_2\}.$$

• Let $\omega > 0$. Assumption 4.3(k, ω) holds if there exists a constant $\kappa(k, \omega) > 0$ such that a.s.:

$$0 < \kappa(k, \omega) \leq \inf_{\mathbf{z} \in \Gamma(k, \omega)} \frac{\mathbf{z}^T \nabla^2 \mathcal{L}(\beta^*) \mathbf{z}}{\|\mathbf{z}\|_2^2},$$

where the cone $\Gamma(k, \omega)$ is defined as:

$$\Gamma(k, \omega) = \left\{ \mathbf{z} \in \mathbb{R}^p : \sum_{j=k+1}^p \lambda_j |z_{(j)}| \leq \omega \sum_{j=1}^k \lambda_j |z_{(j)}| \right\} \text{ where } |z_{(1)}| \geq \dots \geq |z_{(p)}|, \forall \mathbf{z}.$$

• Let $\epsilon_1, \epsilon_2 > 0$. Assumption 4.4(s, ϵ) holds if there exists a constant $\kappa(s, \epsilon_1, \epsilon_2) > 0$ such that a.s.:

$$0 < \kappa(s, \epsilon_1, \epsilon_2) \leq \inf_{|\mathcal{J}| \leq s} \inf_{\mathbf{z} \in \Omega(\mathcal{J}, \epsilon_1, \epsilon_2)} \frac{\mathbf{z}^T \nabla^2 \mathcal{L}(\beta^*) \mathbf{z}}{\|\mathbf{z}\|_2^2},$$

where $\epsilon = (\epsilon_1, \epsilon_2)$ and for every subset $\mathcal{J} \subset \{1, \dots, G\}$, we define $\mathcal{T}(\mathcal{J}) = \cup_{g \in \mathcal{J}} \mathcal{I}_g$ the subset of all indexes accross all the groups in \mathcal{J} . $\Omega(\mathcal{J}, \epsilon_1, \epsilon_2)$ is defined as:

$$\Omega(\mathcal{J}, \epsilon_1, \epsilon_2) = \left\{ \mathbf{z} \in \mathbb{R}^p : \sum_{g \notin \mathcal{J}} \|\mathbf{z}_g\|_2 \leq \epsilon_1 \sum_{g \in \mathcal{J}} \|\mathbf{z}_g\|_2 + \epsilon_2 \|\mathbf{z}_{\mathcal{T}(\mathcal{J})}\|_2 \right\}.$$

In the SVM framework [24], Assumptions (A3) and (A4) are similar to our Assumptions 4.1 and 4.2. For logistic regression [27], Assumptions A1 and A2 similarly define a dependency and incoherence conditions. For quantile regression, Assumption D.4 [4] is equivalent to a uniform restricted eigenvalue condition.

2.2.5 Growth conditions

Since β^* minimizes the theoretical loss, it holds $\nabla L(\beta^*) = 0$. In particular, under Assumption 4, the theoretical loss is lower-bounded by a quadratic function on a certain subset surrounding β^* . By continuity, we define the maximal radius on which the following lower-bound holds:

$$r^* = \max \left\{ r > 0 \mid \mathcal{L}(\beta^* + z) \geq \mathcal{L}(\beta^*) + \frac{1}{4} \kappa^* \|z\|_2^2, \forall z \in \mathcal{C}^*, \|z\|_2 \leq r \right\} \text{ where:}$$

- $\mathcal{C}^* = \bigcup_{S \subset \{1, \dots, p\}: |S| \leq k^*} \Lambda(S, \gamma_1, \gamma_2)$ and $\kappa^* = \kappa(k^*, \gamma_1, \gamma_2)$ for L1 regularization.
- $\mathcal{C}^* = \Gamma(k^*, \omega)$ and $\kappa^* = \kappa(k^*, \omega)$ for Slope regularization.
- $\mathcal{C}^* = \bigcup_{\mathcal{J} \subset \{1, \dots, G\}: |\mathcal{J}| \leq s^*} \Omega(\mathcal{J}, \epsilon_1, \epsilon_2)$ and $\kappa^* = \kappa(s^*, \epsilon_1, \epsilon_2)$ for Group L1-L2 regularization.

r^* depends upon the same parameters than κ^* . We propose the following growth conditions which give a relation between the number of samples n , the dimension space p , the sparsity levels k^* or s^* , the maximal radius r^* , and a parameter δ .

Assumption 5 Let $\delta \in (0, 1)$. Assumptions 5.1(p, k^*, α, δ) and 5.2(p, k^*, α, δ)—respectively defined for L1 and Slope regularizations—are said to hold if:

$$p \leq k^* \sqrt{k^*} \text{ and } \kappa^* r^* \geq 4\sqrt{k^*}(\tau^* + \lambda)$$

where λ and $\tau^* = \tau^*(k^*, k^*, \lceil p/k^* \rceil)$ are respectively defined in the following Theorems 1 and 2. In addition, for Group L1-L2 regularization, Assumption 5.3($G, g_*, s^*, m^*, \alpha, \delta$) is said to hold if:

$$m_0 \leq \gamma m^*, G \leq \sqrt{g_* s^*}, \text{ and } \kappa^* r^* \geq 4(\tau^* \sqrt{m^*} + \lambda_G \sqrt{s^*})$$

where $\gamma \geq 1$ and $m_0, \lambda_G, \tau^* = \tau(g_* s^*, g_*, G)$ are also defined in the following Theorems 1 and 2.

The constants κ^* and r^* depend upon the family of cone corresponding to the regularization used. Note that Assumption 5 is similar to Equation (17) for logistic regression [27]. A similar definition is proposed in the proof of Lemma (3.7) for quantile regression [4]. Our framework can now be used to derive upper bounds for coefficients estimation, scaling with the problem size parameters and the constants introduced.

2.3 Cone conditions

Similarly to the regression cases for L1, Slope and Group L1-L2 regularizations [5, 3, 19], Theorem 1 first derives cone conditions satisfied by a respective solution of Problem (3), (4) or (5). Theorem 1 says that, for each problem, the difference between the theoretical and empirical minimizers belongs to one of the families of cones defined in Assumption 4. The cone conditions are derived by selecting a regularization parameter large enough so that it dominates the sub-gradient of the loss f evaluated at the theoretical minimizer β^* .

Theorem 1 Let $\delta \in (0, 1)$, $\alpha \geq 2$, and assume that Assumptions 1 and 3 are satisfied.

We denote $\lambda_j^{(r)} = \sqrt{\log(2re/j)}, \forall j, \forall r$ and fix the parameters $\eta = 34\alpha LM \sqrt{n^{-1} \log(2/\delta)}$, $\lambda = \eta \lambda_{k^*}^{(p)} = 34\alpha LM \sqrt{n^{-1} \log(2pe/k^*) \log(2/\delta)}$ for Slope and L1 regularizations and $\lambda_G = \eta \lambda_{s^*}^{(G)} + 4\alpha LM \sqrt{\gamma (s^* n)^{-1} m^*} = 34\alpha LM \sqrt{n^{-1} \log(2Ge/s^*) \log(2/\delta)} + 4\alpha LM \sqrt{\gamma (s^* n)^{-1} m^*}$ for Group L1-L2 regularization. The following results hold with probability at least $1 - \frac{\delta}{2}$.

- Let $\hat{\beta}_1$ be a solution of the L1 regularized Problem (3) with parameter λ , and $S_0 \subset \{1, \dots, p\}$ be the subset of indexes of the k^* highest coefficients of $\mathbf{h}_1 := \hat{\beta}_1 - \beta^*$. It holds:

$$\mathbf{h}_1 \in \Lambda \left(S_0, \gamma_1^* := \frac{\alpha}{\alpha - 1}, \gamma_2^* := \frac{\sqrt{k^*}}{\alpha - 1} \right).$$

- Let $\hat{\beta}_S$ be a solution of the Slope regularized Problem (4) with parameter η and the sequence of coefficients $\lambda_j^{(p)} = \sqrt{\log(2pe/j)}, \forall j$. It holds:

$$\mathbf{h}_S := \hat{\beta}_S - \beta^* \in \Gamma \left(k^*, \omega^* := \frac{\alpha + 1}{\alpha - 1} \right).$$

- Let $\hat{\beta}_{L1-L2}$ be a solution of the Group L1-L2 Problem (5) with parameter λ_G , and let $\mathcal{J}_0 \subset \{1, \dots, G\}$ be the subset of indexes of the s^* highest groups of $\mathbf{h}_{L1-L2} := \hat{\beta}_{L1-L2} - \beta^*$ for the L2 norm, and m_0 be the total size of the s^* largest groups. Finally let $\mathcal{T}_0 = \cup_{g \in \mathcal{J}_0} \mathcal{I}_g$ define the subset of size of all indexes across all the s^* groups in \mathcal{J}_0 . It holds:

$$\mathbf{h}_{L1-L2} \in \Omega \left(\mathcal{J}_0, \epsilon_1^* := \frac{\alpha}{\alpha - 1}, \epsilon_2^* := \frac{\sqrt{s^*}}{\alpha - 1} \right).$$

The proof is presented in Appendix B: it uses a new result to control the maximum of sub-Gaussian random variables. As a consequence, for the L1 regularized Problem (3), the parameter λ^2 is of the order of $\log(p/k^*)/n$. In particular, our conditions are stronger than [24], [27] and [33], which all propose a scaling as $\log(p)/n$ for L1 regularized estimator with all three Lipschitz losses considered herein. In addition, for Group L1-L2 regularization, the parameter λ_G^2 is of the order of $\log(G/s^*)/n + m^*/(s^*n)$: our conditions are also stronger than [16], which considers a $\log(G)/n + m^*/n$ scaling for least-squares.

2.4 Restricted strong convexity conditions

The next Theorem 2 says that the loss f satisfies a restricted strong convexity [21] with curvature $\kappa^*/4$ and L1 tolerance function. It is derived by combining (i) a supremum result from Theorem 5 presented in Appendix C (ii) the minimality of β^* and (iii) restricted eigenvalue conditions from Assumption 4.

Theorem 2 *Let $\delta \in (0, 1)$ and assume that Assumptions 1, 2 and 3 hold. In addition, assume that Assumptions 4.1(k^*) and 4.2(k^*, γ) hold for L1 regularization, Assumptions 4.1(k^*) and 4.3(k^*, ω^*) for Slope, Assumptions 4.1($g_* s^*$) and 4.4(s^*, ϵ^*) for Group L1-L2—where γ^*, ω^* and ϵ^* are defined in Theorem 1. Finally, let $\tau(k, m, q) = 14L\mu(k) \sqrt{\frac{\log(7)}{n} + \frac{\log(2q)}{nk} + \frac{\log(2/\delta)}{nk}}$ for all integers k, m, q and let $\mathbf{h} = \hat{\beta} - \beta^*$ be a shorthand for $\mathbf{h}_1, \mathbf{h}_S$, or \mathbf{h}_{L1-L2} .*

Then, it holds with probability at least $1 - \frac{\delta}{2}$:

$$\frac{1}{n} \sum_{i=1}^n f(\langle \mathbf{x}_i, \beta^* + \mathbf{h} \rangle; y_i) - \frac{1}{n} \sum_{i=1}^n f(\langle \mathbf{x}_i, \beta^* \rangle; y_i) \geq \frac{1}{4} \kappa^* \{ \|\mathbf{h}\|_2^2 \wedge r^* \|\mathbf{h}\|_2 \} - \tau^* \|\mathbf{h}\|_1,$$

where $\tau^ = \tau(k^*, k^*, \lceil p/k^* \rceil)$ for L1 and Slope regularizations and $\tau^* = \tau(g_* s^*, g_*, G)$ for Group L1-L2 regularization. κ^*, r^* are shorthands for the restricted eigenvalue constant and maximum radius introduced in Assumptions 4 and 5: they depend upon the regularization used.*

Our cone conditions could be extended to the use of an L2 tolerance function: our parameter $(\tau^*)^2$ would scale as $(k^* + \log(p/k^*))/n$. In contrast, [24], [27] and [21] propose a parameter scaling as $k^* \log(p)/n$ with an L2 tolerance function: our results are stronger than existing works.

2.5 Upper bounds for coefficients estimation

We conclude this section by presenting our main bounds in Theorem 3 and Corollary 1.

Theorem 3 *Let $\delta \in (0, 1)$. We consider the same assumptions and notations than in Theorems 1 and 2. In addition, we assume that the growth conditions 5.1(p, k^*, α, δ), 5.2(p, k^*, α, δ) and 5.3($G, g_*, s^*, m^*, \alpha, \delta$) respectively hold for L1, Slope and Group L1-L2 regularizations. We select $\alpha \geq 2$ so that $\mu(k^*) \leq 2\alpha M$ for L1 and Slope regularizations, and $\mu(g_* s^*) \leq 2\alpha M \sqrt{s^*}$ for Group L1-L2 regularization.*

Then the estimators $\hat{\beta}_1$ and $\hat{\beta}_S$ satisfies with probability at least $1 - \delta$:

$$\|\hat{\beta}_{1,S} - \beta^*\|_2 \lesssim \frac{\alpha LM}{\kappa^*} \sqrt{\frac{k^* \log(p/k^*) \log(2/\delta)}{n}} + \frac{L\mu(k^*)}{\kappa^*} \sqrt{\frac{k^* + \log(p/k^*) + \log(2/\delta)}{n}},$$

In addition, the estimator $\hat{\beta}_{L1-L2}$ satisfies with probability at least $1 - \delta$:

$$\|\hat{\beta}_{L1-L2} - \beta^*\|_2 \lesssim \frac{\alpha LM}{\kappa^*} \sqrt{\frac{s^* \log(G/s^*) \log(2/\delta) + \gamma m^*}{n}} + \frac{L\mu(g_* s^*)}{\kappa^*} \sqrt{\frac{m^* + \log(G) + \log(2/\delta)}{n}}.$$

where $\kappa^* = \kappa(S_0, \gamma_1^*, \gamma_2^*)$ for L1 regularization, $\kappa^* = \kappa(k^*, \omega^*)$ for Slope regularization and $\kappa^* = \kappa(\mathcal{J}_0, \epsilon_1^*, \epsilon_2^*)$ for Group L1-L2 regularization.

The proof is presented in Appendix D. The bounds follow from the cone conditions and the restricted strong convexity conditions derived in Theorems 1 and 2. Theorem 3 holds for any $\delta \leq 1$. Thus, we obtain by integration the following bounds in expectation. The proof is presented in Appendix E.

Corollary 1 *If the assumptions presented in Theorem are satisfied for a small enough δ , then:*

$$\begin{aligned} \mathbb{E}\|\hat{\beta}_{1,S} - \beta^*\|_2 &\lesssim \frac{L}{\kappa^*} \left(\alpha M \sqrt{\frac{k^* \log(p/k^*)}{n}} + \mu(k^*) \sqrt{\frac{k^* + \log(p/k^*)}{n}} \right), \\ \mathbb{E}\|\hat{\beta}_{L1-L2} - \beta^*\|_2 &\lesssim \frac{L}{\kappa^*} \left(\alpha M \sqrt{\frac{s^* \log(G/s^*) + \gamma m^*}{n}} + \mu(g_* s^*) \sqrt{\frac{m^* + \log(G)}{n}} \right). \end{aligned}$$

Discussion for L1 and Slope: For L1 and Slope regularizations, our family of estimators reach a bound scaling as $(k^*/n) \log(p/k^*)$. This bound strictly improves over existing results for L1-regularized versions of all three losses [24, 27, 33, 4] and it matches the best rate known for the least-squares case [3]. We recover our previous result [13] in the more general framework presented herein which also applies to Group L1-L2 regularization. In addition, the L1 regularization parameter λ uses the sparsity k^* . In contrast, similarly to the least-squares case [3], Slope presents the statistical advantage of adapting to unknown sparsity.

Discussion for Group L1-L2: For Group L1-L2, our family of estimators reach a bound scaling as $(s^*/n) \log(G/s^*) + m^*/n$. This bound improves over the regression case [16], which scales as $(s^*/n) \log(G) + m^*/n$. This is due to the stronger cone condition derived in Theorem 1.

Comparison of both bounds for group-sparse signals: We compare the statistical performance and upper bounds of Group L1-L2 regularization to L1 and Slope regularizations when sparsity is structured. Let us first consider two edge case. (i) If all the groups are all of size k^* and the optimal solution is contained

in only one group—that is, $g_* = k^*$, $G = \lceil p/k^* \rceil$, $s^* = 1$, $m^* = k^*$, $\gamma = 1$ —the bound for Group L1-L2 is lower than the ones for L1 and Slope. Group L1-L2 is superior as it strongly exploits the problem structure. (ii) If now all the groups are of size one—that is, $g_* = 1$, $G = p$, $s^* = k^*$, $m^* = k^*$, $\gamma = 1$ —both bounds have a similar first term (due to the cone conditions), however the second term is worse for the group estimator due to of a suboptimal partition choice in Theorem 2 (cf. Appendix C). L1 and Slope are superior. For the general case, when $m_*/k_* \ll \log(p/k^*)$, the signal is efficiently covered by the groups—the group structure is useful—and the upper bound for Group L1-L2 is lower than the one for L1 and Slope. That is, similarly to the regression case [16], Group L1-L2 is superior to L1 for strongly group-sparse signals.³ However, when m_*/k_* is larger, sparsity is not as useful and Group L1-L2 is outperformed by L1 and Slope.

3 Empirical analysis

All the estimators studied are convex. In this section, we study their empirical properties for computational settings where the signal is either sparse or group-sparse, and the number of variables is of the order of 100,000s. To this end, we present a proximal gradient algorithm which solves the tractable Problems (3), (4) and (5).

3.1 Smoothing the loss

We note $g(\boldsymbol{\beta}) = \frac{1}{n} \sum_{i=1}^n f(\langle \mathbf{x}_i, \boldsymbol{\beta} \rangle; y_i)$. Problem (2) can be formulated as: $\min_{\boldsymbol{\beta} \in \mathbb{R}^p} g(\boldsymbol{\beta}) + \Omega(\boldsymbol{\beta})$ —we drop the L1 constraint in the rest of this section. The proximal method we propose requires g to be a differentiable loss with continuous C -Lipschitz gradient. The hinge and quantile regression losses are non-smooth: we propose to use Nesterov’s smoothing method [22] to construct a convex function with continuous Lipschitz gradient g^τ — g_θ^τ for quantile regression—which approximates these losses for $\tau \approx 0$.

Hinge loss: For the hinge loss, let us first note that $\max(0, t) = \frac{1}{2}(t + |t|) = \max_{|w| \leq 1} \frac{1}{2}(t + wt)$ as this maximum is achieved for $\text{sign}(x)$. Consequently, the hinge loss can be expressed as a maximum over the L_∞ unit ball:

$$g(\boldsymbol{\beta}) = \frac{1}{n} \sum_{i=1}^n \max(0, z_i) = \max_{\|\mathbf{w}\|_\infty \leq 1} \frac{1}{2n} \sum_{i=1}^n [z_i + w_i z_i]$$

where $z_i = 1 - y_i \mathbf{x}_i^T \boldsymbol{\beta}$, $\forall i$. We apply the technique suggested by [22] and define for $\tau > 0$ the smoothed version of the loss:

$$g^\tau(\boldsymbol{\beta}) = \max_{\|\mathbf{w}\|_\infty \leq 1} \frac{1}{2n} \sum_{i=1}^n [z_i + w_i z_i] - \frac{\tau}{2n} \|\mathbf{w}\|_2^2. \quad (6)$$

Let $\mathbf{w}^\tau(\boldsymbol{\beta}) \in \mathbb{R}^n$: $w_i^\tau(\boldsymbol{\beta}) = \min(1, \frac{1}{2\tau} |z_i|) \text{sign}(z_i)$, $\forall i$ be the optimal solution of the right-hand side of Equation (6). The gradient of g^τ is expressed as:

$$\nabla g^\tau(\boldsymbol{\beta}) = -\frac{1}{2n} \sum_{i=1}^n (1 + w_i^\tau(\boldsymbol{\beta})) y_i \mathbf{x}_i \in \mathbb{R}^p \quad (7)$$

and its associated Lipschitz constant is derived from the next theorem. The proof is presented in Appendix F. It follows [22] and uses first order necessary conditions for optimality.

³[16] does not discuss the superiority of Group L1-L2 over Slope, which we do

Theorem 4 Let $\mu_{\max}(n^{-1}\mathbf{X}^T\mathbf{X})$ be the highest eigenvalue of $n^{-1}\mathbf{X}^T\mathbf{X}$. Then ∇g^τ is Lipschitz continuous with constant $C^\tau = \mu_{\max}(n^{-1}\mathbf{X}^T\mathbf{X})/4\tau$.

Quantile regression: The same method applies to the non smooth quantile regression loss. We first note that $\max((\theta - 1)t, \theta t) = \frac{1}{2}((2\theta - 1)t + |t|) = \max_{|w| \leq 1} \frac{1}{2}((2\theta - 1)t + wt)$. Hence the smooth quantile regression loss is defined as $g_\theta^\tau(\boldsymbol{\beta}) = \max_{\|\mathbf{w}\|_\infty \leq 1} \frac{1}{2n} \sum_{i=1}^n ((2\theta - 1)\tilde{z}_i + w_i \tilde{z}_i) - \frac{\tau}{2n} \|\mathbf{w}\|_2^2$ and its gradient is:

$$\nabla g_\theta^\tau(\boldsymbol{\beta}) = -\frac{1}{2n} \sum_{i=1}^n (2\theta - 1 + \tilde{w}_i^\tau(\boldsymbol{\beta})) \mathbf{x}_i \in \mathbb{R}^p$$

where we now have $\tilde{w}_i^\tau = \min(1, \frac{1}{2\tau} |\tilde{z}_i|) \text{sign}(\tilde{z}_i)$ with $\tilde{z}_i = y_i - \mathbf{x}_i^T \boldsymbol{\beta}$, $\forall i$. The Lipschitz constant of ∇g_θ^τ is still given by Theorem 4.

3.2 Thresholding operators

We now assume that g is a differentiable loss with C -Lipschitz continuous gradient. Following [23, 2], for $D \geq C$, we upper-bound g around any $\boldsymbol{\alpha} \in \mathbb{R}^p$ with the quadratic form $Q_D(\boldsymbol{\alpha}, \cdot)$ defined as:

$$g(\boldsymbol{\beta}) \leq Q_D(\boldsymbol{\alpha}, \boldsymbol{\beta}) = g(\boldsymbol{\alpha}) + \nabla g(\boldsymbol{\alpha})^T (\boldsymbol{\beta} - \boldsymbol{\alpha}) + \frac{D}{2} \|\boldsymbol{\beta} - \boldsymbol{\alpha}\|_2^2, \quad \boldsymbol{\beta} \in \mathbb{R}^p \quad (8)$$

The proximal gradient method approximates the solution of Problem (2) by solving the problem

$$\hat{\boldsymbol{\beta}} \in \underset{\boldsymbol{\beta}}{\text{argmin}} \{Q_D(\boldsymbol{\alpha}, \boldsymbol{\beta}) + \Omega(\boldsymbol{\beta})\} = \underset{\boldsymbol{\beta}}{\text{argmin}} \frac{1}{2} \left\| \boldsymbol{\beta} - \left(\boldsymbol{\alpha} - \frac{1}{D} \nabla Q_D(\boldsymbol{\alpha}) \right) \right\|_2^2 + \frac{1}{D} \Omega(\boldsymbol{\beta}). \quad (9)$$

Problem (9) can be solved via the the following proximal operator (evaluated at $\mu = \frac{1}{D}$):

$$\mathcal{S}_{\mu\Omega}(\boldsymbol{\eta}) := \underset{\boldsymbol{\beta} \in \mathbb{R}^p}{\text{argmin}} \frac{1}{2} \|\boldsymbol{\beta} - \boldsymbol{\eta}\|_2^2 + \mu\Omega(\boldsymbol{\beta}). \quad (10)$$

We discuss computation of (9) for the specific choices of Ω considered.

L1 regularization: When $\Omega(\boldsymbol{\beta}) = \lambda \|\boldsymbol{\beta}\|_1$, $\mathcal{S}_{\mu\Omega}(\boldsymbol{\eta})$ is available via componentwise softthresholding, where the soft-thresholding operator is: $\underset{u \in \mathbb{R}}{\text{argmin}} \frac{1}{2}(u - c)^2 + \mu\lambda|u| = \text{sign}(c)(|c| - \mu\lambda)_+$.

Slope regularization: When $\Omega(\boldsymbol{\beta}) = \sum_{j=1}^p \tilde{\lambda}_j |\beta_{(j)}|$ —where $\tilde{\lambda}_j = \eta \lambda_j$ —we note that, at an optimal solution to Problem (10), the signs of β_j and η_j are the same [7]. Consequently, we solve the following close relative to the isotonic regression problem [29]:

$$\min_{\mathbf{u}} \frac{1}{2} \|\mathbf{u} - \tilde{\boldsymbol{\eta}}\|_2^2 + \sum_{j=1}^p \mu \tilde{\lambda}_j u_j, \quad \text{s.t.} \quad u_1 \geq \dots \geq u_p \geq 0. \quad (11)$$

where, $\tilde{\boldsymbol{\eta}}$ is a decreasing rearrangement of the absolute values of $\boldsymbol{\eta}$. A solution \hat{u}_j of Problem (11) corresponds to $|\hat{\beta}_{(j)}|$, where $\hat{\boldsymbol{\beta}}$ is a solution of Problem (10). We use the software provided by [7] in our experiments.

Group L1-L2 regularization: For $\Omega(\boldsymbol{\beta}) = \lambda \sum_{g=1}^G \|\boldsymbol{\beta}_g\|_2$, we consider the projection operator onto an

L2-ball with radius $\mu\lambda$:

$$\tilde{\mathcal{S}}_{\frac{1}{\mu\lambda}\|\cdot\|_2}(\boldsymbol{\eta}) \in \underset{\boldsymbol{\beta}}{\operatorname{argmin}} \frac{1}{2} \|\boldsymbol{\beta} - \boldsymbol{\eta}\|_2^2 \quad \text{s.t.} \quad \frac{1}{\mu\lambda} \|\boldsymbol{\beta}\|_2 \leq 1. \quad (12)$$

From standard results pertaining to Moreau decomposition ([20, 1]) we have:

$$\mathcal{S}_{\mu\lambda\|\cdot\|_2}(\boldsymbol{\eta}) = \boldsymbol{\eta} - \tilde{\mathcal{S}}_{\frac{1}{\mu\lambda}\|\cdot\|_2}(\boldsymbol{\eta}) = \left(1 - \frac{\mu\lambda}{\|\boldsymbol{\eta}\|_2}\right)_+ \boldsymbol{\eta} \quad (13)$$

We solve Problem (10) with Group L1-L2 regularization by noticing the separability of the problem across the different groups, and computing $\mathcal{S}_{\mu\lambda\|\cdot\|_2}(\boldsymbol{\eta}_g)$ for every $g = 1, \dots, G$.

3.3 First order algorithm

Let us denote the mapping $\boldsymbol{\alpha} \mapsto \hat{\boldsymbol{\beta}}$ in (9) by the operator: $\hat{\boldsymbol{\beta}} := \Theta(\boldsymbol{\alpha})$. The standard version of the proximal gradient descent algorithm performs the updates: $\boldsymbol{\beta}_{t+1} = \Theta(\boldsymbol{\beta}_t)$ for $T \geq 1$. The accelerated gradient descent algorithm [2], which enjoys a faster convergence rate, performs updates with a minor modification. It starts with $\boldsymbol{\beta}_1 = \tilde{\boldsymbol{\beta}}_0$, $q_1 = 1$ and then performs the updates: $\tilde{\boldsymbol{\beta}}_{t+1} = \Theta(\boldsymbol{\beta}_t)$ where, $\boldsymbol{\beta}_{t+1} = \tilde{\boldsymbol{\beta}}_T + \frac{q_t-1}{q_{t+1}}(\tilde{\boldsymbol{\beta}}_t - \tilde{\boldsymbol{\beta}}_{t-1})$ and $q_{t+1} = (1 + \sqrt{1 + 4q_t^2})/2$. We perform these updates till some tolerance criterion is satisfied, or a maximum number of iterations is reached.

3.4 Simulations

We compare the sparse estimators studied herein with standard baselines when the signal is sparse or group-sparse. We consider the 3 examples below with an increasing number of variables up to 100,000s.

3.4.1 Example 1: sparse binary classification with hinge and logistic losses

Our first experiments compare L1 and Slope estimators with an L2 baseline for sparse binary classification problems. We use both the logistic and hinge losses. Our hypothesis for this case is that (i) the estimators performance will only be affected by the statistical difficulty of the problem, not by the choice of the loss function and (ii) sparse regularizations will outperform their non-sparse opponents.

Data Generation: We consider n samples from a multivariate Gaussian distribution with covariance matrix $\boldsymbol{\Sigma} = ((\sigma_{ij}))$ with $\sigma_{ij} = \rho$ if $i \neq j$ and $\sigma_{ij} = 1$ otherwise. Half of the samples are from the +1 class and have mean $\boldsymbol{\mu}_+ = (\boldsymbol{\delta}_{k^*}, \mathbf{0}_{p-k^*})$ where $\delta > 0$. A smaller δ makes the statistical setting more difficult since the two classes get closer. The other half are from the -1 class and have mean $\boldsymbol{\mu}_- = -\boldsymbol{\mu}_+$. We standardize the columns of the input matrix \mathbf{X} to have unit L2-norm.

Following our high-dimensional study, we set $p \gg n$ and consider a sequence of increasing values of p . We study the effect of making the problem statistically harder by making the classes closer. Hence we consider two settings, with a small and a large δ .

Competing methods: We compare 3 approaches for both the logistic loss and the hinge loss:

- Method **(a)** computes a family of L1 regularized estimators for a decreasing geometric sequence of regularization parameters $\eta_0 > \dots > \eta_M$. We start from $\eta_0 = \max_{j \in [p]} \sum_{i \in [n]} |x_{ij}|$ so that the solution of Problem (3) is $\mathbf{0}$ and we fix $\eta_M < 10^{-4}\eta_0$. When f is the logistic loss, we use the first order algorithm presented in Section 3.3. When f is the hinge loss, we directly solve the Linear Programming (LP) L1-SVM problem with the commercial LP solver GUROBI version 6.5 with Python interface. We present an LP reformulation of the L1-SVM problem in Appendix G.1.

- Method **(b)** computes a family of Slope regularized estimators, using the first order algorithm presented in Section 3.3. The Slope coefficients $\{\lambda_j\}$ are the ones defined in Theorem 3; the sequence of parameters $\{\eta_i\}$ is identical to method **(a)**. When f is the hinge-loss, we consider the smoothing method defined in Section 3.1 with a coefficient $\tau = 0.2$.
- Method **(c)** returns a family of L2 regularized estimators with SCIKIT-LEARN package: we start from $\eta_0 = \max_i \{\|\mathbf{x}_i\|_2^2\}$ as suggested in [12]—and $\eta_M < 10^{-4}\eta_0$.

3.4.2 Example 2: group-sparse binary classification with hinge loss

Our second example considers classification problems where sparsity is structured. We compare the performance of two coefficient regularizations with two group regularizations. Our hypothesis is that (i) group regularizations outperform their coefficient-based opponents (ii) the gap in performance increases with the statistical difficulty of the problem.

Data Generation: The p covariates are drawn from a multivariate Gaussian and divided into G groups of the same size g_* . Covariates have pairwise correlation of ρ within each group, and are uncorrelated across groups. Half of the samples are from the +1 class with mean $\boldsymbol{\mu}_+ = (\boldsymbol{\delta}_{g_*}, \dots, \boldsymbol{\delta}_{g_*}, \mathbf{0}_{g_*}, \dots, \mathbf{0}_{g_*})$ where s^* groups are relevant for classification; the remaining samples from class -1 have mean $\boldsymbol{\mu}_- = -\boldsymbol{\mu}_+$. The columns of the input matrix are standardized to have unit L2-norm. Similarly to Example 1, we consider a sequence of increasing values of p and study the effect of making the problem statistically harder by considering a small and a large δ .

Competing methods: We compare the L1 and Slope regularized methods **(a)** and **(b)** described above with the two following group regularizations:

- Method **(d)** computes a family of Group L1-L2 estimators with the first order algorithm presented in Section 3.3. We use the same sequence of regularization parameters as method **(a)**.
- Method **(e)** considers an alternative Group L1- L_∞ regularization [1]—discussed in Appendix G.2. We start from $\eta_0 = \max_{g \in [G]} \sum_{j \in \mathcal{I}_g} \sum_{i=1}^n |x_{ij}|$ and solve the LP formulation presented in the appendix with the GUROBI solver.

3.4.3 Example 3: sparse linear regression with heteroscedastic noise and quantile loss

Our last experiments compare L1 and Slope regularizations with quantile regression loss with the popular Lasso [31] and Ridge for regression settings. Our experiments draw inspiration from [33]: the authors showed the computational advantages of L1 regularized least-angle deviation (the quantile regression loss evaluated at $\theta = 1/2$) over Lasso for non-Gaussian regimes—the authors studied the noiseless and Cauchy noise cases. They additionally reported that the former is outperformed by Lasso for standard Gaussian linear regression. We consider herein a more challenging heteroscedastic regime—i.e. the noise is not identically distributed. Our hypothesis is that (i) L1 and Slope regularized quantile regression estimators perform similarly than Lasso (ii) Ridge is outperformed by all its sparse opponents.

Data Generation: We consider n samples from a multivariate Gaussian distribution with covariance matrix $\boldsymbol{\Sigma} = ((\sigma_{ij}))$ with $\sigma_{ij} = \rho^{|i-j|}$ if $i \neq j$ and $\sigma_{ij} = 1$ otherwise. The columns of \mathbf{X} are standardized to have unit L2-norm. Half of the noise observations are Gaussian and the rest is set to 0. That is, we generate $\mathbf{y} = \mathbf{X}\boldsymbol{\beta}^* + \boldsymbol{\epsilon}$ where $\epsilon_i \stackrel{\text{iid}}{\sim} N(0, \sigma^2)$ for $N/2$ randomly drawn indexes and $\epsilon_i = 0$ otherwise. We set $\boldsymbol{\beta}^* = (\boldsymbol{\delta}_{k^*}, \mathbf{0}_{p-k^*})$ and define the signal-to-noise (SNR) ratio of the problem as $\text{SNR} = \|\mathbf{X}\boldsymbol{\beta}^*\|_2^2 / \sigma^2$. A low SNR makes the problem statistically harder. Similarly to Examples 1 and 2, we consider two settings with a low and a large SNR.

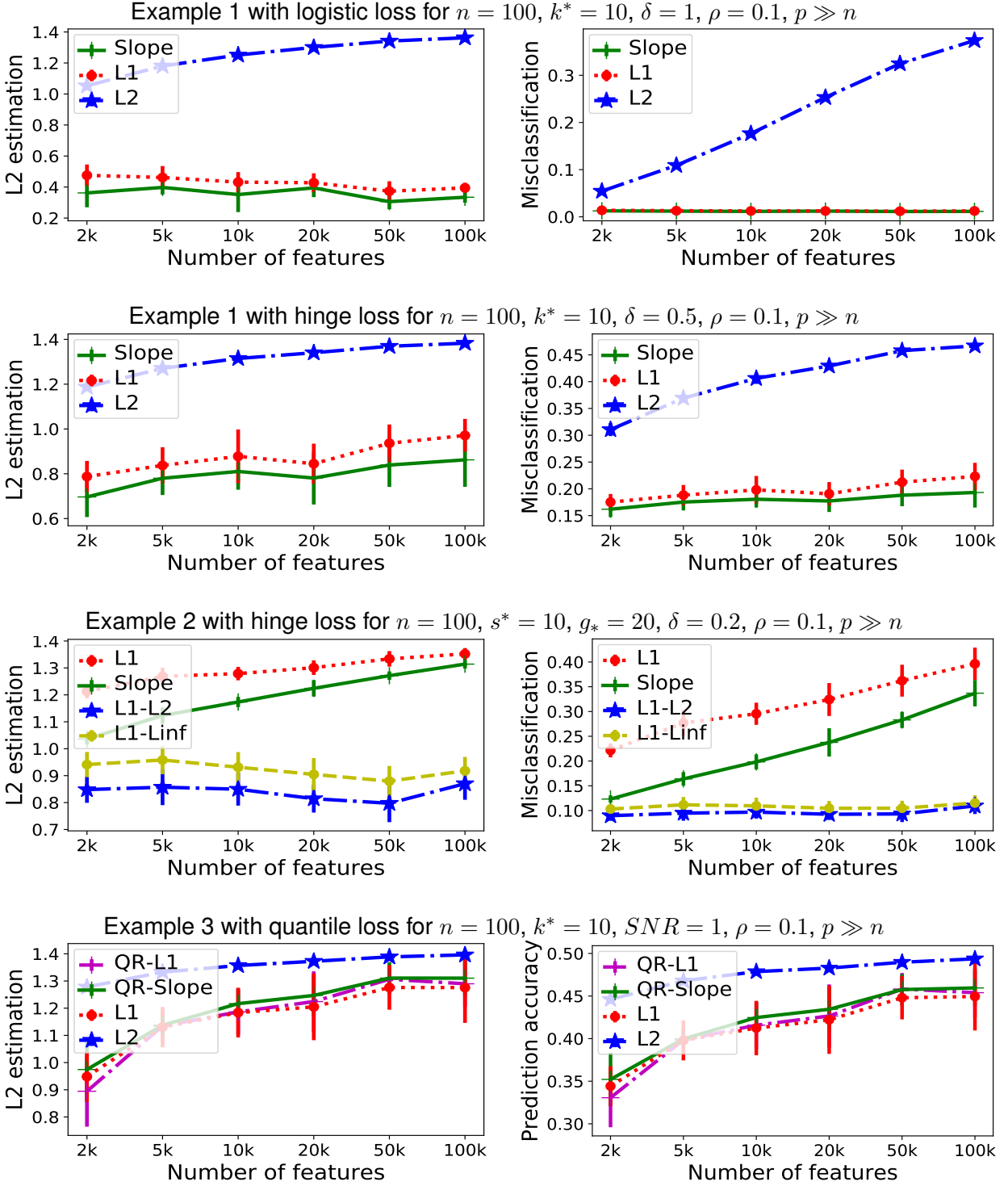


Figure 1: [Top and top middle panels] L1 and Slope outperform L2 and show impressive gains for estimating the theoretical minimizer β^* while achieving lower misclassification errors. Slope slightly performs better than L1 for the statistically hard problems studied. [Bottom middle panel] For small values of p , L1 and Slope compete with group regularizations. As p increases, group regularizations exhibit their statistical superiority and Group L1-L2 appears as the overall winner. [Bottom panel] L1 and Slope regularized quantile regression compete with Lasso in the heteroscedastic regression case, while outperforming Ridge for both L2 estimation and prediction accuracy.

Competing methods: We compare 4 approaches. We first consider L1 and Slope methods **(a)** and **(b)**—where we replace the hinge loss with the least-angle deviation loss. Note that in the case of L1 regularization, we directly solve the LP formulation presented in Appendix G.3. We additionally introduce methods **(e)** and **(f)**, which run Lasso and Ridge using the SCIKIT-LEARN package: we set $\eta_0 = \|\mathbf{X}^T \mathbf{y}\|_\infty$ for Lasso so that the Lasso estimator is $\mathbf{0}$; η_0 is set to be the highest eigenvalue of $\mathbf{X}^T \mathbf{X}$ for Ridge.

3.4.4 Metrics

Our theoretical results suggest to compare the estimators in terms of L2 estimation error $\left\| \frac{\hat{\beta}}{\|\hat{\beta}\|_2} - \frac{\beta^*}{\|\beta^*\|_2} \right\|_2$, where β^* is the theoretical minimizer. β^* is computed on a large test set with 10,000 samples restricted to the k^* columns relevant for classification: we use the loss considered and a very small regularization coefficient. We also report an additional metric, namely the test misclassification performance for classification experiments (Examples 1 and 2) and the prediction accuracy $\frac{1}{n} \|\mathbf{X}\hat{\beta} - \mathbf{X}\beta^*\|_2$ for regression experiments (Example 3). For a given method, we compute both test metrics for the estimator which achieves the lowest score for this additional metric on an independent validation set of size 10,000. Our findings are presented in Figure 1. We report the mean and standard deviations values of each test metrics averaged over 10 iterations.

3.4.5 Results

We derive three main learning from our experiments, which complement our theoretical findings:

- First, for sparse binary classification Example 1, our experiments show that L2 is outperformed by both L1 and Slope. The gap in performance does not depend upon the loss, and all three estimators are affected by the statistical difficulty of the problem. In particular, L2 performs close to random guess for $\delta = 0.5$ and $p > 20k$. Slope seems to achieve slightly better performance than L1 for both L2 estimation and misclassification for the statistical hard problems considered.
- Second, for group-sparse binary classification Example 2, our analysis reveals the computational advantage of group regularizations over L1 and Slope. Interestingly, Slope competes with its group opponents for the simpler statistical regime $\delta = 0.4$ case presented in Figure 2, Appendix H—and for the hard regime when $p < 5k$. However, it is significantly outperformed for hard problems with 10,000s of variable. In addition, Group L1-L2 regularization appears better than its L1- L_∞ opponent, which additionally cannot reach the bounds presented in this paper.
- Finally, for sparse linear regression with heteroscedastic noise Example 3, our findings show the good performance of L1 and Slope regularized quantile regression when the SNR is low. Both methods reach a similar L2 estimation error and prediction accuracy than Lasso and appear as a solid alternative for this heteroscedastic noise regime. Note that all three estimators reach the optimal minimax rate presented above. When the signal increases, Figure 2 (Appendix H) suggests that L1 quantile regression and Lasso still compete with each other, while Slope performance slightly decreases. For both small and large SNR, all sparse estimators significantly outperform Ridge for both L2 estimation and prediction accuracy.

References

- [1] Francis Bach, Rodolphe Jenatton, Julien Mairal, and Guillaume Obozinski. Convex optimization with sparsity-inducing norms. *Optimization for Machine Learning*, 5:19–53, 2011.
- [2] Amir Beck and Marc Teboulle. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM journal on imaging sciences*, 2(1):183–202, 2009.
- [3] Pierre C Bellec, Guillaume Lecué, Alexandre B Tsybakov, et al. Slope meets Lasso: improved oracle bounds and optimality. *The Annals of Statistics*, 46(6B):3603–3642, 2018.
- [4] Alexandre Belloni, Victor Chernozhukov, et al. L1-penalized quantile regression in high-dimensional sparse models. *The Annals of Statistics*, 39(1):82–130, 2011.
- [5] Peter J Bickel, Ya’acov Ritov, and Alexandre B Tsybakov. Simultaneous analysis of Lasso and Dantzig selector. *The Annals of Statistics*, pages 1705–1732, 2009.
- [6] Malgorzata Bogdan, Ewout van den Berg, Weijie Su, and Emmanuel Candes. Statistical estimation and testing via the sorted L1 norm. *arXiv preprint arXiv:1310.1969*, 2013.
- [7] Małgorzata Bogdan, Ewout van den Berg, Chiara Sabatti, Weijie Su, and Emmanuel J Candès. Slopeadaptive variable selection via convex optimization. *The annals of applied statistics*, 9(3):1103, 2015.
- [8] Moshe Buchinsky. Recent advances in quantile regression models: a practical guideline for empirical research. *Journal of human resources*, pages 88–126, 1998.
- [9] Florentina Bunea, Alexandre B Tsybakov, Marten H Wegkamp, et al. Aggregation for Gaussian regression. *The Annals of Statistics*, 35(4):1674–1697, 2007.
- [10] Emmanuel Candes and Mark A Davenport. How well can we estimate a sparse vector? *Applied and Computational Harmonic Analysis*, 34(2):317–323, 2013.
- [11] Emmanuel J Candes and Terence Tao. The Dantzig selector: statistical estimation when p is much larger than n . *The Annals of Statistics*, pages 2313–2351, 2007.
- [12] Bo-Yu Chu, Chia-Hua Ho, Cheng-Hao Tsai, Chieh-Yen Lin, and Chih-Jen Lin. Warm start for parameter selection of linear classifiers. In *Proceedings of the 21th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 149–158. ACM, 2015.
- [13] Antoine Dedieu. Error bounds for sparse classifiers in high-dimensions. *arXiv preprint arXiv:1810.03081*, 2018.
- [14] Antoine Dedieu and Rahul Mazumder. Solving large-scale l_1 -regularized svms and cousins: the surprising effectiveness of column and constraint generation. *arXiv preprint arXiv:1901.01585*, 2019.
- [15] Daniel Hsu, Sham Kakade, Tong Zhang, et al. A tail inequality for quadratic forms of subgaussian random vectors. *Electronic Communications in Probability*, 17, 2012.
- [16] Junzhou Huang and Tong Zhang. The benefit of group sparsity. *The Annals of Statistics*, 38(4):1978–2004, 2010.
- [17] Roger Koenker and Gilbert Bassett Jr. Regression quantiles. *Econometrica: journal of the Econometric Society*, pages 33–50, 1978.

- [18] Ja-Yong Koo, Yoonkyung Lee, Yuwon Kim, and Changyi Park. A Bahadur representation of the linear support vector machine. *Journal of Machine Learning Research*, 9(Jul):1343–1368, 2008.
- [19] Karim Lounici, Massimiliano Pontil, Sara Van De Geer, Alexandre B Tsybakov, et al. Oracle inequalities and optimal inference under group sparsity. *The Annals of Statistics*, 39(4):2164–2204, 2011.
- [20] Jean-Jacques Moreau. Dual convex functions and proximal points in a Hilbert space. *CR Acad. Sci. Paris Ser. At Math.*, 255:2897–2899.
- [21] Sahand Negahban, Bin Yu, Martin J Wainwright, and Pradeep K Ravikumar. A unified framework for high-dimensional analysis of M-estimators with decomposable regularizers. In *Advances in Neural Information Processing Systems*, pages 1348–1356, 2009.
- [22] Yu Nesterov. Smooth minimization of non-smooth functions. *Mathematical programming*, 103(1):127–152, 2005.
- [23] Yurii Nesterov. *Introductory Lectures on Convex Optimization: A Basic Course*. Kluwer, Norwell, 2004.
- [24] Bo Peng, Lan Wang, and Yichao Wu. An error bound for L1-norm support vector machine coefficients in ultra-high dimension. *Journal of Machine Learning Research*, 17:1–26, 2016.
- [25] Yaniv Plan and Roman Vershynin. Robust 1-bit compressed sensing and sparse logistic regression: A convex programming approach. *IEEE Transactions on Information Theory*, 59(1):482–494, 2013.
- [26] Garvesh Raskutti, Martin J Wainwright, and Bin Yu. Minimax rates of estimation for high-dimensional linear regression over l_q -balls. *IEEE transactions on information theory*, 57(10):6976–6994, 2011.
- [27] Pradeep Ravikumar, Martin J Wainwright, John D Lafferty, et al. High-dimensional ising model selection using L1-regularized logistic regression. *The Annals of Statistics*, 38(3):1287–1319, 2010.
- [28] Philippe Rigollet. 18.s997: High dimensional statistics. *Lecture Notes*, Cambridge, MA, USA: MIT OpenCourseWare, 2015.
- [29] Tim Robertson. *Order restricted statistical inference*. Wiley, New York., 1988.
- [30] Bernadetta Tarigan, Sara A Van De Geer, et al. Classifiers of support vector machine type with L1 complexity regularization. *Bernoulli*, 12(6):1045–1076, 2006.
- [31] Robert Tibshirani. Regression shrinkage and selection via the Lasso. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 267–288, 1996.
- [32] Sara A Van de Geer. High-dimensional generalized linear models and the Lasso. *The Annals of Statistics*, pages 614–645, 2008.
- [33] Lie Wang. The L1 penalized LAD estimator for high dimensional linear regression. *Journal of Multivariate Analysis*, 120:135–151, 2013.
- [34] Ming Yuan and Yi Lin. Model selection and estimation in regression with grouped variables. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 68(1):49–67, 2006.
- [35] Xiang Zhang, Yichao Wu, Lan Wang, and Runze Li. Variable selection for support vector machines in high dimensions.

Appendices

A Usefull properties of sub-Gaussian random variables

This section presents useful preliminary results satisfied by sub-Gaussian random variables. In particular, Lemma 4 provides a probabilistic upper-bound on the maximum of sub-Gaussian random variables.

A.1 Preliminary results

Under Assumption 3, the random variables $\partial f(\langle \mathbf{x}_i, \boldsymbol{\beta}^* \rangle; y_i) x_{ij}$, $\forall i, j$ are sub-Gaussian. They all consequently satisfy the next Lemma 1:

Lemma 1 *Let $Z \sim \text{subG}(\sigma^2)$ for a fixed $\sigma > 0$. Then for any $t > 0$ it holds*

$$\mathbb{E}(\exp(tZ)) \leq e^{4\sigma^2 t^2}.$$

In addition, for any positive integer $\ell \geq 1$ we have:

$$\mathbb{E}(|Z|^\ell) \leq (2\sigma^2)^{\ell/2} \ell \Gamma(\ell/2)$$

where Γ is the Gamma function defined as $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$, $\forall t > 0$.

Finally, let $Y = Z^2 - \mathbb{E}(Z^2)$ then we have

$$\mathbb{E}\left(\exp\left(\frac{1}{16\sigma^2} Y\right)\right) \leq \frac{3}{2}, \quad (14)$$

and as a consequence $\mathbb{E}(\exp(\frac{1}{16\sigma^2} Z^2)) \leq 2$.

Proof: The two first results correspond to Lemmas 1.4 and 1.5 from [28]. In particular $\mathbb{E}(|Z|^2) \leq 4\sigma^2$.

In addition, using the proof of Lemma 1.12 we have:

$$\mathbb{E}(\exp(tY)) \leq 1 + 128t^2\sigma^4, \quad \forall |t| \leq \frac{1}{16\sigma^2}.$$

Equation (14) holds in the particular case where $t = 1/16\sigma^2$. The last part of the lemma combines our precedent results with the observation that $\frac{3}{2}e^{1/4} \leq 2$. \square

A.2 Lemma 2

The next Lemma 2 proved below is a first consequence of Lemma 1.

Lemma 2 *Let $\{(\mathbf{x}_i, y_i)\}_{i=1}^n$, $(\mathbf{x}_i, y_i) \in \mathbb{R}^p \times \mathcal{Y}$ be independent samples from an unknown distribution. Let f be a loss satisfying Assumption 1 and $\boldsymbol{\beta}^*$ be a theoretical minimizer of f . If Assumption 3 is satisfied, then it holds*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \partial f(\langle \mathbf{x}_i, \boldsymbol{\beta}^* \rangle; y_i) x_{ij} \sim \text{subG}(8L^2 M^2), \quad \forall j.$$

Proof: We note $S_i = \partial f(\langle \mathbf{x}_i, \boldsymbol{\beta}^* \rangle, y_i)$, $\forall i$. Since $\boldsymbol{\beta}^*$ minimizes the theoretical loss, we have $\mathbb{E}(S_i x_{ij}) = 0$, $\forall i, j$. We fix $M > 0$ such that: $\forall t > 0$,

$$\mathbb{P}(|S_i x_{ij}| > t) \leq 2 \exp\left(-\frac{t^2}{2L^2 M^2}\right), \forall i, j.$$

Then from Lemma 1 it holds:

$$\mathbb{E}(\exp(t S_i x_{ij})) \leq e^{4L^2 M^2 t^2}, \forall t > 0, \forall i, j.$$

Since the samples are independent, it holds $\forall t > 0$,

$$\mathbb{E}\left(\exp\left(\frac{t}{\sqrt{n}} \sum_{i=1}^n S_i x_{ij}\right)\right) = \prod_{i=1}^n \mathbb{E}\left(\exp\left(\frac{t}{\sqrt{n}} S_i x_{ij}\right)\right) \leq \prod_{i=1}^n e^{4L^2 M^2 t^2/n} = e^{4L^2 M^2 t^2}.$$

Let $M_1 = 2\sqrt{2}M$, then with a Chernoff bound:

$$\mathbb{P}\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n S_i x_{ij} > t\right) \leq \min_{s>0} \exp\left(\frac{M_1^2 L^2 s^2}{2} - st\right) = \exp\left(-\frac{t^2}{2L^2 M_1^2}\right), \forall t > 0,$$

which concludes the proof. \square

A.3 A bound for the maximum of sub-Gaussian variables

As a second consequence of Lemma 1, the next two technical lemmas derive a probabilistic upper-bound for the maximum of sub-Gaussian random variables. Lemma 3 is an extension for sub-Gaussian random variables of Proposition E.1 [3].

Lemma 3 *Let g_1, \dots, g_r be sub-Gaussian random variables with variance σ^2 . Denote by $(g_{(1)}, \dots, g_{(r)})$ a non-increasing rearrangement of $(|g_1|, \dots, |g_r|)$. Then $\forall t > 0$ and $\forall j \in \{1, \dots, r\}$:*

$$\mathbb{P}\left(\frac{1}{j\sigma^2} \sum_{k=1}^j g_{(k)}^2 > t \log\left(\frac{2r}{j}\right)\right) \leq \left(\frac{2r}{j}\right)^{1-\frac{t}{16}}.$$

Proof: We first apply a Chernoff bound:

$$\mathbb{P}\left(\frac{1}{j\sigma^2} \sum_{k=1}^j g_{(k)}^2 > t \log\left(\frac{2r}{j}\right)\right) \leq \mathbb{E}\left(\exp\left(\frac{1}{16j\sigma^2} \sum_{k=1}^j g_{(k)}^2\right)\right) \left(\frac{2r}{j}\right)^{-\frac{t}{16}}.$$

We then use Jensen inequality to obtain

$$\begin{aligned} \mathbb{E}\left(\exp\left(\frac{1}{16j\sigma^2} \sum_{k=1}^j g_{(k)}^2\right)\right) &\leq \frac{1}{j} \sum_{k=1}^j \mathbb{E}\left(\exp\left(\frac{1}{16\sigma^2} g_{(k)}^2\right)\right) \leq \frac{1}{j} \sum_{k=1}^r \mathbb{E}\left(\exp\left(\frac{1}{16\sigma^2} g_k^2\right)\right) \\ &\leq \frac{2r}{j} \text{ with Lemma 1.} \end{aligned}$$

\square

Using Lemma 3, we can derive the following bound holding with high probability:

Lemma 4 We consider the same assumptions and notations than Lemma 3. In addition, we define the coefficients $\lambda_j^{(r)} = \sqrt{\log(2r/j)}$, $j = 1, \dots, r$ similarly to Theorem 1. Then for $\delta \in (0, \frac{1}{2})$, it holds with probability at least $1 - \delta$:

$$\sup_{j=1, \dots, r} \left\{ \frac{g_{(j)}}{\sigma \lambda_j^{(r)}} \right\} \leq 12\sqrt{\log(1/\delta)}.$$

Proof: We fix $\delta \in (0, \frac{1}{2})$ and $j \in \{1, \dots, r\}$. We upper-bound $g_{(j)}^2$ by the average of all larger variables:

$$g_{(j)}^2 \leq \frac{1}{j} \sum_{k=1}^j g_{(k)}^2.$$

Applying Lemma 3 gives, for $t > 0$:

$$\mathbb{P} \left(\frac{g_{(j)}^2}{\sigma^2 (\lambda_j^{(r)})^2} > t \right) \leq \mathbb{P} \left(\frac{1}{j\sigma^2} \sum_{k=1}^j g_{(k)}^2 > t (\lambda_j^{(r)})^2 \right) \leq \left(\frac{j}{2r} \right)^{\frac{t}{16} - 1}.$$

We fix $t = 144 \log(1/\delta)$ and use an union bound to get:

$$\mathbb{P} \left(\sup_{j=1, \dots, p} \frac{g_{(j)}}{\sigma \lambda_j^{(r)}} > 12\sqrt{\log(1/\delta)} \right) \leq \left(\frac{1}{2p} \right)^{9 \log(1/\delta) - 1} \sum_{j=1}^r j^{9 \log(1/\delta) - 1}.$$

Since $\delta < \frac{1}{2}$ it holds that $9 \log(1/\delta) - 1 \geq 9 \log(2) - 1 > 0$, then the map $t > 0 \mapsto t^{9 \log(1/\delta) - 1}$ is increasing. An integral comparison gives:

$$\sum_{j=1}^r j^{9 \log(1/\delta) - 1} \leq \frac{1}{2} (r+1)^{9 \log(1/\delta)} = \frac{1}{2} \delta^{-9 \log(r+1)}.$$

In addition $9 \log(1/\delta) - 1 \geq 7 \log(1/\delta)$ and

$$\left(\frac{1}{2r} \right)^{9 \log(1/\delta) - 1} \leq \left(\frac{1}{2r} \right)^{-7 \log(\delta)} = \delta^{7 \log(2r)}.$$

Finally, by assuming $r \geq 2$, then we have $7 \log(2r) - 9 \log(r+1) > 1$ and we conclude:

$$\mathbb{P} \left(\sup_{j=1, \dots, r} \frac{g_{(j)}}{\sigma \lambda_j^{(r)}} > 12\sqrt{\log(1/\delta)} \right) \leq \delta.$$

□

B Proof of Theorem 1

We use the minimality of $\hat{\beta}$ and Lemma 3 to derive the cone conditions.

Proof: We first consider a general solution of Problem (2) with regularization $\Omega(\cdot)$ before specifying the cases of L1, Slope and Group L1-L2 regularizations.

$\hat{\beta}$ is the solution of the learning Problem (2) hence:

$$\frac{1}{n} \sum_{i=1}^n f(\langle \mathbf{x}_i, \hat{\beta} \rangle; y_i) + \Omega(\hat{\beta}) \leq \frac{1}{n} \sum_{i=1}^n f(\langle \mathbf{x}_i, \beta^* \rangle; y_i) + \Omega(\beta^*). \quad (15)$$

Similarly to Theorem 5, we define $\Delta(\beta^*, \mathbf{h}) = \frac{1}{n} \sum_{i=1}^n f(\langle \mathbf{x}_i, \hat{\beta} \rangle; y_i) - \frac{1}{n} \sum_{i=1}^n f(\langle \mathbf{x}_i, \beta^* \rangle; y_i)$.

Equation (15) can be written in a compact form as:

$$\Delta(\beta^*, \mathbf{h}) \leq \Omega(\beta^*) - \Omega(\hat{\beta}).$$

We lower bound $\Delta(\beta^*, \mathbf{h})$ by exploiting the existence of a bounded sub-Gradient ∂f :

$$\Delta(\beta^*, \mathbf{h}) \geq S(\beta^*, \mathbf{h}) := \frac{1}{n} \sum_{i=1}^n \partial f(\langle \mathbf{x}_i, \beta^* \rangle; y_i) \langle \mathbf{x}_i, \mathbf{h} \rangle.$$

We now consider each regularization separately.

L1 regularization: For L1 regularization, we have:

$$\begin{aligned} |S(\beta^*, \mathbf{h})| &= \left| \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^p \partial f(\langle \mathbf{x}_i, \beta^* \rangle; y_i) x_{ij} h_j \right| \\ &\leq \frac{1}{\sqrt{n}} \sum_{j=1}^p \left(\frac{1}{\sqrt{n}} \left| \sum_{i=1}^n \partial f(\langle \mathbf{x}_i, \beta^* \rangle; y_i) x_{ij} \right| \right) |h_j|. \end{aligned}$$

Let us define the random variables $g_j = \frac{1}{\sqrt{n}} \sum_{i=1}^n \partial f(\langle \mathbf{x}_i, \beta^* \rangle; y_i) x_{ij}$, $j = 1, \dots, p$.

Under Assumption 3, Lemma 2 in Appendix A.2 guarantees that g_1, \dots, g_p are sub-Gaussian with variance $8L^2M^2$. A first upper-bound of the quantity $|S(\beta^*, \mathbf{h})|$ could be obtained by considering the maximum of the sequence $\{g_j\}$. However, Lemma 4 gives us a stronger result. We note $\lambda_j = \lambda_j^{(p)}$ where we drop the dependency upon p .

Since $\delta \leq 1$ we introduce a non-increasing rearrangement $(g_{(1)}, \dots, g_{(p)})$ of $(|g_1|, \dots, |g_p|)$. We recall that $S_0 = \{1, \dots, k^*\}$ denotes the subset of indexes of the k^* highest elements of \mathbf{h} and we use Lemma 4 to get, with probability at least $1 - \frac{\delta}{2}$:

$$\begin{aligned} |S(\beta^*, \mathbf{h})| &\leq \frac{1}{\sqrt{n}} \sum_{j=1}^p |g_j| |h_j| = \frac{1}{\sqrt{n}} \sum_{j=1}^p g_{(j)} |h_{(j)}| = \frac{1}{\sqrt{n}} \sum_{j=1}^p \frac{g_{(j)}}{2\sqrt{2}LM\lambda_j} 2\sqrt{2}LM\lambda_j |h_{(j)}| \\ &\leq \frac{1}{\sqrt{n}} \sup_{j=1, \dots, p} \left\{ \frac{g_{(j)}}{2\sqrt{2}LM\lambda_j} \right\} \sum_{j=1}^p 2\sqrt{2}LM\lambda_j |h_{(j)}| \\ &\leq 24\sqrt{2}LM \sqrt{\frac{\log(2/\delta)}{n}} \sum_{j=1}^p \lambda_j |h_{(j)}| \text{ with Lemma 4} \\ &\leq 34LM \sqrt{\frac{\log(2/\delta)}{n}} \sum_{j=1}^p \lambda_j |h_j| \text{ since } \lambda_1 \geq \dots \geq \lambda_p \text{ and } |h_1| \geq \dots \geq |h_p| \end{aligned} \quad (16)$$

To conclude, by pairing Equations (15) and (16) it holds:

$$-34LM\sqrt{\frac{\log(2/\delta)}{n}}\sum_{j=1}^p\lambda_j|h_j|\leq\lambda\|\beta^*\|_1-\lambda\|\hat{\beta}\|_1. \quad (17)$$

We refer to $A = -34LM\sqrt{\frac{\log(2/\delta)}{n}}\sum_{j=1}^p\lambda_j|h_j|$ and $B = \lambda\|\beta^*\|_1 - \lambda\|\hat{\beta}\|_1$ as the respective left and right-hand sides of Equation (17).

We assume without loss of generality that $|h_1| \geq \dots \geq |h_p|$. We define $S_0 = \{1, \dots, k^*\}$ as the set of the k^* highest coefficients of $\mathbf{h} = \hat{\beta} - \beta^*$. Let S^* be the support of β^* . By definition of S_0 it holds:

$$\begin{aligned} B &\leq \lambda\|\beta_{S^*}^*\|_1 - \lambda\|\hat{\beta}_{S^*}\|_1 - \lambda\|\hat{\beta}_{(S^*)^c}\|_1 \leq \lambda\|\mathbf{h}_{S^*}\|_1 - \lambda\|\mathbf{h}_{(S^*)^c}\|_1 \\ &\leq \lambda\|\mathbf{h}_{S_0}\|_1 - \lambda\|\mathbf{h}_{(S_0)^c}\|_1. \end{aligned} \quad (18)$$

In addition, we lower bound the left-hand side of Equation (17) by:

$$-A \leq 34LM\sqrt{\frac{\log(2/\delta)}{n}}\left(\sum_{j=1}^{k^*}\lambda_j|h_j| + \lambda_{k^*}\|\mathbf{h}_{(S_0)^c}\|_1\right). \quad (19)$$

Cauchy-Schwartz inequality leads to:

$$\sum_{j=1}^{k^*}\lambda_j|h_j| \leq \sqrt{\sum_{j=1}^{k^*}\lambda_j^2}\|\mathbf{h}_{S_0}\|_2 \leq \sqrt{k^*\log(2pe/k^*)}\|\mathbf{h}_{S_0}\|_2,$$

where we have used the Stirling formula to obtain

$$\begin{aligned} \sum_{j=1}^{k^*}\lambda_j^2 &= \sum_{j=1}^{k^*}\log(2p/j) = k^*\log(2p) - \log(k^*) \\ &\leq k^*\log(2p) - k^*\log(k^*/e) = k^*\log(2pe/k^*). \end{aligned}$$

In the statement of Theorem 1 we have defined $\lambda = 34\alpha LM\sqrt{n^{-1}\log(2pe/k^*)\log(2/\delta)}$.

Because $\lambda_{k^*} \leq \sqrt{\log(2pe/k^*)}$, Equation (19) leads to:

$$-A \leq \frac{1}{\alpha}\lambda\left(\sqrt{k^*}\|\mathbf{h}_{S_0}\|_2 + \|\mathbf{h}_{(S_0)^c}\|_1\right)$$

Combined with Equation (18), it holds with probability at least $1 - \frac{\delta}{2}$:

$$-\frac{\lambda}{\alpha}\left(\sqrt{k^*}\|\mathbf{h}_{S_0}\|_2 + \|\mathbf{h}_{(S_0)^c}\|_1\right) \leq \lambda\|\mathbf{h}_{S_0}\|_1 - \lambda\|\mathbf{h}_{(S_0)^c}\|_1,$$

which immediately leads to:

$$\|\mathbf{h}_{(S_0)^c}\|_1 \leq \frac{\alpha}{\alpha-1}\|\mathbf{h}_{S_0}\|_1 + \frac{\sqrt{k^*}}{\alpha-1}\|\mathbf{h}_{S_0}\|_2.$$

We conclude that $\mathbf{h} \in \Lambda\left(S_0, \frac{\alpha}{\alpha-1}, \frac{\sqrt{k^*}}{\alpha-1}\right)$ with probability at least $1 - \frac{\delta}{2}$.

Slope regularization: For the Slope regularization, Equation (17) still holds and the quantity A is still defined. We define B by replacing the L1 regularization with Slope. We still assume $|h_1| \geq \dots \geq |h_p|$. To upper-bound B , we define a permutation $\phi \in \mathcal{S}_p$ such that $|\beta^*|_S = \sum_{j=1}^{k^*} \lambda_j |\beta_{\phi(j)}^*|$ and $|\hat{\beta}_{\phi(k^*+1)}| \geq \dots \geq |\hat{\beta}_{\phi(p)}|$. It holds:

$$\begin{aligned}
\frac{1}{\eta} B &= \frac{1}{\eta} \|\beta^*\|_S - \frac{1}{\eta} \|\hat{\beta}\|_S = \sum_{j=1}^{k^*} \lambda_j |\beta_{\phi(j)}^*| - \max_{\psi \in \mathcal{S}_p} \sum_{j=1}^p \lambda_j |\hat{\beta}_{\psi(j)}| \\
&\leq \sum_{j=1}^{k^*} \lambda_j |\beta_{\phi(j)}^*| - \sum_{j=1}^p \lambda_j |\hat{\beta}_{\phi(j)}| \\
&\leq \sum_{j=1}^{k^*} \lambda_j \left(|\beta_{\phi(j)}^*| - |\hat{\beta}_{\phi(j)}| \right) - \sum_{j=k^*+1}^p \lambda_j |\hat{\beta}_{\phi(j)}| \\
&= \sum_{j=1}^{k^*} \lambda_j |h_{\phi(j)}| - \sum_{j=k^*+1}^p \lambda_j |h_{\phi(j)}|.
\end{aligned} \tag{20}$$

Since $\{\lambda_j\}$ is monotonically non decreasing: $\sum_{j=1}^{k^*} \lambda_j |h_{\phi(j)}| \leq \sum_{j=1}^{k^*} \lambda_j |h_j|$. Because $|h_{\phi(k^*+1)}| \geq \dots \geq |h_{\phi(p)}|$: $\sum_{j=k^*+1}^p \lambda_j |h_j| \leq \sum_{j=k^*+1}^p \lambda_j |h_{\phi(j)}|$. It consequently holds:

$$\frac{1}{\eta} B \leq \sum_{j=1}^{k^*} \lambda_j |h_j| - \sum_{j=k^*+1}^p \lambda_j |h_j| \tag{21}$$

In addition, since $\eta = 34\alpha LM \sqrt{\frac{\log(2/\delta)}{n}}$, we obtain with probability at least $1 - \frac{\delta}{2}$:

$$A = -34LM \sqrt{\frac{\log(2/\delta)}{n}} \sum_{j=1}^p \lambda_j |h_j| = -\frac{\eta}{\alpha} \|\mathbf{h}\|_S.$$

Thus, combining this last equation with Equation (21), it holds with probability at least $1 - \frac{\delta}{2}$:

$$-\frac{1}{\alpha} \|\mathbf{h}\|_S \leq \sum_{j=1}^{k^*} \lambda_j |h_j| - \sum_{j=k^*+1}^p \lambda_j |h_j|,$$

which is equivalent to saying that with probability at least $1 - \frac{\delta}{2}$:

$$\sum_{j=k^*+1}^p \lambda_j |h_j| \leq \frac{\alpha + 1}{\alpha - 1} \sum_{j=1}^{k^*} \lambda_j |h_j|, \tag{22}$$

that is $\mathbf{h} \in \Gamma \left(k^*, \frac{\alpha+1}{\alpha-1} \right)$.

Group L1-L2 regularization: For Group L1-L2 regularization we also introduce the vector of sub-Gaussian random variables $\mathbf{g} = (g_1, \dots, g_p)$ with $g_j = \frac{1}{\sqrt{n}} \sum_{i=1}^n \partial f(\langle \mathbf{x}_i, \boldsymbol{\beta}^* \rangle; y_i) x_{ij}, \forall j$. We then have:

$$|S(\boldsymbol{\beta}^*, \mathbf{h})| = \frac{1}{\sqrt{n}} |\langle \mathbf{g}, \mathbf{h} \rangle| \leq \frac{1}{\sqrt{n}} \sum_{g=1}^G |\langle \mathbf{g}_g, \mathbf{h}_g \rangle| \leq \frac{1}{\sqrt{n}} \sum_{g=1}^G \|\mathbf{g}_g\|_2 \|\mathbf{h}_g\|_2, \quad (23)$$

where we have used Cauchy-Schwartz inequality on each group.

We have denoted n_g the cardinality of the set of indexes \mathcal{I}_g of group g and $\mathbf{n} = (n_1, \dots, n_G)$. We aim at using Theorem 2.1 from [15]. To this end, we first need to show that $\forall g, \forall \mathbf{u}_g \in \mathbb{R}^{n_g}$ it holds:

$$\mathbb{E}(\exp(\mathbf{g}_g^T \mathbf{u}_g)) \leq \exp(4L^2 M^2 \|\mathbf{u}_g\|_2^2)$$

Let us fix $g \leq G, \mathbf{u}_g \in \mathbb{R}^{n_g}$. Assumption 3 guarantees that:

$$\mathbb{P}(|\partial f(\langle \mathbf{x}_i, \boldsymbol{\beta}^* \rangle, y_i) (\mathbf{x}_i)_g^T \mathbf{u}_g| > t) \leq 2 \exp\left(-\frac{t^2}{2L^2 M^2 \|\mathbf{u}_g\|_2^2}\right), \quad \forall t > 0.$$

In addition since $\boldsymbol{\beta}^*$ minimizes the theoretical loss, we have $\mathbb{E}(\partial f(\langle \mathbf{x}_i, \boldsymbol{\beta}^* \rangle, y_i) (\mathbf{x}_i)_g) = 0, \forall i$. Consequently, the random variables $\partial f(\langle \mathbf{x}_i, \boldsymbol{\beta}^* \rangle, y_i) (\mathbf{x}_i)_g^T \mathbf{u}_g$ are sub-Gaussian with variance $L^2 M^2 \|\mathbf{u}_g\|_2^2, \forall i$.

Hence from Lemma 1 it holds:

$$\mathbb{E}(\exp(t \partial f(\langle \mathbf{x}_i, \boldsymbol{\beta}^* \rangle, y_i) (\mathbf{x}_i)_g^T \mathbf{u}_g)) \leq \exp(4L^2 M^2 t^2 \|\mathbf{u}_g\|_2^2) \quad \forall t > 0, \forall i.$$

As a consequence, since the rows of the design matrix are independent, it holds:

$$\begin{aligned} \mathbb{E}(\exp(\mathbf{g}_g^T \mathbf{u}_g)) &= \mathbb{E}\left(\exp\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \partial f(\langle \mathbf{x}_i, \boldsymbol{\beta}^* \rangle, y_i) (\mathbf{x}_i)_g^T \mathbf{u}_g\right)\right) \\ &= \prod_{i=1}^n \mathbb{E}\left(\exp\left(\frac{1}{\sqrt{n}} \partial f(\langle \mathbf{x}_i, \boldsymbol{\beta}^* \rangle, y_i) (\mathbf{x}_i)_g^T \mathbf{u}_g\right)\right) \\ &\leq \prod_{i=1}^n \exp\left(\frac{4L^2 M^2 \|\mathbf{u}_g\|_2^2}{n}\right) \\ &= \exp(4L^2 M^2 \|\mathbf{u}_g\|_2^2). \end{aligned} \quad (24)$$

We can then use Theorem 2.1 from [15]. By denoting \mathbf{I}_g the identity matrix of size n_g it holds:

$$\mathbb{P}\left(\|\mathbf{I}_g \mathbf{g}_g\|_2^2 \geq 8L^2 M^2 \left(\text{tr}(\mathbf{I}_g) + 2\sqrt{\text{tr}(\mathbf{I}_g^2)t} + 2\|\mathbf{I}_g\|\right)\right) \leq e^{-t},$$

which can be equivalently expressed as

$$\mathbb{P}\left(\|\mathbf{g}_g\|_2 - 2\sqrt{2}LM\sqrt{n_g} \geq 4LM\sqrt{t}\right) = \mathbb{P}\left(\|\mathbf{g}_g\|_2^2 \geq 8L^2 M^2 \left(\sqrt{n_g} + \sqrt{2t}\right)^2\right) \leq e^{-t},$$

which is equivalent from saying that:

$$\mathbb{P}\left(\|\mathbf{g}_g\|_2^2 - 2\sqrt{2}LM\sqrt{n_g} \geq t\right) \leq \exp\left(\frac{-t^2}{16L^2 M^2}\right). \quad (25)$$

Let us define the random variables $f_g = \max(0, \|\mathbf{g}_g\|_2 - 2\sqrt{2}LM\sqrt{n_g})$, $g = 1, \dots, G$. Equation (25) shows that f_g satisfies the same tail condition than a sub-Gaussian random variable with variance $8L^2M^2$ and we can apply Lemma 4. In addition, following Equation (23) it holds:

$$|S(\boldsymbol{\beta}^*, \mathbf{h})| \leq \frac{1}{\sqrt{n}} \sum_{g=1}^G \left(\|\mathbf{g}_g\|_2 - 2\sqrt{2}LM\sqrt{n_g} \right) \|\mathbf{h}_g\|_2 + \frac{1}{\sqrt{n}} \sum_{g=1}^G 2\sqrt{2}LM\sqrt{n_g} \|\mathbf{h}_g\|_2.$$

We introduce a non-increasing rearrangement $(f_{(1)}, \dots, f_{(G)})$ of $(|f_1|, \dots, |f_G|)$. In addition, we assume without loss of generality that $\|\mathbf{h}_1\|_2 \geq \dots \geq \|\mathbf{h}_G\|_2$. We have defined $\mathcal{J}_0 = \{1, \dots, s^*\}$ as the subset of indexes of the s^* groups of \mathbf{h} with highest L2 norm. We note a permutation ψ such that $n_{\psi(1)} \geq \dots \geq n_{\psi(G)}$. Similarly than above, Lemma 4 gives with probability at least $1 - \frac{\delta}{2}$ —we use the coefficients $\lambda_g^{(G)} = \sqrt{\log(2Ge/g)}$:

$$\begin{aligned} |S(\boldsymbol{\beta}^*, \mathbf{h})| &\leq \frac{1}{\sqrt{n}} \sum_{g=1}^G \left(\|\mathbf{g}_g\|_2 - 2\sqrt{2}LM\sqrt{n_g} \right) \|\mathbf{h}_g\|_2 + \frac{2\sqrt{2}LM}{\sqrt{n}} \sum_{g=1}^G \sqrt{n_g} \|\mathbf{h}_g\|_2 \\ &\leq \frac{1}{\sqrt{n}} \sum_{g=1}^G |f_g| \|\mathbf{h}_g\|_2 + \frac{2\sqrt{2}LM}{\sqrt{n}} \sum_{g=1}^G \sqrt{n_g} \|\mathbf{h}_g\|_2 \\ &= \frac{1}{\sqrt{n}} \sum_{g=1}^G \frac{f_{(g)}}{2\sqrt{2}LM\lambda_g^{(G)}} 2\sqrt{2}LM\lambda_g^{(G)} \|\mathbf{h}_{(g)}\|_2 + \frac{2\sqrt{2}LM}{\sqrt{n}} \sum_{g=1}^G \sqrt{n_g} \|\mathbf{h}_g\|_2 \\ &\leq 24\sqrt{2}LM \sqrt{\frac{\log(2/\delta)}{n}} \sum_{g=1}^G \lambda_g^{(G)} \|\mathbf{h}_{(g)}\|_2 + \frac{2\sqrt{2}LM}{\sqrt{n}} \sum_{g=1}^G \sqrt{n_g} \|\mathbf{h}_g\|_2 \\ &\leq 34LM \sqrt{\frac{\log(2/\delta)}{n}} \sum_{g=1}^G \lambda_g^{(G)} \|\mathbf{h}_g\|_2 + \frac{4LM}{\sqrt{n}} \sum_{g=1}^G \sqrt{n_{\psi(g)}} \|\mathbf{h}_g\|_2 \\ &\quad \text{since } \lambda_1^{(G)} \geq \dots \geq \lambda_G^{(G)}, \|\mathbf{h}_1\|_2 \geq \dots \geq \|\mathbf{h}_G\|_2 \text{ and } n_{\psi(1)} \geq \dots \geq n_{\psi(G)} \\ &\leq 34LM \sqrt{\frac{\log(2/\delta)}{n}} \left(\sqrt{s^* \log(2Ge/s^*)} \left(\sum_{g \in \mathcal{J}_0} \|\mathbf{h}_g\|_2^2 \right)^{1/2} + \lambda_{s^*}^{(G)} \sum_{g \notin \mathcal{J}_0} \|\mathbf{h}_g\|_2 \right) \\ &\quad + \frac{4LM}{\sqrt{n}} \left(\left(\sum_{g \in \mathcal{J}_0} n_{\psi(g)} \right)^{1/2} \left(\sum_{g \in \mathcal{J}_0} \|\mathbf{h}_g\|_2^2 \right)^{1/2} + \max_{g=s^*+1, \dots, G} \sqrt{n_{\psi(g)}} \sum_{g \notin \mathcal{J}_0} \|\mathbf{h}_g\|_2 \right) \\ &\leq 34LM \sqrt{\frac{\log(2Ge/s^*)}{n} \log(2/\delta)} \left(\sqrt{s^*} \|\mathbf{h}_{\mathcal{T}_0}\|_2 + \sum_{g \notin \mathcal{J}_0} \|\mathbf{h}_g\|_2 \right) \\ &\quad + \frac{4LM}{\sqrt{n}} \left(\sqrt{m_0} \|\mathbf{h}_{\mathcal{T}_0}\|_2 + \sqrt{\frac{m_0}{s^*}} \sum_{g \notin \mathcal{J}_0} \|\mathbf{h}_g\|_2 \right) \\ &= \left(34LM \sqrt{\frac{\log(2Ge/s^*)}{n} \log(2/\delta)} + 4LM \sqrt{\gamma \frac{m^*/s^*}{n}} \right) \left(\sqrt{s^*} \|\mathbf{h}_{\mathcal{T}_0}\|_2 + \sum_{g \notin \mathcal{J}_0} \|\mathbf{h}_g\|_2 \right), \end{aligned} \tag{26}$$

where $\mathcal{T}_0 = \cup_{g \in \mathcal{J}_0} \mathcal{I}_g$ has been defined as the subset of all indexes across all the s^* groups in \mathcal{J}_0 , m_0 is the total size of the s^* largest groups, and Cauchy-Schwartz inequality gives: $\sum_{g=1}^{s^*} \left(\lambda_g^{(G)} \right)^2 \leq s^* \log(2Ge/s^*)$.

We have defined $\lambda_G = 34\alpha LM \sqrt{n^{-1} \log(2Ge/s^*) \log(2/\delta)} + 4\alpha LM \sqrt{\gamma(s^*n)^{-1} m^*}$ and $\mathcal{J}^* \subset \{1, \dots, G\}$ as the smallest subset of group indexes such that the support of β^* is included in the union of these groups. By pairing Equations (15) and (26) it holds:

$$\begin{aligned} -\frac{\lambda_G}{\alpha} \left(\sqrt{s^*} \|\mathbf{h}_{\mathcal{T}_0}\|_2 + \sum_{g \notin \mathcal{J}_0} \|\mathbf{h}_g\|_2 \right) &\leq \lambda_G \sum_{g \in \mathcal{J}^*} \|\mathbf{h}_g\|_2 - \lambda_G \sum_{g \notin \mathcal{J}^*} \|\mathbf{h}_g\|_2 \\ &\leq \lambda_G \sum_{g \in \mathcal{J}_0} \|\mathbf{h}_g\|_2 - \lambda_G \sum_{g \notin \mathcal{J}_0} \|\mathbf{h}_g\|_2, \end{aligned} \quad (27)$$

which is equivalent to saying that with probability at least $1 - \frac{\delta}{2}$:

$$\sum_{g \notin \mathcal{J}_0} \|\mathbf{h}_g\|_2 \leq \frac{\alpha}{\alpha - 1} \sum_{g \in \mathcal{J}_0} \|\mathbf{h}_g\|_2 + \frac{\sqrt{s^*}}{\alpha - 1} \|\mathbf{h}_{\mathcal{T}_0}\|_2, \quad (28)$$

that is $\mathbf{h} \in \Omega \left(\mathcal{J}_0, \frac{\alpha}{\alpha-1}, \frac{\sqrt{s^*}}{\alpha-1} \right)$. □

C Proof of Theorem 2

The restricted strong convexity conditions presented in Theorem 2 are a consequence of the following Theorem 5. It derives a control of the supremum of the difference between an empirical random variable and its expectation. This supremum is controlled over a bounded set of sequences of length q of m sparse vectors with disjoint supports.

To motivate this theorem, it helps considering the difference between the usual regression framework and our framework for classification problems. The linear regression case assumes the generative model $\mathbf{y} = \mathbf{X}\beta^* + \epsilon$. Therefore, with the notations of Theorem 5, $\Delta(\beta^*, \mathbf{z}) = \frac{1}{n} \|\mathbf{X}\mathbf{z}\|_2^2 - \frac{2}{n} \epsilon^T \mathbf{X}\mathbf{z}$. By combining a cone condition (similar to Theorem 1) with an upper-bound of the term $\epsilon^T \mathbf{X}\mathbf{z}$, we can obtain a restricted strong convexity similar to Theorem 2. However, in the classification case, β^* is defined as the minimizer of the theoretical risk. Two majors differences appear: (i) we cannot simplify $\Delta(\beta^*, \mathbf{z})$ with basic algebra, (ii) we need to introduce the expectation $\mathbb{E}(\Delta(\beta^*, \mathbf{z}))$ and to control the quantity $|\mathbb{E}(\Delta(\beta^*, \mathbf{z})) - \Delta(\beta^*, \mathbf{z})|$. Theorem 5 helps expliciting the cost to pay for this control.

Theorem 5 We define $\forall \mathbf{w}, \mathbf{z} \in \mathbb{R}^p$:

$$\Delta(\mathbf{w}, \mathbf{z}) = \frac{1}{n} \sum_{i=1}^n f(\langle \mathbf{x}_i, \mathbf{w} + \mathbf{z} \rangle; y_i) - \frac{1}{n} \sum_{i=1}^n f(\langle \mathbf{x}_i, \mathbf{w} \rangle; y_i).$$

Let $k, m, q \in \{1, \dots, p\}$ such that $m \leq k$ and $q \leq \sqrt{k}$. Let S_1, \dots, S_q be partition of $\{1, \dots, p\}$ of size q such that $|S_\ell| \leq m$, $\forall \ell$. We assume that Assumptions 1 and 4.1(k) hold.

Let us note $\tau = \tau(k, m, q) = 14L\mu(k) \sqrt{\frac{\log(7)}{n} + \frac{\log(2q)}{nk} + \frac{\log(2/\delta)}{nk}}$. Then, for any $\delta \in (0, 1)$, it holds with

probability at least $1 - \frac{\delta}{2}$:

$$\sup_{\substack{\mathbf{z}_{S_1}, \dots, \mathbf{z}_{S_q} \in \mathbb{R}^p: \\ \text{Supp}(\mathbf{z}_{S_j}) \subset S_j, \|\mathbf{z}_{S_j}\|_1 \leq 3R, \forall j}} \left\{ \sup_{\ell=1, \dots, q} \{ |\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}) - \mathbb{E}(\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}))| - \tau \|\mathbf{z}_{S_\ell}\|_1 \} \right\} \leq 0,$$

where $\text{Supp}(\cdot)$ refers to the support of a vector and we define $\mathbf{w}_\ell = \boldsymbol{\beta}^* + \sum_{j=1}^{\ell} \mathbf{z}_{S_j}, \forall \ell$.

The proof is presented in Appendix C.1. It uses Hoeffding's inequality to obtain an upper bound of the inner supremum for any sequence of m sparse vectors. The result is extended to the outer supremum with an ϵ -net argument. We first prove Theorem 2 before Theorem 5.

Proof: The proof of Theorem 2 is divided in two steps. First, we lower-bound the quantity $\Delta(\boldsymbol{\beta}^*, \mathbf{h})$ by using a decomposition of $\{1, \dots, p\}$ and applying Theorem 5. Second, we consider the cone conditions derived in Theorem 1 and use the restricted eigenvalue conditions presented in Assumption 4.

Step 1: First, let us fix a partition S_1, \dots, S_q of $\{1, \dots, p\}$ such that $|S_\ell| \leq m, \forall \ell$ and define the corresponding sequence $\mathbf{h}_{S_1}, \dots, \mathbf{h}_{S_q}$ of m sparse vectors corresponding to the decomposition of $\mathbf{h} = \hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*$. We note that:

$$\begin{aligned} \Delta(\boldsymbol{\beta}^*, \mathbf{h}) &= \frac{1}{n} \sum_{i=1}^n f(\langle \mathbf{x}_i, \boldsymbol{\beta}^* + \mathbf{h} \rangle; y_i) - \frac{1}{n} \sum_{i=1}^n f(\langle \mathbf{x}_i, \boldsymbol{\beta}^* \rangle; y_i) \\ &= \frac{1}{n} \sum_{i=1}^n f\left(\langle \mathbf{x}_i, \boldsymbol{\beta}^* + \sum_{j=1}^q \mathbf{h}_{S_j} \rangle; y_i\right) - \frac{1}{n} \sum_{i=1}^n f(\langle \mathbf{x}_i, \boldsymbol{\beta}^* \rangle; y_i) \\ &= \sum_{\ell=1}^q \left\{ \frac{1}{n} \sum_{i=1}^n f\left(\langle \mathbf{x}_i, \boldsymbol{\beta}^* + \sum_{j=1}^{\ell} \mathbf{h}_{S_j} \rangle; y_i\right) - \frac{1}{n} \sum_{i=1}^n f\left(\langle \mathbf{x}_i, \boldsymbol{\beta}^* + \sum_{j=1}^{\ell-1} \mathbf{h}_{S_j} \rangle; y_i\right) \right\} \quad (29) \\ &= \sum_{\ell=1}^q \Delta\left(\boldsymbol{\beta}^* + \sum_{j=1}^{\ell-1} \mathbf{h}_{S_j}, \mathbf{h}_{S_\ell}\right) \\ &= \sum_{\ell=1}^q \Delta(\mathbf{w}_{\ell-1}, \mathbf{h}_{S_\ell}) \end{aligned}$$

where we have defined $\mathbf{w}_\ell = \boldsymbol{\beta}^* + \sum_{j=1}^{\ell} \mathbf{h}_{S_j}, \forall \ell$ and $\mathbf{h}_{S_0} = \mathbf{0}$.

We now consider two such partitions of $\{1, \dots, p\}$ for which we apply Theorem 5. For L1 and Slope regularizations, we fix $k = k^*, m = k^*$, and consider the partition $S_1 = \{1, \dots, k^*\}, S_2 = \{k^* + 1, \dots, 2k^*\}, \dots, S_q, q = \lceil p/k^* \rceil$. It holds $m = k$ and Assumption 5 guarantees $p \leq k^* \sqrt{k^*}$, thus $q \leq \sqrt{k}$. In addition we have:

$$\tau^* = \tau(k^*, k^*, \lceil p/k^* \rceil) = 14L\mu(k^*) \sqrt{\frac{\log(7)}{n} + \frac{\log(2p/k^*)}{nk^*} + \frac{\log(2/\delta)}{nk^*}}.$$

For Group L1-L2 regularization, we fix $k = g_* s^*, m = g_*, q = G$ and we define the partition S_1, \dots, S_G (of size q) as the different groups—hence it holds $|S_\ell| \leq m, \forall \ell$. Assumption 5 guarantees $G \leq \sqrt{g_* s^*}$,

thus $q \leq \sqrt{k}$. In addition, we have:

$$\tau^* = \tau(g_* s^*, g_*, G) = 14L\mu(g_* s^*) \sqrt{\frac{\log(7)}{n} + \frac{\log(2G)}{ng_* s^*} + \frac{\log(2/\delta)}{ng_* s^*}}$$

Consequently, since $\|\mathbf{h}_{S_\ell}\|_1 \geq 3R, \forall \ell$, Theorem 5 guarantees that for all regularization schemes, it holds with probability at least $1 - \frac{\delta}{2}$:

$$|\Delta(\mathbf{w}_{\ell-1}, \mathbf{h}_{S_\ell}) - \mathbb{E}(\mathbf{w}_{\ell-1}, \mathbf{h}_{S_\ell})| \geq \tau^* \|\mathbf{h}_{S_\ell}\|_1, \forall \ell.$$

As a result, following Equation (29), we have:

$$\begin{aligned} \Delta(\boldsymbol{\beta}^*, \mathbf{h}) &\geq \sum_{\ell=1}^q \{\mathbb{E}(\mathbf{w}_{\ell-1}, \mathbf{h}_{S_\ell}) - \tau^* \|\mathbf{h}_{S_\ell}\|_1\} = \mathbb{E} \left(\sum_{\ell=1}^q \Delta(\mathbf{w}_{\ell-1}, \mathbf{h}_{S_\ell}) \right) - \sum_{\ell=1}^q \tau^* \|\mathbf{h}_{S_\ell}\|_1 \\ &= \mathbb{E}(\Delta(\boldsymbol{\beta}^*, \mathbf{h})) - \tau^* \|\mathbf{h}\|_1. \end{aligned} \quad (30)$$

In addition, we have:

$$\mathbb{E}(\Delta(\boldsymbol{\beta}^*, \mathbf{h})) = \frac{1}{n} \sum_{i=1}^n \mathbb{E} \{f(\langle \mathbf{x}_i, \boldsymbol{\beta}^* + \mathbf{h} \rangle; y_i) - f(\langle \mathbf{x}_i, \boldsymbol{\beta}^* \rangle; y_i)\} = \mathcal{L}(\boldsymbol{\beta}^* + \mathbf{h}) - \mathcal{L}(\boldsymbol{\beta}^*).$$

Consequently, we conclude that with probability at least $1 - \frac{\delta}{2}$:

$$\Delta(\boldsymbol{\beta}^*, \mathbf{h}) \geq \mathcal{L}(\boldsymbol{\beta}^* + \mathbf{h}) - \mathcal{L}(\boldsymbol{\beta}^*) - \tau^* \|\mathbf{h}\|_1. \quad (31)$$

Step 2: We now lower-bound the right-hand side of Equation (31). Since \mathcal{L} is twice differentiable, a Taylor development around $\boldsymbol{\beta}^*$ gives:

$$\mathcal{L}(\boldsymbol{\beta}^* + \mathbf{h}) - \mathcal{L}(\boldsymbol{\beta}^*) = \nabla \mathcal{L}(\boldsymbol{\beta}^*)^T \mathbf{h} + \frac{1}{2} \mathbf{h}^T \nabla^2 \mathcal{L}(\boldsymbol{\beta}^*)^T \mathbf{h} + o(\|\mathbf{h}\|_2^2).$$

The optimality of $\boldsymbol{\beta}^*$ implies $\nabla \mathcal{L}(\boldsymbol{\beta}^*) = 0$. In addition, by using Theorem 1, we obtain with probability at least $1 - \frac{\delta}{2}$ that $\mathbf{h} \in \Lambda(S_0, \gamma_1^*, \gamma_2^*)$ for L1 regularization, $\mathbf{h} \in \Gamma(k^*, \omega^*)$ for Slope regularization and $\mathbf{h} \in \Omega(\mathcal{J}_0, \epsilon_1^*, \epsilon_2^*)$ for Group L1-L2 regularization. Consequently, for each regularization, we can use the restricted eigenvalue conditions defined in Assumption 4. However we do not want to keep the term $o(\|\mathbf{h}\|_2^2)$ as it can hide non trivial dependencies.

We use the shorthand κ^* and r^* for the restricted eigenvalue constant and maximum radius introduced in the growth conditions in Assumption 5: $\kappa^* = \kappa(k^*, \gamma_1^*, \gamma_2^*)$ and $r^* = r(k^*, \gamma_1^*, \gamma_2^*)$ for L1 regularization, $\kappa^* = \kappa(k^*, \omega^*)$, $r^* = r(k^*, \omega^*)$ for Slope regularization, $\kappa^* = \kappa(s^*, \epsilon_1^*, \epsilon_2^*)$ and $r^* = r(s^*, \epsilon_1^*, \epsilon_2^*)$ for Group L1-L2 regularization. We consider the two mutually exclusive existing cases separately.

Case 1: If $\|\mathbf{h}\|_2 \leq r^*$, then using Theorem 1 and Assumption 4, it holds with probability at least $1 - \frac{\delta}{2}$:

$$\mathcal{L}(\boldsymbol{\beta}^* + \mathbf{h}) - \mathcal{L}(\boldsymbol{\beta}^*) \geq \frac{1}{4} \kappa^* \|\mathbf{h}\|_2^2. \quad (32)$$

Case 2: If now $\|\mathbf{h}\|_2 \geq r^*$, then using the convexity of \mathcal{L} thus of $t \rightarrow \mathcal{L}(\boldsymbol{\beta}^* + t\mathbf{h})$, we similarly obtain

with the same probability:

$$\begin{aligned}
\mathcal{L}(\beta^* + \mathbf{h}) - \mathcal{L}(\beta^*) &\geq \frac{\|\mathbf{h}\|_2}{r^*} \left\{ \mathcal{L} \left(\beta^* + \frac{r^*}{\|\mathbf{h}\|_2} \mathbf{h} \right) - \mathcal{L}(\beta^*) \right\} \text{ by convexity} \\
&\geq \frac{\|\mathbf{h}\|_2}{r^*} \inf_{\substack{\mathbf{z} \in \Lambda(S_0, \gamma_1^*, \gamma_2^*) \\ \|\mathbf{z}\|_2 = r^*}} \{ \mathcal{L}(\beta^* + \mathbf{z}) - \mathcal{L}(\beta^*) \} \\
&\geq \frac{\|\mathbf{h}\|_2}{r^*} \frac{1}{4} \kappa^* (r^*)^2 = \frac{1}{4} \kappa^* r^* \|\mathbf{h}\|_2.
\end{aligned} \tag{33}$$

where the cone used is for L1 regularization. The same equation holds for Slope and Group L1-L2 regularizations by respectively replacing $\Lambda(S_0, \gamma_1^*, \gamma_2^*)$ with $\Gamma(k^*, \omega^*)$ and $\Omega(\mathcal{J}_0, \epsilon_1^*, \epsilon_2^*)$

Combining Equations (31), (32) and (33), we conclude that with probability at least $1 - \delta$, the following restricted strong convexity holds:

$$\Delta(\mathbf{h}) \geq \frac{1}{4} \kappa^* \|\mathbf{h}\|_2^2 \wedge \frac{1}{4} \kappa^* r^* \|\mathbf{h}\|_2 - \tau^* \|\mathbf{h}\|_1. \tag{34}$$

We now prove Theorem 5. □

C.1 Proof of Theorem 5

Proof: Let $k, m, q \in \{1, \dots, p\}$ be such that $m \leq k, q \leq \sqrt{k}$ and S_1, \dots, S_q be a partition of $\{1, \dots, p\}$ of size q such that $|S_\ell| \leq m, \forall \ell \leq q$. We divide the proof in 3 steps. We first upper-bound the inner supremum for any sequence of m sparse vectors $\mathbf{z}_{S_1}, \dots, \mathbf{z}_{S_q}$. We then extend this bound for the supremum over a compact set of sequences through an ϵ -net argument.

Step 1: Let us fix a sequence $\mathbf{z}_{S_1}, \dots, \mathbf{z}_{S_q} \in \mathbb{R}^p : \text{Supp}(\mathbf{z}_{S_\ell}) \subset S_\ell, \forall \ell$ and $\|\mathbf{z}_{S_\ell}\|_1 \leq 3R, \forall \ell$. In particular, $\|\mathbf{z}_{S_\ell}\|_0 \leq m \leq k, \forall \ell$. In the rest of the proof, we define $\mathbf{z}_{S_0} = \mathbf{0}$ and

$$\mathbf{w}_\ell = \beta^* + \sum_{j=1}^{\ell} \mathbf{z}_{S_j}, \forall \ell, \tag{35}$$

In addition, we introduce $Z_{i\ell}, \forall i, \ell$ as follows

$$Z_{i\ell} = f(\langle \mathbf{x}_i, \mathbf{w}_\ell \rangle; y_i) - f(\langle \mathbf{x}_i, \mathbf{w}_{\ell-1} \rangle; y_i) = f(\langle \mathbf{x}_i, \mathbf{w}_{\ell-1} + \mathbf{z}_{S_\ell} \rangle; y_i) - f(\langle \mathbf{x}_i, \mathbf{w}_{\ell-1} \rangle; y_i).$$

In particular, let us note that:

$$\begin{aligned}
\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}) &= \frac{1}{n} \sum_{i=1}^n f(\langle \mathbf{x}_i, \mathbf{w}_{\ell-1} + \mathbf{z}_{S_\ell} \rangle; y_i) - \frac{1}{n} \sum_{i=1}^n f(\langle \mathbf{x}_i, \mathbf{w}_{\ell-1} \rangle; y_i) \\
&= \frac{1}{n} \sum_{i=1}^n \{ f(\langle \mathbf{x}_i, \mathbf{w}_{\ell-1} + \mathbf{z}_{S_\ell} \rangle; y_i) - f(\langle \mathbf{x}_i, \mathbf{w}_{\ell-1} \rangle; y_i) \} \\
&= \frac{1}{n} \sum_{i=1}^n Z_{i\ell}.
\end{aligned} \tag{36}$$

Assumption 1 guarantees that $f(\cdot, y)$ is L -Lipschitz $\forall y$ then:

$$|Z_{i\ell}| \leq L |\langle \mathbf{x}_i, \mathbf{z}_{S_\ell} \rangle|, \forall i, \ell.$$

Then using Assumption 4.1(k) on the m sparse then k sparse vector \mathbf{z}_{S_ℓ} it holds:

$$|\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell})| \leq \frac{1}{n} \sum_{i=1}^n |Z_{i\ell}| \leq \frac{1}{n} \sum_{i=1}^n L |\langle \mathbf{x}_i, \mathbf{z}_{S_\ell} \rangle| = \frac{L}{n} \|\mathbf{X} \mathbf{z}_{S_\ell}\|_1 \leq \frac{L\mu(k)}{\sqrt{nk}} \|\mathbf{z}_{S_\ell}\|_1, \forall \ell.$$

Hence, with Hoeffding's lemma, the centered bounded random variable $\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}) - \mathbb{E}(\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}))$ is sub-Gaussian with variance $\frac{L^2\mu(k)^2}{nk} \|\mathbf{z}_{S_\ell}\|_1^2$. It then holds, $\forall t > 0$,

$$\mathbb{P}(|\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}) - \mathbb{E}(\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}))| > t \|\mathbf{z}_{S_\ell}\|_1) \leq 2 \exp\left(-\frac{knt^2}{2L^2\mu(k)^2}\right), \forall \ell. \quad (37)$$

Equation (37) holds for all values of ℓ . Thus, an union bound immediately gives:

$$\mathbb{P}\left(\sup_{\ell=1, \dots, q} \{|\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}) - \mathbb{E}(\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}))| - t \|\mathbf{z}_{S_\ell}\|_1\} > 0\right) \leq 2q \exp\left(-\frac{knt^2}{2L^2\mu(k)^2}\right). \quad (38)$$

Step 2: We extend the result to any sequence of vectors $\mathbf{z}_{S_1}, \dots, \mathbf{z}_{S_q} \in \mathbb{R}^p$: $\text{Supp}(\mathbf{z}_{S_\ell}) \subset S_\ell, \forall \ell$ and $\|\mathbf{z}_{S_\ell}\|_1 \leq 3R, \forall \ell$ with an ϵ -net argument.

We recall that an ϵ -net of a set \mathcal{I} is a subset \mathcal{N} of \mathcal{I} such that each element of \mathcal{I} is at a distance at most ϵ of \mathcal{N} . We know from Lemma 1.18 from [28], that for any $\epsilon \in (0, 1)$, the ball $\{\mathbf{z} \in \mathbb{R}^d : \|\mathbf{z}\|_1 \leq R\}$ has an ϵ -net of cardinality $|\mathcal{N}| \leq \left(\frac{2R+1}{\epsilon}\right)^d$ – the ϵ -net is defined in term of L1 norm. In addition, we can create this set such that it contains $\mathbf{0}$.

Consequently, we use Equation (38) on a product of ϵ -nets $\mathcal{N}_{m,R} = \prod_{\ell=1}^q \mathcal{N}_{m,R}^\ell$. Each $\mathcal{N}_{m,R}^\ell$ is an ϵ -net of the bounded sets of m sparse vectors $\mathcal{I}_{m,R}^\ell = \{\mathbf{z}_{S_\ell} \in \mathbb{R}^p : \text{Supp}(\mathbf{z}_{S_\ell}) \subset S_\ell ; \|\mathbf{z}_{S_\ell}\|_1 \leq 3R\}$ which contains $\mathbf{0}_{S_\ell}$. We note $\mathcal{I}_{m,R} = \prod_{\ell=1}^q \mathcal{I}_{m,R}^\ell$. Since $|S_\ell| \leq m, \forall \ell \leq q$, it then holds:

$$\begin{aligned} & \mathbb{P}\left(\sup_{(\mathbf{z}_{S_1}, \dots, \mathbf{z}_{S_q}) \in \mathcal{N}_{m,R}} \left\{ \sup_{\ell=1, \dots, q} \{|\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}) - \mathbb{E}(\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}))| - t \|\mathbf{z}_{S_\ell}\|_1\} > 0 \right\}\right) \\ & \leq 2q \left(\frac{6R+1}{\epsilon}\right)^m q \exp\left(-\frac{knt^2}{2L^2\mu(k)^2}\right) = 2q^2 \left(\frac{6R+1}{\epsilon}\right)^m \exp\left(-\frac{knt^2}{2L^2\mu(k)^2}\right). \end{aligned} \quad (39)$$

Step 3: We now extend Equation (39) to control any vector in $\mathcal{I}_{m,R}$. For $\mathbf{z}_{S_1}, \dots, \mathbf{z}_{S_q} \in \mathcal{I}_{m,R}$, there exists $\tilde{\mathbf{z}}_{S_1}, \dots, \tilde{\mathbf{z}}_{S_q} \in \mathcal{N}_{m,R}$ such that $\|\mathbf{z}_{S_\ell} - \tilde{\mathbf{z}}_{S_\ell}\|_1 \leq \epsilon, \forall \ell$. Similarly to Equation (35), we define:

$$\tilde{\mathbf{w}}_\ell = \boldsymbol{\beta}^* + \sum_{j=1}^{\ell} \tilde{\mathbf{z}}_{S_j}, \forall \ell.$$

For a given t , let us define

$$f_t(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}) = |\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}) - \mathbb{E}(\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}))| - t \|\mathbf{z}_{S_\ell}\|_1, \forall \ell.$$

We fix $\ell_0(t)$ such that $\ell_0 \in \operatorname{argmax}_{\ell=1,\dots,q} \{f_{7t}(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell})\}$. The choice of $7t$ will be justified later. We fix t and

will just note $\ell_0 = \ell_0(t)$ when no confusion can be made.

With Assumption 1 we obtain:

$$\begin{aligned}
& \left| \Delta(\mathbf{w}_{\ell_0-1}, \mathbf{z}_{S_{\ell_0}}) - \Delta(\tilde{\mathbf{w}}_{\ell_0-1}, \tilde{\mathbf{z}}_{S_{\ell_0}}) \right| \\
&= \frac{1}{n} \left| \sum_{i=1}^n f(\langle \mathbf{x}_i, \mathbf{w}_{\ell_0} \rangle; y_i) - \sum_{i=1}^n f(\langle \mathbf{x}_i, \tilde{\mathbf{w}}_{\ell_0} \rangle; y_i) + \sum_{i=1}^n f(\langle \mathbf{x}_i, \tilde{\mathbf{w}}_{\ell_0-1} \rangle; y_i) - \sum_{i=1}^n f(\langle \mathbf{x}_i, \mathbf{w}_{\ell_0-1} \rangle; y_i) \right| \\
&\leq \frac{1}{n} \sum_{i=1}^n L |\langle \mathbf{x}_i, \mathbf{w}_{\ell_0} - \tilde{\mathbf{w}}_{\ell_0} \rangle| + \frac{1}{n} \sum_{i=1}^n L |\langle \mathbf{x}_i, \mathbf{w}_{\ell_0-1} - \tilde{\mathbf{w}}_{\ell_0-1} \rangle| \\
&= \frac{1}{n} \sum_{i=1}^n L \left| \sum_{\ell=1}^{\ell_0} \langle \mathbf{x}_i, \mathbf{z}_{S_\ell} - \tilde{\mathbf{z}}_{S_\ell} \rangle \right| + \frac{1}{n} \sum_{i=1}^n L \left| \sum_{\ell=1}^{\ell_0-1} \langle \mathbf{x}_i, \mathbf{z}_{S_\ell} - \tilde{\mathbf{z}}_{S_\ell} \rangle \right| \\
&\leq \frac{2}{n} \sum_{i=1}^n \sum_{\ell=1}^q L |\langle \mathbf{x}_i, \mathbf{z}_{S_\ell} - \tilde{\mathbf{z}}_{S_\ell} \rangle| \\
&= \frac{2}{\sqrt{n}} \sum_{\ell=1}^q \frac{L}{\sqrt{n}} \|\mathbf{X}(\mathbf{z}_{S_\ell} - \tilde{\mathbf{z}}_{S_\ell})\|_1 \\
&\leq \frac{2}{\sqrt{n}} \sum_{\ell=1}^q \frac{L}{\sqrt{k}} \mu(k) \|\mathbf{z}_{S_\ell} - \tilde{\mathbf{z}}_{S_\ell}\|_1 \\
&\leq \frac{2q}{\sqrt{kn}} L \mu(k) \epsilon = \eta \frac{q}{\sqrt{k}} \epsilon \leq \eta \epsilon.
\end{aligned} \tag{40}$$

where $\eta = \frac{2L\mu(k)}{\sqrt{n}}$ and since $q \leq \sqrt{k}$. It then holds

$$\begin{aligned}
f_t(\tilde{\mathbf{w}}_{\ell_0-1}, \tilde{\mathbf{z}}_{S_{\ell_0}}) &\geq f_t(\mathbf{w}_{\ell_0-1}, \mathbf{z}_{S_{\ell_0}}) - \left| \Delta(\mathbf{w}_{\ell_0-1}, \mathbf{z}_{S_{\ell_0}}) - \Delta(\tilde{\mathbf{w}}_{\ell_0-1}, \tilde{\mathbf{z}}_{S_{\ell_0}}) \right| \\
&\quad - \left| \mathbb{E} \left(\Delta(\mathbf{w}_{\ell_0-1}, \mathbf{z}_{S_{\ell_0}}) - \Delta(\tilde{\mathbf{w}}_{\ell_0-1}, \tilde{\mathbf{z}}_{S_{\ell_0}}) \right) \right| - t \|\mathbf{z}_{S_{\ell_0}} - \tilde{\mathbf{z}}_{S_{\ell_0}}\|_1 \\
&\geq f_t(\mathbf{w}_{\ell_0-1}, \mathbf{z}_{S_{\ell_0}}) - 2\eta\epsilon - t\epsilon.
\end{aligned}$$

Case 1: Let us assume that $\|\mathbf{z}_{S_{\ell_0}}\|_1 \geq \epsilon/2$ and $t \geq \eta$, then we have:

$$f_t(\tilde{\mathbf{w}}_{\ell_0-1}, \tilde{\mathbf{z}}_{S_{\ell_0}}) \geq f_t(\mathbf{w}_{\ell_0-1}, \mathbf{z}_{S_{\ell_0}}) - 2(2\eta + t) \|\tilde{\mathbf{z}}_{S_{\ell_0}}\|_1 \geq f_{7t}(\mathbf{w}_{\ell_0-1}, \mathbf{z}_{S_{\ell_0}}). \tag{41}$$

Case 2: We now assume $\|\mathbf{z}_{S_{\ell_0}}\|_1 \leq \epsilon/2$. Since $\mathbf{0}_{S_{\ell_0}} \in \mathcal{N}_{k,R}$ we derive similarly to Equation (40):

$$\left| \Delta(\mathbf{w}_{\ell_0-1}, \mathbf{z}_{S_{\ell_0}}) - \Delta(\mathbf{w}_{\ell_0-1}, \mathbf{0}_{S_{\ell_0}}) \right| \leq \frac{L\mu(k)}{\sqrt{nk}} \|\mathbf{z}_{S_{\ell_0}}\|_1,$$

which then implies that:

$$f_{7t}(\mathbf{w}_{\ell_0-1}, \mathbf{z}_{S_{\ell_0}}) \leq f_{7t}(\mathbf{w}_{\ell_0-1}, \mathbf{0}_{S_{\ell_0}}) + \frac{2L\mu(k)}{\sqrt{nk}} \|\mathbf{z}_{S_{\ell_0}}\|_1 - 7t \|\mathbf{z}_{S_{\ell_0}}\|_1,$$

and this quantity is smaller than $f_{7t}(\mathbf{w}_{\ell_0-1}, \mathbf{0}_{S_{\ell_0}})$ as long as $7t \geq \frac{2L\mu(k)}{\sqrt{nk}}$, which is true if $t \geq \eta$. In this case, we can define a new $\tilde{\ell}_0$ for the sequence $\mathbf{z}_{S_1}, \dots, \mathbf{z}_{S_{\ell_0-1}}, \mathbf{0}_{S_{\ell_0}}, \mathbf{z}_{S_{\ell_0+1}}, \dots, \mathbf{z}_{S_q}$. After a finite number of iterations, by using the result in Equation (41) and the definition of ℓ_0 , we finally get that $f_{7t}(\mathbf{w}_{\ell_0-1}, \mathbf{z}_{S_{\ell_0}}) \leq f_t(\tilde{\mathbf{w}}_{\ell_0-1}, \tilde{\mathbf{z}}_{S_{\ell_0}})$ for some $\tilde{\mathbf{z}}_{S_1}, \dots, \tilde{\mathbf{z}}_{S_q} \in \mathcal{N}_{m,R}$.

By combining cases 1 and 2, we obtain: $\forall t \geq \eta, \forall \mathbf{z}_{S_1}, \dots, \mathbf{z}_{S_q} \in \mathcal{I}_{m,R}, \exists \tilde{\mathbf{z}}_{S_1}, \dots, \tilde{\mathbf{z}}_{S_q} \in \mathcal{N}_{m,R}$:

$$\sup_{\ell=1, \dots, q} f_{7t}(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}) = f_{7t}(\mathbf{w}_{\ell_0-1}, \mathbf{z}_{S_{\ell_0}}) \leq f_t(\tilde{\mathbf{w}}_{\ell_0-1}, \tilde{\mathbf{z}}_{S_{\ell_0}}) \leq \sup_{\ell=1, \dots, q} f_t(\tilde{\mathbf{w}}_{\ell-1}, \tilde{\mathbf{z}}_{S_\ell}).$$

This last relation is equivalent to saying that $\forall t \geq 7\eta$:

$$\sup_{\mathbf{z}_{S_1}, \dots, \mathbf{z}_{S_q} \in \mathcal{I}_{m,R}} \left\{ \sup_{\ell=1, \dots, q} f_t(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}) \right\} \leq \sup_{\mathbf{z}_{S_1}, \dots, \mathbf{z}_{S_q} \in \mathcal{N}_{m,R}} \left\{ \sup_{\ell=1, \dots, q} f_{t/7}(\tilde{\mathbf{w}}_{\ell-1}, \tilde{\mathbf{z}}_{S_\ell}) \right\}. \quad (42)$$

As a consequence, we have $\forall t \geq 7\eta$:

$$\begin{aligned} & \mathbb{P} \left(\sup_{\mathbf{z}_{S_1}, \dots, \mathbf{z}_{S_q} \in \mathcal{I}_{m,R}} \sup_{\ell=1, \dots, q} \{ |\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}) - \mathbb{E}(\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}))| - t \|\mathbf{z}_{S_\ell}\|_1 \} > 0 \right) \\ & \leq \mathbb{P} \left(\sup_{\mathbf{z}_{S_1}, \dots, \mathbf{z}_{S_q} \in \mathcal{N}_{m,R}} \sup_{\ell=1, \dots, q} \left\{ |\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}) - \mathbb{E}(\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}))| - \frac{t}{7} \|\mathbf{z}_{S_\ell}\|_1 \right\} > 0 \right) \\ & \leq 2q^2 \left(\frac{6R+1}{\epsilon} \right)^m \exp \left(-\frac{kn(t/7)^2}{2L^2\mu(k)^2} \right) \\ & \leq (2q)^2 7^m \exp \left(-\frac{knt^2}{98L^2\mu(k)^2} \right) \text{ by fixing } \epsilon = R. \end{aligned} \quad (43)$$

Thus we select t such that $t \geq 7\eta$ and that $t^2 \geq \frac{98L^2\mu(k)^2}{2kn} [m \log(7) + 2 \log(2q) + \log(\frac{2}{\delta})]$ holds. To this end, since $m \leq k$, we define:

$$\tau = \tau(k, m, q) = 14L\mu(k) \sqrt{\frac{\log(7)}{n} + \frac{\log(2q) + \log(2/\delta)}{nk}} \geq 7\eta.$$

We conclude that with probability at least $1 - \frac{\delta}{2}$:

$$\sup_{\mathbf{z}_{S_1}, \dots, \mathbf{z}_{S_q} \in \mathcal{I}_{m,R}} \left\{ \sup_{\ell=1, \dots, q} \{ |\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}) - \mathbb{E}(\Delta(\mathbf{w}_{\ell-1}, \mathbf{z}_{S_\ell}))| - \tau \|\mathbf{z}_{S_\ell}\|_1 \} \right\} \leq 0.$$

□

D Proof of Theorem 3

Proof: We now prove our main Theorem 3 for the three regularizations considered.

L1 regularization: For L1 regularization, we have proved in Theorem 1 that $\mathbf{h} = \hat{\beta}_1 - \beta^* \in \Lambda(S_0, \gamma_1^*, \gamma_2^*)$ where S_0 has been defined as the subset of the k^* highest elements of \mathbf{h} . We have defined $\kappa^* = \kappa(k^*, \gamma_1^*, \gamma_2^*)$, $r^* = r(k^*, \gamma_1^*, \gamma_2^*)$ and $\tau^* = \tau(k^*, k^*, \lceil p/k^* \rceil)$.

Since $\mu(k^*) \leq 2\alpha M$, then $14L\mu(k^*)\sqrt{\frac{\log(7)}{n} + \frac{\log(2p/k^*) + \log(2/\delta)}{nk^*}} \leq 34\alpha LM\sqrt{\frac{\log(2e)\log(2/\delta)}{n}}$, hence we have $\tau^* \leq \eta\lambda_p^{(p)} \leq \eta\lambda_{k^*}^{(p)} = \lambda$ —where $\lambda_j^{(r)} = \sqrt{\log(2re/j)}$.

Pairing Equation (15) and the restricted strong convexity derived in Theorem 2, it holds with probability at least $1 - \delta$:

$$\begin{aligned}
\frac{1}{4}\kappa^* \{ \|\mathbf{h}\|_2^2 \wedge r^* \|\mathbf{h}\|_2 \} &\leq \tau^* \|\mathbf{h}\|_1 + \lambda \|\mathbf{h}_{S^*}\|_1 - \lambda \|\mathbf{h}_{(S^*)^c}\|_1 \\
&= \tau^* \|\mathbf{h}_{S_0}\|_1 + \tau^* \|\mathbf{h}_{(S_0)^c}\|_1 + \lambda \sqrt{k^*} \|\mathbf{h}_{S_0}\|_2 - \lambda \|\mathbf{h}_{(S^*)^c}\|_1 \\
&\leq \tau^* \|\mathbf{h}_{S_0}\|_1 + \lambda \sqrt{k^*} \|\mathbf{h}_{S_0}\|_2 \text{ since } \tau^* \leq \lambda \\
&\leq (\tau^* + \lambda) \sqrt{k^*} \|\mathbf{h}_{S_0}\|_2 \text{ from Cauchy-Schwartz inequality} \\
&\leq (\tau^* + \lambda) \sqrt{k^*} \|\mathbf{h}\|_2.
\end{aligned} \tag{44}$$

It then holds with probability at least $1 - \delta$:

$$\frac{1}{4}\kappa^* \{ \|\mathbf{h}\|_2 \wedge r^* \} (\tau^* + \lambda) \sqrt{k^*}$$

Exploiting Assumption 5.1(p, k^*, α, δ), and using the definitions of λ and τ^* as in Theorems 1 and 5, Equation (44) leads to:

$$\frac{1}{4}\kappa^* \|\mathbf{h}\|_2 \leq 34\alpha LM \sqrt{\frac{k^* \log(2pe/k^*)}{n} \log(2/\delta)} + 14L\mu(k^*) \sqrt{\frac{k^* \log(7)}{n} + \frac{\log(2p/k^*)}{n} + \frac{\log(2/\delta)}{n}}.$$

Hence we obtain with probability at least $1 - \delta$:

$$\|\mathbf{h}\|_2^2 \lesssim \left(\frac{\alpha LM}{\kappa^*} \right)^2 \frac{k^* \log(p/k^*) \log(2/\delta)}{n} + \left(\frac{L\mu(k^*)}{\kappa^*} \right)^2 \frac{k^* + \log(p/k^*) + \log(2/\delta)}{n}.$$

which concludes the proof.

Slope regularization: For Slope regularization, the cone condition derived in Theorem 1 gives $\mathbf{h} = \hat{\beta}_S - \beta^* \in \Gamma(k^*, \omega^*)$. In addition, we have defined $\kappa^* = \kappa(k^*, \omega^*)$, $r^* = r(k^*, \omega^*)$ and $\tau^* = \tau(k^*, k^*, \lceil p/k^* \rceil)$. Similarly to above, we denote S_0 the subset of the k^* highest elements of \mathbf{h} , and note $\lambda_j = \lambda_j^{(p)}$ where we drop the dependency upon p .

Pairing Equation (15) and the restricted strong convexity derived in Theorem 2, we obtain with probability at least $1 - \delta$:

$$\begin{aligned}
\frac{1}{4}\kappa^* \{ \|\mathbf{h}\|_2^2 \wedge r^* \|\mathbf{h}\|_2 \} &\leq \tau^* \|\mathbf{h}\|_1 + \eta \sum_{j=1}^{k^*} \lambda_j |h_j| - \eta \sum_{j=k^*+1}^p \lambda_j |h_j| \\
&\leq \tau^* \|\mathbf{h}_{S_0}\|_1 + \eta \sum_{j=1}^{k^*} \lambda_j |h_j| + \tau^* \|\mathbf{h}_{(S_0)^c}\|_1 - \eta \sum_{j=k^*+1}^p \lambda_j |h_j| \\
&\leq \tau^* \|\mathbf{h}_{S_0}\|_1 + \eta \sum_{j=1}^{k^*} \lambda_j |h_j| \text{ since } \tau \leq \eta\lambda_p.
\end{aligned} \tag{45}$$

Hence by using Cauchy-Schwartz inequality we obtain:

$$\begin{aligned} \frac{1}{4}\kappa^* \{ \|\mathbf{h}\|_2^2 \wedge r^* \|\mathbf{h}\|_2 \} &\leq \tau^* \sqrt{k^*} \|\mathbf{h}_{S_0}\|_2 + \eta \sqrt{k^* \log(2pe/k^*)} \|\mathbf{h}_{S_0}\|_2 \\ &= \tau^* \sqrt{k^*} \|\mathbf{h}_{S_0}\|_2 + \lambda \sqrt{k^*} \|\mathbf{h}_{S_0}\|_2 \\ &\leq (\tau^* + \lambda) \sqrt{k^*} \|\mathbf{h}\|_2, \end{aligned}$$

which is equivalent to Equation (44). We conclude the proof as above by exploiting Assumption 5.2(p, k^*, α, δ).

Group L1-L2 regularization: For Group L1-L2 regularization, the cone condition proved in Theorem 1 gives $\mathbf{h} = \hat{\beta}_{L1-L2} - \beta^* \in \Omega \left(\mathcal{J}_0, \epsilon_1^* = \frac{\alpha}{\alpha-1}, \epsilon_2^* = \frac{\sqrt{s^*}}{\alpha-1} \right)$, where \mathcal{J}_0 has been defined as the subset of s^* groups with highest L2 norm. We have defined $\kappa^* = \kappa(s^*, \epsilon_1^*, \epsilon_2^*)$, $r^* = r(s^*, \epsilon_1^*, \epsilon_2^*)$ and $\tau^* = \tau(g_* s^*, g_*, G) = 14LM\mu(g_* s^*) \sqrt{\frac{\log(7)}{n} + \frac{\log(2G) + \log(2/\delta)}{ng_* s^*}}$.

In particular, since we have defined $\lambda_G = \lambda_{s^*}^{(G)} + 4\alpha LM \sqrt{\gamma(s^* n)^{-1} m_*} = 34\alpha LM \sqrt{\frac{\log(2Ge/s^*)}{n} \log(2/\delta) + 4\alpha LM \sqrt{\gamma(s^* n)^{-1} m_*}}$ and we have assumed $\mu(g_* s^*) \leq 2\alpha M \sqrt{s^*}$, it then holds $\tau^* \leq \lambda_G$.

Pairing Equation (15) and the restricted strong convexity derived in Theorem 2, we obtain with probability at least $1 - \delta$:

$$\begin{aligned} \frac{1}{4}\kappa^* \{ \|\mathbf{h}\|_2^2 \wedge r^* \|\mathbf{h}\|_2 \} &\leq \tau^* \|\mathbf{h}\|_1 + \lambda_G \sum_{g \in \mathcal{J}^*} \|\beta_g^*\|_2 - \lambda_G \sum_{g \in \mathcal{J}^*} \|\hat{\beta}_g\|_2 - \lambda_G \sum_{g \notin \mathcal{J}^*} \|\hat{\beta}_g\|_2 \\ &\leq \tau^* \sum_{g=1}^G \|\mathbf{h}_g\|_1 + \lambda_G \sum_{g \in \mathcal{J}^*} \|\mathbf{h}_g\|_2 - \lambda_G \sum_{g \notin \mathcal{J}^*} \|\mathbf{h}_g\|_2 \\ &\leq \tau^* \sum_{g \in \mathcal{J}^*} \|\mathbf{h}_g\|_1 + \lambda_G \sum_{g \in \mathcal{J}^*} \|\mathbf{h}_g\|_2 + \tau^* \sum_{g \notin \mathcal{J}^*} \|\mathbf{h}_g\|_1 - \lambda_G \sum_{g \notin \mathcal{J}^*} \|\mathbf{h}_g\|_1 \end{aligned} \quad (46)$$

Since $\tau^* \leq \lambda_G$, we then have with probability at least $1 - \delta$:

$$\begin{aligned} \frac{1}{4}\kappa^* \{ \|\mathbf{h}\|_2^2 \wedge r^* \|\mathbf{h}\|_2 \} &\leq \tau^* \sum_{g \in \mathcal{J}^*} \|\mathbf{h}_g\|_1 + \lambda_G \sum_{g \in \mathcal{J}^*} \|\mathbf{h}_g\|_2 \\ &\leq \tau^* \sqrt{m^*} \|\mathbf{h}_{\mathcal{T}^*}\|_2 + \lambda_G \sum_{g \in \mathcal{J}^*} \|\mathbf{h}_g\|_2, \end{aligned} \quad (47)$$

where we have used Cauchy-Schwartz inequality and denoted $\mathcal{T}^* = \cup_{g \in \mathcal{J}^*} \mathcal{I}_g$ the subset of size m^* of all indexes across all the s^* groups in \mathcal{J}^* . In addition, Cauchy-Schwartz inequality also leads to: $\sum_{g \in \mathcal{J}^*} \|\mathbf{h}_g\|_2 \leq \sqrt{s^*} \|\mathbf{h}_{\mathcal{T}_0}\|_2$ since \mathcal{J}^* is of size s^* . Hence it holds with probability at least $1 - \delta$:

$$\frac{1}{4}\kappa^* \{ \|\mathbf{h}\|_2^2 \wedge r^* \|\mathbf{h}\|_2 \} \leq (\tau^* \sqrt{m^*} + \lambda_G \sqrt{s^*}) \|\mathbf{h}_{\mathcal{T}^*}\|_2 \leq (\tau^* \sqrt{m^*} + \lambda_G \sqrt{s^*}) \|\mathbf{h}\|_2. \quad (48)$$

Then, by using Assumption 5.3($G, g_*, s^*, m^*, \alpha, \delta$) and the fact that $m^* \leq s^* g_*$, we obtain with probability at least $1 - \delta$:

$$\|\mathbf{h}\|_2^2 \lesssim \left(\frac{\alpha LM}{\kappa^*} \right)^2 \frac{s^* \log(G/s^*) \log(2/\delta) + \gamma m^*}{n} + \left(\frac{L\mu(g_* s^*)}{\kappa^*} \right)^2 \frac{m^* + \log(G) + \log(2/\delta)}{n}.$$

□

E Proof of Corollary 1

Proof: In order to derive the bound in expectation, we define the bounded random variable:

$$Z = \frac{\kappa^{*2}}{L^2} \|\hat{\beta} - \beta^*\|_2^2,$$

where κ^* depends upon the regularization used. We assume that Assumptions 5.1(p, k^*, α, δ), 5.2(p, k^*, α, δ) and 5.3($G, g_*, s^*, m^*, \alpha, \delta$) are satisfied for a small enough δ_0 in the respective cases of the L1, Slope and Group L1-L2 regularizations. Hence can fix $C_0 > 0$ such that $\forall \delta \in (0, 1)$, it holds with probability at least $1 - \delta$:

$$Z \leq C_0 H_1 \log(2/\delta) + C_0 H_2,$$

where $H_1 = n^{-1}(\alpha^2 M^2 k^* \log(p/k^*) + \mu(k^*)^2)$ and $H_2 = n^{-1} \mu(k^*)^2 (k^* + \log(p/k^*))$ for L1 and Slope regularizations.

Similarly $H_1 = n^{-1}(\alpha^2 M^2 s^* \log(G/s^*) + \mu(g_* s^*)^2)$ and $H_2 = n^{-1} \alpha^2 \gamma m^* + n^{-1} \mu(g_* s^*)^2 (m^* + \log(G))$ for Group L1-L2 regularization.

Then it holds $\forall t \geq t_0 = \log(4)$:

$$\mathbb{P}(Z/C_0 \geq H_1 t + H_2) \leq 2e^{-t}.$$

Let $q_0 = H_1 t_0$, then $\forall q \geq q_0$

$$\mathbb{P}(Z/C_0 \geq q + H_2) \leq 2 \exp\left(-\frac{q}{H_1}\right).$$

Consequently, by integration we have:

$$\begin{aligned} \mathbb{E}(Z) &= \int_0^{+\infty} C_0 \mathbb{P}(|Z|/C_0 \geq q) dq \\ &\leq \int_{H_2+q_0}^{+\infty} C_0 \mathbb{P}(|Z|/C_0 \geq q) dq + C_0(H_2 + q_0) \\ &\leq \int_{q_0}^{+\infty} C_0 \mathbb{P}(|Z|/C_0 \geq q + H_2) dq + C_0(H_2 + q_0) \\ &\leq \int_{q_0}^{+\infty} 2C_0 \exp\left(-\frac{q}{H_1}\right) dq + C_0 H_2 + C_0 H_1 t_0 \\ &\leq 2C_0 H_1 \exp\left(-\frac{q_0}{H_1}\right) + C_0 H_2 + C_0 H_1 \log(4) \\ &\leq C_1 (H_1 + H_2) \end{aligned} \tag{49}$$

for $C_1 = 2C_0 + \log(4)$. Hence we derive

$$\mathbb{E}\|\hat{\beta} - \beta^*\|_2^2 \lesssim \left(\frac{L}{\kappa^*}\right)^2 (H_1 + H_2),$$

which for L1 and Slope regularizations, can be expressed as:

$$\mathbb{E}\|\hat{\boldsymbol{\beta}}_{1,S} - \boldsymbol{\beta}^*\|_2 \lesssim \left(\frac{L}{\kappa^*}\right)^2 \left(\alpha^2 M^2 \frac{k^* \log(p/k^*)}{n} + \mu(k^*)^2 \frac{k^* + \log(p/k^*)}{n} \right),$$

and in the case of Group L1-L2 regularization, can be expressed as

$$\mathbb{E}\|\hat{\boldsymbol{\beta}}_{L1-L2} - \boldsymbol{\beta}^*\|_2 \lesssim \left(\frac{L}{\kappa^*}\right)^2 \left(\alpha^2 M^2 \frac{s^* \log(G/s^*) + \gamma m^*}{n} + \mu(g_* s^*)^2 \frac{m^* + \log(G)}{n} \right).$$

□

F Proof of Theorem 4

Proof: We fix $\tau > 0$ and denote $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_p) \in \mathbb{R}^{n \times p}$ the design matrix.

For $\boldsymbol{\beta} \in \mathbb{R}^p$, we define $\mathbf{w}^\tau(\boldsymbol{\beta}) \in \mathbb{R}^n$ by:

$$w_i^\tau(\boldsymbol{\beta}) = \min\left(1, \frac{1}{2\tau}|z_i|\right) \text{sign}(z_i), \quad \forall i$$

where $z_i = 1 - y_i \mathbf{x}_i^T \boldsymbol{\beta}$, $\forall i$. We easily check that

$$\mathbf{w}^\tau(\boldsymbol{\beta}) = \underset{\|w\|_\infty \leq 1}{\text{argmax}} \frac{1}{2n} \sum_{i=1}^n (z_i + w_i z_i) - \frac{\tau}{2n} \|w\|_2^2.$$

Then the gradient of the smooth hinge loss is

$$\nabla g^\tau(\boldsymbol{\beta}) = -\frac{1}{2n} \sum_{i=1}^n (1 + w_i^\tau(\boldsymbol{\beta})) y_i \mathbf{x}_i \in \mathbb{R}^p.$$

For every couple $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^p$ we have:

$$\nabla g^\tau(\boldsymbol{\beta}) - \nabla g^\tau(\boldsymbol{\gamma}) = \frac{1}{2n} \sum_{i=1}^n (w_i^\tau(\boldsymbol{\gamma}) - w_i^\tau(\boldsymbol{\beta})) y_i \mathbf{x}_i. \quad (50)$$

For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ we define the vector $\mathbf{a} * \mathbf{b} = (a_i b_i)_{i=1}^n$. Then we can rewrite Equation (50) as:

$$\nabla g^\tau(\boldsymbol{\beta}) - \nabla g^\tau(\boldsymbol{\gamma}) = \frac{1}{2n} \mathbf{X}^T [\mathbf{y} * (\mathbf{w}^\tau(\boldsymbol{\gamma}) - \mathbf{w}^\tau(\boldsymbol{\beta}))]. \quad (51)$$

The operator norm associated to the Euclidean norm of the matrix \mathbf{X} is $\|\mathbf{X}\| = \max_{\|z\|_2=1} \|\mathbf{X}z\|_2$.

Let us recall that $\|\mathbf{X}\|^2 = \|\mathbf{X}^T\|^2 = \|\mathbf{X}^T \mathbf{X}\| = \mu_{\max}(\mathbf{X}^T \mathbf{X})$ corresponds to the highest eigenvalue of the matrix $\mathbf{X}^T \mathbf{X}$. Consequently, Equation (51) leads to:

$$\|\nabla L^\tau(\boldsymbol{\beta}) - \nabla L^\tau(\boldsymbol{\gamma})\|_2 \leq \frac{1}{2n} \|\mathbf{X}\| \|\mathbf{w}^\tau(\boldsymbol{\gamma}) - \mathbf{w}^\tau(\boldsymbol{\beta})\|_2. \quad (52)$$

In addition, the first order necessary conditions for optimality applied to $\mathbf{w}^\tau(\boldsymbol{\beta})$ and $\mathbf{w}^\tau(\boldsymbol{\gamma})$ give:

$$\sum_{i=1}^n \left\{ \frac{1}{2n} (1 - y_i \mathbf{x}_i^T \boldsymbol{\beta}) - \frac{\tau}{n} w_i^\tau(\boldsymbol{\beta}) \right\} \{w_i^\tau(\boldsymbol{\gamma}) - w_i^\tau(\boldsymbol{\beta})\} \leq 0, \quad (53)$$

$$\sum_{i=1}^n \left\{ \frac{1}{2n} (1 - y_i \mathbf{x}_i^T \boldsymbol{\gamma}) - \frac{\tau}{n} w_i^\tau(\boldsymbol{\gamma}) \right\} \{w_i^\tau(\boldsymbol{\beta}) - w_i^\tau(\boldsymbol{\gamma})\} \leq 0. \quad (54)$$

Then by adding Equations (53) and (54) and rearranging the terms we have:

$$\begin{aligned} \tau \|\mathbf{w}^\tau(\boldsymbol{\gamma}) - \mathbf{w}^\tau(\boldsymbol{\beta})\|_2^2 &\leq \frac{1}{2} \sum_{i=1}^n y_i \mathbf{x}_i^T (\boldsymbol{\beta} - \boldsymbol{\gamma}) (w_i^\tau(\boldsymbol{\gamma}) - w_i^\tau(\boldsymbol{\beta})) \\ &\leq \frac{1}{2} \|\mathbf{X} (\boldsymbol{\beta} - \boldsymbol{\gamma})\|_2 \|\mathbf{w}^\tau(\boldsymbol{\gamma}) - \mathbf{w}^\tau(\boldsymbol{\beta})\|_2 \\ &\leq \frac{1}{2} \|\mathbf{X}\| \|\boldsymbol{\beta} - \boldsymbol{\gamma}\|_2 \|\mathbf{w}^\tau(\boldsymbol{\gamma}) - \mathbf{w}^\tau(\boldsymbol{\beta})\|_2, \end{aligned}$$

where we have used Cauchy-Schwartz inequality. We then have:

$$\|\mathbf{w}^\tau(\boldsymbol{\gamma}) - \mathbf{w}^\tau(\boldsymbol{\beta})\|_2 \leq \frac{1}{2\tau} \|\mathbf{X}\| \|\boldsymbol{\beta} - \boldsymbol{\gamma}\|_2. \quad (55)$$

We conclude the proof by combining Equations (52) and (55):

$$\begin{aligned} \|\nabla L^\tau(\boldsymbol{\beta}) - \nabla L^\tau(\boldsymbol{\gamma})\|_2 &\leq \frac{1}{4n\tau} \|\mathbf{X}\|^2 \|\boldsymbol{\beta} - \boldsymbol{\gamma}\|_2 \\ &= \frac{\mu_{\max}(n^{-1} \mathbf{X}^T \mathbf{X})}{4\tau} \|\boldsymbol{\beta} - \boldsymbol{\gamma}\|_2. \end{aligned}$$

□

G LP formulations for Section 3.4

We present below LP formulations for the LP problems studied in the computational experiments presented in Section 3.4. These formulations allows us to leverage the efficiency of modern commercial LP solvers as we solve these problems using GUROBI version 6.5 with Python interface.

G.1 LP formulation for L1-SVM

We first consider L1 regularized SVM Problem (3) when f is the hinge loss. This problem can be expressed as the following LP:

$$\begin{aligned} \min_{\boldsymbol{\xi} \in \mathbb{R}^n, \boldsymbol{\beta}^+, \boldsymbol{\beta}^- \in \mathbb{R}^p} \quad & \sum_{i=1}^n \xi_i + \lambda \sum_{j=1}^p \beta_j^+ + \lambda \sum_{j=1}^p \beta_j^- \\ \text{s.t.} \quad & \xi_i + y_i \mathbf{x}_i^T \boldsymbol{\beta}^+ - y_i \mathbf{x}_i^T \boldsymbol{\beta}^- \geq 1 \quad i \in [n] \\ & \boldsymbol{\xi} \geq 0, \boldsymbol{\beta}^+ \geq 0, \boldsymbol{\beta}^- \geq 0. \end{aligned} \quad (56)$$

G.2 LP formulation for Group L1- L_∞ SVM

The Group L1-L2 regularization considered in Problem (5) has a popular alternative, namely the Group L1- L_∞ penalty [1], which considers the L_∞ norm over the groups. Using this regularization, Problem (2) becomes

$$\min_{\beta \in \mathbb{R}^p: \|\beta\|_1 \leq 2R} \frac{1}{n} \sum_{i=1}^n f(\langle \mathbf{x}_i, \beta \rangle; y_i) + \lambda \sum_{g=1}^G \|\beta_g\|_\infty. \quad (57)$$

When f is the hinge-loss, Problem (57) can be expressed as an LP. To this end, we introduce the variables $\mathbf{v} = (v_g)_{g \in [G]}$ such that v_g refers to the L_∞ norm of the coefficients β_g . Problem (57) can be reformulated as:

$$\begin{aligned} \min_{\xi \in \mathbb{R}^n, \beta^+, \beta^- \in \mathbb{R}^p, \mathbf{v} \in \mathbb{R}^G} \quad & \sum_{i=1}^n \xi_i + \lambda \sum_{g=1}^G v_g \\ \text{s.t.} \quad & \xi_i + y_i \mathbf{x}_i^T \beta^+ - y_i \mathbf{x}_i^T \beta^- \geq 1 \quad i \in [n] \\ & v_g - \beta_j^+ - \beta_j^- \geq 0 \quad j \in \mathcal{I}_g, g \in [G] \\ & \xi \geq 0, \beta^+ \geq 0, \beta^- \geq 0, \mathbf{v} \geq 0. \end{aligned} \quad (58)$$

We solve Problem (58) with Gurobi in our experiments. When f is the logistic loss, a proximal operator for Group L1- L_∞ can be derived [1] using the Moreau decomposition presented in Section 3.2.

G.3 LP formulation for L1 regularized least-angle deviation loss

Finally, when f is the least-angle deviation loss [33] and $\Omega(\cdot)$ is the L1 regularization, Problem (2) is expressed as:

$$\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^n |y_i - \mathbf{x}_i^T \beta| + \lambda \|\beta\|_1, \quad (59)$$

An LP formulation for Problem (59) is:

$$\begin{aligned} \min_{\xi \in \mathbb{R}^n, \beta^+, \beta^- \in \mathbb{R}^p, \mathbf{v} \in \mathbb{R}^G} \quad & \sum_{i=1}^n \xi_i + \lambda \sum_{j=1}^p \beta_j^+ + \lambda \sum_{j=1}^p \beta_j^- \\ \text{s.t.} \quad & \xi_i \geq y_i - \mathbf{x}_i^T \beta^+ + \mathbf{x}_i^T \beta^- \quad i \in [n] \\ & \xi_i \geq \mathbf{x}_i^T \beta^+ - \mathbf{x}_i^T \beta^- - y_i \quad i \in [n] \\ & \xi \geq 0, \beta^+ \geq 0, \beta^- \geq 0. \end{aligned} \quad (60)$$

Specific linear optimization techniques could be used for efficiently solving all three LP Problems (56), (58) and (60). For instance, [14] recently combined first order methods with column-and-constraint generation algorithms to solve Problem (2) when f is the hinge-loss and $\Omega(\cdot)$ is the L1, Slope or Group L1- L_∞ regularization.

H Additional experiments for Section 3.4

The next Figure 2 presents the two additional experiments described in Section H. It considers Examples 2 and 3 when the statistical settings are simpler than the ones in Figure 1—we respectively use a higher δ and a higher SNR .

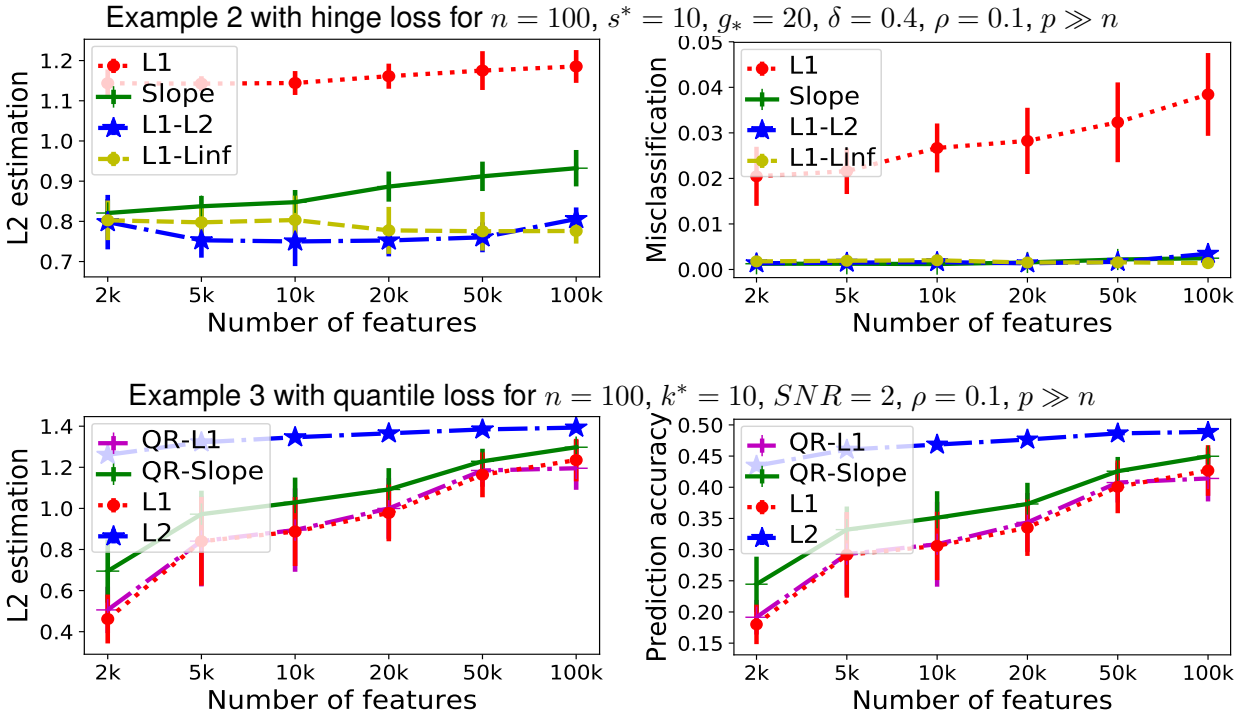


Figure 2: [Top panel] Slope can compete with group regularizations when the distance between the two classes increases. However the gap in performance greatly increases for large values of p . [Bottom panel] When the SNR increases, Slope performance slightly decreases while L1 regularized quantile regression and Lasso exhibit very similar behaviors.